Resolvable group divisible designs with large groups

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Submitted: Jul 23, 2015; Accepted: Oct 29, 2016; Published: Nov 10, 2016
Mathematics Subject Classifications: 05B05,05C70

Abstract

We prove that the necessary divisibility conditions are sufficient for the existence of resolvable group divisible designs with a fixed number of sufficiently large groups. Our method combines an application of the Rees product construction with a streamlined recursion based on incomplete transversal designs. With similar techniques, we also obtain new results on decompositions of complete multipartite graphs into a prescribed graph.

1 Introduction

A *group divisible design*, or GDD, is a triple $(V, \Pi, \mathcal{B})$ which satisfies the following properties.

- Π is a partition of the set V into subsets called *groups*: $\Pi = \{V_1, \ldots, V_u\}$.
- $\mathcal{B}$ is a collection of subsets of V called *blocks* such that a group and a block contain at most one point in common.

*Supported by NSERC grant 312595–2010.

- Every pair of points from distinct groups occurs in precisely λ blocks; λ is called the *index*.

The *type* of a GDD is a list of its group sizes, say $T = (g_1, g_2, \dots, g_u)$, where $|V_i| = g_i$ for $i = 1, 2, \dots, u$. We usually use exponential notation to shorten the type. In particular, if all groups have the same size, say $|V_i| = g$ for each i, then the GDD is called *uniform* and its type is written g^u.

Consider a GDD of type T and index λ and suppose the set of its block sizes belongs to $K \subseteq \mathbb{Z}_{\geqslant 2}$. (Blocks of size one can be discarded without loss.) We write this as a $\mathrm{GDD}_\lambda(T, K)$ for consistency with PBD notation(see below); it is often also denoted as a (K, λ)-GDD of type T. We adopt two standard conventions: if $K = \{k\}$, we write 'k' instead of '$\{k\}$' and if $\lambda = 1$, it can be omitted from the notation.

Group divisible designs are common generalizations of pairwise balanced designs, balanced incomplete block designs, and transversal designs. A *pairwise balanced design*, or $\mathrm{PBD}(v, K)$, is a $\mathrm{GDD}(1^v, K)$. A *balanced incomplete block design*, (v, k, λ)-BIBD, is a $\mathrm{GDD}_\lambda(1^v, k)$. A *transversal design*, or $\mathrm{TD}(k, n)$, is a $\mathrm{GDD}(n^k, k)$. In this case, blocks of the GDD are transversals of the group partition Π.

A *parallel class* or *resolution class* in a GDD $(V, \Pi, \mathcal{B})$ is a subcollection of $\mathcal{B}$ which partitions V. A GDD is *resolvable* if $\mathcal{B}$ can be completely partitioned into parallel classes. A resolvable GDD is abbreviated RGDD and, similarly, a resolvable TD is denoted RTD.

As is natural in the case of resolvable designs and transversal designs, we restrict our attention in this paper to the case where all blocks have a constant size, say k. Then, [17, Lemma 1.1] tells us that in such an RGDD all groups have the same size; that is, they are uniform.

For the existence of an $\mathrm{RGDD}_\lambda(g^u, k)$, it is clearly necessary that

$$gu \equiv 0 \pmod{k}. \tag{1}$$

Also, by considering degrees at each point, another necessary condition is

$$\lambda g(u-1) \equiv 0 \pmod{k-1}. \tag{2}$$

These conditions are not sufficient in general. For example, there does not exist a $\mathrm{GDD}(2^6, 3)$ with index $\lambda = 1$. On the other hand, in [4], it was shown that, given fixed k and g, these conditions are sufficient for $u \gg 0$.

Theorem 1 ([4])**.** *Given $g, \lambda \geqslant 1$ and $k \geqslant 2$, there exists $\mathrm{RGDD}_\lambda(g^u, k)$ for all sufficiently large u satisfying $gu \equiv 0 \pmod{k}$ and $\lambda g(u-1) \equiv 0 \pmod{k-1}$.*

Here, we take the opposite point of view, fixing u and considering large admissible g. This is similar to the existence theory for (resolvable) transversal designs, [2], and GDDs of large order, [12]. It is well known that a $\mathrm{TD}(k, n)$ is equivalent to a set of $k-2$ mutually orthogonal latin squares of order n. Therefore, from the result of Chowla, Erdős and Strauss in [2], a $\mathrm{TD}(k, n)$ exists for sufficiently large integers n. See also [3] for references and further results. Since an $\mathrm{RTD}(k, n)$ is equivalent to a $\mathrm{TD}(k+1, n)$, we also have the existence of RTD(k,n) for all sufficiently large n. Our work in this paper is an extension of this result to RGDDs.

The asymptotic existence of uniform GDDs with large groups and index one is a recent result of Mohácsy, [12]. Note that the necessary divisibility conditions are a bit weaker than in the resolvable case.

Theorem 2 ([12])**.** *Let k and u be positive integers, $2 \leqslant k \leqslant u$. There exists a GDD$(g^u,k)$ for all sufficiently large g satisfying $g(u-1) \equiv 0 \pmod{k-1}$ and $g^2u(u-1) \equiv 0 \pmod{k(k-1)}$.*

This paper obtains a similar theorem for resolvable GDDs. Given integers $u \geqslant k$, let

$$g_{\min} = \frac{k(k-1)}{\gcd(u,k)\gcd(u-1,k-1)}. \tag{3}$$

Then (1) and (2) can be summarized simply as $g_{\min} \mid g$. We now state our main result.

Theorem 3. *Given integers $u \geqslant k \geqslant 2$, there exist RGDD$(g^u,k)$ for all sufficiently large $g \equiv 0 \pmod{g_{\min}}$.*

The rest of the paper is structured as follows. In the next section, we cover the necessary background for our constructions, including some slightly more general objects. Then, in Section 3, we construct the first general examples of RGDDs with a fixed arbitrary number of groups. Section 4 contains the recursive argument used to complete the proof. In Section 5, we extend our results to graph decompositions which we call 'transverse' graph GDDs and RGDDs.

