On universal hypergraphs

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Submitted: Sep 10, 2015; Accepted: Nov 8, 2016; Published: Nov 25, 2016
Mathematics Subject Classifications: 05C60, 05C65

Abstract

A hypergraph H is called universal for a family $\mathcal{F}$ of hypergraphs, if it contains every hypergraph $F \in \mathcal{F}$ as a copy. For the family of r-uniform hypergraphs with maximum vertex degree bounded by Δ and at most n vertices any universal hypergraph has to contain $\Omega(n^{r-r/\Delta})$ many edges. We exploit constructions of Alon and Capalbo to obtain universal r-uniform hypergraphs with the optimal number of edges $O(n^{r-r/\Delta})$ when r is even, $r \mid \Delta$ or $\Delta = 2$. Further we generalize the result of Alon and Asodi about optimal universal graphs for the family of graphs with at most m edges and no isolated vertices to hypergraphs.

1 Introduction

Let $\mathcal{F}^{(r)}(n, \Delta)$ be the family of r-uniform hypergraphs (short r-graphs) on at most n vertices and with maximum vertex degree bounded by Δ. An r-graph is called $\mathcal{F}^{(r)}(n, \Delta)$-universal (or universal for $\mathcal{F}^{(r)}(n, \Delta)$) if it contains every $F \in \mathcal{F}^{(r)}(n, \Delta)$ as a copy. The purpose of this paper is to show that the existence and almost optimal explicit constructions of many universal hypergraphs follow from the corresponding results about universal graphs.

The problem of finding various universal graphs has a long history, see an excellent survey of Alon [1] and the references therein. Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi studied in [5, 6] explicit constructions of $\mathcal{F}^{(2)}(n, \Delta)$-universal graphs and the universality of the random graph $G(n, p)$ as well. Thus, in [6] they constructed first nearly optimal $\mathcal{F}^{(2)}(n, \Delta)$-universal graphs ($\Delta \geqslant 3$) with $O(n)$ vertices and $O(n^{2-2/\Delta} \ln^{1+8/\Delta} n)$ edges, while it was noted by the same authors that any such universal graph has to contain $\Omega(n^{2-2/\Delta})$ edges. Notice further, that in the case $\Delta = 2$ the

[*]The research of Samuel Hetterich leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 278857-PTCC.

[†]The research of Olaf Parczyk and Yury Person leading to these results was supported by DFG grant PE 2299/1-1.

square of a Hamilton cycle is $\mathcal{F}^{(2)}(n,\Delta)$-universal [3] (and thus $2n$ edges are enough in this case). In two subsequent papers, Alon and Capalbo [3, 4] improved the result of [6] and obtained $\mathcal{F}^{(2)}(n,\Delta)$-universal graphs with the optimal number $\Theta(n^{2-2/\Delta})$ of edges and only $O(n)$ vertices and also provided $\mathcal{F}^{(2)}(n,\Delta)$-universal graphs on n vertices with almost optimal number of edges.

Theorem 1 (Alon and Capalbo [3, 4])**.** *For any $\Delta \geqslant 2$ there exist explicitly constructible $\mathcal{F}^{(2)}(n,\Delta)$-universal graphs on $O(n)$ vertices with $O(n^{2-2/\Delta})$ edges and on n vertices with $O(n^{2-2/\Delta}\ln^{4/\Delta} n)$ edges.*

Universality of random graphs has been also a subject of intensive study by various researchers. Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi proved in [5] that the random graph $G((1+\varepsilon)n,p)$ is $\mathcal{F}^{(2)}(n,\Delta)$-universal for $p \geqslant C_{\varepsilon,\Delta}(\ln n/n)^{1/\Delta}$ a.a.s., where $C_{\varepsilon,\Delta}$ is a constant that depends only on Δ and ε. Since then several improvements of this result have been given. So, for example, in the spanning case Dellamonica, Kohayakawa, Rödl and Ruciński [9] showed that $G(n,p)$ is $\mathcal{F}^{(2)}(n,\Delta)$-universal for $p \geqslant C_\Delta(\ln n/n)^{1/\Delta}$ (for $\Delta \geqslant 3$) a.a.s., while the case $\Delta = 2$ was covered by Kim and Lee [11]. In the almost spanning case, Conlon, Ferber, Nenadov and Škorić [8] recently showed that for every $\varepsilon > 0$ and $\Delta \geqslant 3$ the random graph $G((1+\varepsilon)n,p)$ is $\mathcal{F}^{(2)}(n,\Delta)$-universal for $p = \omega(n^{-1/(\Delta-1)}\ln^5 n)$ a.a.s.

The study of universal graphs has been extended recently in [14] to universal hypergraphs by the second and third author, who showed that the random r-graph $\mathcal{H}^{(r)}(n,p)$ is $\mathcal{F}^{(r)}(n,\Delta)$-universal a.a.s. for $p \geqslant C(\ln n/n)^{1/\Delta}$, where C is a constant depending on r and Δ only. On the other hand, it follows from the asymptotic number of Δ-regular r-graphs on n vertices, see e.g. Dudek, Frieze, Ruciński and Šileikis [10], that *any* $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraph must possess $\Omega(n^{r-r/\Delta})$ edges [14]. Moreover, in [14] explicit constructions of $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs on $O(n)$ vertices with $O(n^{r-2/\Delta})$ edges were derived from Theorem 1, and the existence of even sparser universal hypergraphs was obtained from the results on universality of random graphs [8, 9]. For example, it was shown that there exist $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs with n vertices and $\Theta\left(n^{r-\frac{r}{2\Delta}}(\ln n)^{\frac{r}{2\Delta}}\right)$ edges, which shows that the best known lower and upper bounds are at most the multiplicative factor $n^{\frac{r}{2\Delta}} \cdot \mathrm{polylog}(n)$ apart. See the summary of these results in the table below. Here and in the following the constants in the O-terms depend on r and Δ.

Another family of graphs that received attention is the family $\mathcal{E}^{(r)}(m)$ of r-graphs with at most m edges and without isolated vertices. Babai, Chung, Erdős, Graham and Spencer [7] proved that any $\mathcal{E}^{(2)}(m)$-universal graph must contain $\Omega(m^2/\ln^2 m)$ many edges and there exists one on $O(m^2 \ln\ln m/\ln m)$ edges. Alon and Asodi [2] closed this gap by proving the existence of an $\mathcal{E}^{(2)}(m)$-universal graph on $O(m^2/\ln^2 m)$ edges.

1.1 New results

We will prove the following statements that allow us to construct r-uniform universal hypergraphs from universal hypergraphs of smaller uniformity. We use universal graphs from [3, 4] with carefully chosen parameters to provide the best known $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs.

Table 1: Known universal hypergraph results for $r \geqslant 3$ [14].

Explicit constructions of $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs	
$O(n)$ vertices	$O(n^{r-2/\Delta})$ edges
n vertices	$O(n^{r-2/\Delta}\ln^{4/\Delta} n)$ edges
Existence results of $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs	
n vertices	$\Theta\left(n^{r-\frac{r}{2\Delta}}(\ln n)^{\frac{r}{2\Delta}}\right)$ edges
$(1+\varepsilon)n$ vertices	$\omega\left(n^{r-\frac{\binom{r}{2}}{(r-1)\Delta-1}}(\ln n)^{5\binom{r}{2}}\right)$ edges

Theorem 2. *Let $r, r' \geqslant 2$ and $\Delta \geqslant 2$ be integers. If $r' \mid r$ and H' is an $\mathcal{F}^{(r')}(n,\Delta)$-universal hypergraph, then there exists an $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraph H on the same vertex set as H' and $e(H) \leqslant e(H')^{r/r'}$.*

This implies that, whenever $r' \mid r$ and 'almost optimal' $\mathcal{F}^{(r')}(n,\Delta)$-universal hypergraphs are known as is for example the case when r is even due to Theorem 1, this leads to constructions of almost optimal $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs.

