Every graph G is Hall $\Delta(G)$-extendible

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Abstract

In the context of list coloring the vertices of a graph, Hall's condition is a generalization of Hall's Marriage Theorem and is necessary (but not sufficient) for a graph to admit a proper list coloring. The graph G with list assignment L, abbreviated (G, L), satisfies *Hall's condition* if for each subgraph H of G, the inequality $|V(H)| \leqslant \sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, L))$ is satisfied, where $\mathcal{C}$ is the set of colors and $\alpha(H(\sigma, L))$ is the independence number of the subgraph of H induced on the set of vertices having color σ in their lists. A list assignment L to a graph G is called *Hall* if (G, L) satisfies Hall's condition. A graph G is *Hall k-extendible* for some $k \geqslant \chi(G)$ if every k-precoloring of G whose corresponding list assignment is Hall can be extended to a proper k-coloring of G. In 2011, Bobga et al. posed the question: If G is neither complete nor an odd cycle, is G Hall $\Delta(G)$-extendible? This paper establishes an affirmative answer to this question: every graph G is Hall $\Delta(G)$-extendible. Results relating to the behavior of Hall extendibility under subgraph containment are also given. Finally, for certain graph families, the complete spectrum of values of k for which they are Hall k-extendible is presented. We include a focus on graphs which are Hall k-extendible for all $k \geqslant \chi(G)$, since these are graphs for which satisfying the obviously necessary Hall's condition is also sufficient for a precoloring to be extendible.

Keywords: vertex coloring; list coloring; precoloring; extendible; Hall's condition; Hall k-extendible.

1 Introduction

Throughout, G is a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. For $U \subseteq V(G)$, we shall use $G[U]$ to denote the subgraph of G induced on U. Additionally $\alpha(G)$, $\delta(G)$, $\Delta(G)$, $\chi(G)$, shall denote the *independence number, minimum degree, maximum degree,* and *chromatic number* of G respectively. Let $\deg_G(v)$ denote the *degree* of the vertex v in the graph G. For any $U \subseteq V(G)$ and any subgraph H of G, let $N_H(U)$ denote the set of vertices in H that are adjacent to at least one vertex in U. Let $[m]$ denote the set $\{1, \ldots, m\}$. We refer the reader to West [16] for any notation not defined here.

Definition 1. A k**-precoloring** of G is a proper k-coloring of $G[U]$ where $U \subset V(G)$. The coloring, say ϕ, can be **extended** (or is **extendible**) if there exists a proper k-coloring $\theta : V(G) \to [k]$ where $\theta(v) = \phi(v)$ for all $v \in U$.

The Precoloring Extension Problem (PrExt) is a natural generalization of the usual graph coloring problem and has been heavily studied. Of course, any precoloring with $\Delta(G) + 1$ colors can be extended greedily, but a precoloring with $\Delta(G)$ colors that gives each vertex in the neighborhood of a vertex v with $\deg_G(v) = \Delta(G)$ a different color cannot be extended. Therefore PrExt is only interesting if conditions are placed on the precoloring. Most results (see for example [1], [2], [3], [4], [14]) place distance-based conditions on the precolored set U; the precolored vertices need to be "far enough apart."

Our approach to the PrExt problem in this article is entirely different. We guarantee the extension of precolorings through an obvious necessary condition that is based on Hall's condition for matching extensions. As is common, this condition views the PrExt problem as a *list coloring* problem. Vizing [15] introduced the notion of list coloring. It was further developed by Erdős, Rubin, and Taylor [9], and has been studied extensively since. If $\mathcal{C}$ is an infinite set of colors (the *palette*) and $\mathcal{L}$ is a set of finite subsets of $\mathcal{C}$, then a *list assignment* to G is a function $L : V(G) \to \mathcal{L}$. The list L is a k*-assignment* to G if $|L(v)| \geqslant k$ for all $v \in V(G)$. Given a list assignment L of G with color palette $\mathcal{C}$, an L*-coloring* of G is a function $\phi : V(G) \to \mathcal{C}$ such that $\phi(v) \in L(v)$ for every vertex v. An L-coloring ϕ is *proper* if each color class induces an independent set. If G has a proper L-coloring, we say G is L*-colorable.*

In 1990, Hilton and Johnson [11] introduced the following concept (also see [5]), which was a generalization of Philip Hall's 1935 Marriage Theorem ([10]) applied to list assignments of graphs. Suppose that ϕ is an L-coloring of G for some list assignment L with a color palette $\mathcal{C}$ and let H be any subgraph of G. For each $\sigma \in \mathcal{C}$, consider $\phi^{-1}(\sigma) \mid_H$, the set of all vertices in H given color σ under ϕ, and let $H(\sigma, L)$ be the subgraph of H induced on all vertices of H having σ in their lists. Then $\phi^{-1}(\sigma) \mid_H$ is an independent set of vertices contained inside $H(\sigma, L)$. Naturally, if G is L-colorable, then for every subgraph H, we must have

$$|V(H)| = \sum_{\sigma \in C} [\phi^{-1}(\sigma) \mid_H] \leqslant \sum_{\sigma \in C} \alpha(H(\sigma, L))$$

This motivated the following definition:

Definition 2 ([11])**.** The graph G with list assignment L satisfies *Hall's condition* if for each subgraph H of G, the inequality

$$|V(H)| \leqslant \sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, L)) \qquad (*)$$

is satisfied. For brevity we say (G, L) satisfies Hall's condition provided that $(*)$ is satisfied for every subgraph of G. If H is a subgraph of G, then (H, L) will denote the natural restriction of L to $V(H)$. If (G, L) does not satisfy (i.e., fails) Hall's condition, then there exists some subgraph H of G such that (H, L) does not satisfy the inequality $(*)$.

Theorem 3 ([11])**.** *If G has a proper L-coloring, then (G, L) satisfies Hall's condition. Also, (G, L) satisfies Hall's condition if and only if $(*)$ holds for each connected, induced subgraph H of G.*

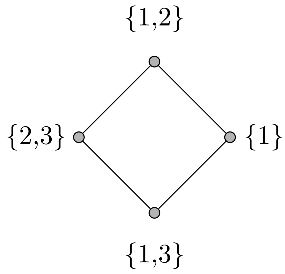

Figure 1: (G, L) satisfy Hall's condition, yet G is not L-colorable.

While satisfying Hall's condition is a necessary condition for G to have a proper L-coloring, it is not sufficient; see Figure 1 for an example of a list assignment L to a 4-cycle in which all subgraphs satisfy $(*)$, but there is no proper L-coloring.

Verifying that a pair (G, L) satisfy Hall's condition is difficult, as generally all subsets of G must be considered. There are several results that assist in this verification process.

Theorem 4 ([11]). *If L is a $\chi(G)$-assignment to G, then (G, L) satisfy Hall's condition.*

The following is proved from a simple extension of the proof of Theorem 4:

Theorem 5. *If L is a list assignment to a graph G and for every subgraph H, the average list cardinality when restricted to $V(H)$ is at least $\chi(G)$, then (G, L) satisfies Hall's condition.*

Proof. Suppose that for each $H \leqslant G$,

$$\frac{1}{|V(H)|} \sum_{v \in V(H)} |L(v)| \geqslant \chi(G) \tag{1}$$

but (G, L) fails Hall's condition. Then for some subgraph K of G, we have

$$\begin{aligned} |V(K)| > \sum_{\sigma \in \mathcal{C}} \alpha(K(\sigma, L)) \geqslant \sum_{\sigma \in \mathcal{C}} \frac{|V(K(\sigma, L))|}{\chi(K(\sigma, L))} \geqslant \frac{1}{\chi(G)} \sum_{\sigma \in \mathcal{C}} |V(K(\sigma, L))| \\ = \frac{1}{\chi(G)} \sum_{v \in V(K)} |L(v)| \geqslant |V(K)|, \end{aligned}$$

a contradiction. The last inequality follows from the assumption that (1) holds for every subgraph of G. □

Definition 6. [11] The *Hall number* of G is the smallest positive integer $h(G) = k$ such that whenever L is a k-assignment to G and (G, L) satisfies Hall's condition, G is L-colorable.

In other words, $h(G)$ is the smallest positive integer such that Hall's condition on k-assignments is both necessary and sufficient for the existence of a proper L-coloring of G.