We conclude with a discussion of some related results and questions, including a few remarks on the existence problem for graph RGDDs with g fixed and u large.

2 Background

We introduce some variant designs and prior results which are useful for our constructions to follow.

2.1 Holey GDDs

A (uniform) *holey group divisible design* is a quadruple $(V,\Pi,\Xi,\mathcal{B})$, where

- V is a set of v points;
- $\Pi = \{V_1,\ldots,V_u\}$ is a partition of V into *groups* of size mt for each i;
- $\Xi = \{W_1,\ldots,W_t\}$ is a partition of V into *holes*, where $|V_i \cap W_j| = m$ for each i,j.
- $\mathcal{B} \subseteq \binom{V}{k}$ is a set of of blocks which meet each group and each hole in at most one point; and

- any two points from distinct groups and distinct holes appear together in exactly one block.

With these parameters, the above design is abbreviated as an HGDD of type $u \times m^t$. To indicate the block size, we write HGDD$(u \times m^t, k)$. It should be remarked that letting $m = 1$ in the definition leads to 'modified group divisible designs', also known as 'grid designs'. An HGDD is also a special type of 'double group divisible design', or DGDD; see [11] for a definition. Resolvability of HGDDs is defined similarly as for GDDs; see [18] for a definition and an existence theorem for $k = 3$.

Example 4. The pattern of groups and holes for an HGDD of type 5×2^4 is shown in Figure 1. The underlying graph is 24-regular; admissible block sizes are $2, 3$ and 4.

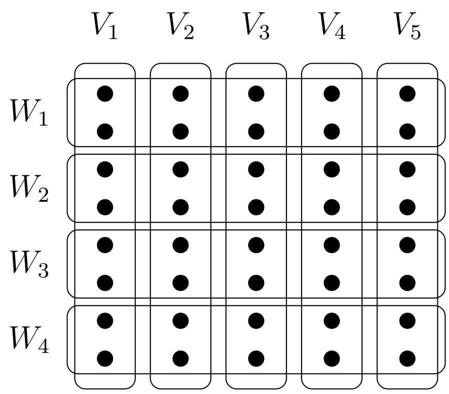

Figure 1: Groups and holes for an HGDD of type 5×2^4

The holes (or groups) of an HGDD can be filled to create a GDD. The proof of the following is straightforward and omitted.

Construction 5. *If there exists an HGDD$(u \times m^t, k)$ and a GDD(m^t, k), then there exists a GDD$((mu)^t, k)$.*

2.2 Index reduction

We point out two constructions which are used to reduce the index of a design; they take as inputs designs of index λ and produce designs of index one.

Construction 6. *Suppose there exists an RGDD$_\lambda(g^u, k)$, and let $q \equiv 1 \pmod{\lambda}$ be a large prime power. Then there exists a resolvable HGDD$(q \times g^u, k)$.*

Proof sketch. This is a routine extension of Lemma 3.2 in [6], which is for the case $g = 1$. Suppose our ingredient design is $(X, \Pi, \mathcal{A})$. The construction has point set $X \times \mathbb{F}_q$ and partitions induced by Π and $\mathbb{F}_q$. We first set up a uniform choice function from blocks in $\mathcal{A}$ on each pair of points to the (multiplicative) cosets of index l in $\mathbb{F}_q$. This defines a lifting of each block in $\mathcal{A}$ onto $X \times \mathbb{F}_q$. From the structure of the choice function, the lifted blocks can be developed such that they cover two points (x, a), (y, b) (once) if and only if x and y are in different groups of Π and $a \neq b$. It follows that we have an HGDD of type $q \times g^u$. Resolvability of this HGDD arises from additively developing the parallel classes induced from $\mathcal{A}$. □

Remark 7. Implicit here and in what follows is Dirichlet's theorem asserting the existence of infinitely many primes in arithmetic progressions of gcd 1.

Construction 8. *Suppose there exists a* $GDD_q(g^u, k)$ *of index* q*, a prime power. Then there exists a* $GDD((gq^*)^u, k)$*, where* q^* *is a sufficiently large power of* q*.*

Proof. This extends the main lemma in [21] from the case $g = 1$. Suppose the ingredient design is $(X, \Pi, \mathcal{A})$. Each block $\{x, y, \dots\} \in \mathcal{A}$ is lifted onto a base block $\{(x, a), (y, b), \dots\}$ of $X \times \mathbb{F}_{q^*}$ in the same way as in [21]. After developing additively in $\mathbb{F}_{q^*}$, two points $(x, a), (y, b)$ appear together in exactly one block if x and y are in distinct groups of Π, and in zero blocks otherwise. It follows that we have a GDD with group type $(gq^*)^u$ and index one. □

2.3 Wilson's fundamental construction

The following construction for group divisible designs illustrates the flexibility and utility of these objects. R.M. Wilson made crucial use of this in his existence theory for pairwise balanced designs, and we will use several variations of this construction.

Construction 9 ([22])**.** *Let* $(V, \Pi = \{V_1, \dots, V_u\}, \mathcal{B})$ *be a GDD, and consider a function* $\omega : V \to \mathbb{N} \cup \{0\}$*. Suppose, for each block* $B \in \mathcal{B}$*, that there exists a* K*-GDD of type* $(\omega(x) : x \in B)$*. Then there exists a* K*-GDD of type* $(\sum_{x \in V_i} \omega(x) : i = 1, \dots, u)$*.*

Remark 10. It is typical to call the starting GDD $(V, \Pi, \mathcal{B})$ the 'master' design, ω a 'weighting', and the K-GDDs of type $(\omega(x) : x \in B)$ the 'ingredients'. The construction itself is abbreviated 'WFC'.

In this paper, our use of Construction 9 has constant weighting (so that the ingredients are uniform GDDs).

