Modification of Griffiths' result for even integers

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Abstract

For a finite abelian group G with $\exp(G) = n$, the arithmetical invariant $\mathsf{s}_A(G)$ is defined to be the least integer k such that any sequence S with length k of elements in G has a A weighted zero-sum subsequence of length n. When $A = \{1\}$, it is *the Erdős-Ginzburg-Ziv constant* and is denoted by $\mathsf{s}(G)$. For certain class of sets A, we already have some general bounds for these weighted constants corresponding to the cyclic group $\mathbb{Z}_n$, which was given by Griffiths. For odd integer n, Adhikari and Mazumdar generalized the above mentioned results in the sense that they hold for more sets A. In the present paper we modify Griffiths' method for even n and obtain general bound for the weighted constants for certain class of weighted sets which include sets that were not covered by Griffiths for $n \equiv 0 \pmod 4$.

Keywords: the zero-sum problem; Kneser's theorem.

1 Introduction

Let G be a finite abelian group (written additively). By a sequence over G we mean a finite sequence of terms from G which is unordered and repetition of terms is allowed and we view sequences over G as elements of the free abelian monoid $\mathcal{F}(G)$ and use multiplicative notation. So, our notation is consistent with [8], [9] and [11].

For $S \in \mathcal{F}(G)$, if

$$S = x_1 x_2 \cdots x_t = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

then $\mathsf{v}_g(S) \geqslant 0$ is the *multiplicity* of g in S, and

$$|S| = t = \sum_{g \in G} \mathsf{v}_g(S)$$

is the *length* of S. The sequence S *contains* some $g \in G$ if $\mathsf{v}_g(S) \geqslant 1$. If S and T are sequences over G, then T is said to be a *subsequence* of S if $\mathsf{v}_g(T) \leqslant \mathsf{v}_g(S)$ for every $g \in G$.

For a non-empty subset A of $\{1, 2, \ldots, n-1\}$, where n is the exponent of G (denoted by $\exp(G)$), a sequence $S = x_1 x_2 \cdots x_t$ of length t over G is said to be an *A-weighted zero-sum sequence,* if there exists $\bar{a} = (a_1, a_2, \ldots, a_t) \in A^t$ such that $\sum_{i=1}^{t} a_i x_i = 0$.

For integers $m < n$, we shall use the notation $[m, n]$ to denote the set $\{m, m+1, \ldots, n\}$. For a finite set A, we denote its size by $|A|$, which is the number of elements of A. If G is a finite abelian group with $\exp(G) = n$, then for a non-empty subset A of $[1, n-1]$, one defines $\mathsf{s}_A(G)$ to be the least integer k such that any sequence S with length k of elements in G has an *A-weighted zero-sum subsequence* of length $\exp(G) = n$. Taking $A = \{1\}$, one recovers the classical Erdős-Ginzburg-Ziv constant $\mathsf{s}(G)$. The above weighted versions and some other invariants with weights were introduced by Adhikari, Chen, Friedlander, Konyagin and Pappalardi [3], Adhikari and Chen [2] and Adhikari, Balasubramanian, Pappalardi and Rath [1]. For developments regarding bounds on the constant $\mathsf{s}_A(G)$ in the case of abelian groups G with higher rank and related references, we refer to the recent paper of Adhikari, Grynkiewicz and Sun [5].

When $A = \mathbb{Z}_n^* = \{a \in [1, n-1] | (a, n) = 1\}$, the set of units of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, Luca [13] and Griffiths [10] proved independently the following result which had been conjectured in [3]:

$$\mathsf{s}_A(\mathbb{Z}_n) \leqslant n + \Omega(n), \tag{1}$$

where $\Omega(n)$ denotes the number of prime factors of n, counted with multiplicity. An example in [3] had already established the inequality in the other direction:

$$\mathsf{s}_A(\mathbb{Z}_n) \geqslant n + \Omega(n).$$

Now we state the following results of Griffiths [10] which generalizes result (1) for integer n:

Theorem 1. *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with its size $|A_s| > p_s^{a_s}/2$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > a$, every sequence $x_1 \cdots x_{m+a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