Corollary 3. *Let $r, r' \geqslant 2$ and $\Delta \geqslant 2$ be integers. If $r' \mid r$ and there exists an $\mathcal{F}^{(r')}(n,\Delta)$-universal hypergraph H' with $O(n^{r'-r'/\Delta})$ edges, then there exists an $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraph on the same vertex set $V(H')$ with $O(n^{r-r/\Delta})$ edges. In particular, if r is even then there exist explicitly constructible $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs on $O(n)$ vertices with $O(n^{r-r/\Delta})$ edges and on n vertices with $O(n^{r-r/\Delta}\ln^{2r/\Delta}(n))$ edges.*

In the case of odd r we can not apply Theorem 2 and we prove the following.

Theorem 4. *Let $r \geqslant 3$ and $\Delta \geqslant 2$ be integers. Then there exist explicitly constructible $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs on $O(n)$ vertices with $O(n^{r-(r+1)/\Delta'})$ edges and on n vertices with $O(n^{r-(r+1)/\Delta'}\ln^{2(r+1)/\Delta'}(n))$ edges, where $\Delta' = \lceil (r+1)\Delta/r \rceil$. In particular, if $r \mid \Delta$ this leads to almost optimal $O(n^{r-r/\Delta}\mathrm{polylog}(n))$ edges.*

By estimating Δ' we see that in any case the lower and upper bounds on the edge densities of optimal universal hypergraphs differ by at most a factor of n^{r/Δ^2}. By applying a graph decomposition result of Alon and Capalbo from [3] we obtain yet another case when constructed universal hypergraphs match the lower bound.

Theorem 5. *Let r be an integer. Then there exists an explicitly constructible $\mathcal{F}^{(r)}(n,2)$-universal hypergraph on $O(n)$ vertices and $O(n^{r/2})$ edges.*

Finally we briefly study $\mathcal{E}^{(r)}(m)$-universal hypergraphs. It can be shown for fixed $r \geqslant 3$ that any $\mathcal{E}^{(r)}(m)$-universal hypergraph must contain at least $\Omega(m^r/\ln^r m)$ many edges. This can be seen by a simple counting argument as in [7] or by counting $(r\ln m)$-regular r-graphs on $m/\ln m$ vertices as was done in the graph case in [2]. We prove that the optimal existence result of Alon and Asodi gives rise to optimal $\mathcal{E}^{(r)}(m)$-universal hypergraphs.

Theorem 6. *There exist $\mathcal{E}^{(r)}(m)$-universal hypergraphs with $O(m^r/\ln^r m)$ edges.*

1.2 Organization of the paper

In the next section we introduce a very useful concept of hitting graphs, which we use in Section 3 to prove Theorem 2, Corollary 3 and Theorem 4 and in Section 4 along with a graph decomposition result from [3] to prove Theorem 5. In the last section we discuss $\mathcal{E}^{(r)}(m)$-universal hypergraphs and prove Theorem 6. We make no effort in optimizing the constants depending on r and Δ hidden in the O-notation.

2 Hitting graphs

Here we define a concept of hitting graphs first introduced in [14]. This will allow us later to obtain r-uniform universal hypergraphs out of universal hypergraphs of smaller uniformity.

Let $r \geqslant 3$ and $2 \leqslant s < r$ be integers. Given two s-graphs G and F and an r-graph H on the same vertex set as G, we say that G *hits* H *on* F if for all edges $f \in E(H)$ there is a copy of F in G induced on f, i.e. in $G[f]$. A family of s-graphs $\mathcal{G}$ *hits* a family of r-graphs $\mathcal{F}$ *on* F if for every $H \in \mathcal{F}$ there is a $G \in \mathcal{G}$ such that G hits H on F.

This concept allows us to reduce the uniformity from r to s keeping at the same time much of the information about H. This motivates a definition that allows us to recover all the edges of the hypergraph H which is being hit by G on F. For given s-graphs G and F let $\mathcal{H}_{(F,r)}(G)$ be the r-graph on the vertex set $V(G)$ whose edges $f \in \binom{V(G)}{r}$ are such that a copy of F is contained in $G[f]$. Then G hits H on F if and only if $H \subseteq \mathcal{H}_{(F,r)}(G)$.

The following lemma establishes the connection between hitting hypergraphs and $\mathcal{H}_{(F,r)}(G)$. It is an extension of Lemma 5.2 from [14]. For completeness we include its easy proof.

Lemma 7. *Let $r > s \geqslant 2$, $\Delta \geqslant 1$ be integers and F be an s-graph on at most r vertices. Further let $\mathcal{F}$ be a family of r-graphs and $\mathcal{G}$ a family of s-graphs hitting $\mathcal{F}$ on F. If G' is a $\mathcal{G}$-universal s-graph, then $\mathcal{H}_{(F,r)}(G')$ is $\mathcal{F}$-universal.*

Proof. Let $H \in \mathcal{F}$ be an r-graph together with the s-graph $G \in \mathcal{G}$ that hits H on F. Since G' is $\mathcal{G}$-universal, there exists an embedding $\varphi\colon V(G) \to V(G')$ of G into G'.

It is now easy to see that φ is an embedding of H into $\mathcal{H}_{F,r}(G')$, and thus, $\mathcal{H}_{F,r}(G')$ is $\mathcal{F}$-universal. This can be seen as follows. For any edge $f \in E(H)$ there is a copy of F in $G[f]$. Since φ is an embedding of G into G', there is a copy of F in $G'[\varphi(f)]$. By the definition of $\mathcal{H}_{F,r}(G')$, $\varphi(f)$ is a hyperedge in $\mathcal{H}_{F,r}(G')$. Thus, φ is an embedding of H into $\mathcal{H}_{F,r}(G')$. □

The lemma above suggests a way of obtaining r-uniform universal hypergraphs out of hypergraphs of smaller uniformity. This will be exploited for particular choices of F in the following sections.

3 Proofs for general Δ

In this section we provide proofs of Theorem 2, Corollary 3 and Theorem 4, which are valid for all $\Delta \geqslant 2$.

3.1 Proof of Theorem 2

Let $r > r' \geqslant 2$ and $\Delta \geqslant 1$ be integers such that $r' \mid r$. We take F to be the r'-uniform perfect matching on r vertices (and thus with r/r' edges). Let $H \in \mathcal{F}^{(r)}(n,\Delta)$. Since every vertex lies in at most Δ edges there is an r'-graph $H' \in \mathcal{F}^{(r')}(n,\Delta)$ hitting H on F. Such an H' can be obtained from H by replacing every edge f of H with an arbitrary perfect r'-uniform matching on f. Therefore, $\mathcal{F}^{(r')}(n,\Delta)$ hits $\mathcal{F}^{(r)}(n,\Delta)$ on F.

Now if G' is $\mathcal{F}^{(r')}(n,\Delta)$-universal then, by Lemma 7, $\mathcal{H}_{(F,r)}(G')$ is $\mathcal{F}^{(r)}(n,\Delta)$-universal. Moreover, since any collection of r/r' independent edges from G' forms an r-edge in $\mathcal{H}_{(F,r)}(G')$, we have $e(\mathcal{H}_{(F,r)}(G')) \leqslant e(G)^{r/r'}$.