2.4 Thick classes and the Rees construction

Consider any design defined on a set V of points and with blocks $\mathcal{B}$. A *σ-parallel class* in $\mathcal{B}$ is a subset $\mathcal{R} \subseteq \mathcal{B}$ such that every element of V appears exactly σ times. Let Σ be some list of positive integers $\sigma_1, \dots, \sigma_t$. A *$\Sigma$-resolution* of $\mathcal{B}$ is a partition $\mathfrak{r} = \{\mathcal{R}_1, \dots, \mathcal{R}_t\}$ of $\mathcal{B}$ into classes, where $\mathcal{R}_i$ is an σ_i-parallel class for $i = 1, \dots, t$. Finally, a design which admits a Σ-resolution is called *Σ-resolvable*. This coincides with the usual notion of resolvability when $\Sigma = (1, 1, \dots, 1)$.

We now develop a condition on transversal designs for the purpose of 'spreading out' σ-parallel classes in a direct product. A *σ-group partition* of a TD(k, n) is a triple $(\mathbb{G}, \mathbb{H}, \mathfrak{c})$, where $\mathbb{G}$ is an algebraic group of order n used to label each group of the TD, $\mathbb{H} \subseteq \mathbb{G}$

with $|\mathbb{H}| = \sigma$, and $\mathfrak{c}$ is a partition of the blocks into aggregates so that, for every class $\mathcal{C} \in \mathfrak{c}$, the set $\{h * C : h \in \mathbb{H}, C \in \mathcal{C}\}$ is a parallel class. Here, $h * C$ denotes the natural (componentwise) action of h on (the group labels in) C. Note that $\mathbb{H}$ need not consist of automorphisms of the TD.

Any TD(k, n) admits a trivial n-group partition with $\mathbb{H} = \mathbb{G}(= \mathbb{Z}_n$, say) and singleton aggregates. An RTD(k, n) admits a 1-group partition by taking $\mathbb{H}$ to be the identity and each parallel class as an aggregate. Intermediate examples arise naturally from taking direct products; we will see an example of this later in Lemma 17.

The following construction modifies a construction of Rolf Rees, [15], to allow the use of TDs with smaller block sizes.

Construction 11. *Let $(V, \mathcal{R})$ be a σ-parallel class of blocks of order k. Suppose there exists a TD$((k-1)\sigma+1, n)$ which admits a σ-group partition based on the group $\mathbb{G}$. Then there exist σn disjoint parallel classes on $\mathbb{G} \times V$ that cover all pairs between $\mathbb{G} \times \{i\}$ and $\mathbb{G} \times \{j\}$ the same number of times as $\mathcal{R}$ covers $\{i, j\} \in \binom{V}{2}$.*

This is essentially recasting Construction 2 of [15] to fit the language of σ-group partitions. However, some comments are needed. First, observe that the graph of pairwise coverages by blocks in $\mathcal{R}$ on the points of V has maximum degree at most $(k-1)\sigma$. It follows that we may (greedily) color V, using at most $(k-1)\sigma + 1$ colors, so that the blocks of $\mathcal{R}$ are transverse to the color classes. (This is sometimes called a "strong" vertex coloring relative to $\mathcal{R}$.) Using this coloring, we can now view $\mathcal{R}$ as a partial GDD with at most $(k-1)\sigma + 1$ groups. This is what allows us to use a smaller block size for the TDs (instead of the block size $\sim v$ used in the original construction in [15]).

Next, like the Rees constructions, this construction is also a direct product. In this case, the σ-group partition is used to construct the parallel classes: $\mathcal{R}$ is spread out into a parallel class on $V \times \mathbb{H}$ and then $\mathbb{H}$ acts on aggregates of the TD. The remark on pairwise coverage by the product then follows.

Finally, at the extremes for σ, Construction 11 reduces to known constructions. If σ is as large as the replication number, then we are back in the case of the original Rees construction, [15]. If $\sigma = 1$, then the TD is resolvable and we are back in the case of the standard 'direct product'. Since this case is important later, we state it here separately.

Construction 12 (Direct Product)**.** *If there exists an RGDD(g^u, k) and an RTD(k, n), then there exists an RGDD$((gn)^u, k)$ which contains as a subdesign an RGDD(g^u, k).*

A crucial observation for this type of construction is that if we have a Σ-resolvable GDD, Construction 11 or Construction 12 can be applied one class at a time. Then, the resulting parallel classes can be assembled into a resolvable GDD in which the group sizes have been inflated by a factor of n.

3 First Examples

To construct our first examples of RGDDs of large order, we begin with existence for some large index. Recall $g_{\min}$, defined as in (3), is a function of the desired block size k and number of groups u.

Lemma 13. *Given integers $u \geqslant k \geqslant 2$, there exists RGDD$_\Lambda(g_{\min}^u, k)$ for some positive integer Λ.*

Proof. Let Γ denote the complete multipartite graph with u parts of size $g_{\min}$. Note that k divides the number of vertices of Γ. Now, take our block collection $\mathcal{B}$ to be the (multiset) union of all possible parallel classes with block size k embedded on Γ. Given any two edges e, e' of Γ, there is a natural bijection (simply a relabelling) mapping the parallel classes covering e to those covering e'. It follows that every edge of Γ gets covered the same number of times by blocks in $\mathcal{B}$. Moreover, $\mathcal{B}$ is automatically resolvable by construction. □

Remark 14. There are in general many repeated blocks in the above construction, and the Λ obtained in this way is surely far larger than needed. Also, observe that we can obtain the same result for the smaller group size $k/\gcd(u,k)$, which divides $g_{\min}$. More importantly, with this same Λ, we obtain RGDD$_\Lambda(g^u, k)$ for all sufficiently large $g \equiv 0 \pmod{g_{\min}}$ from a direct product using an RTD with u groups.

Next, we inflate to obtain a resolvable HGDD, and then fill holes to get a GDD with many parallel classes and a few thick classes.