Theorem 2. *Let $n = 2^{a_1} \cdots p_k^{a_k}$ be an even integer and let $a = \sum_s a_s$. Let $A_1 \subset \mathbb{Z}_{2^{a_1}}$ be such that $|A_1| > 2^{a_1-1}$ or $|A_1| > 2^{a_1-2}$ and $A_1 \subset \mathbb{Z}_{2^{a_1}}^*$. For each $s \geqslant 2$, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with $|A_s| > (1/2)p_s^{a_s}$, and let $A = A_1 \times \cdots \times A_k$. Then for any even m, every sequence $x_1 \cdots x_{m+a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

For odd integer n with suitable modifications in the method of Griffiths [10], Adhikari and Mazumdar established the following result in [6]:

Theorem 3. *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with $|A_s| > (4/9)p_s^{a_s}$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > 2a$, every sequence $x_1 \cdots x_{m+2a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

For general n, consider the set A of squares in the group of units in the cyclic group $\mathbb{Z}_n$, it was proved by Adhikari, Chantal David and Urroz [4] that if n is a square-free integer, coprime to 6, then

$$\mathsf{s}_A(\mathbb{Z}_n) = n + 2\Omega(n). \tag{2}$$

Later, removing the requirement that n is square-free, Chintamani and Moriya [7] showed that if n is a power of 3 or n is coprime to $30 = 2 \times 3 \times 5$, then the result (2) holds, where A is again the set of squares in the group of units in $\mathbb{Z}_n$.

But still we lack any information on bounds in case when n is an even integer. In this paper we mainly focus on the case when n is an even integer and get the following theorem:

Theorem 4. *Let $n = 2^{a_1} \cdots p_k^{a_k}$ be an even integer and let $a = \sum_s a_s$. Let $A_1 \subset \mathbb{Z}_{2^{a_1}}$ such that $|A_1| \geqslant 2^{a_1-1}$. For each $s \geqslant 2$, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with $|A_s| > (5/12)p_s^{a_s}$, and let $A = A_1 \times \cdots \times A_k$. Then for any integer m multiple of 4, every sequence $x_1 \cdots x_{m+3a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

For $a_1 \geqslant 2$ from Theorem 4 it follows that any sequence of length $n + 3a$ of elements of $\mathbb{Z}_n$ has 0 as an A-weighted n-sum. In other words, if A is as in Theorem 4,

$$\mathsf{s}_A(\mathbb{Z}_n) \leqslant n + 3\Omega(n),$$

for $n \equiv 0 \pmod 4$.

It has been observed in [3] that if we consider $n \equiv 0 \pmod 4$ such that n is the product of $\Omega(n)$ (not neccessarily distinct) primes $n = q_1 \cdots q_{\Omega(n)}$ and $A = \mathbb{Z}_n^*$, then the sequence $1, q_1, q_1q_2, \ldots, q_1q_2 \cdots q_{\Omega(n)-1}$ has no non-empty subset with weighted sum 0. Thus adjoining $n-1$ zeros we have a sequence of length $n+\Omega(n)-1$ which does not have 0 as a A-weighted n-sum. So in this case we can conclude that

$$n + \Omega(n) \leqslant \mathsf{s}_A(\mathbb{Z}_n) \leqslant n + 3\Omega(n),$$

for $n \equiv 0 \pmod 4$.

Clearly, in the Theorem 4 we include more sets which are not covered by the results of Griffiths.

In the next section we shall give the proof of Theorem 4 and discuss some results. But before that let us first state the Kneser's theorem ([12], also see [14, Chapter 4]), which is one of the most useful tools for Section 2.

Recall that the stabilizer of a subset S of an abelian group G is defined as

$$stab(S) = \{x \in G : x + S = S\}.$$

Theorem 5 (Kneser's Theorem)**.** *Let G be an abelian group and A, B be finite, non-empty subsets of G. Then*

$$|A + B| \geqslant |A + H| + |B + H| - |H|,$$

where H is the stabilizer of $A + B$.

2 Proof of Theorem 4

To prove Theorem 4 we need the following lemmas.