□

3.2 Proof of Corollary 3

If $r' \mid r$ and there exists an $\mathcal{F}^{(r')}(n,\Delta)$-universal hypergraph H' with $O(n^{r'-r'/\Delta})$ edges, then we immediately obtain an $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraph with the vertex set $V(H')$ and with

$$O\left((n^{r'-r'/\Delta})^{r/r'}\right) = O(n^{r-r/\Delta})$$

edges.

By Theorem 1 there exist optimal explicitly constructible $\mathcal{F}^{(2)}(n,\Delta)$-universal graphs on $O(n)$ vertices with $O(n^{2-2/\Delta})$ edges. This yields for even r an explicitly constructible optimal $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraph with $O(n^{r-r/\Delta})$ edges. A similar argument applies also for the case of explicitly constructible $\mathcal{F}^{(2)}(n,\Delta)$-universal graphs on n vertices with $O(n^{2-2/\Delta}\ln^{4/\Delta} n)$ edges, giving $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs on n vertices with $O(n^{r-r/\Delta}\ln^{2r/\Delta}(n))$ edges. □

Remark 8. We remark, that obtaining $\mathcal{F}^{(r')}(n,\Delta)$-universal hypergraphs on $O(n)$ vertices with $O(n^{r'-r'/\Delta})$ edges for r' being prime would provide then the conjectured optimal upper bound $O(n^{r-r/\Delta})$ for all r and Δ.

3.3 Proof of Theorem 4

In the case when r is odd, our hitting r'-graphs will be simply graphs, i.e. $r' = 2$. Moreover, the graph F can no longer be perfect matching, and thus we take F as the disjoint union of a matching on $r-3$ vertices and a path P_3 of length 2, i.e. a path with 2 edges. We remark, that the cases when $F = K_2$ (a single edge) and $F = K_r$ were considered in [14]. We use the following lemma which asserts that one can find a family of graphs with not too large maximum degree which hits $\mathcal{F}^{(r)}(n,\Delta)$ on F.

Lemma 9. *Let $r \geqslant 3$ be odd and $\Delta \geqslant 1$. Let F be the disjoint union of a matching on $r-3$ vertices and a path P_3. Then $\mathcal{F}^{(2)}(n,\lceil (r+1)\Delta/r\rceil)$ hits $\mathcal{F}^{(r)}(n,\Delta)$ on F.*

Proof. Let $H \in \mathcal{F}^{(r)}(n,\Delta)$. One defines an auxiliary bipartite incidence graph B as follows. The first class V_1 consists of $\lceil \Delta/r\rceil$ copies of $V(H)$ and the second class V_2 is equal to $E(H)$, while an edge of B corresponds to a pair (v,f), where v is some copy of a vertex from $V(H)$ and $f \in E(H)$ is such that $v \in f$. The vertices in V_1 have degree at most Δ and every hyperedge is connected to all $\lceil \Delta/r\rceil$ copies of its r vertices, i.e. the vertices from V_2 have degree $r\lceil \Delta/r \rceil \geqslant \Delta$. By Hall's condition, there is then a matching M covering V_2 and thus of size $e(H)$.

We build the hitting graph H' on the vertex set $V(H)$ by replacing edges $f \in E(H)$ through copies of F as follows. For every edge f in $E(H)$ we use the edge (v, f) of the matching M and place a copy of F on f such that the vertex v is the degree 2 vertex of the path P_3 from F while the other vertices are placed on $f \setminus \{v\}$ arbitrary. We see that each 'placed' copy of F that contains v contributes 1 (in case $(v, f) \notin M$) or 2 (in case $(v, f) \in M$) to $\deg_{H'}(v)$. Since there are $\lceil \Delta/r \rceil$ copies of every vertex v and every vertex v lies in at most Δ edges of H, the maximum degree in H' is at most $\Delta + \lceil \frac{\Delta}{r} \rceil$ and therefore $\Delta(H') \leqslant \lceil (r+1)\Delta/r \rceil$. This implies $H' \in \mathcal{F}^{(2)}(n, \lceil (r+1)\Delta/r \rceil)$. □

For any $\mathcal{F}^{(2)}(n, \lceil (r+1)\Delta/r \rceil)$-universal graph G we use Lemma 7 to get an $\mathcal{F}^{(r)}(n, \Delta)$-universal hypergraph $H = \mathcal{H}_{(F,r)}(G)$ on the same number of vertices with at most $2|E(G)|^{(r-1)/2}\Delta(G)$ many edges, where the bound comes from first choosing a matching on $r-1$ vertices and then one of the two possible endpoints enlarging one edge to a P_3. The maximum degree of universal graphs G in the constructions of Alon and Capalbo from Theorem 1 is $O(|E(G)|/|V(G)|)$, and thus we obtain Theorem 4 with $\mathcal{F}^{(2)}(n, \lceil (r+1)\Delta/r \rceil)$-universal graph G on $O(n)$ vertices with $O(n^{2-2/\lceil (r+1)\Delta/r \rceil})$ edges since

$$O\left((n^{2-2/\lceil (r+1)\Delta/r \rceil})^{(r-1)/2} \cdot n^{1-2/\lceil (r+1)\Delta/r \rceil}\right) = O\left(n^{r-(r+1)/\lceil (r+1)\Delta/r \rceil}\right).$$

A similar calculation yields $\mathcal{F}^{(r)}(n, \Delta)$-universal hypergraphs on n vertices with

$$O(n^{r-(r+1)/\lceil (r+1)\Delta/r \rceil} \ln^{2(r+1)/\lceil (r+1)\Delta/r \rceil} n)$$

edges, which we obtain from $\mathcal{F}^{(2)}(n, \lceil (r+1)\Delta/r \rceil)$-universal graphs G on n vertices with $O(n^{2-2/\lceil (r+1)\Delta/r \rceil} \log^{4/\lceil (r+1)\Delta/r \rceil} n)$ edges. □

Remark 10. In contrary to the F chosen as a matching plus P_3 we could work with any forest F. To find hitting graphs of small maximum degree we can use similar matching techniques and counting arguments, but in general it is not clear how low we can get. For example, if F is the path P_r on r vertices one can show that $\mathcal{F}^{(2)}(n, \lceil 2(r-1)\Delta/r \rceil)$ hits $\mathcal{F}^{(r)}(n, \Delta)$ on F. This leads to an $\mathcal{F}^{(r)}(n, \Delta)$-universal hypergraph on $O(n)$ vertices with $O(n^{r-2(r-1)/\lceil 2(r-1)\Delta/r \rceil})$ edges. It depends on the values of r and Δ, which bound is better, but one does not get anything significantly better than $O\left(n^{r-(r+1)/\lceil (r+1)\Delta/r \rceil}\right)$ edges and therefore we do not further pursue this here.

3.4 Reducing the number of vertices

Note that it is possible to reduce the number of vertices from $O(n)$ to $(1+\varepsilon)n$ in Theorems 1, 4, 5 and Corollary 3, for any fixed $\varepsilon > 0$, by using a *concentrator* as was done in [6]. Consider the $\mathcal{F}^{(r)}(n, \Delta)$-universal hypergraph H on $O(n)$ vertices and with m edges. A *concentrator* is a bipartite graph C on the vertex sets $V(H)$ and Q, where $|Q| = (1+\varepsilon)n$ such that for every $S \subseteq V(H)$ with $|S| \leqslant n$ we have $|N(S)| \geqslant |S|$ and every vertex from $V(H)$ has $O_\varepsilon(1)$ neighbours in C. We define a new hypergraph H' on Q by taking all sets $f' \in \binom{Q}{r}$ as edges for which there exists a perfect matching in C from an edge $f \in E(H)$ to f'. Since every vertex from $V(H)$ has $O_\varepsilon(1)$ degree in C, the hypergraph

H' has $O_\varepsilon(m)$ edges. It is also not difficult to see that H' is $\mathcal{F}^{(r)}(n,\Delta)$-universal. Indeed, let $F \in \mathcal{F}^{(r)}(n,\Delta)$ and let $\varphi\colon V(F) \to V(H)$ be its embedding into H. By the property of the concentrator C, there is a matching of $\varphi(V(F))$ in C which we can describe by an injection $\psi\colon \varphi(V(F)) \to V(H')$. But now, by construction of H', $\psi \circ \varphi$ is an embedding of F into H'.