Lemma 15. *Given integers $u \geqslant k \geqslant 2$ and $g \equiv 0 \pmod{g_{\min}}$, there exists, for some large prime power q, a resolvable HGDD$(q \times g^u, k)$. Consequently, for large such g, there exists a Σ-resolvable GDD$((gq)^u, k)$, where the list Σ of class thicknesses contains $g(q-1)(u-1)/(k-1)$ occurrences of 1 and one occurrence of $g(u-1)/(k-1)$.*

Proof. For the resolvable HGDD, we simply apply Construction 6 to the output of Lemma 13, choosing $q \equiv 1 \pmod{\Lambda}$. It is easy to count that this HGDD has $g(q-1)(u-1)/(k-1)$ parallel classes. Now, use Construction 5 and the main result of [12], which gives the existence of GDD(g^u, k) for large $g \equiv 0 \pmod{g_{\min}}$. The q copies of this latter GDD themselves form a σ-parallel class for $\sigma = g(u-1)/(k-1)$, this simply being its replication number. □

Remark 16. By amalgamating single parallel classes with the thick one, the remaining thickness can be assumed to equal some prime, say p, between σ and 2σ. The existence of such a prime follows from Bertrand's postulate.

Now, we apply Construction 11 to get three examples with 'additively independent' group sizes. Together with our recursive constructions, these examples are enough to prove Theorem 3. In order to apply Construction 11, we need the existence of the ingredient TDs with σ-group partitions. We use the natural partitioning from a standard direct product to help construct the necessary σ-group partitions.

Lemma 17. *Let n and k be positive integers. Suppse that p is a prime with $p \geqslant k$ and suppose that n is sufficiently large with $p^2 \mid n$. Then there exists an RTD$(p(k-1)+1, n)$ admitting a p-group partition.*

Proof. Since $p \geqslant k$, it easily follows that $p^2 \geqslant p(k-1)+1$. Therefore, there exists an RTD$(p(k-1)+1, p^2)$ arising from the standard finite field construction. So we may assume the points of each group are identified with $\mathbb{Z}_p \times \mathbb{Z}_p$, and that the blocks of this RTD are developed and resolve under this group action. Each parallel class can therefore be partitioned into p sub-classes, each of p blocks, according to (say) the first $\mathbb{Z}_p$-coordinate in the first group. Now take a direct product of each sub-class with an RTD$(p(k-1)+1, n/p^2)$. The resulting RTD$(p(k-1)+1, n)$ can have its groups identified with $\mathbb{G} = \mathbb{Z}_p \times \mathbb{Z}_{n/p}$. To verify the p-group partition, aggregates arise from the direct product with individual sub-classes (i.e. are constant within each group on the first $\mathbb{Z}_p$-coordinate of $\mathbb{G}$). The subgroup $\mathbb{H} = \mathbb{Z}_p \times \{0\}$ develops each such aggregate into a parallel class. $\square$

Proposition 18. *Given $u \geqslant k \geqslant 2$, there exist RGDD$(g_i^u, k)$ for $i = 1, 2, 3$ for arbitrarily large integers g_1, g_2, g_3 satisfying* $\gcd(g_1, g_2) = \gcd(g_1 g_2, g_3) = g_{\min}$.

Proof. This amounts to a selection of several large and distinct primes for the preceding constructions. First, choose distinct primes a_1, a_2, a_3 large enough for the existence of GDD$((a_i g_{\min})^u, k)$, appealing to [12]. Working from these groupsizes, we next choose, using Dirichlet's Theorem, three new large primes $q_1, q_2, q_3 \equiv 1 \pmod{\Lambda}$ for the application of Lemma 15. Finally, by amalgamating classes, the nontrivial class thicknesses can be assumed to be three new primes p_1, p_2, p_3 to facilitate an easy application of Construction 11 with Lemma 17. We obtain RGDD(g_i^u, k) having groupsizes $g_i = a_i q_i p_i^2 g_{\min}$ for $i = 1, 2, 3$. The numerical requirements are easy to fulfill if we simply avoid in our selection any prime factors of $g_{\min}$. $\square$

4 Recursion and Proof

We set up some recursive constructions with the goal of combining our examples from Proposition 18.

4.1 Incomplete designs and recursive constructions

An *incomplete group divisible design*, or IGDD, is a quadruple $(V, \Pi, \Pi', \mathcal{B})$ such that V is a set of v points, $\Pi = \{V_1, \ldots, V_u)$ is a partition of V into 'groups', $\Pi' = \{W_1, \ldots, W_u\}$ with $W_i \subseteq V_i$ for each i, and $\mathcal{B} \subseteq \binom{V}{\geqslant 2}$ is a set of blocks such that

- two points get covered by a block (exactly one block) if and only if they come from different groups, say V_i and V_j, $i \neq j$, **and** they do not both belong to the corresponding holes W_i and W_j.

As with GDDs, the type of an IGDD can be written by listing, using exponential notation when appropriate, the pairs $(|V_i|; |W_i|)$ of group size and corresponding hole size. So, for example, a (uniform) *incomplete group divisible design* of type $(g; h)^u$ with block size k is denoted IGDD$((g; h)^u, k)$.

In the special case when $u = k$, $g = n$ and $h = m$, an incomplete GDD is an *incomplete transversal design* and we write ITD$(k, n; m)$. These objects are equivalent to $k - 2$ incomplete mutually orthogonal latin squares of order n with aligned $m \times m$ holes.

The following existence result for ITDs with small holes is sufficient for our applications.

Lemma 19 ([5]). *Let k and i be integers with $k \geqslant 2$ and $0 \leqslant i \leqslant k$. For all sufficiently large n, an ITD$(k, n + i; i)$ exists.*

In a uniform IGDD$((g; h)^u, k)$, there are two replication numbers: $r_1 := (g - h)(u - 1)/(k - 1)$ for points in a hole W_i, and $r_2 := g(u - 1)/(k - 1)$ for points in some $V_i \setminus W_i$. Let us call such an IGDD *resolvable* if it partitions into r_1 full parallel classes and $r_2 - r_1$ partial classes missing the union $\cup_{i=1}^{u} W_i$ of all holes. As is standard, we use IRGDD as an abbreviation.

It is useful to recall that an ITD$(k+1, n+i; i)$ is equivalent to an IRGDD$((n+i; i)^k, k)$. The next construction is an application of Construction 9 where the master design is an RGDD and copies of an ITD (IRGDD) are used as ingredients. More details can be found in the book [8].