The following result characterizes graphs with Hall number 1.

Theorem 7 ([11], Hilton et al. [12]). *The following statements are equivalent:*

1. $h(G) = 1$.

2. *Every block (maximally 2-connected subgraph) of G is a clique.*

3. *G contains no induced cycle C_n, $n \geqslant 4$, nor an induced copy of $K_4 - e$ (that is, K_4 with an edge deleted).*

The extension of a partial coloring of G can be viewed as a list coloring problem, where the lists on precolored vertices have cardinality one, and the list on any other vertex contains the colors that do not appear on precolored vertices in its neighborhood. In this paper, we study Hall's condition in the context of extensions of partial colorings.

Definition 8. A list assignment L of a graph G is a *Hall assignment* if (G, L) satisfy Hall's condition. For $V_0 \subseteq V(G)$, a k-precoloring $\phi : V_0 \to [k]$ of a graph G is a *Hall k-precoloring* if L_ϕ is a Hall assignment, where L_ϕ is the natural list assignment associated with ϕ:

$$L_\phi(x) = \begin{cases} \{\phi(x)\} & \text{if } x \in V_0 \\ [k] \setminus \{\phi(y) \colon y \in N_G(x) \cap V_0\} & \text{if } x \notin V_0. \end{cases}$$

A graph G is *Hall k-extendible* if every Hall k-precoloring is extendible. The graph G is *Hall chromatic extendible* if G is Hall $\chi(G)$-extendible and G is *total Hall extendible* if G is Hall k-extendible for all $k \geqslant \chi(G)$.

The definition of "Hall k-extendible," was first stated in [5] and was called "Hall k-*completable*," due to the relationship that paper explored with completing partial Latin squares. We have adopted the term "extendible" as this is more common in the PrExt literature, and precoloring extensions are the focus of this paper. The following basic results regarding k-precolorings were also established in [5].

Theorem 9 ([5]). *Let G be a graph.*

1. *G is Hall k-extendible for all $k \geqslant \Delta(G) + 1$.*

2. *G is Hall k-extendible if and only if every component of G is Hall k-extendible.*

3. *Let $\phi : V_0 \to [k]$ be a k-precoloring of G, and let $G' = G[V \setminus V_0]$.*

   (a) *G is L_ϕ-colorable if and only if G' is L_ϕ-colorable.*

   (b) *(G, L_ϕ) satisfy Hall's condition if and only if (G', L_ϕ) satisfy Hall's condition.*

The main result of this paper addresses a question asked in [5] that was motivated by Brooks' theorem ([8]): *If G is a graph that is neither complete nor an odd cycle, is G Hall $\Delta(G)$-extendible?* In Section 2, we provide a fully affirmative answer to this question. This proves a natural precoloring extension version of Brooks' theorem: when precoloring a graph with a palette of $\Delta(G)$ available colors, one is guaranteed an extension provided the obvious necessary condition in Definition 2 is not violated.

The following theorems highlight the fact that the concept of Hall extendibility diverges considerably from that of colorability.

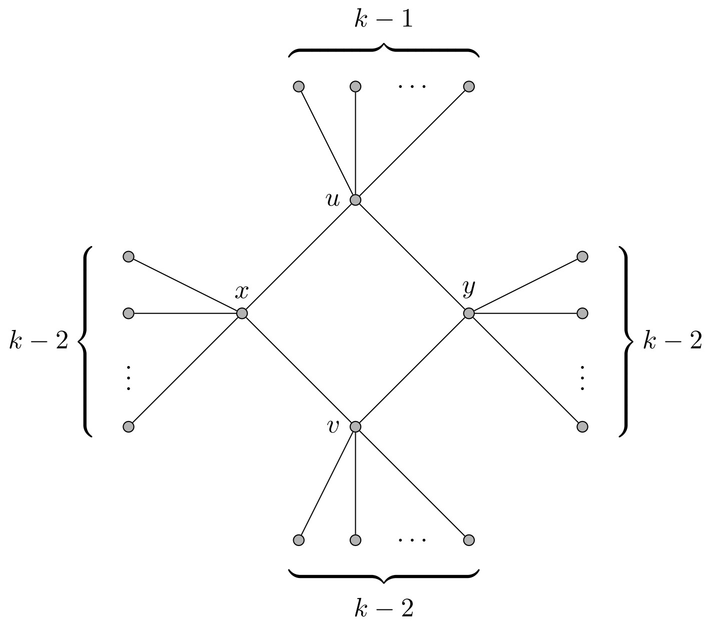

Figure 2: A family of graphs $\mathcal{G} = \{G_k : k \geqslant 3\}$ that are Hall 2-extendible because they are bipartite, but for any $k \geqslant 3$, the graph G_k is not Hall k-extendible.

Theorem 10 ([5])**.** *Every bipartite graph is Hall chromatic extendible, but for every $k \geqslant 3$, there exists a bipartite graph which is not Hall k-extendible.*

Figure 2 illustrates this; we may precolor the vertices of degree one in the graph G_k so as to produce the Hall assignment shown in Figure 1. (We shall refer frequently to the family of graphs in Figure 2 throughout, especially the graph G_3.) One might conjecture that this is due to the existence of a list of cardinality one on an uncolored vertex, but in fact, bipartite graphs may fail to be Hall 3-extendible, even if all lists of non-precolored vertices have cardinality at least two. Consider the graph in Figure 3. The 3-precoloring is Hall by Theorem 4, but it is not extendible.

This behavior is not limited to bipartite graphs.

Theorem 11 (Holliday et. al [13])**.** *For all $k \geqslant 2$, there exists a k-chromatic graph that is Hall k-extendible but not Hall $(k+1)$-extendible.*

In Section 3, we further investigate surprising ways in which this graph parameter behaves, looking at how the Hall k-extendibility of G relates to the Hall k-extendibility of its subgraphs. In Section 4, we discuss how increasing the number of colors can affect the Hall extendibility of various graphs.

2 Every graph G is Hall $\Delta(G)$-extendible

In [13], the authors established the following in response to the question of Bobga et. al [5] on whether any graph G is Hall $\Delta(G)$-extendible:

Theorem 12 ([13])**.** *If G is a bipartite graph with $h(G) \leqslant 3$, then G is Hall $\Delta(G)$-extendible.*

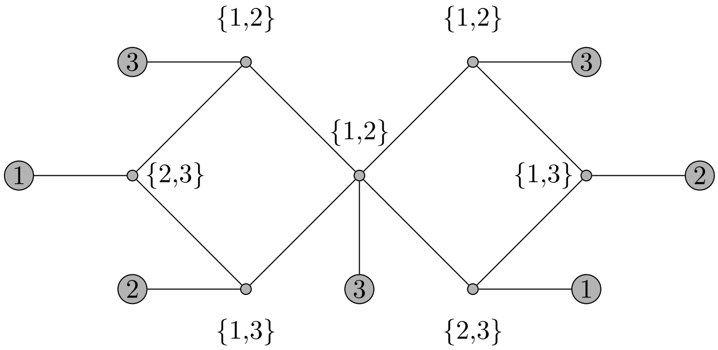

Figure 3: A Hall 3-precoloring that is not extendible.

We now completely answer this question in the affirmative. Some preliminaries are required. A list assignment L is a *degree-assignment* to G if $|L(v)| \geqslant \deg_G(v)$ for all $v \in V(G)$. The graph G is called *degree-choosable* if G has a proper L-coloring for every degree-assignment L. A graph is a *Gallai tree* if every block of the graph is either a clique or an odd cycle.

Theorem 13 (Borodin [6], Erdős et. al [9]). *A connected graph is degree-choosable if and only if it is not a Gallai tree.*

Furthermore, the degree-assignments L under which Gallai trees fail to be proper L-colorable have a restricted form:

Theorem 14 (Borodin [6, 7], Erdős et. al [9]). *If L is a degree-assignment for a connected graph G and there is no L-coloring of G, then*

(i) $|L(v)| = \deg_G(v)$ *for all* $v \in V(G)$.

(ii) *G is a Gallai tree.*

(iii) *For every $v \in V(G)$,*
$$L(v) = \bigcup_{B \in \mathcal{B}(v)} L_B,$$
where $\mathcal{B}(v)$ is the set of blocks containing v, and for each block B, L_B is a set of $\chi(B) - 1$ colors.