4 Proof of Theorem 5

At this point in all cases where r is not even and r does not divide Δ we do not have constructions of $\mathcal{F}^{(r)}(n,\Delta)$-universal hypergraphs that match the lower bound $\Omega(n^{r-r/\Delta})$ on the number of edges. In this section we will deal with the 'smallest' open case $\Delta = 2$ by constructing optimal $\mathcal{F}^{(r)}(n,2)$-universal hypergraphs on $O(n)$ vertices with $O(n^{r/2})$ edges. So, for example, if $r = 3$ then Theorem 4 yields $\mathcal{F}^{(3)}(n,2)$-universal hypergraphs on $O(n)$ vertices with $O(n^{3-4/\lceil 8/3\rceil}) = O(n^{5/3})$ edges, while the lower bound is $\Omega(n^{3/2})$.

We will first deal with the case $r = 3$ and $\Delta = 2$ and then reduce the case of general r and $\Delta = 2$ to this one. Let us say a few words how an improvement from $O(n^{5/3})$ to $O(n^{3/2})$ can be accomplished. We will use the concept of a graph G that hits some hypergraph H on P_3 (the path on 3 vertices). If we would follow the arguments in the previous section, then we see that taking a hypergraph $H \in \mathcal{F}^{(3)}(n,2)$ and replacing every hyperedge by P_3 we can obtain a hitting graph G of maximum degree 3 and of average degree 8/3. Thus, if we would like to use Theorem 1 we need to consider $\mathcal{F}^{(2)}(n,3)$-universal graphs, which results in the loss of some $n^{1/6}$-factor in the edge density. Instead, we will seek to decompose the hitting graph G into appropriate subgraphs G_1, G_2, G_3 and G_4 such that every edge of G lies in *exactly* three of the graphs G_i. A decomposition result of Alon and Capalbo from [3] will assist us in this. Finally, following closely the arguments again due to Alon and Capalbo but now from [4] will allow us to construct a universal graph $\mathcal{G}$ on $O(n)$ vertices and with maximum degree $O(n^{1/4})$ for a carefully chosen family $\mathcal{F}'$ of graphs allowing a decomposition as above, which hits $\mathcal{F}^{(3)}(n,2)$ on P_3. Lemma 7 implies then that $\mathcal{H}_{P_3,3}(G)$ is $\mathcal{F}^{(3)}(n,2)$-universal and has $O(n^{3/2})$ edges.

4.1 A graph decomposition result

The following notation is from [3]. Let G be a graph and $S \subseteq V(G)$ be a subset of its vertices. A graph G' which is obtained from G by adding additionally $|S|$ new vertices to G and placing an (arbitrary) matching between these new vertices and the vertices from S is called an *augmentation* of G. We call a graph *thin* if every of its components is an augmentation of a path or a cycle, or if they contain at most two vertices of degree 3. We also call any subgraph of a thin graph thin.

The following decomposition theorem may be seen as a generalization of Petersen's Theorem to graphs of odd degree. It was proved in [3, Theorem 3.1].

Theorem 11. *Let Δ be an integer and G a graph with maximum degree Δ. Then there are Δ spanning subgraphs $G_1, \ldots, G_\Delta$ such that each G_i is thin and every edge of G appears in precisely two graphs G_i.*

Its proof is built on the Gallai-Edmonds decomposition theorem, and is implied by the following lemma.

Lemma 12 (Lemma 3.3 from [3])**.** *Let $\Delta \geqslant 3$ be an odd integer and G a Δ-regular graph. Then G contains a spanning subgraph in which every vertex has degree 2 or 3 and every connected component has at most 2 vertices of degree 3.*

We will use the two results above to prove the existence of a hitting graph G with nice properties so that we can later take advantage of them when constructing a universal graph for the family of such 'nice' hitting graphs.

Lemma 13. *Let $H \in \mathcal{F}^{(3)}(n,2)$. Then there exists a graph G that hits H on P_3 with the following properties:*

(i) there are spanning subgraphs G_1, G_2, G_3 and G_4 of G such that every G_i is an augmentation of a thin graph, and

(ii) every edge lies in exactly *three of the G_i.*

Proof. Let $H \in \mathcal{F}^{(3)}(n,2)$. We assume first that H is linear, i.e. edges are always intersecting in at most one vertex. Further we assume that H is 2-regular (otherwise we add 'dummy' vertices and edges and obtain a 2-regular hypergraph, and, once the desired graph G is constructed, we delete these dummy vertices from G).

The rough outline of the proof is to find a graph G that hits H on P_3 and such that G contains a matching M so that $G \setminus M$ is an augmentation of a thin graph and if we contract the matching edges from M in G we obtain a graph of maximum degree at most 3. Decomposing such contracted graph via Theorem 11 into thin graphs G_1', G_2' and G_3' and then 'recontracting' edges yields the desired family $G_1, \ldots, G_4$ (where $G_4 = G \setminus M$).

Let H^* be the line graph of H, that is $V(H^*) = E(H)$ and $e \neq f \in E(H)$ form an edge ef in H^* if $e \cap f \neq \varnothing$. Thus, H^* is a 3-regular graph on $2n/3$ vertices. Lemma 12 asserts then the existence of a matching M^* in H^* such that in $H^* \setminus M^*$ every component has at most 2 vertices of degree 3 and all other vertices have degree 2. Such a decomposition implies thus that every component of $H^* \setminus M^*$ is either a cycle, or has exactly two vertices, say a and b, of degree 3, so that either there are 3 internally vertex-disjoint paths between a and b or there is one path between a and b and, additionally, a and b lie on vertex-disjoint cycles (which also do not contain inner vertices from the path between a and b). We assume that a and b are not adjacent, because otherwise we could add the edge ab to M^*, splitting this component into two cycles.

From the matching M^* we define a subset $D := \{v \colon e \cap f = \{v\} \text{ where } ef \in E(M^*)\}$. Since M^* is a matching in the line graph of H it follows that no two vertices from D lie in an edge from H.

We denote by H_D the hypergraph which we obtain from H if we delete from the edges of H the vertices in D but we keep the edges, obtaining thus a hypergraph on the vertex set $V(H) \setminus D$, whose edges have cardinality 2 or 3. Thus, if ef is an edge in H^* and $e \cap f = \{v\}$ then the deletion of v from e and f implies that the edges $e \setminus \{v\}$ and $f \setminus \{v\}$ are no longer adjacent in the line graph $(H_D)^*$, which corresponds to the deletion of the edge ef in H^*. This implies that every component of $H^* \setminus M^*$ corresponds to a component of H_D, and therefore in every component of H_D there are at most two edges of cardinality 3 and all other edges have cardinality exactly 2. Again, the structure of every component of H_D is thus either a (graph) cycle, or there are exactly two edges, say g and h, of cardinality 3, with $g \cap h = \varnothing$ and there are three vertex-disjoint (graph) paths that connect the vertices from $g \cup h$.