Lemma 20 ([8]). *If there exists an RGDD(g^u, k) and an ITD$(k + 1, n + j; j)$, then there is an IRGDD$((gn + gj; gj)^u, k)$.*

Our main recursive construction is a generalization of Wilson's construction for TDs, [19]. The IRGDDs used in our application come from Lemma 20.

Lemma 21 ([8, 9]). *Suppose that there exists an RTD$(u + 1, t)$ and an IRGDD$((m + e_i; e_i)^u, k)$ for each i, $1 \leqslant i \leqslant t$. Then there is an IRGDD$((mt + e; e)^u, k)$ with $e = \sum_{1 \leqslant i \leqslant t} e_i$. Furthermore if there exists an RGDD(e^u, k), then there exists an RGDD$((mt + e)^u, k)$.*

We will also use a special case of this construction which sets $e_i = 0$ and w and uses the existence of the subdesign and the RTD to get the existence of the last subdesign. This corollary without the resolvability conditions was used in [12].

Corollary 22. *If there exists an RGDD(n^u, k), an RGDD$((n + w)^u, k)$ containing an RGDD(w^u, k), an RTD$(u + 1, s)$, and an RTD(k, p) with $p \leqslant s$, then there exists an RGDD$((sn + pw)^u, k)$ which contains an RGDD$((pw)^u, k)$ as a subdesign.*

4.2 Proof of Theorem 3.

The broad idea is to get the two ingredient RGDDs of Corollary 22, where $\gcd(n, w) = g_{\min}$, using a sequence of earlier constructions. We thus obtain constructions for groupsizes which are various integral linear combinations of n and w. Such combinations are shown to be sufficient with a representation lemma for large integers due to Mohácsy.

Lemma 23 ([12]). *Let n and w be positive integers such that gcd$(n, w) = c$. Then all sufficiently large multiples of c can be represented in the form $sn + pw$, where s and p are integers with $n < p \leqslant s$.*

We begin the proof of Theorem 3. Using Proposition 18, choose large integers g_i such that there exist RGDD(g_i^u, k) for $i = 1, 2, 3$ satisfying $\gcd(g_1, g_2) = \gcd(g_1 g_2, g_3) = g_{\min}$.

In particular, we take g_1 and g_2 sufficiently large so that there exist RTD(k, g_i) and ITD$(k+1, g_i+1; 1)$ for $i = 1, 2$. In fact, in view of the theorem of Chowla, Erdős and Strauss, we may assume similar designs exist with larger groupsizes. We also choose g_3 large enough to apply Lemma 23 with $n = g_1$ and $w = g_2$, writing $g_3 = sg_1 + pg_2$ where s and p are non-negative integers such that $g_1 < p \leqslant s$.

Now use Lemma 20 twice: once with $n = g_2$ and $g = g_1$ to construct an IRGDD$((g_1 g_2 + g_1; g_1)^u, k)$ and then again with $n = g_1$ and $g = g_2$ to construct an IRGDD$((g_1 g_2 + g_2; g_2)^u, k)$. Observe also that Construction 12 gives an RGDD$((g_1 g_2)^u, k)$.

Next, choose t to be a positive integer relatively prime to g_3 and also sufficiently large so that $t \geqslant s + p$, and also there exists an RTD$(u+1, x)$ for any $x \geqslant t$.

With the preceding ingredient designs, apply Lemma 21 with $m = g_1 g_2$, and $e_i \in \{0, g_1, g_2\}$. This gives an IRGDD$((g_1 g_2 t + e; e)^u, k)$ where $e = sg_1 + pg_2$. Recall that $g_3 = sg_1 + pg_2$ and there exists an RGDD(g_3^u, k). So we have constructed an RGDD$((g_1 g_2 t + g_3)^u, k)$ with a sub-RGDD(g_3^u, k).

Since t and g_3 were chosen relatively prime, we have $\gcd(g_1 g_2 t, g_3) = \gcd(g_1 g_2, g_3) = g_{\min}$.

Let g be large and divisible by $g_{\min}$. Use Lemma 23 again, this time with $n = g_1 g_2 t$ and $w = g_3$ to write $g = s'n + p'w$, where s' and p' are non-negative integers and $n < p' \leqslant s'$. These integers are large enough for the existence of RTD$(u+1, s')$ and RTD(k, p'). We also have an RGDD(n^u, k), from Construction 12. It follows from Corollary 22 that there exists an RGDD(g^u, k). This completes the proof.

5 Graph RGDDs

In this section, we extend Theorem 3 to the more general case of graph decompositions.

5.1 Graph decompositions

Consider the complete multipartite graph K_T, where T denotes the partition type. A group divisible design GDD(T, k) is equivalent to an edge-decomposition of K_T into cliques K_k. A clique in this decomposition corresponds with a block of size k. If, more generally, we wish to edge-decompose the same graph K_T into copies of some fixed graph G, we adopt similar notation: GDD(T, G) instead of GDD(T, k), and so on. Sometimes, even a set of allowed graphs is desired. We refer the reader to [4, 6, 10] for more details on the notation and terminology for graph decompositions which we use below.

In what follows, let G be a simple undirected graph with k vertices, e edges, and degree sequence $d_1, \dots, d_k$, and let $d = \gcd(d_1, \dots, d_k)$. For an ordinary (not necessarily resolvable) GDD(g^u, G), the divisibility conditions generalizing those in Theorem 2 are $g(u-1) \equiv 0 \pmod{d}$ and $g^2 u(u-1) \equiv 0 \pmod{2e}$. These are known to be sufficient for fixed g and large u; see for instance [1]. In this section, we are interested in fixed u and large g.

For an RGDD(g^u, G), we must also have

$$gu \equiv 0 \pmod{k}; \tag{4}$$

this simply reiterates (1) since a parallel class of G-blocks again partitions the gu points. The necessary divisibility condition extending (2) is discussed in [4]. We have

$$g(u-1) \equiv 0 \pmod{\alpha^*(G)}, \tag{5}$$

where where $\alpha^*(G)$ is the least positive integer A so that the system

$$\begin{aligned} x_1 + \cdots + x_k &= A\frac{k}{2e}, \\ d_1x_1 + \cdots + d_kx_k &= A \end{aligned}$$

has an integer solution $(x_1, \ldots, x_k)$. For example, in the case when G is d-regular, we have $\alpha^*(G) = d$, seen by taking $x_1 = 1$ and $x_i = 0$ for each $i = 2, 3, \ldots, k$. In general, $\gcd(d_1, \ldots, d_k) \mid \alpha^*(G)$.