Lemma 15. *If L is the list assignment to Gallai tree G as described in Theorem* 14*, then (G, L) fails Hall's condition.*

Proof. We prove by induction on b, the number of blocks of G, that any list assignment to a Gallai tree satisfying (i) and (iii) from Theorem 14 fails $(*)$ on the entire graph G. If $b = 1$, then G is an odd cycle or a clique. It is routine to check that Hall's condition fails on G, establishing a basis. Now suppose that B is a leaf block of G (a block with exactly one cut-vertex in G) and v is the unique cut-vertex in B. Delete all vertices in B except v, and remove the colors in $L(v)$ that are assigned to B. Let L' be the resulting list assignment to $G' = G - (B - v)$. Note that properties (i) and (iii) of the hypothesis still hold for G'. By the induction hypothesis, (G', L') fails $(*)$ and so

$$|V(G')| > \sum_{\sigma \in \mathcal{C}} \alpha(G'(\sigma, L')).$$

Returning to G itself, we have two cases:

Case 1: $B = K_m$. If σ is any of the $m-1$ colors assigned to B, then σ can contribute at most one more to $\alpha(G(\sigma, L))$. Thus,

$$\sum_{\sigma \in \mathcal{C}} \alpha(G(\sigma, L)) \leqslant \sum_{\sigma \in \mathcal{C}} \alpha(G'(\sigma, L')) + (m-1) < |V(G')| + (m-1) = |V(G)|.$$

Case 2: $B = C_{2k+1}$ for some $k > 0$. If σ is either of the 2 colors assigned to B, then σ can contribute at most k more to $\alpha(G(\sigma, L))$. Thus,

$$\sum_{\sigma \in \mathcal{C}} \alpha(G(\sigma, L)) \leqslant \sum_{\sigma \in \mathcal{C}} \alpha(G'(\sigma, L')) + 2k < |V(G')| + 2k = |V(G)|.$$

In either case, (G, L) fails to satisfy $(*)$. □

The following observation will be used in the proof of Lemma 17.

Observation 16. *Suppose ϕ is a precoloring of $V_0 \subset V(G)$ and θ is an extension of ϕ to V_1 with $V_0 \subset V_1 \subset V(G)$. If (G, L_θ) satisfies Hall's condition, then (G, L_ϕ) satisfies Hall's condition.*

Proof. Viewing the precolored vertices $V_1 - V_0$ as vertices with lists of cardinality 1, we see that $L_\theta(v) \subseteq L_\phi(v)$ for all $v \in V(G)$, hence

$$\sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, L_\phi)) \geqslant \sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, L_\theta)) \geqslant |V(H)|$$

for all subgraphs H of G. □

Lemma 17. *If G is a complete graph or an odd cycle, then any $\Delta(G)$-precoloring to G is not Hall. Thus, all graphs of this type are Hall $\Delta(G)$-extendible by default.*

Proof. By the contrapositive of Observation 16, it suffices to show that the empty $\Delta(G)$-precoloring, that is, with $V_0 = \varnothing$, is not Hall. If $G = C_{2k+1}$, then $\Delta(G) = 2$, and each of the two colors contributes exactly k to the right side of $(*)$. If $G = K_m$, then each of the $m-1$ colors contributes exactly one to the right side of $(*)$. In either case, the inequality is violated. □

Theorem 18 (Main Result)**.** *Every graph G is Hall $\Delta(G)$-extendible.*

Proof. By Lemma 17 and Brooks' Theorem (see [8]), we may assume $\chi(G) \leqslant \Delta(G)$. Suppose that the theorem is false, and let G be a graph of minimum order where the theorem fails. Let ϕ be a Hall $\Delta(G)$-precoloring of G that is not extendible. Suppose

Graphs with Hall number 1	by definition
Complete graphs K_n; $n \geqslant 1$	Bobga et. al [5]
Trees	Bobga et. al [5]
Complete multipartite graphs $K_{n,m}$; $n, m \geqslant 1$	Bobga et. al [5]
Odd cycle graphs C_n; $n \geqslant 3$	Bobga et. al [5]
Even cycle graphs C_n; $n \geqslant 4$	Holliday et. al [13]
Prism graphs $P_2 \square C_n$; $n \geqslant 3$	Holliday et. al [13]
Ladder graphs $P_2 \square P_n$; $n \geqslant 2$	Holliday et. al [13]
The Petersen graph	Holliday et. al [13]
Wheels with even order $W_n = C_{n-1} \vee K_1$; $n \geqslant 3$	Corollary 38
All subcubic graphs	Theorem 18
Bipartite tori $C_{2k} \square C_{2\ell}$; $k, \ell \geqslant 3$	Corollary 33
Graphs with $\chi(G) = \Delta(G)$	Theorem 18

Table 1: A table of some graph families that are total Hall extendible.

$V_0 \subset V$ is the set of vertices that are precolored by ϕ, and L_ϕ the corresponding Hall list assignment to G. Let $G' = G[V \setminus V_0]$.

Claim: G' is connected.

Proof of Claim: If G' is not connected, then let H be an arbitrary component and let $U \subset V_0$ be its precolored neighborhood. Because $|V(H+U)| < |V(G)|$ and $\Delta(H+U) \leqslant \Delta(G)$ we conclude by minimality of G that $H+U$ is Hall $\Delta(H+U)$-extendible and thus by Theorem 9 (statement 1), $H+U$ is Hall $\Delta(G)$-extendible. So, any Hall $\Delta(G)$-precoloring of $H+U$ is extendible. Precolor U to match ϕ and then color H. Because H was arbitrary and no two components of G' share any edges, iterating this creates a proper $\Delta(G)$-coloring of G'. This is a contradiction to the claim that ϕ was not extendible. Thus, the Claim is established.

For every $v \in V(G')$, we have that $|L_\phi(v)| \geqslant \deg_{G'}(v)$, i.e., L_ϕ is a degree-assignment to G'. By Theorem 14, because G' does not have an L_ϕ-coloring, we must have $|L_\phi(v)| = \deg_{G'}(v)$ for all $v \in V(G')$ and $L_\phi(v) = \bigcup_{B \in \mathcal{B}(v)} L_B$ for all $v \in V(G')$, where $\mathcal{B}(v)$ is the set of blocks containing v, and for each block B, L_B is a set of $\chi(B) - 1$ colors. By Lemma 15, L_ϕ cannot be Hall, a contradiction because by hypothesis and Theorem 9 (statement 3(b)), the restriction of L_ϕ to G' must be a Hall-assignment. Hence, no such counterexample G can exist to the theorem. □

Theorem 18 adds substantially to the categories of graphs that are total Hall extendible, including graphs with $\chi(G) = \Delta(G)$ and graphs with $\Delta(G) \leqslant 3$. We close this section with a table of graphs (see Table 1) that have been shown to be total Hall extendible. Note that several of these categories that were previously shown to be total Hall extendible, such as the Petersen graph, are included in the category of subcubic graphs, demonstrating the strength of Theorem 18. The graphs in Table 1 can be viewed as graphs on which precolorings with at least $\chi(G)$ colors can "easily" be extended. That is, any precoloring with $k \geqslant \chi(G)$ colors that does not violate the obvious necessary condition in Definition 2 can be extended to a proper k-coloring of the graph. We believe further expansion of this list would be an interesting contribution to the PrExt problem.

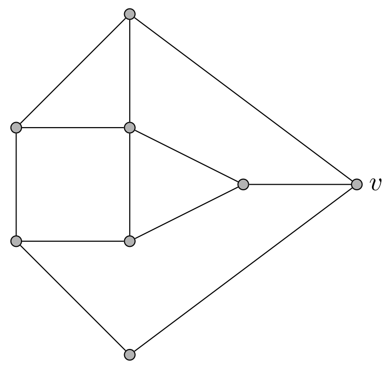

Figure 4: G is Hall 3-extendible but $G - v$ is not Hall 3-extendible.