Finally we come to the definition of the hitting graph G. For every component C of H_D, let D_C be the vertices that have been deleted from the hyperedges in H that lie now in H_D. Thus, there is a (natural) map ψ_C between the edges from C of cardinality 2 and D_C: $\psi_C(f) = v$ if $\{v\} \cup f \in E(H)$. Note that this map is not necessarily injective. Since every vertex from D lies in exactly two edges of H, it will suffice to explain how we replace the 3-uniform edges of H_D and the edges of H incident with D by paths P_3. If C is the (graph) cycle, then we replace every edge of the form $\{v\} \cup f$, where $\psi_C(f) = v$, by P_3 so that the graph G_C obtained contains all the edges from $E(C)$ and is such that $\Delta(G_C) \leqslant 3$ and the vertices from D_C have degree at most 2 in G_C. If C contains exactly two 3-uniform edges (say g and h), then it is possible to replace the edges g, h and every edge of the form $\{v\} \cup f$, where $\psi_C(f) = v$, by P_3 such that the graph G_C satisfies the following: It contains all 2-uniform edges of C, is such that $\Delta(G_C) \leqslant 3$, the vertices from D_C have degree at most 2 in G_C and $G_C \setminus D_C$ is connected and has exactly two vertices of degree 3 (this is easily done by considering the structure of the components C from H_D described in the previous paragraph). The graph G is then the union of all G_C and observe that G_C and $G_{C'}$ intersect in $D_C \cap D_{C'}$ for $C \neq C'$ and in particular have no common edges. Furthermore, every vertex from D has degree 2 in G, since it is an image of ψ_C precisely twice.

Let M be a matching in G that saturates D. Such a matching exists since D is independent in G (no two vertices from D lie in an edge from H), every vertex of D is connected to a vertex of degree 2 in $G \setminus D$ and $\deg(G) \leqslant 3$. By the definition of G above, every component in $G \setminus M$ is an augmentation of a graph with at most two vertices of degree 3, and thus an augmentation of a thin graph. We set $G_4 := G \setminus M$. Next we contract the edges of M in G obtaining the graph G/M. Since M saturates D, which are vertices of degree 2 in G, it follows that G/M has maximum degree at most 3. Theorem 11 yields a decomposition of G/M into thin graphs G_1', G_2', G_3' such that every edge of G/M appears in precisely two of the graphs. Now we reverse the recontraction procedure. This leads to three graphs G_1, G_2 and G_3 where every edge of $G \setminus M$ appears in exactly two of the graphs, every edge from M appears in all three of them, and each of the G_1, G_2 and G_3 is an augmentation of a thin graph. Together with the graph $G_4 = G \setminus M$ we thus constructed the desired decomposition of a hitting graph G.

If H is not linear, then things get in some sense even easier, so we shall be brief. We proceed essentially in the same way. That is, we define the line graph H^* of H, which is now not necessarily 3-regular, but whose maximum degree is at most 3. Again, Lemma 12 asserts then the existence of a matching M^* in H^* such that in $H^* \setminus M^*$ every component has at most 2 vertices of degree 3 and all other vertices have degree at most 2. We then define the set D as before but in the case that the edge $ef \in M^*$ with, say, $e = \{a, b, c\}$ and $f = \{b, c, d\}$ we simply replace the edge e by $\{a, b\}$ and f by $\{c, d\}$ without putting anything into D. Once the components of H_D are identified and the graphs G_C are defined we add the edge bc (which we call nonlinear) to those graphs G_C, which contain either b or c (or both). Then we choose edges into the matching M as before and add all nonlinear edges such as bc to M. The rest of the argument remains the same. □

An ℓ-th power of a graph G, denoted by G^ℓ, is the graph on $V(G)$, whose vertices at distance at most ℓ in G are connected. It is not difficult to see that a thin graph on n vertices can be embedded into P_n^4, and thus, an augmentation of a thin graph into P_n^8. This motivates the following general definition.

Definition 14 (**(k, r, ℓ)**-decomposable graphs)**.** Let k, r and ℓ be integers. A graph G on n vertices is called (k, r, ℓ)*-decomposable* if there exist k graphs G_i with the following properties. Every edge of G appears in exactly r of the G_i and there are maps $g_i\colon G_i \to [n]$, which are injective homomorphisms from G_i into P_n^ℓ. Then we denote by $\mathcal{F}_{k,r,\ell}(n)$ the family of (k, r, ℓ)-decomposable graphs on n vertices.

We can restate our Lemma 13 in the following slightly weaker form.

Lemma 15. *The family $\mathcal{F}_{4,3,8}(n)$ hits $\mathcal{F}^{(3)}(n, 2)$ on a path P_3.*

This lemma implies that it is the family $\mathcal{F}_{4,3,8}(n)$ for which a universal graph is needed. This graph will be constructed in the section below and briefly explained why a desired embedding works, which will follow from the results of Alon and Capalbo from [4].

4.2 Constructions of universal graphs

First we briefly describe the construction from [4] of $\mathcal{F}^{(2)}(n, k)$-universal graphs on $O(n)$ vertices with $O(n^{2-2/k})$ edges. One chooses $m = 20n^{1/k}$, a fixed $d > 720$ and a graph R to be a d-regular graph on m vertices with the absolute value of all but the largest eigenvalues at most λ (such graphs are called (n, d, λ)-graphs). One can assume that $\lambda \leqslant 2\sqrt{d-1}$ (then R is called Ramanujan) and $\mathrm{girth}(R) \geqslant \frac{2}{3}\log m/\log(d-1)$. Explicit constructions of such Ramanujan graphs have been found first for $d - 1$ being a prime congruent to 1 mod 4 in [12, 13]. Finally, the graph $G_{k,n}$ is defined on the vertex set $V(R)^k$ where two vertices $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are adjacent if and only if there are at least two indices i such that x_i and y_i are within distance 4 in R. It is easily seen that such a graph $G_{k,n}$ has $O(n)$ vertices, $O(n^{2-2/k})$ edges and maximum degree $O(n^{1-2/k})$.

The first step in the proof of $\mathcal{F}^{(2)}(n, k)$-universality of $G_{k,n}$ is Theorem 11 implying that any graph F with $\Delta(F) \leqslant k$ is $(k, 2, 4)$-decomposable. In what follows we summarize a straightforward generalization of the central claim from [4] (which is inequality (3.1) there), from which an existence of embedding of any graph $G \in \mathcal{F}^{(2)}(n, k)$ into $G_{k,n}$ follows. Its proof can be taken almost verbatim from [4].

Lemma 16. *Let $k \geqslant 3$, r and ℓ be natural numbers. For any choice of k permutations $g_i\colon [n] \to [n]$ there are k homomorphisms $f_i\colon [n] \to V(R)$ from the path P_n to the Ramanujan graph R introduced above such that the map $f\colon [n] \to V(G_{k,r,\ell}(n))$ defined by $f(v) = (f_1(g_1(v)), \ldots, f_k(g_k(v)))$ is injective.*

More precisely, the f_i's are inductively constructed as non-returning walks preserving the property that for any i vertices $v_1, \ldots, v_i \in V(G)$, $i \leqslant k$, one has

$$|\{v \in [n] : f_1(g_1(v)) = v_1, \ldots, f_i(g_i(v)) = v_i\}| \leqslant n^{(k-i)/k}.$$

For the last step $i = k$ this is equivalent to injectivity.