Recall that for a GDD(g^u, k), there is a necessary inequality $u \geqslant k$ on the number of groups. In the case of graph decompositions GDD(g^u, G), this is relaxed to $u \geqslant \chi(G)$, since now vertices of some block can coexist in the same group provided they form an independent set in G.

Example 24. Let $G = K_{a,b}$, the complete bipartite graph. We have $\chi(G) = 2$ as the minimum possible number of groups. It is shown in [16] that $K_{n,n}$ edge-decomposes into $K_{a,b}$ if and only if $ab \mid n^2$, $a, b \leqslant n$ and there exist at least two distinct expressions $n = ax + by$ for nonnegative integers x, y. It follows that the necessary conditions on g for GDD$(g^2, K_{a,b})$ are sufficient if g is (mildly) large relative to a, b. We can join copies of such a GDD to obtain GDD$(g^u, K_{a,b})$ for any $u \geqslant 2$ (although the necessary conditions on g may weaken). As a side note, the question of edge-decomposing $K_{n,n}$ into $K_{a,b}$ is closely related with the famous problem of decomposing an $n \times n$ square into $a \times b$ rectangles, except that we do not demand that such rectangles be 'contiguous'.

We remark that, in general, allowing $u < k$ increases the complexity of the degree condition on $g(u-1)$. In this case, degrees $g(u-1)$ in some group of the GDD must be formed as an integral linear combination of the degree 'vectors' of entire colour classes placed on that group. For this reason, we are mainly interested in the case where $u \geqslant k$ and where graph blocks are only placed across the groups.

For a graph G and positive integers g, u, define a *transverse* graph GDD, denoted by GDD$^!_\lambda(T, G)$, as an edge-decomposition of the λ-fold multigraph $K_T^{(\lambda)}$ into copies of G such that each G-block and each group (partite set) share at most one vertex in common. Clearly, a GDD$_\lambda(T, K_k)$ is equivalent to a GDD$^!_\lambda(T, K_k)$ but the latter is stronger for general graphs. We point out that standard construction methods for large u, such as edge-colored graph decompositions (see [1, 10]) yield transverse graph GDDs. In other words, in many cases there is no loss in discarding those G-blocks with two points together in a group.

A resolvable transverse graph GDD is denoted $\mathrm{RGDD}^{\mathrm{!}}(T,G)$. We use similar notation for IRGDDs.

Recall that our proof of Theorem 3 used the existence of three 'numerically independent' examples, Proposition 18, and then applied the recursive constructions from Section 4. In each of the cases for graph designs, we construct an analogue of Proposition 18 by extending our techniques from Section 3 or from [12] to construct a set of examples for graph GDDs. Once we show that the recursive constructions in Section 4 can also be extended with little difficulty to the more general setting of graphs, we use the same proof to establish the existence results. Instead of repeating all proofs, we point out the key differences in working with graph GDDs.

5.2 Theorem 2 for graphs

We first consider the existence question for $\mathrm{GDD}^{\mathrm{!}}(g^u,G)$; that is, without resolvability imposed. In this case, necessary conditions are $g(u-1) \equiv 0 \pmod{d}$ and $g^2u(u-1) \equiv 0 \pmod{2e}$. We begin with a few results that roughly correspond with the treatment for blocks ($G = K_k$) in [12].

The starting point is a 'large index' result. For this, it is technically convenient to assume that our graph G satisfies the *four point condition*: for every edge $\{x,y\} \in E(G)$, there exist two other vertices w,z of G such that none of $\{w,y\}$, $\{w,z\}$, $\{x,z\}$ are edges of G. If necessary, we can include two additional isolated vertices to G so that it satisfies this condition. One impact is that, for transverse GDDs, it is then necessary that the number of groups satisfies $u \geqslant |V(G)|+2$. Of course, when considering the existence of a decomposition into G, isolated vertices are immaterial.

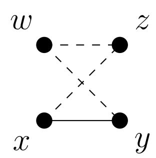

Figure 2: The four point condition

Lemma 25. *Let G be a simple graph with k vertices, e edges, and degree sequence $d_1,\dots,d_k$. Suppose that G satisfies the four point condition and let $u \geqslant k \geqslant 2$ and $g \geqslant 1$. Then there exists $\mathrm{GDD}^{\mathrm{!}}_{\Lambda}(g^u,G)$ for all sufficiently large Λ satisfying $\gcd(d_1,\dots,d_k) \mid \Lambda g(u-1)$ and $e \mid \Lambda g^2\binom{u}{2}$.*

Proof sketch. Let $\lambda_{\min}$ denote the positive generator of the ideal of integers Λ satisfying the two divisibility conditions. Following [10, 12, 23], we proceed in two steps. First, we prove the existence of a 'signed GDD' with index $\lambda_{\min}$, and then we add sufficiently many multiples of a 'complete GDD' so that all G-blocks occur with nonnegative multiplicity. Since the latter step is quite standard and occurs in nearly identical form in the references, we focus on the signed decomposition.

Consider the family $\mathcal{G}$ of all edge-g^2-colored bi-directed graphs isomorphic to G, where edge colors are induced by vertex labellings $V(G) \to [g]$. That is, if $\{x, y\}$ is an edge of G with x labelled i and y labelled j, then (x, y) is colored (i, j) and (y, x) is colored (j, i) in this graph in $\mathcal{G}$. Our 'signed GDD' is a solution to the equations $\sum_{H \in \mathcal{G}: \epsilon \in H} X_H = \lambda$, where there is one variable X_H for each $H \in \mathcal{G}$ and one equation for each (colored) edge $\epsilon \in K_u^{g^2}$. To obtain the desired integral solution $\{X_H\}$, we apply [10, Lemma 5.4] to this system. (Note that the four point condition essentially appears in the argument on [10, p. 167].)