3 Hall-Extendibility Under Subgraph Containment

In this section, we briefly discuss how Hall extendibility behaves under subgraph containment. Again, the divergence from colorability is remarkably strong. If G is k-colorable, then any subgraph of G is k-colorable, since a k-coloring of G restricted to $H \leqslant G$ is a k-coloring of H. However, the same behavior is not observed for Hall k-extendibility. For example, consider the graph G in Figure 4. Deleting the vertex v from G results in a subgraph H that can be precolored to obtain the graph with the Hall list assignment shown in Figure 1, so G contains a subgraph that is not Hall 3-extendible. However, the Hall 3-precoloring of $G - v$ (which is not extendible) viewed as a precoloring of G would leave v with an empty list, and therefore is not a Hall 3-precoloring of G. Furthermore, it has been verified through case analysis that G is, in fact, Hall 3-extendible. One might conjecture that this behavior is due to the fact that the precoloring is not Hall on $G - H \cong K_1$; however, the following proposition illustrates that is not the case.

Proposition 19. *For every $k \geqslant 3$, there exists a graph G such that G contains a subgraph H with a Hall k-precoloring $\phi : V_0 \to [k]$ that is not extendible and $(G - H, L_\phi)$ satisfies Hall's condition, but (G, L_ϕ) does not satisfy Hall's condition.*

Proof. Let $k \geqslant 3$. We claim the graph $G = J_k$ shown in Figure 5 satisfies the proposition. The graph $H = G - v$ is a subgraph of G that has a Hall k-precoloring $\phi = \phi_k$ (which is not extendible) shown in Figure 5; note that the lists on a, b, c, and d correspond to the list assignment shown in Figure 1. Furthermore, because $V(G - H) = \{v\}$ and $L(v) = \{1, 2, 3\}$, we have that $(G - H, L_\phi)$ trivially satisfies Hall's condition. However, it is routine to verify that Hall's condition fails on the subgraph induced by $\{a, b, c, d, v\}$. Therefore, (G, L_ϕ) does not satisfy Hall's condition. □

It is obvious that if one or more of the components of G are not Hall k-extendible, then G itself is not Hall k-extendible. The next theorem indicates a nontrivial sufficient condition for G to inherit a Hall k-precoloring from a subgraph H that is not extendible.

Theorem 20. *Suppose H is not Hall k-extendible, and G is a supergraph of H such that each vertex in $G - H$ has at most t neighbors in H, where $t \in [k]$. If G is $(k - t)$-colorable, then G is not Hall k-extendible.*

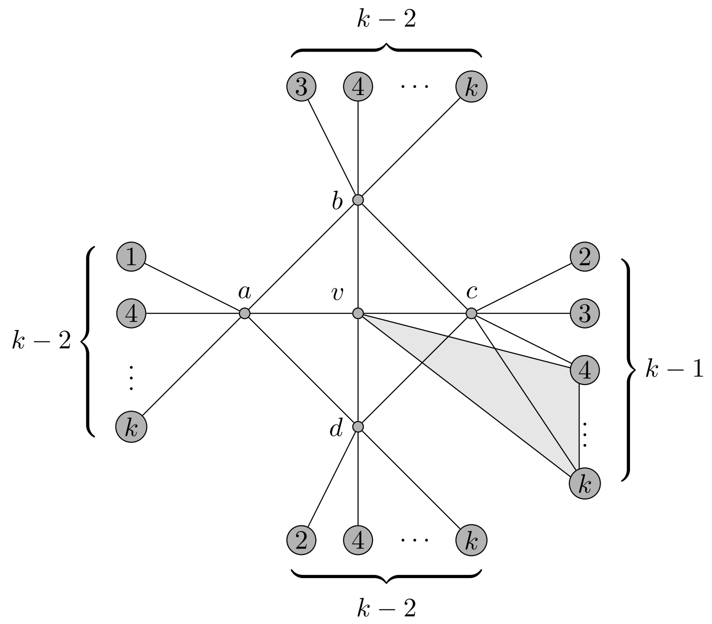

Figure 5: A family of graphs $\mathcal{J} = \{J_k : k \geqslant 3\}$ shown with a non-extendible k-precoloring ϕ_k from Proposition 19. The shaded triangular region indicates that the vertex $v \in V(J_k)$ is adjacent to the $k-3$ vertices colored 4 through k when $k \geqslant 4$; when $k = 3$, the vertex $v \in V(J_3)$ is adjacent only to the vertices a, b, c, d.

Proof. Let ϕ be a Hall k-precoloring of H that is not extendible and view ϕ as a k-precoloring of G. It suffices to show that ϕ is Hall on G. Let F be an arbitrary subgraph of G; we show that F satisfies $(*)$. By Theorem 9 (statement 3), we may assume that F contains no precolored vertices. If F is entirely contained in H, then F satisfies $(*)$ by hypothesis. If F is entirely contained in $G - H$, then since each vertex in $G - H$ has at most t neighbors in H, $|L(v)| \geqslant k - t \geqslant \chi(F)$ for all $v \in V(F)$. Hence F satisfies $(*)$ by Theorem 4.

Therefore we may assume that F contains vertices of H and $G - H$. Let H_F and G_F be the subgraphs of F induced on H and $G-H$, respectively. Let X be a maximum independent set in G_F (refer to Figure 6). Since G_F is $(k-t)$-colorable, $|X| \geqslant |V(G_F)|/(k-t)$. Let H'_F be the subgraph induced by $V(H_F) - N_{H_F}(X)$ and F' be the subgraph induced on the remaining vertices of F.

Since there are no edges between H'_F and X, and X is an independent set,

$$\sum_{\sigma \in [k]} \alpha(F(\sigma, L_\phi)) \geqslant \sum_{\sigma \in [k]} \alpha(H'_F(\sigma, L_\phi)) + \sum_{\sigma \in [k]} \alpha(F[X](\sigma, L_\phi)) \geqslant |V(H'_F)| + \sum_{x \in X} |L(x)|$$

because by hypothesis H'_F satisfies Hall's condition. As $|V(F)| = |V(H'_F)| + |V(F')|$, it remains to verify $\sum_{x \in X} |L(x)| \geqslant |V(F')|$.

Since only vertices of H are precolored and any vertex in X is adjacent to at most t vertices in H, we have that $|L(x)| \geqslant k - t$ for every $x \in X$. However, F itself contains no precolored vertices, so if $x \in X$, then each $y \in N_{H_F}(x)$ contributes one to $|L(x)|$ as each such y reduces the potential number of precolored vertices to which x can be adjacent.

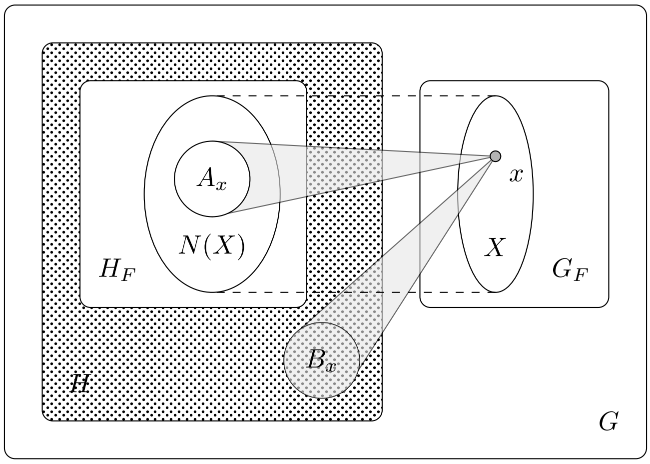

Figure 6: Illustration for the proof of Theorem 20. All precolored vertices of G must lie in $H - H_F$ (dotted region). Here, $N(X) = N_{H_F}(X)$, $A_x = N_{H_F}(x)$ and $B_x = N_H(x) - A_x$. By hypothesis, $|A_x| + |B_x| \leqslant t$.

Hence $|L(x)| \geqslant k-t+|N_{H_F}(x)|$ for every $x \in X$. This together with $|X| \geqslant |V(G_F)|/(k-t)$ yields

$$\sum_{x\in X} |L(x)| \geqslant (k-t)|X| + |N_{H_F}(X)| \geqslant |V(G_F)| + |N_{H_F}(X)| = |V(F')|$$

as needed. □

Recall that if G and H are graphs, then the *cartesian product* of G and H is the graph $G \,\square\, H$ having vertex set $V(G) \times V(H)$ and edge set $E(G \,\square\, H)$ defined as

$$\{\{(g_1, h_1), (g_2, h_2)\} : h_1 = h_2 \text{ and } \{g_1, g_2\} \in E(G) \text{ or } g_1 = g_2 \text{ and } \{h_1, h_2\} \in E(H)\}.$$

The following corollary will be used in Section 4.