Lemma 6. *Let $A \subset \mathbb{Z}_{2^a}$ be a subset such that $|A| \geqslant 2^{a-1}$. If $x, y, z, w \in \mathbb{Z}_{2^a}^*$, the group of units in $\mathbb{Z}_{2^a}$, then given any even $t \in \mathbb{Z}_{2^a}$, there exist $\alpha, \beta, \gamma, \delta \in A$ such that*

$$\alpha x + \beta y + \gamma z + \delta w = t.$$

Proof. There are two cases.

- When $|A| = 2^{a-1}$ and A contains all odd elements or all even elements of $\mathbb{Z}_{2^a}$. Given $x, y, z, w \in \mathbb{Z}_{2^a}^*$ and $A \subset \mathbb{Z}_{2^a}$ with $|A| = 2^{a-1}$, $Ax + Ay + Az + Aw$ will contain only even elements from $\mathbb{Z}_{2^a}$ which will imply

  $$|Ax + Ay + Az + Aw| \leqslant 2^{a-1}.$$

  Now we calculate the cardinality $|Ax + Ay + Az + Aw|$. As we observed that

  $$|Ax + Ay + Az + Aw| \geqslant |Ax| = |A| = 2^{a-1},$$

  it follows that $Ax + Ay + Az + Aw = Z_{2^{a-1}}$. Thus we have the theorem in this case.

- When A contains even as well as odd elements of $\mathbb{Z}_{2^a}$. Using Theorem 5 we get

  $$|Ax + Ay| \geqslant |Ax + H_1| + |Ay + H_1| - |H_1|,$$

  where H_1 is the stabilizer of $Ax+Ay$. For $|H_1| = 2^a$, $Ax+Ay = \mathbb{Z}_{2^a}$. For $|H_1| = 2^{a-1}$, H_1 contains all even elements of $\mathbb{Z}_{2^a}$ and the order of $\mathbb{Z}_{2^a}/H_1$ will be 2. For an even $u \in A$ we have $u + H_1 = H_1$ and for an odd $v \in A$ we have $v + H_1 \neq H_1$. Therefore we get

  $$\mathbb{Z}_{2^a} = H_1 \bigcup (v + H_1).$$

  Thus by Theorem 5 we have

  $$|Ax + Ay| \geqslant |Ax + H_1| + |Ay + H_1| - |H_1| \geqslant |Ax + H| = 2^a.$$

  and hence $Ax + Ay = \mathbb{Z}_{2^a}$. If $|H_1| \leqslant 2^{a-2}$, then

  $$|Ax + Ay| \geqslant |Ax + H_1| + |Ay + H_1| - |H_1| \geqslant 3.2^{a-2}.$$

  Similarly, using Theorem 5 we have

  $$|t - Az - Aw| \geqslant |t - Az + H_2| + |(-Aw) + H_2| - |H_2|,$$

  where H_2 is the stabilizer of $t - Az - Aw$. Now, for $|H_2| = 2^a$ or 2^{a-1}, $t - Az - Aw = \mathbb{Z}_{2^a}$. For $|H_2| \leqslant 2^{a-2}$, $|t - Az - Aw| \geqslant 3.2^{a-2}$. In all the cases we will find that

  $$|Ax + Ay| + |t - Az - Aw| > 2^a.$$

  So, by the pigeonhole principle, we conclude that any even number t can be written as $\alpha x + \beta y + \gamma z + \delta w$ for some $\alpha, \beta, \gamma, \delta \in A$. □

Lemma 7. *Let p^a be an odd prime power and $A \subset \mathbb{Z}_{p^a}$ be a subset such that $|A| > \frac{5}{12}p^a$. If $x, y, z, w \in \mathbb{Z}_{p^a}^*$, the group of units in $\mathbb{Z}_{p^a}$, then given any $t \in \mathbb{Z}_{p^a}$, there exist $\alpha, \beta, \gamma, \delta \in A$ such that*

$$\alpha x + \beta y + \gamma z + \delta w = t.$$

Proof. Consider the following sets

$$A_1 = \{\alpha x : \alpha \in A\},\ B_1 = \{\beta y : \beta \in A\},$$

$$C_1 = \{-\gamma z : \gamma \in A\},\ D_1 = \{t - \delta w : \delta \in A\}.$$

Observe that $|A_1| = |B_1| = |C_1| = |D_1| = |A|$. Therefore by using Theorem 5, we have