Admissibility of $\mathcal{G}$ follows by symmetry considerations. The calculation of $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$ proceeds as a straightforward combination of [6, Lemma 2.1] and [10, Theorem 8.1]. □

Remark 26. If G is edgeless, that is if $e = 0$, then only $\Lambda = 0$ is admissible.

Next, we adapt Construction 8 to G-blocks. There is no sustantial difference in the proof, since the key idea is just the scheme for lifting blocks in the inflation.

Construction 27. *Suppose there exists a $GDD_q(g^u, G)$ of index q, a prime power. Then there exists a $GDD((gq^*)^u, G)$, where q^* is a sufficiently large power of q.*

As in [12, Lemma 6.2], for fixed G and u, the necessary numerical conditions on g for existence of a $\mathrm{GDD}^!(g^u, G)$ generate an ideal for admissible g, summarized as $g \equiv 0 \pmod{g_{\min}}$. In this case, $g_{\min}$ has a complicated form; however, for our purposes, it is sufficient to know that $g_{\min}$ exists.

At this point, we can apply Lemma 25 with $g = g_{\min}$ to obtain three large and distinct prime values of Λ, say p_i for $i = 1, 2, 3$. After inflating via Construction 27, we have mirrored Proposition 18 for transverse GDDs and have constructed the three examples needed for the recursive constructions. There is no need in this case to use HGDDs and the Rees product, since resolvability is not (yet) being considered.

The graph-adapted recursive constructions, whose details we reference forward to the next subsection, together with these three examples yield the following existence theorem for graph GDDs.

Theorem 28. *Let G be a simple graph on k vertices, e edges, and gcd of degrees d. Suppose either $u \geqslant k + 2$, or that G satisfies the four point condition and $u \geqslant k$. Then there exists a $GDD^!(g^u, G)$ for all sufficiently large g satisfying $g(u-1) \equiv 0 \pmod{d}$ and $g^2u(u-1) \equiv 0 \pmod{2e}$.*

For our application in the resolvable case, we will need a slightly stronger condition. A decomposition into simple graphs G is called *equireplicate* if every point belongs to the same number of G-blocks. A large-order existence theory for general equireplicate G-decompositions was established in [7] and used in constructions of resolvable G-decompositions in [6]. We note that Lemma 25 is easily adapted to equireplicate GDDs, with the only change being that $\gcd(d_1, \dots, d_k)$ becomes $\alpha^*(G)$. Also, Construction 25 respects the equireplicate property due to the transitive action of the finite field on G-blocks.

The numerically allowed group sizes for a (uniform) equireplicate graph GDD with given parameters again form an ideal. In the equireplicate case, $d = \gcd(d_1, \dots, d_k)$ is replaced by $\alpha^*(G)$. The equireplicate version of Theorem 28 is stated below, with further details omitted.

Theorem 29. *Let G be a simple graph on k vertices. Suppose either $u \geqslant k+2$, or that G satisfies the four point condition and $u \geqslant k$. Then there exists an equireplicate GDD$^!(g^u, G)$ for all sufficiently large g satisfying $g(u-1) \equiv 0 \pmod{\alpha^*(G)}$ and $g^2u(u-1) \equiv 0 \pmod{2e}$.*

5.3 Resolvability and recursive constructions

We turn now to constructions for graph RGDDs. We omit details of the essentially similar steps and focus on the key differences. In this case, we adapt the techniques we used in Section 3 to graphs to construct our basic three examples. Again, we note that for fixed G and u, the necessary numerical conditions for the existence of an RGDD$^!(g^u, G)$, (4) and (5), generate an ideal for admissable g, summarized as $g \equiv 0 \pmod{g_{\min}}$.

First, it is routine to extend Lemma 13 and Constructions 5 and 6 to graphs; in fact, [6, Lemma 3.2] used in 6 was originally proved for graphs. Using our own graph GDD result Theorem 29 in place of k-GDDs to fill in the holes of the HGDD, we obtain an extension of Lemma 15 to the graph case. Instead of $g(q-1)(u-1)/(k-1)$ parallel classes, we have $g(q-1)(u-1)k/2e$ parallel classes, and instead of one class of thickness $g(u-1)/(k-1)$, we have one of thickness $g(u-1)k/2e$. It is important to note that the latter class has constant thickness because equireplicate GDD$^!(g^u, G)$ were used. So with the hypothesis of Theorem 29 for G and fixed u, we have constructed a Σ-resolvable GDD$((gq)^u, k)$, where the list Σ contains one thick class which is a p-paralllel class for some prime p. Our next step is to apply the Rees construction.

The Rees product, Construction 11, extends with no difficulty to the case of G-blocks. We remark that a G-block is acted on by the transversal design group as a labeled graph; that is, adjacencies between vertices get preserved.

Construction 30. *Let H be a simple undirected graph with k vertices. Let $(V, \mathcal{R})$ be a σ-parallel class of H-blocks. Suppose there exists a TD$((k-1)\sigma + 1, n)$ which admits a σ-group partition based on the group $\mathbb{G}$. Then there exist σn disjoint parallel classes on $\mathbb{G} \times V$ that cover all pairs between $\mathbb{G} \times \{i\}$ and $\mathbb{G} \times \{j\}$ the same number of times as $\mathcal{R}$ covers $\{i,j\} \in \binom{V}{2}$.*

It is helpful to recall the special case of resolvable GDDs, in which $\sigma = 1$ for all classes. This direct product simply results from replacing each G-block with a copy of the graph Cartesian product $G \times \overline{K_n}$, and using labels of an RTD(k, n) to (resolvably) G-decompose this product.

Construction 31. *Let G be a simple undirected graph with k vertices. If there exists an RGDD$^!(g^u, G)$ and an RTD(k, n), then there exists a RGDD$^!((gn)^u, G)$.*

It is important to distinguish Construction 31 from a standard direct product, which replaces each block of an RTD(u, n) with a copy of some RGDD(g^u, G). This latter product is actually an instance of WFC, where it is straightforward to see that if each of the ingredients is a graph GDD with G-blocks, then the so is the resulting design. It is important here that the master be an 'ordinary' GDD with blocks (rather than G-blocks). We state this adaptation without proof.