Corollary 21. *Suppose H is not Hall k-extendible, and $G \,\square\, H$ is $k-1$ colorable. Then $G \,\square\, H$ is not Hall k-extendible.*

Hall extendibility is poorly behaved under edge deletions. Consider first the case where G is not Hall k-extendible; must $G - e$ maintain this property? Consider the graph H formed by adding an extra neighbor a of degree 1 to vertex u in the graph G_k in Figure 2. Both H and $H - \{ua\}$ fail to be Hall t-extendible for any $3 \leqslant t \leqslant k$. Hence it is possible to delete an edge from a graph that is not Hall k-extendible and retain this property. However, deleting an edge from the graph G_k in Figure 2 connecting vertex u to one of its leaves produces a graph that is Hall k-extendible (since any k-precoloring leaves either a forest induced on the precolored vertices or bipartite graph with a 2-assignment), but is not Hall t-extendible for any $3 \leqslant t \leqslant k-1$. Finally, deleting an edge of the 4-cycle results in a tree, and hence a graph that is Hall t extendible for all $t \geqslant 2$. Therefore deleting an edge may have a drastic effect on the extendibility of the graph.

Note that if G is not Hall k-extendible, but $G - e$ is for any $e \in E(G)$, then G may be considered Hall k-(edge)-critical. An example of such a graph would be G_3 from Figure 2. Deleting any single edge results in a graph that is Hall 3-extendible.

Hall extendibility also behaves unpredictably under edge contraction, as the following shows:

Proposition 22. *Let $G = G_3$ be the graph shown in Figure 2 (where $k = 3$). Although G is not Hall 3-extendible, contracting any edge of G produces a graph that is Hall 3-extendible.*

Proof. Let $e \in E(G)$ and let H be the graph obtained from G by contracting e. Clearly $\Delta(H) \geqslant 3$. If $\Delta(H) = 3$, then H is Hall 3-extendible by Theorem 18. Thus, we may assume that $\Delta(H) \in \{4, 5\}$. Let C' denoted the unique 4-cycle (if e is a bridge in G) or 3-cycle (if e is a non-bridge in G) in H. Take any Hall 3-precoloring $\phi : V_0 \to [3]$ of H. If $V_0 \cap V(C') \neq \varnothing$, then $H' = H[V - V_0]$ is a forest, thus H' has Hall number 1 by Theorem 7, and since $L_\phi \mid_{H'}$ is a Hall 1-assignment, ϕ is extendible. Therefore, we shall assume that $V_0 \cap V(C') = \varnothing$, i.e., ϕ precolors only vertices of degree 1 in H. Clearly if ϕ extends to a proper 3-coloring of C', then ϕ also extends to a proper 3-coloring of H' since every vertex of $v \in V(H') \setminus V(C')$ has only one neighbor in $V(C')$.

Thus we need only verify that ϕ extends to a proper 3-coloring of C'. If C' is a 3-cycle, then L_ϕ restricted to C' is a Hall 1-assignment of a clique, which has Hall number 1 by Theorem 7, and ϕ extends. It remains to consider C' a 4-cycle. Let $V(C') = \{a, b, c, d\}$ and suppose that a is the unique vertex on C' which had two neighbors of degree 1 in G: $\{a_1, a_2\}$. If at most one of a_1 and a_2 are precolored, if $\phi(a_1) = \phi(a_2)$, or if one of the edges aa_1 or aa_2 was contracted, then $L_\phi \mid_{C'}$ is a Hall 2-assignment to C', which is bipartite, and thus ϕ extends to a proper 3-coloring of C'. If both are precolored and $\phi(a_1) \neq \phi(a_2)$, then since we are guaranteed that C' has at least one vertex among $\{b, c, d\}$ with a full list and the other two vertices have lists of cardinality two, we can color a with $[3] \setminus \{\phi(a_1), \phi(a_2)\}$ and are guaranteed a way to extend ϕ. □

Definition 23. Let L be a Hall assignment to a graph G and let $\sigma \in \mathcal{C}$. A vertex $v \in G(\sigma, L)$ is called a *mandatory witness* to color σ for the list L if the list assignment L' created from L by removing σ from $L(v)$ is not a Hall assignment to G.

Observation 24. *If any vertex is a mandatory witness to a color in a Hall assignment L to G, then*

$$|V(G)| = \sum_{\sigma \in \mathcal{C}} \alpha(G(\sigma, L)).$$

Lemma 25. *For any Hall assignment to a graph, a vertex can be a mandatory witness to at most one color.*

Proof. Let L be a Hall assignment to G and suppose some vertex $v \in V(G)$ is a mandatory witness to colors $\sigma_1, \sigma_2 \in \mathcal{C}$. Let $G' = G - v$. Then for $i \in \{1, 2\}$, since $\alpha(G'(\sigma_i, L)) = \alpha(G(\sigma_i, L)) - 1$:

$$\left(\sum_{\sigma \in \mathcal{C}} \alpha(G(\sigma, L))\right) - 1 = |V(G')| \leqslant \sum_{\sigma \in \mathcal{C}} \alpha(G'(\sigma, L)) \leqslant \left(\sum_{\sigma \in \mathcal{C}} \alpha(G(\sigma, L))\right) - 2$$

a contradiction. □

Theorem 26. *Let $k \geqslant 2$. If G is Hall k-extendible and u is a vertex of degree one in G, then $G - u$ is Hall k-extendible.*

Proof. Suppose otherwise, that G is Hall k-extendible, but $G-u$ has a Hall k-precoloring ϕ that is not extendible. Since ϕ is a valid precoloring of G that clearly cannot be extended, (G, L_ϕ) does not satisfy Hall's condition, so there must exist a subgraph H of G such that (H, L_ϕ) does not satisfy $(*)$. If $u \notin V(H)$, then H is a subgraph of $G - u$, but then $(G - u, L_\phi)$ would not satisfy Hall's condition, a contradiction. Hence, $u \in V(H)$. Note $(G - u, L_\phi)$ satisfies Hall's condition but (H, L_ϕ) does not satisfy $(*)$. Therefore, if $H' = H - u$, then (H', L_ϕ) must satisfy $(*)$ with equality. Furthermore, u cannot contribute at all to the summation on the right side of $(*)$ for H, for otherwise (H, L_ϕ) would satisfy $(*)$. Thus the unique neighbor of u, say v, is not precolored by ϕ. Hence u has a full list, but v is a mandatory witness for all k colors. As $k \geqslant 2$, this violates Lemma 25. □

Corollary 27. *If G is Hall k-extendible and uv is an edge in G with $\deg_G(u) = 1$, then $G - uv$ is Hall k-extendible.*

Proof. This follows from Theorem 26 and Theorem 9 (statement 2). □

The conclusion of Corollary 27 is not necessarily true if u is a vertex of degree 2. The graph in Figure 4 is Hall 3-extendible, but deleting the edge connecting v to its unique neighbor of degree 2, say u, leaves a graph that is not Hall 3-extendible because the neighbors of the 4-cycle can be 3-colored to produce the lists in Figure 1. However the deletion of the edge uv prevents the list on v from ever being empty, and one can verify that Hall's condition is satisfied with uv removed.

Recall the *center* of a graph G is the graph $C(G)$ obtained by iteratively deleting vertices of degree one until none exist.

Corollary 28. *If G is Hall k-extendible, then $C(G)$ is Hall k-extendible.*

4 Hall Spectrum for Certain Graph Classes

In light of the fact that Hall k-extendibility does not imply Hall $(k+1)$-extendibility, we introduce the following definition.

Definition 29. The *Hall spectrum* of a graph G is a binary vector $\overline{h}(G) = [h_0, \ldots, h_\beta]$ where $\beta = \Delta(G) - \chi(G)$ and

$$h_i(G) = \begin{cases} 1 & \text{if } G \text{ is Hall } (\chi(G) + i)\text{-extendible} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 18 implies that $h_\beta(G) = 1$ for every graph G. Total Hall extendible graphs have a spectrum consisting entirely of 1's. In this section, we discuss the variety one finds in the behavior of the Hall spectrum.

