Hadwiger's conjecture for 3-arc graphs*

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Abstract

The 3-arc graph of a digraph D is defined to have vertices the arcs of D such that two arcs uv, xy are adjacent if and only if uv and xy are distinct arcs of D with $v \neq x$, $y \neq u$ and u, x adjacent. We prove Hadwiger's conjecture for 3-arc graphs.

Keywords: Hadwiger's conjecture, graph colouring, graph minor, 3-arc graph

1 Introduction

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. An H*-minor* is a minor isomorphic to H. The *Hadwiger number* $h(G)$ of G is the maximum integer k such that G contains a K_k-minor, where K_k is the complete graph with k vertices.

In 1943, Hadwiger [10] posed the following conjecture, which is thought to be one of the most important problems in graph theory:

Hadwiger's Conjecture. For every graph G, $h(G) \geqslant \chi(G)$.

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Hadwiger's conjecture has been proved for graphs G with $\chi(G) \leqslant 6$ [19], and is open for graphs with $\chi(G) \geqslant 7$. This conjecture also holds for particular classes of graphs, including powers of cycles [14], proper circular arc graphs [2], line graphs [18], quasi-line graphs [6] and complements of Kneser graphs [24]. See [21] or more recently [20] for a survey.

In this paper we prove Hadwiger's conjecture for a large family of graphs. Such graphs are defined by means of a graph operator, called the 3-arc graph construction (see Definition 1), which bears some similarities with the line graph operator and path graph operator [4, 16]. This construction was first introduced by Li, Praeger and Zhou [15] in the study of a family of arc-transitive graphs whose automorphism group contains a subgroup acting imprimitively on the vertex set. (A graph is *arc-transitive* if its automorphism group is transitive on the set of oriented edges.) It was used in classifying or characterizing certain families of arc-transitive graphs [9, 11, 15, 17, 25, 26, 27]. Recently, various graph-theoretic properties of 3-arc graphs have been investigated [1, 12, 13, 23].

The original 3-arc graph construction [15] was defined for a finite, undirected and loopless graph $G = (V(G), E(G))$. In G, an *arc* is an ordered pair of adjacent vertices. Denote by $A(G)$ the set of arcs of G. For adjacent vertices u, v of G, we use uv to denote the arc from u to v, and $\{u, v\}$ the edge between u and v. We emphasise that each edge of G gives rise to two arcs in $A(G)$. A *3-arc* of G is a 4-tuple of vertices (v, u, x, y), possibly with $v = y$, such that both (v, u, x) and (u, x, y) are paths of G. The 3-arc graph of G is defined as follows:

Definition 1. [15, 26] Let G be an undirected graph. The *3-arc graph* of G, denoted by $X(G)$, has vertex set $A(G)$ such that two vertices corresponding to arcs uv and xy are adjacent if and only if (v, u, x, y) is a 3-arc of G.

The 3-arc graph construction can be generalised for a *digraph* $D = (V(D), A(D))$ as follows [12], where $A(D)$ is a multiset of ordered pairs (namely, arcs) of distinct vertices of $V(D)$. Here a digraph allows parallel arcs but not loops.

Definition 2. Let $D = (V(D), A(D))$ be a digraph. The *3-arc graph* of D, denoted by $X(D)$, has vertex set $A(D)$ such that two vertices corresponding to arcs uv and xy are adjacent if and only if $v \neq x$, $y \neq u$ and u, x are adjacent.

Let D be the digraph obtained from an undirected graph G by replacing each edge $\{x, y\}$ by two opposite arcs xy and yx. Then, $X(D) = X(G)$.

Knor, Xu and Zhou [12] introduced the notion of 3-*arc colouring* of a digraph, which can be defined as a proper vertex-colouring of $X(D)$. The minimum number of colours in a 3-arc colouring of D is called the 3-*arc chromatic index* of D, and is denoted by $\chi'_3(D)$. Then $\chi(X(D)) = \chi'_3(D)$.

The main result of this paper is the following:

Theorem 3. *Let D be a digraph without loops. Then $h(X(D)) \geqslant \chi(X(D))$.*

Note that in the case of the 3-arc graph of an undirected graph, we have obtained a much simpler proof of Theorem 3.

2 Preliminaries

We need the following notation. Let $D = (V(D), A(D))$ be a digraph. We denote by $A_D\{x, y\}$ the set of arcs between vertices x and y, and by $A_D(x)$ the set of arcs outgoing from x. Then vertices x and y are adjacent if and only if $A_D\{x, y\} \neq \emptyset$. When $|A_D\{x, y\}| = 1$, we misuse the notation $A_D\{x, y\}$ to indicate the arc between x and y. An *in-neighbour* (respectively, *out-neighbour*) of a vertex x of D is a vertex y such that $yx \in A(D)$ (respectively, $xy \in A(D)$). The set of all in-neighbours (respectively, out-neighbours) of x is denoted by $N^-_D(x)$ (respectively, $N^+_D(x)$). The *in-degree* $d^-_D(x)$ (respectively, *out-degree* $d^+_D(x)$) is defined to be the number of in-neighbours (respectively, out-neighbours) of x. A vertex x is called a *sink* if $d^+_D(x) = 0$. A digraph is *simple* if $|A_D\{x, y\}| \leqslant 1$ for all distinct vertices x and y of D. A *tournament* is a simple digraph whose underlying undirected graph is complete.

For an undirected graph G, the degree of a vertex v in G is denoted by $d_G(v)$, and the minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We omit the subscript when there is no ambiguity. For notation not given here we refer to [3].

A K_t-minor in G can be thought of as t connected subgraphs in G that are pairwise disjoint such that there is at least one edge of G between each pair of subgraphs. Each such subgraph is called a *branch set*.

Lemma 4. *Let D be a tournament on $n \geqslant 5$ vertices. Then $h(X(D)) \geqslant n$.*

Proof. Since D is a tournament, $A\{x, y\}$ is interpreted as a single arc. Denote $V(D) = \{x, v_0, v_1, \ldots, v_{n-2}\}$. We now construct a collection of n branch sets. For $0 \leqslant i \leqslant n-2$, let $B_i := \{A\{x, v_i\}, A\{v_{i+1}, v_{i+2}\}\}$. Let $U := \{A\{v_i, v_{i+2}\} \mid 0 \leqslant i \leqslant n-2\}$, where all subscripts are taken modulo $n-1$. Clearly, these branch sets are pairwise disjoint.

Now we show that each branch set is connected. Note that each B_i induces K_2 in $X(D)$. Since $A\{v_i, v_{i+2}\}$ is adjacent to $A\{v_{i+1}, v_{i+3}\}$ in $X(D)$, U induces a subgraph that contains an $(n-1)$-cycle passing through each element of U.

Next we show that these branch sets are pairwise adjacent. For each pair of distinct B_i, B_j, if $j \neq i+1$ and $j \neq i+2$, then B_i and B_j are adjacent since $A\{v_{i+1}, v_{i+2}\}$ is adjacent to $A\{x, v_j\}$. If $j = i+1$, then $i \neq j+1$ and $i \neq j+2$ because $n-1 \geqslant 4$, so $A\{x, v_i\}$ is adjacent to $A\{v_{j+1}, v_{j+2}\}$. If $j = i+2$, then $A\{v_{j+1}, v_{j+2}\}$ is adjacent to $A\{v_{i+1}, v_{i+2}\}$ since $\{v_{j+1}, v_{j+2}\} \cap \{v_{i+1}, v_{i+2}\} = \emptyset$. Thus, B_i is adjacent to B_j as well. Since $A\{x, v_i\} \in B_i$ is adjacent to $A\{v_{i+1}, v_{i+3}\} \in U$, each B_i is adjacent to U. □

Let v be a vertex of a digraph D. Let $A \subseteq A(v)$. An arc xy is said to be *A-feasible* if $vx \in A$, $y \neq v$ and (v, x, y) is a directed path. A set $A^f \subseteq A(D)$ is *A-feasible* if each arc in A^f is A-feasible and no two arcs in A^f share a tail. An arc xy of D is said to be *A-compatible* if $y \neq v$, $A\{v, x\} \neq \emptyset$ and $vx \notin A$. A set $A^c \subseteq A(D)$ is *A-compatible* if each arc in A^c is A-compatible. Note that each feasible arc xy is adjacent in $X(D)$ to each arc in A except vx, and each compatible arc xy is adjacent to each arc in A. For example, let $A = \{vv_0, vv_1, vv_2\}$ (see Fig. 1). Then each of v_0v_0', v_1v_1' and v_2v_2' is A-feasible, and each of v_3v_3' and ww' is A-compatible.

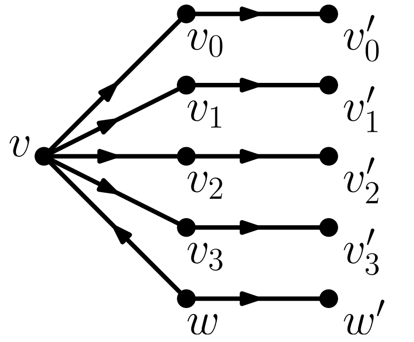

Figure 1: An illustration for A-feasibility and A-compatibility. Let $A = \{vv_0, vv_1, vv_2\}$, then each of v_0v_0', v_1v_1' and v_2v_2' is A-feasible, and each of v_3v_3' and ww' is A-compatible.

Let A^f be an A-feasible set, and A^c be an A-compatible set. An (A, A^f, A^c)*-net* of *size* p is a K_p-minor in $X(D)$ using only arcs in $A \cup A^f \cup A^c$ such that $p := |A|$ and each branch set has exactly one arc in A. An (A, A^f, A^c)-net is called a *net at* v if v is the common tail of all arcs in A. It may happen that one of A^f and A^c is empty. The following lemma provides some sufficient conditions for the existence of an (A, A^f, A^c)-net.