Finally, we explain, how we obtain $\mathcal{F}_{k,r,\ell}(n)$-universal graphs. The choice of the Ramanujan graph R along with the parameters m and d remains the same. The graph

$G_{k,r,\ell}(n)$ is defined on the vertex set $V(R)^k$ and two vertices $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are adjacent if and only if there are at least r indices i such that x_i and y_i are within distance ℓ in R. It is then an easy calculation to show that $G_{k,r,\ell}(n)$ has $O(n)$ vertices, at most $n\binom{k}{r}d^{r\ell}m^{k-r} = O(n^{2-r/k})$ edges and maximum degree $O(n^{1-r/k})$, where the constants in O-notation depend on k, r, ℓ and d. Lemma 16 implies then the following.

Theorem 17. *Let $k \geqslant 3$, r and ℓ be natural numbers. The graph $G_{k,r,\ell}(n)$ is $\mathcal{F}_{k,r,\ell}(n)$-universal.*

Proof. Let G be a (k, r, ℓ)-decomposable graph on n vertices together with the decomposition $G_1, \ldots, G_k$ and injective homomorphisms $g_i \colon V(G_i) \to [n]$ from G_i into P_n^ℓ. Lemma 16 asserts the existence of the homomorphisms $f_i \colon [n] \to V(R)$ from P_n to R for every $i \in [k]$, so that the map $f \colon V(G) \to V(G_{k,r,\ell}(n))$ given by $f(v) = (f_1(g_1(v)), \ldots, f_k(g_k(v)))$ is injective.

It is clear that the composition of f_i with g_i is a homomorphism from G_i to R^ℓ. Furthermore, every edge $\{u, v\}$ from G lies in r graphs G_i. Thus, there are r indices i such that $g_i(u)$ and $g_i(v)$ are distinct and within distance ℓ in P_n. This implies that $f_i(g_i(u))$ and $f_i(g_i(v))$ are also distinct and within distance ℓ in G. By the definition of $G_{k,r,\ell}(n)$ this implies that $f(u)$ and $f(v)$ are adjacent in $G_{k,r,\ell}(n)$ and f is the desired embedding of G into $G_{k,r,\ell}(n)$. □

From this, Theorem 5 follows immediately for $r = 3$.

Proof of Theorem 5, case $r = 3$. Note, that the graph $G_{4,3,8}(n)$ has $m^4 = O(n)$ vertices and $O(nm) = O(n^{5/4})$ edges. By Theorem 17 $G_{4,3,8}(n)$ is $\mathcal{F}_{4,3,8}(n)$-universal, and since $\mathcal{F}_{4,3,8}(n)$ hits $\mathcal{F}^{(3)}(n, 2)$ on P_3, Lemma 7 implies that $\mathcal{H}_{P_3,3}(G_{4,3,8}(n))$ is $\mathcal{F}^{(3)}(n, 2)$-universal, has $O(n)$ vertices and $O(n^{3/2})$ edges. This proves the case $r = 3$. □

Remark 18. We believe that the constructions from [3] can also be adapted to work with (k, r, ℓ)-decomposable graphs. For the cases discussed here this would lead to universal graphs on n vertices, where the number of edges is some polylog factor larger.

4.3 $\mathcal{F}^{(r)}(n, 2)$-universal hypergraphs of uniformity $r \geqslant 5$

Proof of Theorem 5 for odd $r \geqslant 5$. First we define the hypergraph $\mathcal{H}$ which will turn out to be $\mathcal{F}^{(r)}(n, 2)$-universal. Let $t = (r - 3)/2$. Let $G_1, \ldots, G_{t+1}$ be vertex-disjoint graphs, where $G_1, \ldots, G_t$ are copies of C_n^4 (the fourth power of the cycle C_n) and G_{t+1} is a copy of the graph $G_{4,3,8}(n)$, introduced in the previous section. Furthermore we add on top of G_{t+1} another graph G^*_{t+1} containing as edges all pairs of vertices which have a common neighbour in G_{t+1}. We define $\mathcal{H}$ to be the r-graph on the vertex set $\dot\bigcup_{i=1}^{t+1} V(G_i)$, and the edges are r-element subsets f such that, with $f_i := f \cap V(G_i)$, we have $|f_i| \leqslant 3$ and each $G_i[f_i]$ contains a copy of $P_{|f_i|}$, a path on $|f_i|$ vertices (thus, P_0 is the empty graph, $P_1 = K_1$ and $P_2 = K_2$). Additionally, in the case $|f_{t+1}| = 2$, we allow f_{t+1} to be an edge (i.e. P_2) in G^*_{t+1} instead of G_{t+1}.

Certainly, $\mathcal{H}$ has $O(n)$ vertices. How many edges does the hypergraph $\mathcal{H}$ contain? For this we need to choose paths P_{ℓ_i} from every G_i (resp. G^*_{t+1}) such that $\ell_i \in \{0, 1, 2, 3\}$ and $\sum_{i=1}^{t+1} \ell_i = r$. Because $G_1, \ldots, G_t$ have maximum degree 8, G_{t+1} has maximum degree

$O(n^{1/4})$, and G^*_{t+1} has maximum degree $n^{1/2}$, we compute the number of edges of $\mathcal{H}$ to be $O(n^{t+1}n^{2/4}) = O(n^{r/2})$, as desired.

Given a hypergraph H and a subset of vertices $X \subseteq V$, we denote through $H(X)$ the (not necessarily uniform) hypergraph on the vertex set X, whose edges are restrictions to X, i.e. $E(H(X_i)) = \{f \cap X_i \colon f \in E(H)\}$.

The rest of the proof hinges on the following auxiliary lemma (whose proof can be found below) and the case $r = 3$ of Theorem 5 shown in the previous section.

Lemma 19. *Let $H \in \mathcal{F}^{(r)}(n, 2)$ and $t = (r-3)/2$. Then there exists a partition of the vertex set of H into disjoint subsets $X_1, \ldots, X_{t+1}$, such that $H(X_1), \ldots, H(X_{t+1})$ have maximum vertex degree 2 and contain hyperedges of cardinality at most 3. Moreover in $H(X_1), \ldots, H(X_t)$ every component contains at most 2 hyperedges of size 3.*

Let us see how then H can be embedded into the hypergraph $\mathcal{H}$. Owing to the structure of $H(X_1), \ldots, H(X_t)$, one can easily find injective maps $g_i \colon X_i \to V(G_i)$, such that every hyperedge $f \in E(H(X_i))$ is such that $G_i\,[g_i(f)]$ contains a path $P_{|f|}$. This can be seen by replacing f in $H(X_i)$ through an arbitrary path $P_{|f|}$ obtaining thus the graph G'_i on the vertex set X_i. Then, since in every component of $H(X_i)$ there are at most two edges of size 3, it is easy to find an injective graph homomorphism from G'_i into G_i.

For $H(X_{t+1})$ we can assume first that it is 3-uniform and lies in $\mathcal{F}^{(3)}(n, 2)$ by adding some 'dummy' vertices appropriately (but still using the notation $H(X_{t+1})$). The $\mathcal{F}_{4,3,8}(n)$-universality of $G_{t+1} = G_{4,3,8}(n)$ and the fact that $\mathcal{F}_{4,3,8}(n)$ hits $H(X_{t+1})$ on P_3 yields an injective map $g_{t+1} \colon X_{t+1} \to V(G_{t+1})$ such that $G_{t+1}\,[g_{t+1}(f)]$ contains P_3 for every $f \in E(H(X_{t+1}))$. Deleting the dummy vertices (but keeping the edges) we see that g_{t+1} remains injective and $G_{t+1}\,[g_{t+1}(f)]$ contains $P_{|f|}$ for every $f \in E(H(X_{t+1}))$ except possibly for the case, when the center vertex of some P_3 was deleted (being a dummy vertex). But in this case we observe that $G^*_{t+1}[g_{t+1}(f)]$ induces P_2 instead, because both vertices of $g_{t+1}(f)$ were incident to the deleted vertex in G_{t+1}.