$$|A_1 + B_1| \geqslant |A_1| + |B_1| - |H_1|, \tag{3}$$

where H_1 is the stabilizer of $A_1 + B_1$. Now if $H_1 = \mathbb{Z}_{p^a}$ then it would imply that $A_1 + B_1 = \mathbb{Z}_{p^a}$. Thus we get $A_1 + B_1 + \gamma z + \delta w = \mathbb{Z}_{p^a}$ and we are through. Otherwise, we have

$$|H_1| \leqslant p^{a-1} = \frac{p^a}{p} \leqslant \frac{p^a}{3}.$$

Hence using (3) we have

$$|A_1 + B_1| \geqslant |A_1| + |B_1| - |H_1| > \frac{5p^a}{6} - \frac{p^a}{3} = \frac{p^a}{2}.$$

Similarly, if the stabilizer of $C_1 + D_1$ has cardinality p^a then we are through. Otherwise

$$|C_1 + D_1| > \frac{p^a}{2},$$

and therefore we get

$$|A_1 + B_1| + |C_1 + D_1| > \frac{p^a}{2} + \frac{p^a}{2} = p^a,$$

which implies that the sets $A_1 + B_1$ and $C_1 + D_1$ intersect and we are done. □

Lemma 8. *Let $A \subset \mathbb{Z}_{2^a}$ be such that $|A| \geqslant 2^{(a-1)}$. Let $x_1 \cdots x_m$ be a sequence over $\mathbb{Z}_{2^a}$ such that for each $b \in [1, a]$, the cardinality of the set $\{i \mid x_i \neq 0 \pmod{2^b}\}$ is divisible by 4. Then $x_1 \cdots x_m$ is an A-weighted zero-sum sequence.*

Proof. Let c be minimal such that $\{i \mid x_i \neq 0 \pmod{2^c}\}$ is non-empty. If no such c exists then all terms are 0 and we are done instantly. Therefore, $\{i \mid x_i \neq 0 \pmod{2^c}\}$ has at least four elements; without loss of generality let $x_1, x_2, x_3, x_4 \neq 0 \pmod{2^c}$. Also there are only even number of elements in $x_5, \ldots, x_m$ which are $\neq 0 \pmod{2^c}$ and $\equiv 0 \pmod{2^{c-1}}$. Others are congruent to 0 modulo 2^c. Set

$$x_i' = x_i / 2^{c-1} \in \mathbb{Z}_{2^{a-(c-1)}},$$

for $i \in [1, m]$. Therefore x_i' for $i = 1, 2, 3, 4$ are odd elements and $-x_5' - \cdots - x_m'$ is even element in $\mathbb{Z}_{2^{a-(c-1)}}$. Also if elements of A meet less than $2^{a-(c-1)-1}$ congruence classes modulo $2^{a-(c-1)}$, then $|A| < 2^{a-(c-1)-1} \times 2^{(c-1)} = 2^{a-1}$, which is a contradiction to our assumption. Therefore, the elements of A must meet at least $2^{a-(c-1)-1}$ congruence classes modulo $2^{a-(c-1)}$. Let $\alpha \in A$ be any element. Therefore, $-\alpha x_5' - \cdots - \alpha x_m'$ is an even element in $\mathbb{Z}_{2^{a-(c-1)}}$. Now using Lemma 6, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in A$ such that

$$\alpha_1 x_1' + \alpha_2 x_2' + \alpha_3 x_3' + \alpha_4 x_4' = -\alpha x_5' - \cdots - \alpha x_m'$$

in $\mathbb{Z}_{2^{a-(c-1)}}$, and we are through in $\mathbb{Z}_{2^a}$ taking $\alpha = \alpha_5 = \cdots = \alpha_m$. □

Remark. For an odd prime power $n = p^a$, the same condition on a sequence $x_1 \cdots x_m$ in $\mathbb{Z}_{p^a}$ (namely that $A \subset \mathbb{Z}_{p^a}$ be such that $|A| > \frac{5}{12}p^a$ and the cardinality of the set $\{i \mid x_i \neq 0 \pmod{p^b}\}$ is divisible by 4 for all $b = 1, \dots, a$) is sufficient to imply that $x_1 \cdots x_m$ is an A-weighted zero-sum sequence. For a proof we should proceed as above. Only at the last part we need to apply Lemma 7 instead of Lemma 6.