Lemma 5. *Let v be a vertex of a digraph D. Let $A \subseteq A(v)$ and $p := |A|$. Let A^f be an A-feasible set. Let A^c be an A-compatible set. Then, in the following cases, D contains an (A, A^f, A^c)-net.*

(1) $p = 1$;

(2) $|A^c| \geqslant 1$ *and* $p = 2$;

(3) $|A^f| = 3$ *and* $p = 3$;

(4) $|A^f| \geqslant 1$ *and* $|A^c| \geqslant 1$ *and* $p = 3$;

(5) $|A^c| \geqslant 2$ *and* $p = 3$;

(6) $|A^f| + |A^c| \geqslant p - 1$ *and* $p \geqslant 4$.

Proof. Denote $A = \{vv_0, vv_1, \ldots, vv_{p-1}\}$, and without loss of generality, assume that $A(v_j) - \{v_jv\} \neq \emptyset$ for $0 \leqslant j \leqslant |A^f| - 1$. Denote the elements of A^f by

$$v_0v_0', v_1v_1', \ldots, v_{|A^f|-1}v_{|A^f|-1}'.$$

Note that (v, v_j, v_j') is a directed path for $0 \leqslant j \leqslant |A^f| - 1$. Consider the following possibilities:

(1) $p = 1$: Then $\{vv_0\}$ is a trivial $(A, \emptyset, \emptyset)$-net of size 1.

(2) $|A^c| \geqslant 1$ and $p = 2$: Let ww' be an A-compatible arc and $A^c := \{ww'\}$. Since ww' is adjacent to each arc of A, $\{vv_0\}$, $\{vv_1, ww'\}$ form an $(A, \emptyset, A^c)$-net of size 2. See Fig 1.

(3) $|A^f| = 3$ and $p = 3$: Then $\{vv_0, v_1v_1'\}$, $\{vv_1, v_2v_2'\}$ and $\{vv_2, v_0v_0'\}$ form an $(A, A^f, \emptyset)$-net of size 3. See Fig 1.

(4) $|A^f| \geqslant 1$ and $|A^c| \geqslant 1$ and $p = 3$: Let ww' be an A-compatible arc and $A^c := \{ww'\}$. Note that ww' is adjacent to each vv_i, and v_0v_0' is adjacent to vv_2 in $X(D)$. So $\{vv_0, ww'\}$, $\{vv_1, v_0v_0'\}$ and $\{vv_2\}$ form an (A, A^f, A^c)-net of size 3.

(5) $|A^c| \geqslant 2$ and $p = 3$: Similar to case (4), $\{vv_0, ww'\}$, $\{vv_1, yy'\}$ and $\{vv_2\}$ form an (A, A^f, A^c)-net of size 3, where A^c contains two A-compatible arcs yy' and ww'.

(6) $|A^f| + |A^c| \geqslant p - 1$ and $p \geqslant 4$: Let $\beta_j := v_jv_j'$ for $0 \leqslant j \leqslant |A^f| - 1$. Since $|A^c| \geqslant p - 1 - |A^f|$, we can choose $p - 1 - |A^f|$ arcs from A^c and name them as $\beta_{|A^f|}$, $\beta_{|A^f|+1}$, $\ldots$, β_{p-2}. Define $B_j := \{vv_j, \beta_{j+1}\}$ for $0 \leqslant j \leqslant p-3$, $B_{p-2} := \{vv_{p-2}, \beta_0\}$, and $B_{p-1} := \{vv_{p-1}\}$. For $0 \leqslant i < j \leqslant p-2$, observe that in $X(D)$, $vv_j \in B_j$ is adjacent to α_i if $i \neq j-1$; and $vv_i \in B_i$ is adjacent to α_j if $i = j-1$, where $\alpha_j \in B_j - \{vv_j\}$ and $\alpha_i \in B_i - \{vv_i\}$. Thus, B_j and B_i are adjacent. In addition, since $vv_{p-1} \in B_{p-1}$ is adjacent in $X(D)$ to every β_j, B_{p-1} is adjacent to B_j with $j \leqslant p-2$. Thus, $B_0, \ldots, B_{p-1}$ form an (A, A^f, A^c)-net of size p. □

Note that if D contains an (A, A^f, A^c)-net of size p, then $X(D)$ contains a K_p-minor and $h(X(D)) \geqslant p$.

A graph G with chromatic number k is called *k-critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. The following result is well known:

Lemma 6. *Let G be a k-critical graph. Then*

(a) *G has minimum degree at least $k-1$, when $k \geqslant 2$* [7]*;*

(b) *no vertex-cut of G induces a clique when $k \geqslant 3$ and G is noncomplete* [8].

Let D be a simple digraph. For each arc $uv \in A(D)$, define $S_D(uv) := d^+(u) + d^+(v) - 1$.

Lemma 7. *For a simple digraph D,*

$$\sum_{uv \in A(D)} S_D(uv) \quad = \quad \sum_{v \in V(D)} d^+(v)(d(v)-1),$$

where $d(v) = d^+(v) + d^-(v)$.

Proof.

$$\begin{aligned}
\sum_{uv \in A(D)} S_D(uv) &= \sum_{uv \in A(D)} (d^+(u) + d^+(v) - 1) \\
&= \sum_{uv \in A(D)} d^+(u) + \sum_{uv \in A(D)} d^+(v) - \sum_{uv \in A(D)} 1 \\
&= \sum_{u \in V(D)} d^+(u)d^+(u) + \sum_{v \in V(D)} d^+(v)d^-(v) - \sum_{u \in V(D)} d^+(u) \\
&= \sum_{w \in V(D)} d^+(w)(d^+(w) + d^-(w) - 1) \\
&= \sum_{w \in V(D)} d^+(w)(d(w) - 1).
\end{aligned}$$

□

3 Proof of Theorem 3

In this proof, we assume that, for every pair of distinct vertices u and v of D, there is at most one arc from u to v and at most one arc from v to u. That is, $A_D\{u,v\} \subseteq \{uv, vu\}$. That is because all the arcs from u to v can be assigned the same colour and deleting an arc does not increase $h(X(D))$.

Let D be a digraph. An arc uv of D is called *redundant* if $A_D(u) \subseteq A_D\{u,v\}$ or $A_D(v) \subseteq A_D\{u,v\}$. Note that if uv is redundant then so is vu if it exists. Let D' be the digraph obtained from D by deleting all redundant arcs. Let G be the (simple) underlying undirected graph of D'. We have the following claim:

Claim 1. $\chi(X(D)) \leqslant \chi(G)$.

Proof. Since G is the underlying undirected graph of D', $V(G) = V(D') = V(D)$. Let $c : V(G) \to \{1, 2, \ldots, \chi(G)\}$ be a $\chi(G)$-colouring of G. For each arc $uv \in A(D)$, define $f(uv) := c(u)$. We now show that f is a 3-arc colouring of D. For every pair of arcs $uv, xy \in A(D)$ adjacent in $X(D)$, we have that $A_D\{u,x\} \neq \emptyset$ (that is, u, x are adjacent), and both uv and xy are not in $A_D\{u,x\}$. Thus, some arc between u and x is not redundant, and u and x are adjacent in G. So, $f(uv) = c(u) \neq c(x) = f(xy)$. It follows that f is a 3-arc colouring of D and $\chi(X(D)) \leqslant \chi(G)$. □

Hadwiger's conjecture is true for k-chromatic graphs with $k \leqslant 6$. So assume that $\chi(X(D)) \geqslant 7$. Let $k := \chi(G)$ and let H be a k-critical subgraph of G. By Lemma 6(a), $\delta(H) \geqslant k - 1$.

Let F be an orientation of H such that each arc uv of F inherits the orientation of an arc in $A_D\{u,v\}$ and the number of out-degree 1 vertices in F is minimized. An arc $xy \in A(D)$ is called *potential* if $xy \notin A(F)$. In particular, every redundant arc is potential. F has the following property:

Property A. If $d^+_F(v) = 1$ and $A_F(v) = \{vw\}$, then there exists one potential arc vz outgoing from v in D such that $vz \neq vw$. If further $zv \in A(F)$, then $d^+_F(z) = 2$.

Proof. Since vw is not redundant, $A_D(v) \not\subseteq A_D\{v,w\}$. Let $vz \in A_D(v) - A_D\{v,w\}$. Then $vz \neq vw$. Since vw is the unique outgoing arc from v in F, vz is potential. Suppose that $zv \in A(F)$. If $d^+_F(z) \neq 2$, let F' be obtained from F by replacing zv by vz. Then $d^+_{F'}(z) \neq 1$, $d^+_{F'}(v) = 2$ and the out-degree of every other vertex remains unchanged. Hence F' is an orientation of H with fewer out-degree 1 vertices than F, which is a contradiction. □

In addition, for each arc xy of F, by the definition of D', $A_D(y) \not\subseteq A_D\{x,y\}$. That is, there is an arc other than yx outgoing from y (hence, $d^+_D(y) \geqslant 1$) and there is a directed path in D of length 2 starting from the arc xy, even if $d^+_F(y) = 0$. Note that F is a simple digraph and $d_F(v) = d^+_F(v) + d^-_F(v) = d_H(v) \geqslant k - 1$ by Lemma 6(a).

By Claim 1, it suffices to prove that $h(X(D)) \geqslant k = \chi(G) \geqslant \chi(X(D))$.

Let $v \in V(F)$ be a vertex with maximum out-degree $\Delta^+_F(v)$. If $\Delta^+_F(v) \geqslant k$, let $A \subseteq A_F(v)$ with $|A| = k$, and let A^f be a maximal A-feasible set. Then $|A^f| = k \geqslant 6$ since there exists a directed path of length 2 starting from every arc of A. By Lemma

5(6) with $p = k$, there exists an $(A, A^f, \emptyset)$-net of size k. Thus, $h(X(D)) \geqslant k$, and the result holds.