The following result shows that although Theorem 10 implies that $h_0(G) = 1$ whenever G is a bipartite graph, this fails for graphs with higher chromatic number.

Proposition 30. *For every $k \geqslant 3$, there exists a graph G where $\chi(G) = k$ but G is not Hall k-extendible.*

Proof. Let G be the graph obtained from G_k shown in Figure 2 by adding an edge between each pair of the $k-1$ neighbors of u that have degree one. It is routine to verify this graph is not Hall chromatic extendible. □

Theorem 20 implies that many bipartite graphs are not Hall 3-extendible. Before we discuss this further, we prove a helpful lemma that provides a sufficient condition for a precoloring of a bipartite graph to satisfy Hall's condition.

Lemma 31. *Suppose G is a bipartite graph. If ϕ is a k-precoloring of G for $k \geqslant 3$ such that $|L_\phi(v)| \geqslant 2$ for all but at most one vertex and no lists are empty, then ϕ is Hall.*

Proof. Assume otherwise, that for some subgraph H of G, (H, L_ϕ) does not satisfy $(*)$. By Theorem 4 we may assume there exists some vertex x with $|L_\phi(x)| = 1$. Let $[H_1, H_2]$ be the bipartition of $V(H)$, and assume by symmetry that $x \in H_1$. For each $\sigma \in [k]$, we have $\alpha(H(\sigma, L_\phi)) \geqslant \alpha(H_i(\sigma, L_\phi))$ for $i \in \{1, 2\}$. Since H_1 and H_2 are independent sets, for $i \in \{1, 2\}$ we have

$$|V(H)| > \sum_{\sigma \in [k]} \alpha(H(\sigma, L_\phi)) \geqslant \sum_{\sigma \in [k]} \alpha(H_i(\sigma, L_\phi)) = \sum_{v \in H_i} |L_\phi(v)|.$$

(The strict inequality comes from the assumption that (H, L_ϕ) does not satisfy $(*)$.) We consider two cases.

Case 1: $|H_2| \geqslant \frac{|V(H)|}{2}$. Since all vertices in H_2 have lists of cardinality at least 2,

$$\sum_{v \in H_2} |L_\phi(v)| \geqslant 2 \cdot |H_2| \geqslant 2 \cdot \frac{|V(H)|}{2} = |V(H)|,$$

contradicting H violating $(*)$.

Case 2: $|H_1| \geqslant \frac{|V(H)|+1}{2}$. All vertices in H_1 have lists of cardinality at least two, except for $L_\phi(x)$. Hence

$$\sum_{v \in H_1} |L_\phi(v)| \geqslant 1 + 2(|H_1| - 1) \geqslant 1 + 2\left(\frac{|V(H)|+1}{2} - 1\right) = |V(H)|$$

again contradicting H violating $(*)$. □

By Theorem 20, any bipartite graph G containing the graph $H = G_3$ from Figure 2 (where $k = 3$) as an induced subgraph and having the property that no vertex in $G - H$ has more than one precolored neighbor in H will fail to be Hall 3-extendible. In fact, we can extend this result to bipartite graphs with arbitrarily large girth in the following way.

Theorem 32. *Let G be a bipartite graph with girth $g \geqslant 6$ and $\delta(G) \geqslant k+1$ where $k \geqslant 3$. If G contains a cycle C of length g such that C is the only g-cycle containing v for all $v \in V(C)$, then G is not Hall k-extendible.*

Proof. Let $V(C) = \{v_0, v_1, \ldots, v_{g-1}\}$. For any v_i and v_j in $V(C)$, the distance, $d_C(v_i, v_j)$, along C from v_i to v_j satisfies $d_C(v_i, v_j) \leqslant g/2$.

Claim: For every v_i and v_j in $V(C)$, the length of a shortest path from v_j to v_i whose internal vertices belong to $G - C$ is at least four.

Proof of Claim: Suppose that the length of such a path, call it P', was less than four for some v_i, v_j in $V(C)$. Let P be a shortest path on C from v_i to v_j. Then PP' is a cycle different from C of length $\ell \leqslant g/2 + 3$. As $g \geqslant 6$, we have $\ell \leqslant g$, contradicting the hypotheses. Thus, the Claim has been established.

Since $\delta(G) \geqslant k + 1$ and C has no chords (since its length is the girth), v_0 has at least $k - 1$ neighbors outside C; let $Y = \{y_1, y_1, \ldots, y_{k-1}\}$ be $k - 1$ such neighbors. For each $i \geqslant 1$, let $\{x_1^i, x_2^i, \ldots, x_{k-2}^i\}$ be $k-2$ neighbors of v_i outside C. By the observation above, note that $x_j^i \neq x_l^k$ for any i, j, k, l. Let

$$X = \bigcup_{i=1}^{g-1} \{x_1^i, x_2^i, \ldots, x_{k-2}^i\}$$

and let $H = G[V(C) \cup X \cup Y]$. We claim that H is not Hall k-extendible. Precolor y_i with color $i + 1$ for each $i \in [k - 1]$. Precolor x_i^1 with color $i + 1$ for each $i \in [k - 2]$. Precolor x_i^{g-1} with $i+1$ for all $i \in [k-3]$, and precolor x_{k-2}^{g-1} with color k. Finally, precolor x_i^j with color i for all $2 \leqslant j \leqslant g - 2$. Figure 7 illustrates H and its precoloring when $k = 5$. By Lemma 31 the coloring is Hall but not extendible, therefore H is not Hall k-extendible.

By Theorem 20, it suffices to show that every vertex in $G-H$ has at most one neighbor in H. By the observation, the distance between any two vertices in H is at most $g/2+2$. If $w \in V(G - H)$ has two neighbors z_1 and z_2 in H, then a shortest path between z_1 and z_2 in C together with the path z_1, w, z_2 forms a cycle of length at most $g/2 + 4$. When $g \geqslant 8$, $g/2 + 4 \leqslant g$, a contradiction. When $g = 6$, $g/2 + 4$ is odd, so in fact the cycle has length at most 6, again a contradiction. □

Notice that the hypotheses in Theorem 32 are much stronger than needed. Cycles smaller than length g that are sufficiently far from C can be permitted, and only the vertices of C need to satisfy the minimum degree condition. The neighbors of the cycle vertices can overlap, since many of them are precolored the same colors; for example, $v_1, v_2, \ldots, v_{g-2}$ could be adjacent to a common set of $k - 1$ vertices. This illustrates how rare it is for a bipartite graph to be Hall k-extendible for some $k \leqslant \delta(G)$. The behavior of the Hall spectrum beyond coordinate $\delta(G) - 2$ is still poorly understood.

Combining Theorem 9 (statement 1), Theorem 18, and Lemma 31, it is straightforward to show the following:

Corollary 33. *For all $k, \ell \geqslant 3$, the bipartite torus $C_{2k} \,\square\, C_{2\ell}$ is total Hall-extendible.*

For some graph families in addition to those that are total Hall extendible, we have completely determined the Hall spectrum. For example:

Theorem 34. *Let $n \geqslant 6$. The Hall spectrum of the hypercube Q_n is*

$$\overline{h}(Q_n) = [1, \underbrace{0, \ldots, 0}_{n-3}, 1].$$

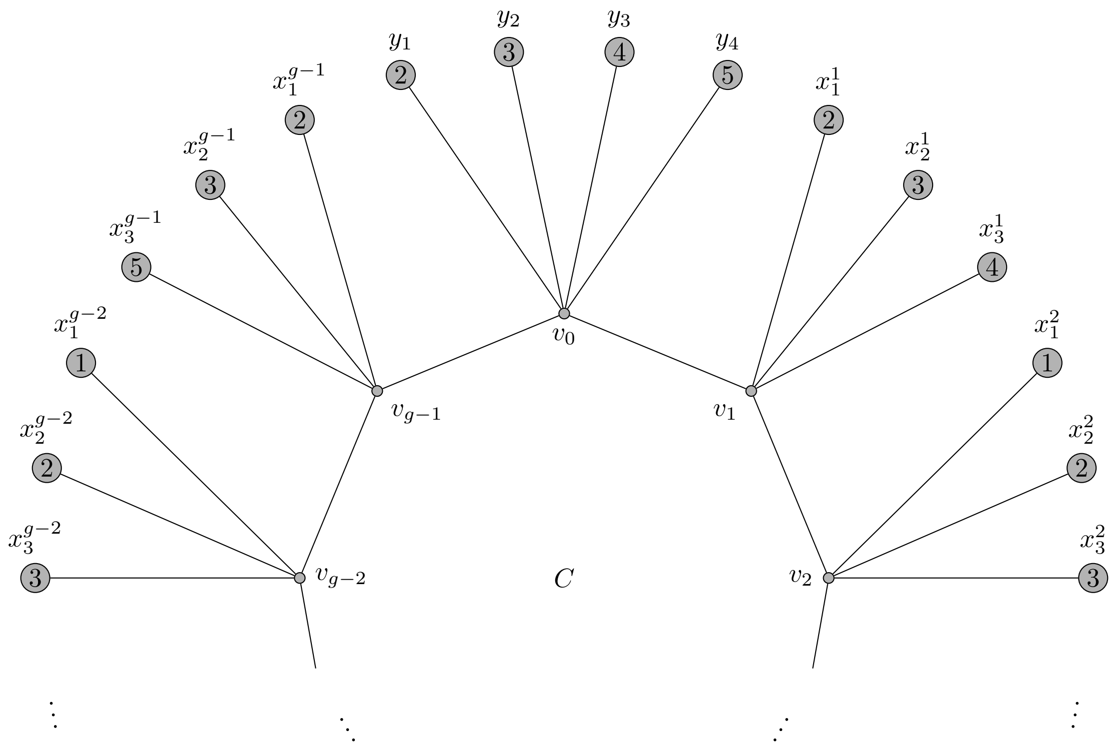

Figure 7: A Hall 5-precoloring of the graph H from the proof of Theorem 32 which is not extendible.

We will prove this theorem through a series of lemmas.

Lemma 35. *For every $n \geqslant 6$, the graph Q_n is not Hall 3-extendible.*

Proof. We induct on n. When $n = 6$, view $V(Q_6)$ as the bitstrings of length six. Let ϕ precolor 100100 and 110010 with color 1, 100010 and 000100 with color 2, and 010001 with color 3. It is routine to verify that L_ϕ restricted to the 4-cycle (000000, 100000, 110000, 010000) has lists matching Figure 1 and hence ϕ is not extendible. Furthermore, vertex 100000 is the only vertex with a list of cardinality less than two, so Lemma 31 implies that Q_6 is not Hall 3-extendible.

Now assume Q_n is not Hall 3-extendible for some $n \geqslant 6$. By Corollary 21, $Q_{n+1} = Q_n \square K_2$ is not Hall 3-extendible. $\square$

It is unknown whether Q_4 or Q_5 are Hall 3-extendible. However, the next lemma shows that Q_5 is not Hall 4-extendible.

Lemma 36. *For every $n \geqslant 5$, the graph Q_n is not Hall $(n-1)$-extendible.*

Proof. We view $V(Q_n)$ as the set of bitstrings of length $n \geqslant 1$. Let $X_n, Y_n \in V(Q_n)$ be defined as

$$X_n = \underbrace{0 \cdots 0}_{n \text{ zeros}} \text{ and } Y_n = \underbrace{0 \cdots 0}_{n-1 \text{ zeros}} 1.$$

Throughout the proof we shall construct new bitstrings via concatenating X_n or Y_n on the right. For each $n \geqslant 5$, we recursively define an $(n-1)$-precoloring ϕ_n of Q_n that has the following three properties, which we shall refer to in the remainder of the proof:

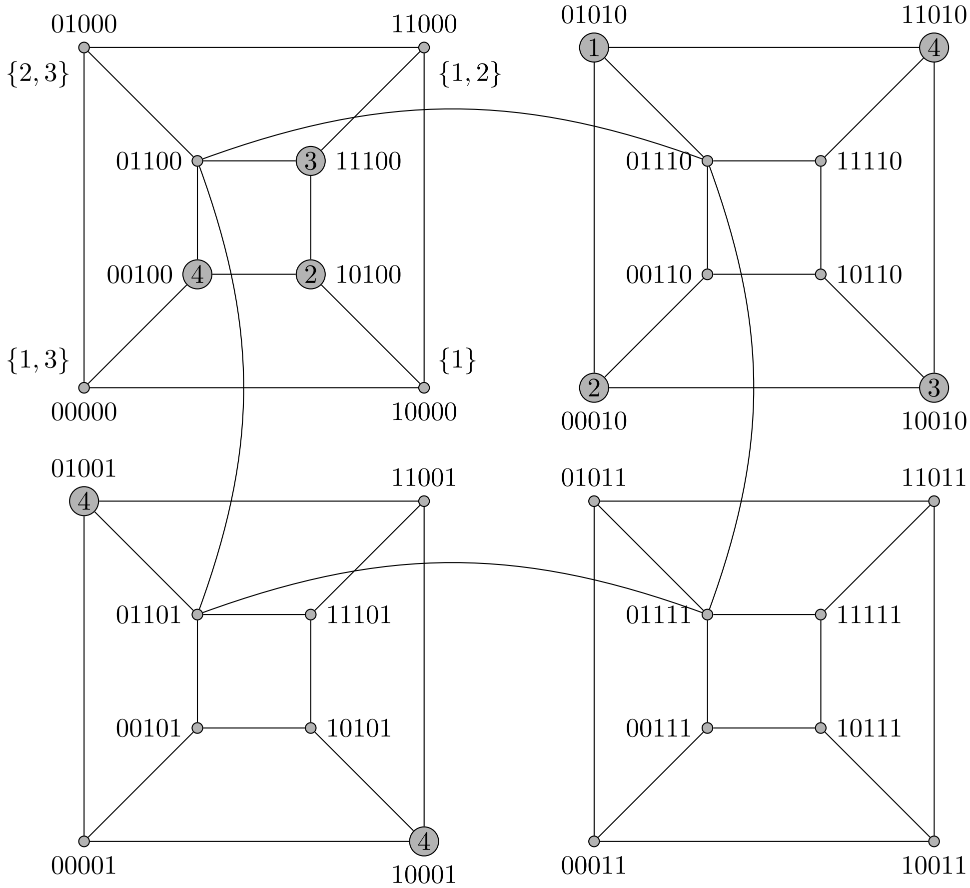

Figure 8: Illustration for the 4-precoloring ϕ_5 of Q_5 (not all edges or lists shown) in the proof of Theorem 36. Notice $u' = 1100Y_{5-4} = 11001$ and $v' = 0000Y_{5-4} = 00001$ are uncolored by ϕ_5, so the vertices $u'0 = 1100Y_{6-5}0 = 110010$ and $v'0 = 0000Y_{6-5}0 = 000010$ (as well as $10000Y_{6-5} = 100001$ and $01000Y_{6-5} = 010001$) will all be assigned color 5 by the 5-precoloring ϕ_6 of Q_6. This will preserve the lists on the four outer vertices in the upper left block.

(1) The 4-cycle $(0000X_{n-4}, 1000X_{n-4}, 1100X_{n-4}, 0100X_{n-4})$ has the lists in Figure 1.

(2) There are no empty lists and $1000X_{n-4}$ is the only vertex with a list of cardinality one.

(3) Either both $0000Y_{n-4}$ and $1100Y_{n-4}$ are uncolored, or both $1000Y_{n-4}$ and $0100Y_{n-4}$ are uncolored.

For $n = 5$, let $\phi_5(01010) = 1$, $\phi(10100) = \phi_5(00010) = 2$, $\phi_5(11100) = \phi_5(10010) = 3$, and $\phi_5(00100) = \phi_5(11010) = \phi_5(10001) = \phi_5(01001) = 4$. It is routine to verify that ϕ_5 satisfies the three properties (refer to Figure 8).

For $n \geqslant 6$, suppose the precoloring ϕ_{n-1} of Q_{n-1} satisfies properties (1) through (3). Let

$$A = \{0000Y_{n-5}0, 1000Y_{n-5}0, 1100Y_{n-5}0, 0100Y_{n-5}0\}$$

and

$$B = \{00000Y_{n-5}, 10000Y_{n-5}, 11000Y_{n-5}, 01000Y_{n-5}\}$$

and $S = A \cup B$. We shall now define a precoloring ϕ_n of Q_n as follows:

Case 1: $\boldsymbol{v} \notin \boldsymbol{S}$. If v ends in a 1, then ϕ_n does not precolor v. If v ends in a 0, $v = v'0$, where $v' \in V(Q_{n-1})$. Then ϕ_n precolors v if and only if ϕ_{n-1} precolors v', in which case $\phi_n(v) = \phi_{n-1}(v')$. That is, ϕ_n restricted to the subgraph $K \cong Q_{n-1}$ of Q_n consisting of all vertices ending in a 0 is the same as ϕ_{n-1} for vertices not in S.