Lemma 9. *Given disjoint subsets $X_1, \dots, X_a$ of the set $V = [1, m + 3a]$, where m is a multiple of 4, there exists a set $I \subset [1, m+3a]$ with $|I| = m$ such that $|I \cap X_s|$ is a multiple of 4, for all $s = 1, \dots, a$.*

Proof. Let $X_{a+1} = V \setminus \cup_{b=1}^{a} X_b$. For $b = 1, \dots, a + 1$, let $W_b \subset X_b$, be a maximal sized subset whose cardinality is a multiple of 4. Then, $|W_b| \in \{|X_b|, |X_b|-1, |X_b|-2, |X_b|-3\}$. So, $| \cup_{b=1}^{a+1} W_b| = \sum_{b=1}^{a+1} |W_b| \geqslant \sum_{b=1}^{a+1}(|X_b| - 3) = (m + 3a) - 3(a + 1) = m - 3$. For each $b \in [1, a + 1]$, $|W_b|$ is divisible by 4 and also it is given that m is divisible by 4. Therefore, $| \cup_{b=1}^{a+1} W_b| \geqslant m$. Thus we can obtain an I with cardinality m having the required property. □

Remark. The same is true for a nested family $X_1 \subset X_2 \subset \cdots \subset X_a$. To see this, set $Y_1 = X_1$, and $Y_b = X_b \setminus X_{b-1}$, for $b = 2, \dots, a$. These sets are disjoint. Now by applying Lemma 9 to $Y_1, \dots, Y_a$, we obtain $I \subset \{1, 2, \dots, m+3a\}$ with $|I| = m$ and $|I \cap Y_b|$ multiple of 4, for $b = 1, \dots, a$. Hence we are done because this implies $|I \cap X_b|$ is multiple of 4 for all $b = 1, \dots, a$.

Observation : Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be a positive integer. Then, $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_k^{a_k}}$ and an element $x \in \mathbb{Z}_n$ can be written as $x = (x^{(1)}, \dots, x^{(k)})$, where $x^{(s)} \equiv x \pmod{p_s^{a_s}}$ for each s. It has been observed in [10] that if $A = A_1 \times A_2 \times \cdots \times A_k$ is a subset of $\mathbb{Z}_n$, where $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ for each $s \in [1, k]$, then a sequence $x_1 \cdots x_m$ over $\mathbb{Z}_n$ is an A-weighted zero-sum sequence in $\mathbb{Z}_n$ if and only if for each $s \in [1, k]$, the sequence $x_1^{(s)} \cdots x_m^{(s)}$ is an A_s-weighted zero-sum sequence in $\mathbb{Z}_{p_s^{a_s}}$.

Proof of Theorem 4. Given a sequence $x_1 \cdots x_{m+3a}$ over $\mathbb{Z}_n$, we define $X_b^{(s)} \subset [1, m + 3a]$ for $s \in [1, k]$ and $b \in [1, a_s]$ by

$$X_b^{(s)} = \{i : x_i \neq 0 \pmod{p_s^b}\}.$$

Using Lemma 9 and the Remark just after that, we can say that there exists $I \subset [1, m+3a]$ such that $|I| = m$ and $|I \cap X_b^{(s)}|$ is a multiple of 4 for all s, b. Let $I = \{i_1, \ldots, i_m\}$. For $s = 1$, by Lemma 8, $x_{i_1}^{(1)} \cdots x_{i_m}^{(1)}$ is an A_1- weighted zero-sum sequence. For $s \geqslant 2$, by remark just after Lemma 8 $x_{i_1}^{(s)} \cdots x_{i_m}^{(s)}$ is a A_s- weighted zero-sum sequence. Now using the above observation it follows that $x_{i_1}, \ldots, x_{i_m}$ is an A-weighted zero-sum sequence. □

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