Now assume that $\Delta^+(F) \leqslant k-1$. By Lemma 7 and since F has minimum degree at least $k-1$,

$$\sum_{uv \in A(F)} S_F(uv) = \sum_{v \in V(F)} d_F^+(v)(d_F(v)-1) \geqslant (k-2) \sum_{v \in V(F)} d_F^+(v) = (k-2)e(F), \quad (1)$$

where $e(F)$ is the number of arcs of F.

If $\sum_{uv\in A(F)} S_F(uv) = (k-2)e(F)$, then $d_H(x) = d_F(x) = k-1$ for every $x \in V(F)$. Since $\chi(H) = k$, by Brooks' Theorem [5], $H \cong K_k$ and F is a tournament. By Lemma 4, $h(X(D)) \geqslant h(X(F)) \geqslant k$, the result follows.

Now assume that $\sum_{uv\in A(F)} S_F(uv) > (k-2)e(F)$. We call a vertex v of F *special* if $d_F^+(v) = k-2$ and $d_F^-(v) = 1$ and $d_F^+(v') = 0$ for each $vv' \in A_F(v)$. Let W be the set of all special vertices of F, and let $W^+ := \{xy \in A(F) \mid x \in W\}$. Let F' be the digraph obtained from F by deleting the arcs in W^+. Then, for each vertex v of F' with $d_{F'}^+(v) = d_{F'}(v) - 1 = k-2$, the head of (at least) one arc $vv' \in A(F')$ is not a sink in F; that is, $d_F^+(v') \geqslant 1$. Since this outgoing arc at v' in F is not redundant, $|d_D^+(v')| \geqslant 2$.

Denote by Q the set of sinks of F. Then each arc of W^+ has its tail in W and head in Q. Note that W is independent in F, and $W \cap Q = \emptyset$. By Lemma 7,

$$\begin{aligned}
&(k-2)e(F)\\
<\ & \sum_{uv\in A(F)} S_F(uv)\\
=\ & \sum_{v\in V(F)} d_F^+(v)(d_F(v)-1)\\
=\ & \sum_{v\in V(F)-(W\cup Q)} d_F^+(v)(d_F(v)-1) + \sum_{v\in Q} d_F^+(v)(d_F(v)-1) + \sum_{v\in W} d_F^+(v)(d_F(v)-1)\\
=\ & \Bigg(\sum_{v\in V(F')-(W\cup Q)} d_{F'}^+(v)(d_{F'}(v)-1)\Bigg) + 0 + (k-2)\Bigg(|W^+| + \sum_{v\in W} d_{F'}^+(v)\Bigg).
\end{aligned}$$

Since vertices in $W \cup Q$ have outdegree 0 in F',

$$\begin{aligned}
(k-2)e(F) &< \Bigg(\sum_{v\in V(F')} d_{F'}^+(v)(d_{F'}(v)-1)\Bigg) + |W^+|(k-2)\\
&= \Bigg(\sum_{uv\in A(F')} S_{F'}(uv)\Bigg) + |W^+|(k-2).
\end{aligned}$$

Thus $\sum_{uv\in A(F')} S_{F'}(uv) > (k-2)(e(F) - |W^+|) = (k-2)e(F')$. Let uv be an arc of F' with maximum $S_{F'}(uv)$. Thus, $S_F(uv) \geqslant S_{F'}(uv) \geqslant k-1$. If $v \in W$, then $d_{F'}^+(v) = 0$ and $d_{F'}^+(u) \geqslant k$, which contradicts the assumption that $\Delta^+(F) \leqslant k-1$. Hence $v \notin W$.

Denote $A_F(u) = \{uv, uu_1, uu_2, \ldots, uu_i\}$ and $A_F(v) = \{vv_1, vv_2, \ldots, vv_j\}$, where $i+j = S_F(uv) \geqslant k-1$. Set $T := \{u_1, u_2, \ldots, u_i\} \cap \{v_1, v_2, \ldots, v_j\}$. Denote $N_1 := N_F(u) - \{v\}$

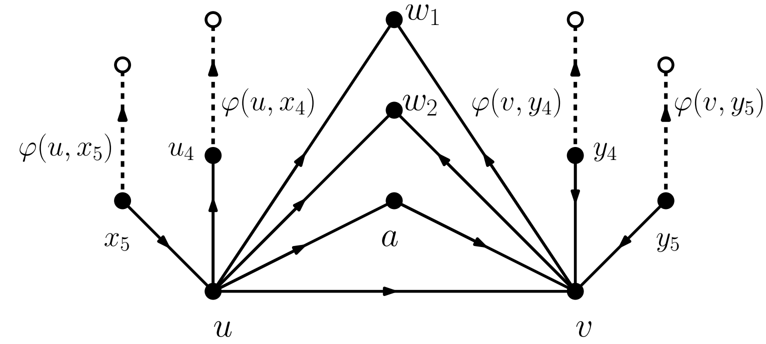

Figure 2: An illustration for $A_F(u), A_F(v), \varphi(u, x_l)$ and $\varphi(v, y_l)$ for a case with $i + j = S_F(uv) \geqslant 6$, where $w_1 = u_1 = x_1 = v_1 = y_1$, $w_2 = u_2 = x_2 = v_2 = y_2$, $a = u_3 = x_3 = y_3$ and $u_4 = x_4$.

and $N_2 := N_F(v) - \{u\}$. Say $N_1 = \{x_1, x_2, \ldots, x_r\}$, and $N_2 = \{y_1, y_2, \ldots, y_s\}$. Since F has minimum degree at least $k-1$, both r and s are at least $k-2$. See Fig. 2 for an illustration for a case with $k = 7$, in which $A_F(u) = \{uv, uu_1 = uw_1, uu_2 = uw_2, uu_3 = ua, uu_4\}$, $A_F(v) = \{vv_1 = vw_1, vv_2 = vw_2\}$, $T = \{w_1, w_2\}$, $N_1 = \{x_1 = w_1, x_2 = w_2, x_3 = a, x_4 = u_4, x_5\}$ and $N_2 = \{y_1 = w_1, y_2 = w_2, y_3 = a, y_4, y_5\}$.

Since the arc $A_F\{u, x_l\}$ is not redundant, $A_D(x_l) \nsubseteq A_D\{u, x_l\}$. Thus, for each $x_l \in N_1$, to arc $A_F\{u, x_l\} \in A(F)$ we can associate an arc, denoted $\varphi(u, x_l)$, which is chosen from $A_D(x_l) - A_D\{u, x_l\}$. Similarly, for each $y_l \in N_2$, associate an arc, denoted $\varphi(v, y_l)$, in $A_D(y_l) - A_D\{v, y_l\}$ to arc $A_F\{v, y_l\} \in A(F)$. An illustration for the definition of $\varphi(u, x_l)$ and $\varphi(v, y_l)$ is given in Fig. 2.

Choose these arcs $\varphi(u, x_l)$ and $\varphi(v, y_l)$ such that if $\Sigma := \cup_{l=1}^{r} \varphi(u, x_l)$ and $\Pi := \cup_{l=1}^{s} \varphi(v, y_l)$ then $t := |\Sigma \cap \Pi|$ is minimized. We now prove that, for each $ww' \in \Sigma \cap \Pi$, ww' is the unique arc outgoing from w in D, $A_F\{u, w\} = uw$, $A_F\{v, w\} = vw$ and $w' \notin \{u, v\}$. Since $ww' = \varphi(u, w) = \varphi(v, w)$, we have $w' \notin \{u, v\}$. Suppose that $|A_D(w)| \geqslant 2$, and ww'' is an arc outgoing from w other than ww' in D. Then at least one of u and v, say u, is not equal to w''. Now set $\varphi(u, w) := ww''$ and keep $\varphi(v, w) = ww'$. Then $|\Sigma \cap \Pi|$ is decreased. Thus, ww' is the unique arc outgoing from w in D. Since $A_D(w) = \{ww'\}$, we have that $A_F\{u, w\} = uw$ and $A_F\{v, w\} = vw$.

Denote $\Sigma \cap \Pi = \{w_1 w_1', w_2 w_2', \ldots, w_t w_t'\}$. Then $w_l \in T$ for each $l \in [1, t]$ and $t \leqslant |T| \leqslant \min\{i, j\}$. Consider the following cases:

Case 1. $S_F(uv) \geqslant k$.

In this case, we will construct an (A, A^f, A^c)-net $\mathcal{A}$ and a (B, B^f, B^c)-net $\mathcal{B}$, for some $A \subseteq A_F(u) - \{uv\}$ and $B \subseteq A_F(v)$, such that $(A \cup A^f \cup A^c) \cap (B \cup B^f \cup \mathrm{B}^c) = \emptyset$. Since each branch set in $\mathcal{A}$ contains an outgoing arc at u other than uv, and each branch set in $\mathcal{B}$ contains an outgoing arc at v other than vu, each branch set in $\mathcal{A}$ is adjacent in $X(D)$ to each branch set in $\mathcal{B}$. Since each branch set in $\mathcal{A}$ is contained in $A \cup A^f \cup A^c$, and each branch set in $\mathcal{B}$ is contained in $B \cup B^f \cup \mathrm{B}^c$, no branch set in $\mathcal{A}$ intersects a branch set in

$\mathcal{B}$. Hence $\mathcal{A} \cup \mathcal{B}$ defines a complete minor in $X(D)$ on $|\mathcal{A}| + |\mathcal{B}|$ vertices. In most cases we construct $\mathcal{A}$ and $\mathcal{B}$ such that $|\mathcal{A}| + |\mathcal{B}| \geqslant k$, giving a K_k-minor in $X(D)$, as desired. Finally, we always choose $A^c \subseteq \Sigma$ and $B^c \subseteq \Pi$ in such a way that $A^c \cap B^c = \emptyset$.