Case 2: $\boldsymbol{v} \in \boldsymbol{S}$. Either both elements in $S_1 = \{0000Y_{n-5}, 1100Y_{n-5}\}$ or both elements in $S_2 = \{1000Y_{n-5}, 0100Y_{n-5}\}$ are uncolored by ϕ_{n-1} as it satisfies property (3). Consider two possibilities:

(a) S_1 is uncolored by ϕ_{n-1}: let

$$\phi_n(0000Y_{n-5}0) = \phi_n(1100Y_{n-5}0) = \phi_n(10000Y_{n-5}) = \phi_n(01000Y_{n-5}) = n-1$$

(b) S_2 is uncolored by ϕ_{n-1}: let

$$\phi_n(1000Y_{n-5}0) = \phi_n(0100Y_{n-5}0) = \phi_n(00000Y_{n-5}) = \phi_n(11000Y_{n-5}) = n-1$$

The remaining vertices of S are uncolored by ϕ_n.

We verify that ϕ_n satisfies the necessary properties (again refer to Figure 8). It is clear that property (3) is satisfied, since exactly two vertices with a 1 in the final position are colored. Property (1) is maintained since the four vertices given color $n-1$ dominate the 4-cycle ($00000X_{n-5}$, $10000X_{n-5}$, $11000X_{n-5}$, $01000X_{n-5}$). Finally, property (2) is maintained since at each iteration, all additional precolored vertices are given color $n-1$, and hence $\varnothing \neq L_{\phi_{n-1}}(v') \subseteq L_{\phi_n}(v)$ for each vertex $v \notin S$.

Finally for any fixed n, the $(n-1)$-precoloring ϕ_n is Hall because property (2) holds, yet it is not extendible because property (1) holds. Thus Q_n is not Hall $(n-1)$-extendible. □

Theorem 34 now follows easily from these lemmas.

Proof of Theorem 34. Let $n \geqslant 6$. The Hall spectrum $\overline{h}(Q_n)$ has length $\beta + 1 = n - 1$. Since Q_n is bipartite, it is clear that $h_0(Q_n) = 1$. By Theorem 18, $h_{n-2}(Q_n) = 1$. It remains to verify that $h_k(Q_n) = 0$ for all $k \in \{1, \ldots, n-3\}$. By Lemmas 35 and 36, $h_1(Q_n) = h_{n-3}(Q_n) = 0$. Now consider Q_n for some $n > 5$ and $4 \leqslant k \leqslant n-1$; we wish to show that Q_n is not Hall k-extendible. By Lemma 36, Q_{k+1} is not Hall k-extendible, and $k + 1 < n$. As in the proof of Lemma 36, applying Corollary 21 inductively shows that Q_n is not Hall k-extendible. □

The Hall spectrum of wheel graphs with even cycles differs from any other graphs discussed so far. The wheel graph W_n is defined as an n-cycle $v_1, v_2, \ldots, v_n$ with a dominating vertex v_0.

Theorem 37. *Let $n \geqslant 10$. The wheel graph W_n is Hall k-extendible for all $k \geqslant 4$. Furthermore, if n is even, then W_n is not Hall 3-extendible.*

Proof. We first show that W_n is Hall k-extendible for all $k \geqslant 4$. Let $\phi : V_0 \to [k]$ be a Hall k-precoloring of W_n. If the vertex $v_0 \in V_0$, then the uncolored vertices of W_n form a cycle or collection of paths, and ϕ is extendible by Theorem 7. Otherwise, since (W_n, L_ϕ) satisfies Hall's condition, $L_\phi(v_0) \neq \varnothing$. Extend ϕ to ϕ' by coloring v_0 with a color from its list, and update the lists on its neighbors. Since each vertex $v_i \in V(W_n) \setminus \{v_0\}$ has degree 3, $|L_{\phi'}(v_i)| \geqslant 1$, with equality only if v_i is isolated in the graph induced by the uncolored vertices. Hence vertices with $|L_{\phi'}(v_i)| = 1$ can be colored easily. The remainder of the vertices uncolored under ϕ' form a bipartite graph, and $L_{\phi'}$ is a 2-assignment to those vertices. Therefore ϕ' is a Hall-assignment by Theorem 4 and thus ϕ' can be extended by Theorem 7.

It remains to show that when n is even, W_n (which satisfies $\chi(W_n) = 3$) is not Hall 3-extendible. Let θ be a precoloring of W_n that colors v_1 and v_6 with color 1. Since θ does not respect the bipartition of the even outer n-cycle, colors 2 and 3 will be needed to extend the coloring of v_1 through v_n, and hence there will be no color available for the dominating vertex v_0. Therefore θ is not extendible with three colors; it remains to show that it is Hall. Let H be an arbitrary subgraph of W_n. By Theorem 9 (statement 3), we may assume $v_1, v_6 \notin V(H)$. If $v_0 \notin V(H)$, or if $V(H) \subseteq \{v_0, v_2, v_5, v_7, v_n\}$, then H is bipartite (since $n \geqslant 10$), and since all lists have cardinality at least 2, (H, L_θ) satisfies $(*)$ by Theorem 4. Otherwise, there is at least one vertex v_i such that $1 \in L(v_i)$. Since $H - v_0$ is bipartite and colors 2 and 3 appear in each list, $V(H) \setminus \{v_0\}$ contribute at least $|V(H)| - 1$ to the summation on the right side of inequality $(*)$ for $\sigma \in \{2, 3\}$. Furthermore, v_i contributes 1 to this sum for $\sigma = 1$. Therefore (H, L_θ) satisfies $(*)$. Hence, (W_n, L_θ) satisfies Hall's condition. $\square$

Corollary 38. *When n is odd, W_n is total Hall extendible. When n is even and $n \geqslant 10$, the Hall spectrum of W_n satisfies $h_k(W_n) = 1$, unless $k = 0$; i.e.,*

$$\overline{h}(W_n) = \begin{cases} [\underbrace{1, \dots, 1}_{n-3}] & \text{if } n \text{ is odd} \\ [0, \underbrace{1, \dots, 1}_{n-4}] & \text{if } n \geqslant 10 \text{ is even} \end{cases}$$

Completely resolving Hall-extendibility for hypercubes and wheels should be straightforward enough, though some case analysis may be required. The question of whether Q_n is Hall k-extendible is only open when $n \in \{4, 5\}$ and $k = 3$. Similarly, the question of whether W_n is Hall k-extendible is only open when $n \in \{4, 6, 8\}$ and $3 \leqslant k \leqslant n - 1$.

5 Conclusion

Many open questions remain in this area. One question was posed by Bobga et al. [5]:

Question 1: If G is a graph that is not Hall k-extendible for some $k \geqslant \chi(G)$, but is Hall $(k + 1)$-extendible, is it possible that G could fail to be Hall $(k + m)$-extendible for some $m \geqslant 2$? We previously conjectured that the answer to this question is no, and we still believe that to be true, but it has not been proven. Note that if the answer is no, then this would imply that if $h_t(G) = 1$ for some $t \geqslant 1$, then $h_i(G) = 1$ for all $i \geqslant t$.

The behavior of this parameter under edge deletion is poorly understood.

Question 2: If G is Hall k-extendible, then must there exist an edge $e \in E(G)$ such that $G - e$ is also Hall k-extendible? Note that Corollary 28 implies that any Hall k-extendible graph G in which $G - e$ fails to be Hall k-extendible for every e satisfies $C(G) = G$.

There are many similar questions that could be asked regarding edge and vertex deletion.

Most importantly, as total Hall extendible graphs can be viewed as graphs in which precolorings with at least $\chi(G)$ can easily be extended (provided the obviously necessary Hall's condition is satisfied), perhaps the most important question to answer would be the following:

Question 3: Does there exist a characterization of total Hall extendible graphs?

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