It should be clear that $g \colon V(H) \to V(\mathcal{H})$ with $g|_{X_i} = g_i$, for all $i \in [t+1]$, is injective. It remains to show that g is a homomorphism into $\mathcal{H}$. Given an edge e of H, by the definition of $H(X_i)$ and the choices of g_i's, we see that $e \cap X_i \in E(H(X_i))$ and $G_i\,[g_i(e \cap X_i)]$ contains a path $P_{|e \cap X_i|}$ for all i, except possibly for the case when $|g_{t+1}(e \cap X_{t+1})| = 2$. But in this case one must necessarily have $g_{t+1}(e \cap X_{t+1}) \in E(G^*_{t+1})$. These conditions fulfill exactly the requirement for $g(e)$ to be the edge in $\mathcal{H}$. Thus, g embeds H into $\mathcal{H}$. □

Finally we provide the proof for the auxiliary lemma above, Lemma 19.

Proof of Lemma 19. Let $H \in \mathcal{F}^{(r)}(n, 2)$. Again we assume first that H is linear and 2-regular. We consider, as in the case $r = 3$, the line graph H^*, which is r-regular now. Hence Lemma 12 yields a spanning subgraph H^*_1, in which every vertex has degree 2 or 3 and every component has at most 2 vertices of degree 3.

If C is a component of H^*_1, then we define V_C as all vertices v such that $\{v\} = e \cap f$ for some $ef \in E(C)$ (recall that H is assumed to be a linear hypergraph). We set $X_1 = \cup V_C$ where the union is over all components C of H^*_1 and then for every edge $f \in E(H)$ the set $\{v \colon \{v\} = e \cap f \text{ for some } ef \in E(C)\}$ is an edge of $H(X_1)$. Observe, that these edges have cardinality either 2 or 3. Indeed, a vertex of degree j in some component C is the edge of H that intersects j other edges of H in different vertices, which give rise to a

j-uniform edge in $H(X_1)$. By construction, $H(X_1)$ is linear and 2-regular. Crucially, the components of $H(X_1)$ have simple structure, since these are 'inherited' from the components C. More precisely, each component of $H(X_1)$ has at most two 3-uniform edges and all other edges have cardinality 2.

We denote by $\tilde{H}_1 = H(V(H) \setminus X_1)$ the hypergraph obtained from H by deleting from its edges all vertices from X_1 (we call this procedure as 'reducing uniformity'). It should be clear that, in this way every edge of H can be written uniquely as the union of one edge of $H(X_1)$ and the other from $\tilde{H}_1$. Since $H(X_1)$ is not necessarily uniform, the hypergraph $\tilde{H}_1$ is now a not necessarily uniform hypergraph as well, but its edges have cardinalities either $r-3$ or $r-2$.

The next step calls for an inductive procedure with a blemish, that $\tilde{H}_1$ is not necessarily uniform. But this can be remedied by adding 'dummy' vertices and edges to $\tilde{H}_1$ and obtaining an $(r-2)$-uniform linear hypergraph still denoted by $\tilde{H}_1$ which is 2-regular (once we are finished with decomposition, we will reduce the uniformity by deleting these dummy vertices from edges, but keeping the altered edges). We keep doing this reduction until we arrive at the hypergraph $\tilde{H}_t$ where $t = (r-3)/2$, thereby generating $X_2, \ldots, X_t$ and $\tilde{H}_2(X_2) \ldots, \tilde{H}_{t-1}(X_t)$. Finally we get $X_{t+1} := V(H) \setminus \cup_{i=1}^{t} X_i$ and a 3-uniform linear hypergraph $\tilde{H}_t$ on X_{t+1}, which is 2-regular.

Before we proceed, let us summarize what we achieved so far. We have found hypergraphs $H(X_1), \tilde{H}_2(X_2) \ldots, \tilde{H}_{t-1}(X_t)$, so that each of them is linear, 2-regular and its edge uniformities are either 2 or 3 and each of its components has simple structure (recall: each component has at most two 3-uniform edges and all other edges have cardinality 2). Furthermore $\tilde{H}_t$ is a 3-uniform linear hypergraph, which is 2-regular, and the vertex sets $X_1, \ldots, X_{t+1}$ are a partition of $V(H)$.

We finally obtain the promised family $H(X_1), \ldots, H(X_{t+1})$. This can be seen as reducing uniformities of the hypergraphs $H(X_1), \tilde{H}_2(X_2), \ldots, \tilde{H}_{t-1}(X_t)$ and $\tilde{H}_t$ by deleting dummy edges and dummy vertices from the edges. In this way it may happen, that the uniformity of some edges of the hypergraph family will be reduced to 0 (in which case they disappear from that particular hypergraph), while some others will be reduced to 1, in which case we get edges of the type $\{v\}$, which we will use.

The case when H is not a linear hypergraph can be treated similarly. We slightly extend the definition of the line graph H^* such that it contains multiple edges, i.e. for $e, f \in E(H)$ there are $|e \cap f|$ many edges between e and f in H^* and we label each of them with a distinct vertex from $e \cap f$. Then H^* is again r-regular and we can again apply Lemma 12, because the proof from [3] extends verbatim to multigraphs. In this way we obtain a multigraph H_1^* and for every component C we define the vertex set V_C as follows: for a given edge $g \in E(C)$, the set V_C contains $e \cap f$ where the edge g connects e and f and it holds $|e \cap f| = 1$, and otherwise (i.e. there are parallel edges to g) the vertex set V_C contains precisely the vertex of the label that the edge g carries. The set X_1 is then the union of the V_C over all components C from H_1^*. The construction of X_2, $\ldots$, X_t is similar to above. The rest of the proof proceeds along the lines of the linear case and we omit further details. □

4.4 A general problem

To prove the embedding for other parameters of r and Δ we would need the analogue of Lemma 15, that is, a solution to the following problem.

Problem 20. Let $r \geqslant 3$ and $\Delta \geqslant 3$ be integers. Find ℓ such that $\mathcal{F}_{(r-1)\Delta,r,\ell}(n)$ hits $\mathcal{F}^{(r)}(n,\Delta)$ on P_r.

It is immediate that, with $k = (r-1)\Delta$, Theorem 17 yields $\mathcal{F}_{(r-1)\Delta,r,\ell}(n)$-universal graphs $G = G_{(r-1)\Delta,r,\ell}$ on $O(n)$ vertices with $O(n^{2-r/((r-1)\Delta)})$ edges and maximum degree $O(n^{1-r/((r-1)\Delta)})$. From this the solution to Problem 20 would yield optimal universal hypergraphs on $O(n)$ vertices with $|V(G)|(|E(G)|/|V(G)|)^{r-1} = O(n^{r-r/\Delta})$ edges. Clearly, the interesting cases are $\Delta \geqslant 3$, $r \nmid \Delta$ and r odd.

Remark 21. An alternative to our approach is to extend the constructions for universal graphs from [3, 4, 6] to hypergraphs. To follow a similar embedding scheme one would ask for appropriate decomposition results for hypergraphs. For example, for $H \in \mathcal{F}^{(3)}(n,2)$ the task is to find subhypergraphs $H_1,\dots,H_4$ which are 'thin' and such that every hyperedge appears in exactly three of them.