Note that $i + j \geqslant k$. By the assumption that $\Delta^+(F) \leqslant k - 1$, we have $1 \leqslant i \leqslant k - 2$ and $2 \leqslant j \leqslant k - 1$.

Case 1.1. $j = k - 1$: Then $i \geqslant 1$. Let $B := A_F(v)$, and B^f be a maximal B-feasible set in D. For $y_l \in N_2$, since $A_D\{y_l, v\}$ is not redundant, $A_D^+(y_l) - A_D\{y_l, v\} \neq \emptyset$. Thus, $|B^f| = |B| = k - 1 \geqslant 4$. By Lemma 5(6) with $p = |B^f| = k - 1$ and $|B^c| = 0$, there exists in D a $(B, B^f, \emptyset)$-net $\mathcal{B}$ of size $k - 1$. Then $\mathcal{B} \cup \{\{uu_1\}\}$ forms the k branch sets of a K_k-minor in $X(D)$, since each branch set of $\mathcal{B}$ contains an outgoing arc at v other than vu and is thus adjacent to uu_1 in $X(D)$ (since $vu \notin B$).

Case 1.2. $j \leqslant k - 2$: Then $0 \leqslant t \leqslant k - 2$. Recall that $t = |\Sigma \cap \Pi| \leqslant |T|$.

Case 1.2.1. $t = k - 2 \geqslant 3$: Suppose first that $\Sigma - \Pi \neq \emptyset$. Let $x_l x_l' \in \Sigma - \Pi$. Since $|A_F(u) - \{uv\}| = i \geqslant t \geqslant 3$, there are distinct arcs uu_a, uu_b in $A_F(u) - \{uv\}$ with $x_l \notin \{u_a, u_b\}$. Let $A := \{uu_a, uu_b\}$. Note that $x_l x_l'$ is A-compatible. Then $\mathcal{A} := \{\{uu_a\}, \{uu_b, x_l x_l'\}\}$ is an $(A, \emptyset, \{x_l x_l'\})$-net of size 2. Let B be a set of $k - 2$ arcs in $A_F(v)$. Then $B^f := \{\varphi(v, y) : vy \in B\}$ is a B-feasible set of $k - 2$ arcs in Π. By Lemma 5(6) with $p = |A^f| = k - 2$ and $|A^c| = 0$, there is a $(B, B^f, \emptyset)$-net $\mathcal{B}$ of size $k - 2$. Each branch set in $\mathcal{A}$ contains an outgoing arc at u other than uv, and each branch set in $\mathcal{B}$ contains an outgoing arc at v other than vu. Thus each branch set in $\mathcal{A}$ is adjacent in $X(D)$ to each branch set in $\mathcal{B}$. Since $x_l x_l' \notin \Pi$ and $B^f \subseteq \Pi$, we have $(A \cup \{x_l x_l'\}) \cap (B \cup B^f) = \emptyset$. Thus, no branch set in $\mathcal{A}$ intersects a branch set in $\mathcal{B}$. Hence $\mathcal{A} \cup \mathcal{B}$ is a K_k-minor in $X(D)$.

By symmetry and since uv is not used in this case, if $\Pi - \Sigma \neq \emptyset$, then we obtain a K_k-minor in $X(D)$.

Now assume that $\Sigma = \Pi$. Then $|\Sigma| = |\Pi| = t = k - 2$. Set $w_0 := v$ and $w_0' := w_1$. For $0 \leqslant l \leqslant t$, let $B_l := \{uw_l, w_{l+1}w_{l+1}'\}$, where subscripts are taken modulo $t + 1$; and let $B_{t+1} := \{vw_2\}$. For $0 \leqslant l < l' \leqslant t$, either uw_l is adjacent to $w_{l'+1}w_{l'+1}'$ or $uw_{l'}$ is adjacent to $w_{l+1}w_{l+1}'$. Thus B_l is adjacent to $B_{l'}$. Note that $vw_2 \in B_{t+1}$ is adjacent to $w_1 w_1' \in B_0$ and $uw_l \in B_l$ with $1 \leqslant l \leqslant t$. Thus B_{t+1} is adjacent to every B_l with $0 \leqslant l \leqslant t$. Therefore, $B_0, B_1, \ldots, B_{t+1}$ form the $t + 2 = k$ branch sets of a K_k-minor in $X(D)$.

Case 1.2.2. $\lceil \frac{k}{2} \rceil \leqslant t \leqslant k - 3$: For $k - t \leqslant l \leqslant t$, set $\alpha_l := w_l w_l'$. Choose $k - 2 - t$ arcs $\alpha_{t+1}, \alpha_{t+2}, \ldots, \alpha_{k-2}$ from $\Sigma - \Pi$ (which exist since $|\Sigma - \Pi| = r - t \geqslant k - 2 - t$). Denote $A := \{uw_1, uw_2, \ldots, uw_t\}$. Then, α_l is A-feasible when $k - t \leqslant l \leqslant t$, and α_l is A-compatible when $t + 1 \leqslant l \leqslant k - 2$. Let $A^f := \{\alpha_{k-t}, \alpha_{k-t+1}, \ldots, \alpha_t\}$ and $A^c := \{\alpha_{t+1}, \alpha_{t+2}, \ldots, \alpha_{k-2}\}$. Note that A^f is A-feasible and A^c is A-compatible. By Lemma 5(6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size t in $X(D)$.

Next, for $1 \leqslant l \leqslant k - t - 1$, set $\beta_l := w_l w_l'$. Choose $k - 2 - t$ arcs β_{k-t}, β_{k-t+1}, $\ldots, \beta_{2k-2t-3}$ from $\Pi - \Sigma$ (which exist since $|\Pi - \Sigma| = s - t \geqslant k - 2 - t$). Note that $|\Sigma \cap \Pi| = t \geqslant k - t$ and $2k - 2t - 3 \geqslant k - t$. Let $B := \{vw_1, vw_2, \ldots, vw_{k-t}\}$. Then β_l is B-feasible when $1 \leqslant l \leqslant k - t - 1$, and β_l is B-compatible when $k - t \leqslant l \leqslant 2k - 2t - 3$. Let $B^f := \{\beta_1, \beta_2, \ldots, \beta_{k-t-1}\}$, and $B^c := \{\beta_{k-t}, \beta_{k-t+1}, \ldots, \beta_{2k-2t-3}\}$. Note that B^f is B-feasible and B^c is B-compatible. If $t = k - 3$, then by Lemma 5(4), there exists a

(B, B^f, B^c)-net $\mathcal{B}$ of size $k-t$ in $X(D)$. Otherwise $t \leqslant k-4$ and by Lemma 5(6) with $p = k-t \geqslant 4$ and $|A^f| = k-t-1$ and $|A^c| = k-t-2$, there exists a (B, B^f, B^c)-net $\mathcal{B}$ of size $k-t$ in $X(D)$.

Case 1.2.3. $t \leqslant \lceil \frac{k}{2} \rceil - 1$: Let $j' := k-i$. Since $i+j = S_F(uv) \geqslant k$, we have $j' \leqslant j$.

If $t = 0$, then $\Sigma \cap \Pi = \emptyset$. Let $A := \{uu_1, uu_2, \ldots, uu_i\}$. Note that each arc in Σ is either A-feasible or A-compatible, and no two arcs in Σ share a tail. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible, respectively) arcs in Σ. Then A^f is A-feasible and A^c is A-compatible. Note that $|A^f|+|A^c| = |\Sigma| \geqslant i$, and $A^c \neq \emptyset$ if $i \leqslant 2$ (Σ contains an A-compatible arc since $|\Sigma| = r \geqslant k-2 \geqslant 3$). If $i \geqslant 3$, then by Lemma 5(3) or Lemma 5(6) with $p = |A^f| = i$, there is an $(A, A^f, \emptyset)$-net $\mathcal{A}$ of size i. If $i \leqslant 2$, then $A^c \neq \emptyset$ (since $|\Sigma| = r \geqslant k-2 \geqslant 3 > i$). By Lemma 5(1) or (2) with $p = |A^f| = i$ and $|A^c| \geqslant 1$, there is an $(A, \emptyset, A^c)$-net $\mathcal{A}$ of size i. Similarly, let $B \subseteq A_F(v)$ with $|B| = j'$. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible, respectively) arcs in Π. Note that $|B^f| + |B^c| = |\Pi| = s \geqslant k-2 \geqslant j \geqslant j'$. As in the construction of $\mathcal{A}$, by Lemma 5, there exists a (B, B^f, B^c)-net $\mathcal{B}$ of size j'. $\mathcal{A} \cup \mathcal{B}$ forms a k branch sets of a K_k-minor in $X(D)$.

Suppose that $t \geqslant 1$ and $j = k-2$. If $t = 1$, then let A be a subset of $A_F(u) - \{uv\}$ with $uw_1 \in A$ and $|A| = 2$. Note that $|\Sigma - \Pi| = r - t \geqslant k-3 \geqslant 3$. Then at least one arc in $\Sigma - \Pi$ is A-compatible. If $t \geqslant 2$, then let $A := \{uw_1, uw_2\}$. Then $|\Sigma - \Pi| = r - t \geqslant k - 2 - \lceil \frac{k}{2} \rceil + 1 = \lfloor \frac{k}{2} \rfloor - 1 \geqslant 2$ because $k \geqslant 6$. Again, at least one arc in $\Sigma - \Pi$ is A-compatible. In both cases, by Lemma 5(2), there exists an $(A, \emptyset, A^c)$-net $\mathcal{A}$ of size 2, where A^c is the set of A-compatible arcs in $\Sigma - \Pi$. Let $B := A_F(v)$. Note that each arc in Π is either B-feasible or B-compatible, and no two arcs in Π share a tail. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible, respectively) arcs in Π. Since $|\Pi| = s \geqslant k-2 = j \geqslant 4$, by Lemma 5(6), there is a (B, B^f, B^c)-net $\mathcal{B}$ of size j. Then $\mathcal{A} \cup \mathcal{B}$ forms the k branch sets of a K_k-minor in $X(D)$.

