Inclusion Matrices and the MDS Conjecture

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Abstract

Let $\mathbb{F}_q$ be a finite field of order q with characteristic p. An arc in $\mathbb{F}_q^k$ is an ordered family of at least k vectors in which every subfamily of size k is a basis of $\mathbb{F}_q^k$. The MDS conjecture, which was posed by Segre in 1955, states that if $k \leqslant q$, then an arc in $\mathbb{F}_q^k$ has size at most $q+1$, unless q is even and $k = 3$ or $k = q-1$, in which case it has size at most $q+2$.

We propose a conjecture which would imply that the MDS conjecture is true for almost all values of k when q is odd. We prove our conjecture in two cases and thus give simpler proofs of the MDS conjecture when $k \leqslant p$, and if q is not prime, for $k \leqslant 2p-2$. To accomplish this, given an arc $G \subset \mathbb{F}_q^k$ and a nonnegative integer n, we construct a matrix $M_G^{\uparrow n}$, which is related to an inclusion matrix, a well-studied object in combinatorics. Our main results relate algebraic properties of the matrix $M_G^{\uparrow n}$ to properties of the arc G and may provide new tools in the computational classification of large arcs.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of order q with characteristic p. An *arc* in $\mathbb{F}_q^k$ is an ordered family of at least k vectors in which every subfamily of size k is a basis of $\mathbb{F}_q^k$. Most authors define an arc, equivalently, as an unordered set of points in the corresponding projective space. For the techniques developed in this article, however, we find it more convenient to define arcs as ordered families of vectors. On the other hand, we will denote arcs with set notation rather than tuple notation as this is more natural.

Given an arc $G \subset \mathbb{F}_q^k$ and a basis B of $\mathbb{F}_q^k$, let $M(G, B)$ be the matrix whose columns are the vectors in G written with respect to the basis B in the order given by G. If $G, G' \subset \mathbb{F}_q^k$ are two arcs, then we say that G is *linearly equivalent* to G' if the matrix $M(G, B)$ can

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be transformed into the matrix $M(G', B)$ using only elementary row operations, column permutations, and multiplication of columns by nonzero scalars.

A natural question is to determine how large an arc in $\mathbb{F}_q^k$ can be.

Question 1.1 What is the maximum size $g(k, q)$ of an arc in $\mathbb{F}_q^k$?

Question 1.1 interests the coding theory, algebraic geometry, and finite geometry communities, and its importance is highlighted by a $1000 prize offered for its solution by the Information Theory and Applications (ITA) center at UCSD [18].

If $(e_1, \ldots, e_k)$ is a basis for $\mathbb{F}_q^k$, then a natural arc in $\mathbb{F}_q^k$ of size $k+1$ is given by

$$\{e_1, \ldots, e_k, e_1 + \cdots + e_k\}, \tag{1.1}$$

which proves that $g(k, q) \geqslant k+1$. A straightforward argument shows that $g(k, q) = k+1$ when $k \geqslant q$, and moreover if $S \subset \mathbb{F}_q^k$ is an arc of size $k+1$, then S is linearly equivalent to (1.1). This result was first proved by Bush [5] in 1952.

Question 1.1 becomes difficult to answer, however, when $k < q$. In this case, we can construct arcs that are larger than the arc in (1.1). For example, the *normal rational curve* $\mathcal{R}_k \subset \mathbb{F}_q^k$, which is defined by

$$\mathcal{R}_k = \{(1, t, t^2, \ldots, t^{k-1}) \,|\, t \in \mathbb{F}_q\} \cup \{(0, \ldots, 0, 1)\}, \tag{1.2}$$

is an arc of size $q+1$. The normal rational curve $\mathcal{R}_k$ shows that $g(k, q) \geqslant q+1$, and in 1955, Segre [16] conjectured that this lower bound is tight in most cases when $k \leqslant q$.

Conjecture 1.2 (Segre, [16]) If $k \leqslant q$, then the maximum size $g(k, q)$ of an arc in $\mathbb{F}_q^k$ is

$$g(k, q) = \begin{cases} q+1 & \text{if } q \text{ is odd or } k \notin \{3, q-1\} \\ q+2 & \text{if } q \text{ is even and } k \in \{3, q-1\}. \end{cases}$$

Conjecture 1.2 is called the MDS conjecture or the main conjecture for maximum distance separable codes, and was first posed by Segre as a question.