Construction 32. *Let $(V, \Pi = \{V_1, \ldots, V_u\}, \mathcal{B})$ be an ordinary GDD and $\omega : V \to \mathbb{N} \cup \{0\}$ a weighting. Suppose, for each block $B \in \mathcal{B}$, that there exists a GDD$^!(T_B, G)$, where $T_B = (\omega(x) : x \in B)$. Then there exists a GDD$^!(T, G)$, where $T = (\sum_{x \in V_i} \omega(x) : i = 1, \ldots, u)$.*

We have now shown that the techniques of Section 3 can be adapted to construct our three basic examples for RGDD$^!(g^u, G)$. Next, we turn to the recursive constructions used in Section 4.

First we recall that the key Lemma 20 comes from WFC. As a minor variation, we can reverse the role of master and ingredient designs to produce the same IRGDD using an ITD$(u+1, n+j; j)$ and an RGDD(g^u, k). This is similar to [8, Theorem 3.4.4]. Now, using Construction 32, this is easily adapted to G-blocks.

Lemma 33. *Let G be a simple undirected graph with k vertices. If there exists an ITD$(u+1, n+j; j)$ and an RGDD$^!(g^u, G)$, then there is an IRGDD$^!((gn+gj; gj)^u, G)$.*

Since Wilson's construction for transversal designs uses the same idea as WFC (where the RTD acts as is the 'master' design), Lemma 21 can also be adapted for graph designs.

Lemma 34. *Let G be a simple undirected graph with k vertices. Suppose that there exists an RTD$(u+1, t)$. If there exists an IRGDD$^!((m+e_i; e_i)^u, G)$ for any i, $1 \leqslant i \leqslant t$, then there is an IRGDD$^!((mt+e; e)^u, G)$ with $e = \sum_{1 \leqslant i \leqslant t} e_i$. Furthermore if there exists an RGDD$^!(e^u, G)$, then there exists an RGDD$^!((mt+e)^u, G)$.*

Corollary 22 constructs an RGDD$((sn+pw)^u, k)$ with a subdesign RGDD$((pw)^u, k)$. A direct product is used to construct the missing subdesign. In order to generalize Corollary 22 to accommodate G-blocks, we use the direct product in Construction 31 (rather than the standard one that arises from applying WFC).

Corollary 35. *Let G be a simple graph with k vertices. If there exists an RGDD$^!(n^u, G)$, an RGDD$^!((n+w)^u, G)$ containing an RGDD$^!(w^u, G)$, an RTD$(u+1, s)$, and an RTD(k, p) with $p \leqslant s$, then there exists an RGDD$^!((sn+pw)^u, G)$ which contains as a subdesign an RGDD$^!((pw)^u, G)$.*

Finally we note that Lemma 33, Lemma 34, and Corollary 35 all hold without the resolvability conditions. In particular, they can be used to construct GDD(g^u, G). It is also important to note that if all of the graph GDDs used in the hypotheses have transverse G-blocks, then the resulting designs will as well. Thus, we have verified that all of the constructions used in Section 4 can be adapted to accommodate G-blocks. This gives us all of the machinery needed to complete the proof of Theorem 28.

Returning to the resolvable case, we have adapted the constructions in Section 3 to construct our necessary basic examples, and we have now shown that all of the constructions used in Section 4.2 (the proof of Theorem 3) carry over to transverse G-blocks. This gives us the following for the graph case.

Theorem 36. *Let G be a simple graph on k vertices. Suppose that either $u \geqslant k+2$ or G satisfies the four point condition and $u \geqslant k$. Then there exists an RGDD$^{!}(g^u, G)$ for all sufficiently large g satisfying* (4) *and* (5).

6 Future directions

We would like to be able to drop the four point condition in Theorem 36 and especially for the case of what might be called 'graph transversal designs' GDD$^{!}(n^{|V(G)|}, G)$. These directions may require new ideas for constructing the underlying signed decompositions used in the proof of Lemma 25.

There remains considerable work in the cases when G-blocks get packed more tightly than in transverse GDDs. Even the normally obvious divisibility conditions appear to be nontrivial when the number of groups is as small as, say, the chromatic number of G.

It should be remarked that our existence theory for RGDD(g^u, G) having u fixed and g large can lead to a new construction for the reverse problem, in which g is fixed and u is large. This latter graph decomposition problem was first considered in [4] following a complete solution for blocks. However, the case of general graphs G was only solved in the case when $\gcd(|V(G)|, \alpha^*(G)) = 1$. The main idea converting RGDDs with large group size into many groups of fixed size is to use block coloring and 'weaving' parallel classes; see [4, Lemma 2.1]. Our transverse graph RGDDs from Section 5 can take the place of transversal designs. However, this approach is not yet sufficiently general to settle the full conjecture which extends Theorem 1 to graphs.

Conjecture 37. Given a positive integer g and simple graph G on k vertices and at least one edge, there exists RGDD(g^u, G) for all sufficiently large u satisfying $gu \equiv 0 \pmod{k}$ and $g(u-1) \equiv 0 \pmod{\alpha^*(G)}$.

Returning to the case of blocks K_k, it is very natural to consider arbitrary index λ. The necessary condition (2) weakens to

$$\lambda g(u-1) \equiv 0 \pmod{k-1}. \tag{6}$$

Although many steps in our method easily accommodate general index λ, the existence problem for GDD$_\lambda(g^u, k)$ and GDD$_\lambda(g^u, G)$ with fixed u, resolvable or not, remains open. Recent progress has been made extending the 'block spreading construction' to general λ in [13], and we feel safe with the guess that (1), (6) and $g \gg 0$ are sufficient conditions.

Conjecture 38. Given integers $u \geqslant k \geqslant 2$ and $\lambda \geqslant 1$, there exist RGDD$_\lambda(g^u, k)$ for all sufficiently large g satisfying $gu \equiv 0 \pmod{k}$ and $\lambda g(u-1) \equiv 0 \pmod{k-1}$.

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