Suppose now that $t \geqslant 1$ and $j \leqslant k-3$. Note that $i \geqslant t$. Consider two possibilities: (i) $i = t$, and (ii) $i \geqslant t+1$. If $i = t$, then $t = i \geqslant k - j \geqslant 3$. Let $A := \{uu_1, uu_2, \ldots, uu_t\} = \{uw_1, uw_2, \ldots, uw_t\}$. Note that $|\Sigma - \Pi| = r - t \geqslant k-2-t \geqslant (2t+1)-2-t = t-1 \geqslant 2$. Since $\Sigma - \Pi \neq \emptyset$, at least one arc in $\Sigma - \Pi$ is A-compatible. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible, respectively) arcs in $\Sigma - \Pi$. By Lemma 5(2), (4) or (6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i. Let $B := \{vv_1, vv_2, \ldots, vv_{j'}\}$. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible, respectively) arcs in Π. Note that $j' = k - t \geqslant k - \lceil \frac{k}{2} \rceil + 1 = \lfloor \frac{k}{2} \rfloor + 1 \geqslant 3$ and $|\Pi| = s \geqslant k-2 \geqslant j \geqslant j'$. By Lemma 5, there is a (B, B^f, B^c)-net $\mathcal{B}$ of size j'.

If $i \geqslant t+1$, then $j' = k - i \leqslant k-t-1$. Let $B := \{vv_1, vv_2, \ldots, vv_{j'}\}$ be a subset of $A_F(v)$ with $vw_1 \in B$. By the assumption that $j \leqslant k-3$, there is at least one incoming arc other than uv at v. Thus, at least one arc in $\Pi - \Sigma$ is B-compatible. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible, respectively) arcs in $\Pi - \Sigma$. Note that $|\Pi - \Sigma| = s - t \geqslant k-t-2 \geqslant j'-1$. By Lemma 5(2), (4) or (6), there is a (B, B^f, B^c)-net $\mathcal{B}$ of size j'. Let $A := \{uu_1, uu_2, \ldots, uu_i\}$. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible, respectively) arcs in Σ. Since $|\Sigma| = r \geqslant k-2 \geqslant i$, by Lemma 5(2), (3) or (6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i.

In each case above, $\mathcal{A} \cup \mathcal{B}$ forms a K_k-minor in $X(D)$.

Case 2. $S(uv) = k-1$: Then $i+j = k-1$.

In this case, we construct an (A, A^f, A^c)-net $\mathcal{A}$ and a (B, B^f, B^c)-net $\mathcal{B}$ as in Case 1, except that $|\mathcal{A}|+|\mathcal{B}| = k-1$. We then define one further branch set B_0 that, with $\mathcal{A}$ and $\mathcal{B}$, forms the desired K_k-minor in $X(D)$.

Case 2.1. $j = 1$: Then $i = k-2$. Let $A := A_F(u) - \{uv\}$. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma - \Pi$. Since $t \leqslant \min\{i, j\} = 1$ and $r \geqslant k-2$, we have $|\Sigma - \Pi| \geqslant r - t \geqslant k-3$. Since $|A^f| + |A^c| = |\Sigma - \Pi| \geqslant k-3$ and $i = k-2 \geqslant 5$, by Lemma 5(6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i. By Property A, there exists a potential arc $vz \neq vv_1$ outgoing from v in D, such that $d_F^+(z) = 2$ if $zv \in A(F)$. Clearly, $z \neq u$ since $d_F^+(u) = i+1 > 3$. Let $B := \{vv_1, vz\}$, and τ be an arc in $\Pi - \Sigma$ such that $\tau \neq \varphi(v, v_1)$ and $\tau \neq \varphi(v, z)$. τ exists because $|\Pi - \Sigma| = s - t \geqslant k-2-t \geqslant k-3 \geqslant 3$. Then $\mathcal{B} := \{\{vv_1\}, \{vz, \tau\}\}$ is a $(B, \emptyset, \{\tau\})$-net of size 2. Thus, $\mathcal{A} \cup \mathcal{B}$ forms a K_k-minor in $X(D)$.

Case 2.2. $2 \leqslant j \leqslant k-3$: Then $2 \leqslant i \leqslant k-3$. Let $U := N_1 \cap N_2$ be the common neighbourhood of u and v in F. Say $U = \{a_1, a_2, \ldots, a_{|U|}\}$. Then $T \subseteq U$ and $t \leqslant |T| \leqslant |U|$. Recall that $t = |\Sigma \cap \Pi|$.

Case 2.2.1. $t \geqslant 2$: Let $A := A_F(u) - \{uv\}$. Since $2 \leqslant t \leqslant \min\{i, j\}$, we have $i = k-1-j \leqslant k-1-t$. Since there is at least one incoming arc at u (because $i \leqslant k-3$), at least one arc in $\Sigma - \Pi$ is A-compatible. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma - \Pi$. Note that $|A^f| + |A^c| = |\Sigma - \Pi| = r - t \geqslant k-2-t \geqslant i-1$. By Lemma 5(2), (4), (5) or (6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i. Let $B := A_F(v)$. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible) arcs in $\Pi - \Sigma$. Similarly, a (B, B^f, B^c)-net $\mathcal{B}$ of size j exists (since $2 \leqslant i, j \leqslant k-3$ and uv is not in $\mathcal{A}$).

Let $B_0 := \{w_1w_1', w_2w_2', uv\}$. Then B_0 induces a connected subgraph in $X(D)$ by noting that uv is adjacent to both w_1w_1' and w_2w_2'. Each branch set of $\mathcal{A}$ and $\mathcal{B}$ contains an arc outgoing from u or v, which is adjacent to w_1w_1' or w_2w_2'. Thus B_0 is adjacent to each branch set of $\mathcal{A} \cup \mathcal{B}$. Hence $\mathcal{A} \cup \mathcal{B} \cup \{B_0\}$ forms a K_k-minor in $X(D)$.

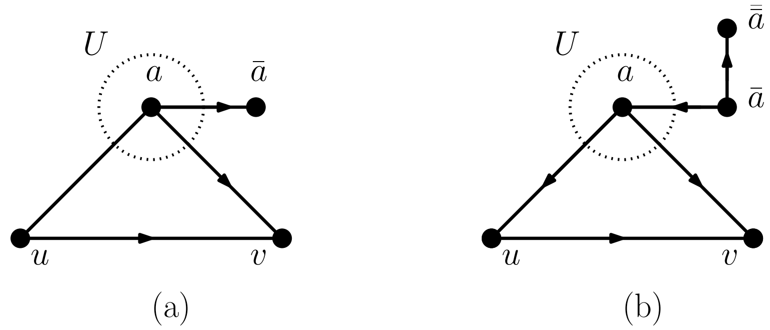

Figure 3: An illustration for the construction of B_0 in Case 2.2.2.

Case 2.2.2. $t \leqslant 1$ and $U \cap N_F^-(v) \neq \emptyset$: That is, there is an arc av in F for some vertex $a \in U$. If there exists an arc $a\bar{a}$ in D with $\bar{a} \notin \{u, v\}$, then let $B_0 := \{uv, a\bar{a}\}$ (see Fig. 3(a)).

Suppose that there is no such an arc $a\bar{a}$. That is, $A_D(a) \subseteq \{au, av\}$. Clearly, $av \in A_D(a)$. Since $A_F\{v, a\}$ is not redundant in F, we have $A_D(a) - A_F\{v, a\} \neq \emptyset$. Thus $au \in A_D(a)$ and $A_D(a) = \{au, av\}$. Let $\bar{a}$ be an in-neighbour other than u, v of a in F. Then $A_F\{a, \bar{a}\} = \bar{a}a$. Let $\bar{\bar{a}} \neq a$ be an out-neighbour of $\bar{a}$ in F. Note that $\bar{\bar{a}}$ exists since $\bar{a}a$ is not redundant. Then, by the minimality of $|\Sigma \cap \Pi|$, we have $\bar{a}\bar{\bar{a}} \notin \Sigma \cap \Pi$. Let $B_0 := \{uv, au, av, \bar{a}\bar{\bar{a}}\}$ (see Fig. 3(b)). Then $\max\{|B_0 \cap \Sigma|, |B_0 \cap \Pi|\} \leqslant 2$ and $|B_0 \cap \Sigma| + |B_0 \cap \Pi| \leqslant 3$.

Let $A := A_F(u) - \{uv\}$ and $B := A_F(v)$. We show that there is a net $\mathcal{A}$ at u of size i, and a net $\mathcal{B}$ at v of size j, such that $\mathcal{A} \cup \mathcal{B} \cup \{B_0\}$ forms a K_k-minor in $X(D)$.

First suppose that $3 \leqslant i, j \leqslant k - 4$. If $|B_0 \cap \Sigma| \leqslant 1$, let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma - \Pi - B_0$. If $|B_0 \cap \Sigma| = 2$, then $|B_0| = 4$ and $\bar{a}\bar{\bar{a}} \in \Sigma \cap B_0$. Thus, $\bar{a}$ is a neighbour of u in F. Note that $\bar{a}a \notin \Sigma$ and $\bar{a}a$ is A-feasible or A-compatible. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $(\Sigma - \Pi - B_0) \cup \{\bar{a}a\}$. In both cases, $|A^f| + |A^c| \geqslant r - t - 1 \geqslant k - 2 - 2 \geqslant i$. By Lemma 5(3), (4), (5) or (6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible) arcs in $\Pi - (B_0 \cup \{\bar{a}a\})$. Note that all arcs of $B_0 \cup \{\bar{a}a\}$ except uv are outgoing from at most two vertices (that is, a and $\bar{a}$). We have $|B^f| + |B^c| = |\Pi - (B_0 \cup \{\bar{a}a\})| \geqslant s - 2 \geqslant k - 4 \geqslant j$. Similarly, by Lemma 5, a (B, B^f, B^c)-net $\mathcal{B}$ of size j exists.