5 Proof of Theorem 6

Proof of Theorem 6. To prove the existence of optimal $\mathcal{E}^{(r)}(m)$-universal hypergraphs we exploit the proof of Alon and Asodi [2].

Take any $H \in \mathcal{E}^{(r)}(m)$ and replace all edges of H by cliques of size r. This gives a graph with at most $\binom{r}{2}m$ edges and thus there exists a graph G with $O(m^2/\ln^2 m)$ edges which is $\mathcal{E}^{(2)}(\binom{r}{2}m)$-universal. We define the r-graph $\mathcal{K}_r(G)$ on the vertex set $V(G)$ with edges being the vertex sets of the copies of K_r in G. It is straightforward to see that $\mathcal{K}_r(G)$ is $\mathcal{E}^{(r)}(m)$-universal and thus it remains to estimate the number of edges in $\mathcal{K}_r(G)$.

The $\mathcal{E}^{(2)}(m)$-universal graph G of Alon and Asodi [2] is defined on the vertex set $V = V_0 \cup V_1 \cup \dots \cup V_k$ where $k = \lceil \log_2 \log_2 m \rceil$, $|V_0| = 4m/\log_2^2 m$ and $|V_i| = 4m2^i/\log_2 m$ for $i \in [k]$. A vertex in V_0 is connected to any other vertex and the graph induced on V_1 is a clique. For any $u \in V_i$, $i \geqslant 2$, and $v \in V_1 \cup V_2 \cup \dots \cup V_i$ with $u \neq v$ the edge uv is present independently with probability $\min(1, 8^{3-i})$. It is shown in [2] that with probability at least $1/4$ the graph G has $O(m^2/\ln^2 m)$ edges and is $\mathcal{E}^{(2)}(m)$-universal. We count the expected number of copies of K_r in G, i.e. $\mathbb{E}(|E(\mathcal{K}_r(G))|)$.

There are several possible types of cliques K_r in G. Indeed, we need to choose r vertices from $V_0,\dots,V_k$, and a particular *type* of a possible r-clique K in G is specified by α, which is the number of its vertices in V_0 and by numbers $t_1 \leqslant \dots \leqslant t_\gamma$ (all from $[k]$), which specify to which sets V_i the remaining $\gamma = r - \alpha$ vertices belong to. There are at most $|V_0|^\alpha \prod_{j=1}^{\gamma} |V_{t_j}|$ cliques of a particular type, and each such clique occurs with probability $\prod_{j=1}^{\gamma} [\min(1, 8^{3-t_j})]^{j-1}$. It is clear that there are at most $|V_0|^{r-1}|V(G)| \leqslant \frac{(4m)^{r-1}\cdot(32m)}{(\log_2 m)^{2(r-1)}} = o\left(\frac{m^r}{\log_2^r m}\right)$ cliques K_r in G that intersect V_0 in at least $r-1$ vertices.

Next we upper bound the expected number of edges in $\mathcal{K}_r(G)$ as follows:

$$\mathbb{E}(|E(\mathcal{K}_r(G))|) \leqslant |V_0|^{r-1}|V(G)| + \sum_{\substack{\alpha+\gamma=r \\ \gamma\geqslant 2}} \sum_{1\leqslant t_1\leqslant\dots\leqslant t_\gamma\leqslant k} |V_0|^\alpha \prod_{j=1}^{\gamma} |V_{t_j}| \cdot \prod_{j=1}^{\gamma} \left[\min\left(1, 8^{3-t_j}\right)\right]^{j-1}$$

$$\leqslant o\left(\frac{m^r}{\log_2^r m}\right) + \sum_{\gamma\geqslant 2}^{r}\left(\frac{4m}{\log_2 m}\right)^r \frac{1}{\log_2^{r-\gamma} m}\sum_{1\leqslant t_1\leqslant\ldots\leqslant t_\gamma\leqslant k} 2^{\sum_{j=1}^{\gamma} t_j}\cdot 2^{\sum_{j=1}^{\gamma}\min\{0,(9-3t_j)(j-1)\}}, \tag{1}$$

and in order to simplify it further we first estimate the inner sum of the second summand by splitting it according to t_1 as follows:

$$\begin{aligned}
&\sum_{1\leqslant t_1\leqslant\ldots\leqslant t_\gamma\leqslant k} 2^{\sum_{j=1}^{\gamma} t_j}\cdot 2^{\sum_{j=1}^{\gamma}\min\{0,(9-3t_j)(j-1)\}}\\
&\leqslant \sum_{t_1\leqslant 19}\sum_{\substack{t_j\geqslant 1\\ j=2,\ldots,\gamma}} 2^{\sum_{j=1}^{\gamma} t_j+\sum_{j=1}^{\gamma}\min\{0,(9-3t_j)(j-1)\}} + \sum_{t_1\geqslant 20}\sum_{\substack{t_j\geqslant t_1\\ j=2,\ldots,\gamma}} 2^{\sum_{j=1}^{\gamma} t_j+\sum_{j=1}^{\gamma}\min\{0,(9-3t_j)(j-1)\}}\\
&\leqslant 2^{20}\sum_{\substack{t_j\geqslant 1\\ j=2,\ldots,\gamma}} 2^{\sum_{j=2}^{\gamma}(t_j+\min\{0,(9-3t_j)(j-1)\})} + \sum_{t_1\geqslant 20} 2^{t_1}\sum_{\substack{t_j\geqslant t_1\\ j=2,\ldots,\gamma}} 2^{\sum_{j=2}^{\gamma}(t_j+(9-3t_j)(j-1))}\\
&\leqslant 2^{20}\left(\sum_{t\geqslant 1} 2^{t+\min\{0,(9-3t)\}}\right)^{\gamma-1} + \sum_{t_1\geqslant 20} 2^{t_1}\left(\sum_{t\geqslant t_1} 2^{t+(9-3t)}\right)^{\gamma-1}\\
&\leqslant 2^{20}\left(6+\sum_{t\geqslant 3} 2^{9-2t}\right)^{\gamma-1} + \sum_{t_1\geqslant 20} 2^{t_1}\left(\sum_{t\geqslant t_1} 2^{-3t/2}\right)^{\gamma-1} \leqslant 2^{20+5\gamma} + \sum_{t_1\geqslant 20} 2^{t_1-\frac{3t_1(\gamma-1)}{2}+2(\gamma-1)}\\
&\qquad\leqslant 2^{20+5\gamma} + 2^{2(\gamma-1)}\sum_{t_1\geqslant 20} 2^{-t_1/2} \leqslant 2^{21+5\gamma} \leqslant 2^{21+5r}.
\end{aligned}$$

This allows us to further upper bound (1) by

$$\mathbb{E}(|E(\mathcal{K}_r(G))|) \leqslant r2^{21+5r}\left(\frac{4m}{\log_2 m}\right)^r.$$

By Markov's inequality, the probability that $|E(\mathcal{K}_r(G))|$ is at least $5r2^{21+5r}\left(\frac{4m}{\log_2 m}\right)^r$ is at most $1/5$. Thus, taking $\hat{m} = \binom{r}{2}m$, there exists an $\mathcal{E}^{(2)}(\hat{m})$-universal graph with $O\left(\frac{\hat{m}^r}{\log_2^r \hat{m}}\right)$ copies of K_r. This implies that there exists an $\mathcal{E}^{(r)}(m)$-universal hypergraph H with $O(m^r/\ln^r m)$ edges. □

It is possible to prove that there exist such hypergraphs H with rm vertices which is optimal. However, no explicit construction is known.

Acknowledgements

The authors would like to thank Rajko Nenadov for useful discussions, and the referees for helpful suggestions.

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