By the well-known principle of duality, if $S \subset \mathbb{F}_q^k$ is an arc of size $s > k$, then up to linear equivalence, we can associate a unique dual arc $S^\perp \subset \mathbb{F}_q^{s-k}$ of size s. This has two immediate implications. First, it explains why in Conjecture 1.2, exceptions occur for both $k = 3$ and $k = q-1$ when q is even. Second, it shows that if $g(k, q) = q+1$, then $g(q+2-k, q) = q+1$. As a result, if q is odd and $g(k, q) = q+1$ when $k \leqslant (q+2)/2$, then $g(k, q) = q+1$ for all $k \leqslant q$. Duality thus allows us to prove Conjecture 1.2 when q is odd by restricting to the case $k \leqslant (q+2)/2$.

Ball [1] proved that $g(k, q) = q+1$ when $k \leqslant p = \text{char}\,(\mathbb{F}_q)$, and thus verified Conjecture 1.2 when q is prime. For a complete list of when Conjecture 1.2 is known to hold for q non-prime, see [10] and [11]. The best-known bounds up to first-order of magnitude (c_i are constants), are that for q an odd non-square, we have $g(k, q) = q+1$ when $k < \sqrt{pq}/4 + c_1 p$, which was proved by Voloch [19]. For $q = p^{2h}$, where $p \geqslant 5$ is a prime, we have $g(k, q) = q+1$ when $k \leqslant \sqrt{q}/2 + c_2$, which was proved by Hirschfeld and

Korchmáros [9]. Ball and De Buele [4] proved that $g(k,q) = q+1$ when $k \leqslant 2\sqrt{q}-2$ and $q = p^2$.

If $k \leqslant q$ and q is odd or $k \notin \{3, q-1\}$, it is natural to ask if the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$ up to linear equivalence. By results of Kaneta and Maruta [12] and Seroussi and Roth [17], a positive answer to this question would imply Conjecture 1.2. For many values of k and q, the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$ up to linear equivalence [10], but Glynn [7] showed that this is not always true. The Glynn arc $\mathcal{G} \subset \mathbb{F}_9^5$ is an arc of size 10 and is defined by

$$\mathcal{G} = \{(1, t, t^2 + \eta t^6, t^3, t^4) \,|\, t \in \mathbb{F}_9\} \cup \{(0,0,0,0,1)\}, \tag{1.3}$$

where $\eta \in \mathbb{F}_9$ satisfies $\eta^4 = -1$. Remarkably, the Glynn arc $\mathcal{G}$ is the only known arc in $\mathbb{F}_q^k$ of size $q+1$ that is not linearly equivalent to the normal rational curve $\mathcal{R}_k$ when $k \leqslant q$ and q is odd.

1.1 New Results

We propose a conjecture, Conjecture 1.9, which would imply that $g(k,q) = q+1$ when

$$k \leqslant \left(\frac{p-2}{2p-3}\right) q + \left(3 - \frac{p-1}{2p-3}\right), \tag{1.4}$$

where $p = \mathrm{char}\,(\mathbb{F}_q)$. In Section 1, we noted that to prove Conjecture 1.2 when q is odd, it suffices to restrict to the case $k \leqslant (q+2)/2$ by duality. As p grows, the right hand side of (1.4) becomes very close to $(q+2)/2$. Consequently, if Conjecture 1.9 is true, then Conjecture 1.2 is true for almost all values of k when q is odd.

To state Conjecture 1.9, given an arc $G \subset \mathbb{F}_q^k$ and a nonnegative integer n, we define a matrix $M_G^{\uparrow n}$ whose algebraic properties are related to properties of G.

Definition 1.3 Let $G \subset \mathbb{F}_q^k$ be an arc and let $0 \leqslant n \leqslant |G| - k + 1$. Let B be a basis of $\mathbb{F}_q^k$ and let $M_G^{\uparrow n}$ be a matrix whose rows are indexed by $\binom{G}{k-1}$, whose columns are indexed by ordered pairs (U, A) where $U \in \binom{G}{n}$ and $A \in \binom{G \setminus U}{k-2}$, and whose $(C,(U,A))$-entry is

$$M_G^{\uparrow n}(C,(U,A)) = \begin{cases} \prod_{u \in U} \det(u, C)_B & \text{if } A \subset C \subset G \setminus U \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

In (1.5), $\det(u, C)_B$ denotes the determinant of the matrix whose first row is u written with respect to the basis B and whose last $k-1$ rows are the elements of C written with respect to the basis B in the order inherited from G.