Next suppose that $i = k - 3$ and $j = 2$. If $|B_0 \cap \Sigma| \leqslant 1$, let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma - B_0$. If $|B_0 \cap \Sigma| = 2$, let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $(\Sigma - B_0) \cup \{\bar{a}a\}$, where $a, \bar{a}$ are as above. In both cases, we have $|A^f| + |A^c| \geqslant r - 1 \geqslant k - 3 = i$. By Lemma 5 (6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible) arcs in $\Pi - \Sigma - (B_0 \cup \{\bar{a}a\})$. Since v has in F at least $k - 3 \geqslant 4$ in-neighbours, one of which is not in $\{u, a, \bar{a}\}$. Thus $B^c \neq \emptyset$. By Lemma 5(2), a (B, B^f, B^c)-net $\mathcal{B}$ of size 2 exists.

Suppose that $i = 2$ and $j = k - 3$. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible) arcs in $\Pi - B_0$. Then $|B^f| + |B^c| = |\Pi - B_0| \geqslant s - 2 \geqslant k - 4 = j - 1$. By Lemma 5(6), there exists a (B, B^f, B^c)-net $\mathcal{B}$ of size j. If $|B_0 \cap \Sigma| \leqslant 1$, let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma - \Pi - B_0$. If $|B_0 \cap \Sigma| = 2$, let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $(\Sigma - \Pi - B_0) \cup \{\bar{a}a\}$, where $a, \bar{a}$ are as above. In both cases, $|A^f| + |A^c| \geqslant r - t - 1 \geqslant k - 2 - 2 \geqslant 3$. Recall that $A = \{uu_1, uu_2\}$. Note that $|(A^f \cup A^c) - \{\varphi(u, u_1)\}| \geqslant 2$. Let τ_1, τ_2 be two arcs in $(A^f \cup A^c) - \{\varphi(u, u_1)\}$. Then, at least one arc, τ_2 say, of τ_1, τ_2 is not equal to $\varphi(u, u_2)$. Note that τ_2 is adjacent to both uu_1 and uu_2, and τ_1 is adjacent to uu_1 in $X(D)$. Let $\mathcal{A} := \{\{uu_1, \tau_1\}, \{uu_2, \tau_2\}\}$. Then, $\mathcal{A}$ is a (A, A^f, A^c)-net of size 2.

In each case, B_0 induces a connected subgraph in $X(D)$. And $uv \in B_0$ is adjacent to each branch set of $\mathcal{A}$, and an arc outgoing from a other than av is adjacent to each branch set of $\mathcal{B}$. Hence $\mathcal{A} \cup \mathcal{B} \cup \{B_0\}$ forms a K_k-minor in $X(D)$.

Case 2.2.3. $t \leqslant 1$ and $U \cap N_F^-(v) = \emptyset$ and $|U| \geqslant 2$: That is, each arc in F between a vertex of U and v is outgoing at v. Let $A := A_F(u) - \{uv\}$ and $B := A_F(v)$. We consider two situations.

First suppose that U is not independent in F. That is, there is an arc τ in F joining two vertices in U. Say, $\tau = a_1a_2$. Since $A_F\{u, a_2\}$ is not redundant, in D there is an arc $\gamma \neq a_2u$ outgoing from a_2. (It may happen that $\gamma \in \{a_2a_1, a_2v\}$.) Let $B_0 := \{uv, \tau, \gamma\}$. Since uv is adjacent to both τ and γ, B_0 induces a connected subgraph in $X(D)$. Note that $\max\{|B_0 \cap \Sigma|, |B_0 \cap \Pi|\} \leqslant 2$.

If $i > j$, then $j < \frac{k-1}{2} \leqslant k - 4$ and $i \geqslant 4$. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma - B_0$; and, let B^f (B^c, respectively) be the set of B-feasible (B-compatible) arcs in $\Pi - \Sigma - B_0$. Then $|A^f| + |A^c| \geqslant r - 2 \geqslant k - 2 - 2 \geqslant i - 1$. By Lemma 5(6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i. Also, $|B^f| + |B^c| = |\Pi - \Sigma - B_0| \geqslant s - t - 2 \geqslant k - 5 \geqslant j - 1$. Note that there is at least one (in fact many) incoming arc v_lv at v with $\varphi(v_l, v) \notin \Sigma \cup B_0$. Thus $\varphi(v_l, v) \in B^c$ and $|B^c| \geqslant 1$. By Lemma 5(2), (4) or (6), a (B, B^f, B^c)-net $\mathcal{B}$ of size j exists. If $i \leqslant j$, then $i \leqslant \frac{k-1}{2} \leqslant k - 4$. Now let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma - \Pi - B_0$; and let B^f (B^c, respectively) be the set of B-feasible (B-compatible) arcs in $\Pi - B_0$. Similarly, we obtain an (A, A^f, A^c)-net $\mathcal{A}$ of size i and a (B, B^f, B^c)-net $\mathcal{B}$ of size j.

Since each arc outgoing from u or v is adjacent to τ or γ, each branch set of $\mathcal{A} \cup \mathcal{B}$ is adjacent to B_0. Thus, $\mathcal{A} \cup \mathcal{B} \cup \{B_0\}$ forms a K_k-minor in $X(D)$.

Next suppose that U is independent in F. For each $a_l \in U$, if in D there is an arc $a_la'_l$ other than a_lu or a_lv, let $Q_l := \{a_la'_l\}$. Otherwise suppose that a_l has no out-neighbours other than u, v in D. Since $A_F\{\bar{a}_l, u\}$ is not redundant, $a_lv \in A(D)$; similarly, $A_F\{\bar{a}_l, v\}$ is not redundant, $a_lu \in A(D)$. Therefore, we have $A_D(a_l) = \{a_lu, a_lv\}$. Let $\bar{a}_l$ be an in-neighbour other than u, v of a_l in F. Then $A_F\{\bar{a}_l, a_l\} = \bar{a}_la_l$. Let $\bar{\bar{a}}_l \neq a_l$ be an out-neighbour of $\bar{a}_l$ in F (such $\bar{\bar{a}}_l$ exists as $\bar{a}_la_l$ is not redundant). Let $Q_l := \{a_lu, a_lv, \bar{a}_l\bar{\bar{a}}_l\}$. Let a_l, a_m be distinct vertices in U such that $w_1 \in \{a_l, a_m\}$ when $t = 1$ and $|Q_l \cup Q_m|$ is minimised. Let $B_0 := \{uv\} \cup Q_l \cup Q_m$. Note that in $X(D)$ each of the subgraphs induced on Q_l and Q_m is connected and adjacent to uv, B_0 induces a connected subgraph.

Note that for each $p \in \{l, m\}$, $|Q_p \cap \Sigma| \leqslant 2$ and $|Q_p \cap \Pi| \leqslant 2$. If $|Q_p \cap \Sigma| = 2$, then $Q_p := \{a_pu, a_pv, \bar{a}_p\bar{\bar{a}}_p\}$ and $\bar{a}_p\bar{\bar{a}}_p \in \Sigma$ and $\bar{a}_p$ is adjacent to u (but not v because U is independent) in F. Thus $\bar{a}_pa_p$ is A-feasible (A-compatible) if $\bar{a}_p\bar{\bar{a}}_p$ is A-feasible (A-compatible). Let Σ' be obtained from Σ by replacing $\bar{a}_p\bar{\bar{a}}_p$ with $\bar{a}_pa_p$. Then $|Q_p \cap \Sigma'| \leqslant 1$ and $|B_0 \cap \Sigma'| \leqslant 2$. In addition, each element in Σ' is A-feasible or A-compatible, and no two share a tail. Similarly, we can obtain Π' such that each of its elements is A-feasible or A-compatible, no two elements share a tail and $|B_0 \cap \Pi'| \leqslant 2$.

Let A^f (A^c, respectively) be the set of A-feasible (A-compatible) arcs in $\Sigma' - B_0$; and let B^f (B^c, respectively) be the set of B-feasible (B-compatible) arcs in $\Pi' - B_0$. Then, $|A^f| + |A^c| \geqslant r - 2 \geqslant k - 2 - 2 \geqslant i - 1$. Also, $|B^f| + |B^c| = |\Pi' - B_0| \geqslant s - 2 \geqslant k - 4 \geqslant j - 1$. When $i = 2$, since $|A^f| + |A^c| \geqslant k - 4 \geqslant 3$, we have $A^c \neq \emptyset$. Analogously, we have that $B^c \neq \emptyset$ when $j = 2$. By Lemma 5(2)-(6), there exist an (A, A^f, A^c)-net $\mathcal{A}$ of size i and a (B, B^f, B^c)-net $\mathcal{B}$ of size j.

Since each arc outgoing from u or v is adjacent to an arc in Q_l or Q_m, each branch set of $\mathcal{A} \cup \mathcal{B}$ is adjacent to B_0. Thus, $\mathcal{A} \cup \mathcal{B} \cup \{B_0\}$ forms a K_k-minor in $X(D)$.

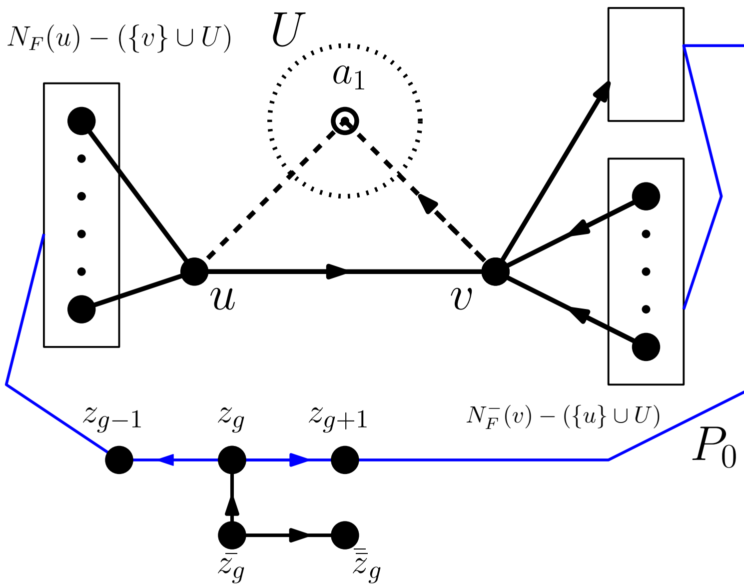

Figure 4: An illustration for Case 2.2.4.

Case 2.2.4. $U \cap N_F^-(v) = \emptyset$ and $|U| \leqslant 1$ (hence $t \leqslant 1$): That is, u and v share at most one neighbour a_1 in F. If a_1 exists, the arc between a_1 and v in F is va_1. Let $A := A_F(u) - \{uv\}$ and $B := A_F(v)$.

Since $\delta(F) \geqslant k-1$ and $j \leqslant k-3$, v has at least $k-1-j \geqslant 2$ in-neighbours in F. Say, $N_F^-(v) = \{u, y_{j+1}, y_{j+2}, \ldots, y_{k-2}\}$. Note that $N_F^-(v) - \{u\} \neq \emptyset$. Recall that $N_F^+(v) = \{v_1, v_2, \ldots, v_j\}$.

Let $\bar{H}$ be obtained from H by deleting vertices in $U \cup \{u, v\}$. By Lemma 6(b), $\bar{H}$ is connected. Let $P_0 := (z_1, z_2, \ldots, z_m)$ be a shortest path in $\bar{H}$ between $N_F(u) - (\{v\} \cup U)$ and $N_F^-(v) - (\{u\} \cup U)$, where $m \geqslant 2$ (because u and v share no common neighbour in $\bar{H}$), $z_1 \in N_F(u) - (\{v\} \cup U)$ and $z_m \in N_F^-(v) - (\{u\} \cup U)$. See Fig. 4. Then each internal vertex of P_0 is not adjacent to u in F.

If $|V(P_0) \cap N_F(v)| = 1$, then z_m is the only neighbour of v in F which is on P_0. Let $P := P_0$ and set $z_l := z_m$. If $|V(P_0) \cap N_F(v)| \geqslant 2$, let $P = (z_1, z_2, \ldots, z_l)$ be the subpath of P_0 such that $z_l \in N_F(v)$ and $|V(P) \cap N_F(v)| = 2$.

We shall construct a branch set P' consisting of arcs alongside P. Let $z_0 = u$ and $z_{l+1} = v$.

For $1 \leqslant g \leqslant l$, we associate to z_g the set Q_g of arcs as follows. If $A_D(z_g) - (A_D\{z_{g-1}, z_g\} \cup A_D\{z_g, z_{g+1}\}) \neq \emptyset$, then let Q_g be a singleton set that contains exactly one arc, say, $z_g z_g' \in A_D(z_g) - (A_D\{z_{g-1}, z_g\} \cup A_D\{z_g, z_{g+1}\})$. Otherwise, $A_D(z_g) - (A_D\{z_{g-1}, z_g\} \cup A_D\{z_g, z_{g+1}\}) = \emptyset$. Since the arc $A_F\{z_g, z_{g+1}\} \in A(F)$ is not redundant, $z_g z_{g-1} \in A_D(z_g)$. Similarly, $z_g z_{g+1} \in A_D(z_g)$ since $A_F\{z_{g-1}, z_g\} \in A(F)$ is not redundant. Let $\bar{z}_g$ be an in-neighbour of z_g in F. Then $\bar{z}_g z_g \in A(F)$. Let $\bar{z}_g \bar{\bar{z}}_g$ with $\bar{\bar{z}}_g \neq z_g$ be an arc outgoing from $\bar{z}_g$ in D (which exists because $\bar{z}_g z_g$ is not redundant). Set $Q_g := \{z_g z_{g-1}, z_g z_{g+1}, \bar{z}_g \bar{\bar{z}}_g\}$ (see Fig. 4). Note that Q_g induces a connected subgraph in $X(D)$ since $\bar{z}_g\bar{\bar{z}}_g$ is adjacent to both z_gz_{g-1} and z_gz_{g+1}.

In the case where $V(P) \cap N_F(v) = \{z_p, z_l\}$ $(p < l)$ and $Q_p = \{z_pv\}$, we slightly modify Q_p as $\{z_pv, \gamma\}$, where $\gamma \in A_D(z_p) - \{z_pv\}$ (which exists because $A_F(z_p, v)$ is not redundant).

Let $P' := \cup_{g=1}^{l} Q_g$. Then, for $1 \leqslant g \leqslant l-1$, since Q_g contains an arc outgoing from z_g other than z_gz_{g+1} and Q_{g+1} contains an arc outgoing from z_{g+1} other than $z_{g+1}z_g$, each Q_g is adjacent to Q_{g+1} in $X(D)$. Thus, P' induces a connected subgraph in $X(D)$. We call P' a *parallel set* of P.

Let Σ and Π be as above. We have the following claim:

Claim 2. (a) There is a set Σ' such that $|\Sigma'| \geqslant |\Sigma| - 1$ and $P' \cap \Sigma' = \emptyset$, and each element of which is A-feasible or A-compatible and no two elements share a tail;
(b) There is a set Π' such that $|\Pi'| \geqslant |\Pi| - 2$ and $P' \cap \Pi' = \emptyset$, and each element of which is B-feasible or B-compatible and no two elements share a tail.

Proof. (a) Initially, set $\Sigma' := \Sigma - P'$. Clearly, all properties except $|\Sigma'| \geqslant |\Sigma| - 1$ in (a) are satisfied. If $|P' \cap \Sigma| \leqslant 1$, then we are done. Suppose that $|P' \cap \Sigma| \geqslant 2$. Since P_0 is a shortest path in $\bar{H}$ between $N_F(u) - (\{v\} \cup U)$ and $N_F^-(v) - (\{u\} \cup U)$, each vertex z_g on P with $g \geqslant 3$ is not adjacent to a vertex of $N_F(u) - (\{v\} \cup U)$. Thus, $Q_g \cap \Sigma = \emptyset$ for each $g \geqslant 3$. We now consider $g = 2$. Since z_2 is not adjacent to u in $\bar{H}$, we have $|Q_2 \cap \Sigma| \leqslant 1$ and if $|Q_2 \cap \Sigma| = 1$ then $|Q_2| = 3$ and $Q_2 := \{z_2z_1, z_2z_3, \bar{z}_2\bar{\bar{z}}_2\}$, where $\bar{z}_2$ is an in-neighbour of z_2 in F. Since z_2 is not adjacent to u, $Q_2 \cap \Sigma = \{\bar{z}_2\bar{\bar{z}}_2\}$, which means that $\bar{z}_2$ is adjacent to u in F and $\varphi(u, \bar{z}_2) = \bar{z}_2\bar{\bar{z}}_2$. In this case, update $\Sigma' := \Sigma' \cup \{\bar{z}_2z_2\}$. Note that $\bar{z}_2z_2$ is A-feasible or A-compatible.

If $|Q_1 \cap \Sigma| \leqslant 1$, then Σ' is the desired set. Suppose that $|Q_1 \cap \Sigma| = 2$. Let $Q_1 := \{z_1u, z_1z_2, \bar{z}_1\bar{\bar{z}}_1\}$, where $\bar{z}_1$ is an in-neighbour of z_1 in F. Then, $Q_1 \cap \Sigma = \{z_1z_2, \bar{z}_1\bar{\bar{z}}_1\}$, which means $\varphi(u, z_1) = z_1z_2$ and $\varphi(u, \bar{z}_1) = \bar{z}_1\bar{\bar{z}}_1$. Note that $\bar{z}_1z_1$ is A-feasible or A-compatible. By adding $\bar{z}_1z_1$ into Σ', we get that $|Q_1 \cap \Sigma'| \leqslant 1$. Then $|\Sigma'| \geqslant |\Sigma| - 1$, as desired.

(b) Initially, set $\Pi' := \Pi - P'$. Recall that P contains at most two neighbours, z_{g_1} and z_{g_2} say, of v. Let γ be an arc in $\Pi \cap P'$ such that there is a Q_g containing γ (there may be more than one Q_g containing γ) and $g \notin \{g_1, g_2\}$. Since z_g is not adjacent to v in $\bar{H}$, we have $|Q_g| = 3$ and $Q_g = \{z_gz_{g-1}, z_gz_{g+1}, \bar{z}_g\bar{\bar{z}}_g\}$, where $\bar{z}_g$ is an in-neighbour of z_g in F and $\bar{z}_g\bar{\bar{z}}_g \neq \bar{z}_gz_g$ is an arc outgoing from $\bar{z}_g$ in D. Further, $\bar{z}_g$ is a neighbour of v in F and $\varphi(v, \bar{z}_g) = \bar{z}_g\bar{\bar{z}}_g$. Note that $\bar{z}_gz_g \notin \Pi$ is B-feasible or B-compatible. Now update Π' by adding $\bar{z}_gz_g$. That is, $\Pi' := \Pi' \cup \{\bar{z}_gz_g\}$. By repeating this procedure for all such γ, we obtain a Π' with the same size as $\Pi - (Q_{g_1} \cup Q_{g_2})$.

For each $g \in \{g_1, g_2\}$, if $|\Pi \cap Q_g| = 2$, we will add a B-feasible or B-compatible arc into Π'. Then $|\Pi'| \geqslant |\Pi| - 2$, as desired. Suppose that $|\Pi' \cap Q_g| = 2$ for some $g \in \{g_1, g_2\}$. Then $Q_g = \{z_gz_{g-1}, z_gz_{g+1}, \bar{z}_g\bar{\bar{z}}_g\}$, where $\bar{z}_g$ is an in-neighbour of z_g in F and $\bar{z}_g\bar{\bar{z}}_g \neq \bar{z}_gz_g$ is an arc outgoing from $\bar{z}_g$ in D. And, $\bar{z}_g$ is a neighbour of v in F with $\varphi(v, \bar{z}_g) = \bar{z}_g\bar{\bar{z}}_g$. Note that $\bar{z}_gz_g \notin \Pi$ is B-feasible or B-compatible. Set $\Pi' := \Pi' \cup \{\bar{z}_gz_g\}$. Then $|\Pi'| \geqslant |\Pi| - 2$. Consequently, we get the desired Π'. □

Let $B_0 := \{uv\} \cup P'$. Then B_0 induces a connected subgraph in $X(D)$ since uv is adjacent to Q_1.

Next we show that there exists a net of size i at u and a net of size j at v such that none of their branch sets intersects B_0.

If $j = 2$ (hence $i = k - 3$), then at least one arc, say γ, in $\Pi' - \Sigma'$ is B-compatible (since there are more incoming arcs at v). Let $B^c := \{\gamma\}$. Since $|\Pi' - \Sigma'| \geqslant s - 2 - 1 \geqslant k - 5 \geqslant j = 2$, by Lemma 5(2), there exists a $(B, \emptyset, \mathrm{B}^c)$-net $\mathcal{B}$ of size $j = 2$. Similarly, let A^f (A^c, respectively) be the set of A-feasible (A-compatible, respectively) arcs in Σ'. Note that $|\Sigma'| \geqslant r - 1 \geqslant k - 3 = i \geqslant 4$. By Lemma 5(6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i.

Suppose that $3 \leqslant j \leqslant k - 3$ (hence $2 \leqslant i \leqslant k - 4$). Let B^f (respectively, B^c) be the set of B-feasible (B-compatible) arcs in Π'. Since $|\Pi'| \geqslant s - 2 \geqslant k - 4 \geqslant j - 1$ and $B^c \neq \emptyset$ when $j = 3$, by Lemma 5(4) or (6), there exists a (B, B^f, B^c)-net $\mathcal{B}$ of size j. Let A^f (A^c, respectively) be the set of A-feasible (A-compatible, respectively) arcs in $\Sigma' - \Pi'$. We now show that there exists a net of size i at u. If $i \geqslant 3$, then $|\Sigma' - \Pi'| \geqslant r - 1 - 1 \geqslant k - 4 \geqslant i \geqslant 3$. By Lemma 5(3) or (6), there exists an (A, A^f, A^c)-net $\mathcal{A}$ of size i. Suppose that $i = 2$. Note that $|\Sigma' - \Pi'| \geqslant k - 4 \geqslant 3$ (because $k \geqslant 7$) and there are at least three incoming arcs at u in F. $\Sigma' - \Pi'$ contains at least two A-compatible arcs, say, γ_1 and γ_2. Let $\mathcal{A} := \{\{uu_1, \gamma_1\}, \{uu_2, \gamma_2\}\}$. Then $\mathcal{A}$ is a net of size 2 at u.

Since each element of $\mathcal{A}$ constructed above contains an arc xx', which is outgoing from a neighbour $x \neq v$ of u and $x' \neq u$, each element of $\mathcal{A}$ is adjacent to B_0 because $uv \in B_0$ is adjacent to each xx'. Note that $|V(P) \cap N_F(v)| \in \{1, 2\}$. In the case when $|V(P) \cap N_F(v)| = 1$, P' contains an arc yy', which is outgoing from an in-neighbour $y \neq u$ of v and $y' \neq v$. Since such a yy' is adjacent to every arc of $A_F(v)$, it is adjacent to every element of $\mathcal{B}$ constructed above. In the case when $|V(P) \cap N_F(v)| = 2$, P' contains two arcs α and β, each of them is outgoing from a neighbour of v other than u and heading to a vertex other than v. Then each arc of $A_F(v)$ is adjacent to either α or β. So every element of $\mathcal{B}$ is adjacent to $P' \subseteq B_0$. Therefore, $\{B_0\} \cup \mathcal{A} \cup \mathcal{B}$ forms a K_k-minor in $X(D)$.

Case 2.3. $j = k - 2$: Then $i = 1$. Suppose first that $d_F^-(v) = 1$; that is, uv is the only incoming arc at v and $d_F(v) = k - 1$. Since v is not special, one out-neighbour v' of v in F is not a sink. Now consider the arc vv'. If $d_F^+(v') \geqslant 2$, then $S_F(vv') = d_F^+(v) + d_F^+(v') - 1 \geqslant k - 2 + 2 - 1 = k - 1$. This is a special case of Case 2.2 and thus can be treated similarly. If $d_F^+(v') = 1$, then by Property A, one potential arc $v'v''$ ($\neq v'v$ as $d_F^+(v) > 2$) is outgoing from v' in D but not present in F (since $d_F^+(v) = 1$). Let F' be obtained from F by adding $v'v''$. Again we have $S_{F'}(vv') = d_{F'}^+(v) + d_{F'}^+(v') - 1 \geqslant k - 2 + 2 - 1 = k - 1$, and this can also be treated similarly. Suppose next that $d_F^-(v) \geqslant 2$. Then $t \leqslant 1$. This case can be dealt with by a similar way as in Cases 2.2.3 or 2.2.4.

Case 2.4. $j = k - 1$: Then $i = 0$, which implies $d_F^+(u) = 1$. By Property A, there exists a potential arc $uz \neq uv$ in D. Then $\mathcal{A} := \{\{uz\}\}$ is a $(\{uz\}, \emptyset, \emptyset)$-net. Let $B := A_F(v)$. Let B^f (B^c, respectively) be the set of B-feasible (B-compatible, respectively) arcs in Π. By Lemma 5(6), a (B, B^f, B^c)-net $\mathcal{B}$ of size j exists. It is not hard to see that $\mathcal{A} \cup \mathcal{B}$ forms a K_k-minor in $X(D)$.

This completes the proof of Theorem 1. □

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References

[1] C. Balbuena, P. García-Vázquez and L. P. Montejano, On the connectivity and restricted edge-connectivity of 3-arc graphs, *Discrete Appl. Math.*, **162** (2014), 90–99.

[2] N. Belkale and L. S. Chandran, Hadwiger's conjecture for proper circular arc graphs, *European J. Combin.* **30** (2009), 946–956.

[3] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, New York, 2008.

[4] H. J. Broersma and C. Hoede, Path graphs, *J. Graph Theory* **13** (1989), 427–444.

[5] R. L. Brooks, On coloring the nodes of a network, *Proc. Cambridge Philos. Soc.* **37** (1941), 194–197.

[6] M. Chudnovsky and A. O. Fradkin, Hadwiger's conjecture for quasi-line graphs, *J. Graph Theory* **59** (2008), 17–33.

[7] G. A. Dirac, The structure of k-chromatic graphs, *Fund. Math.* **40** (1953), 42–55.

[8] G. A. Dirac A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27** (1952), 85–92.

[9] A. Gardiner, C. E. Praeger and S. Zhou, Cross-ratio graphs, *J. London Math. Soc.* (2) **64** (2001), 257–272.

[10] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich* **88** (1943), 133–142.

[11] M. A. Iranmanesh, C. E. Praeger and S. Zhou, Finite symmetric graphs with two-arc transitive quotients, *J. Combin. Theory (Ser. B)* **94** (2005), 79–99.

[12] M. Knor, G. Xu and S. Zhou, A study of 3-arc graphs, *Discrete Appl. Math.* **159** (2011), 344–353.

[13] M. Knor and S. Zhou, Diameter and connectivity of 3-arc graphs, *Discrete Math.* **310** (2010), 37 – 42.

[14] D. Li and M. Liu, Hadwiger's conjecture for powers of cycles and their complements, *European J. Combin.* **28** (2007), 1152–1155.

[15] C. H. Li, C. E. Praeger and S. Zhou, A class of finite symmetric graphs with 2-arc transitive quotients, *Math. Proc. Cambridge Phil. Soc.* **129** (2000), 19–34.

[16] H. E. Li and Y. X. Lin, On the characterization of path graphs, *J. Graph Theory* **17** (1993), 463–466.

[17] Z. Lu and S. Zhou, Finite symmetric graphs with 2-arc transitive quotients (II), *J. Graph Theory*, **56** (2007), 167–193.

[18] B. Reed and P. Seymour, Hadwiger's conjecture for line graphs, *European J. Combin.* 25 (2004), 873–876.

[19] N. Robertson, P. Seymour and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993), 279–361.

[20] P. Seymour, Hadwiger's conjecture. In J. F. Nash, Jr., M. T. Rassias, editors, *Open Problems in Mathematics*, pages 417-438. 1st ed., Springer, 2016.

[21] B. Toft, A survey of Hadwiger's conjecture, *Congr. Numer.* **115** (1996), 249–283.

[22] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, New York, 2001.

[23] G. Xu and S. Zhou, Hamiltonicity of 3-arc graphs, *Graphs Combin.* **30** (2014), 1283–1299.

[24] G. Xu and S. Zhou, Hadwiger's conjecture for the complements of Kneser graphs, *J. Graph Theory*, 2015, doi:10.1002/jgt.22007.

[25] S. Zhou, Constructing a class of symmetric graphs, *European J. Combin.* **23** (2002), 741–760.

[26] S. Zhou, Imprimitive symmetric graphs, 3-arc graphs and 1-designs, *Discrete Math.* **244** (2002), 521–537.

[27] S. Zhou, Almost covers of 2-arc transitive graphs, *Combinatorica* **24** (2004), 731-745. [Erratum: **27** (2007), 745–746.]