Although the matrices $M_G^{\uparrow n}$ may seem unfamiliar, we claim that they are related to inclusion matrices, which are well-studied in combinatorics. Recall that the inclusion matrix $I_r(a,b)$ has its rows indexed by $\binom{\{1,\ldots,r\}}{a}$, its columns indexed by $\binom{\{1,\ldots,r\}}{b}$, and (A,B)-entry

$$I_r(a,b)_{(A,B)} = \begin{cases} 1 & \text{if } B \subset A \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

For example, when $n = 0$, the matrix $M_G^{\uparrow 0}$ is the inclusion matrix $I_{|G|}(k-1, k-2)$. When $n > 0$, the matrix $M_G^{\uparrow n}$ is formed by gluing together matrices which are equivalent to inclusion matrices. For a fixed $U \in \binom{G}{n}$, let D_U be a diagonal matrix whose rows and columns are indexed by $\binom{G\setminus U}{k-1}$ and whose (C, C)-entry is $\prod_{u\in U} \det(u, C)_B$. We then have that the submatrix $M_G^{\uparrow n}(U)$ of $M_G^{\uparrow n}$ whose rows are indexed by $\binom{G\setminus U}{k-1}$ and whose columns are indexed by ordered pairs (U, A), where $A \in \binom{G\setminus U}{k-2}$, equals $D_U I_{|G\setminus U|}(k-1, k-2)$.

It is easy to see that linear equivalence of the arcs G and G' induces equivalence of the corresponding matrices $M_G^{\uparrow n}$ and $M_{G'}^{\uparrow n}$. More precisely, if $G, G' \subset \mathbb{F}_q^k$ are linearly equivalent arcs and B and B' are the bases of $\mathbb{F}_q^k$ used in the construction of the matrices $M_G^{\uparrow n}$ and $M_{G'}^{\uparrow n}$ respectively, then there exist invertible matrices N_1 and N_2 so that $M_G^{\uparrow n} = N_1 M_{G'}^{\uparrow n} N_2$.

Our main results relate algebraic properties of the matrix $M_G^{\uparrow n}$ to properties of the arc G. For example, our first main result says that if G is an arc whose matrix $M_G^{\uparrow n}$ has full row rank, then G cannot be extended to a larger arc of a specific size.

Theorem 1.4 *Let $G \subset \mathbb{F}_q^k$ be an arc and let $n \in \mathbb{N}$ be a natural number such that*

$$n + k - 1 \leqslant |G| \leqslant \frac{q + 2k - 2 + n}{2}. \tag{1.7}$$

If the matrix $M_G^{\uparrow n}$ has full row rank, then the arc G cannot be extended to an arc of size $q + 2k - 1 + n - |G|$.

The left-hand and right-hand sides of (1.7) respectively are required so that the matrix $M_G^{\uparrow n}$ exists and so that the arc G has size strictly smaller than $q + 2k - 1 + n - |G|$.

Suppose $0 \leqslant n \leqslant q - 2k + 4$ so that $2k - 3 + n \leqslant q + 1$. Also, suppose we can show that for all arcs $G \subset \mathbb{F}_q^k$ of size $2k - 3 + n$, the matrix $M_G^{\uparrow n}$ has full row rank. If an arc of size $q + 2$ exists in $\mathbb{F}_q^k$, then it would contain a subarc G of size $2k - 3 + n$ that can be extended to an arc of size $q + 2k - 1 + n - |G|$, which contradicts Theorem 1.4. Consequently, Theorem 1.4 allows us to eliminate the existence of arcs of size $q + 2$ in $\mathbb{F}_q^k$ by proving that for all arcs $G \subset \mathbb{F}_q^k$ of size $2k - 3 + n$, the matrix $M_G^{\uparrow n}$ has full row rank.

Corollary 1.5 *If $0 \leqslant n \leqslant q - 2k + 4$ and for every arc $G \subset \mathbb{F}_q^k$ of size $2k - 3 + n$, the matrix $M_G^{\uparrow n}$ has full row rank, then $g(k, q) = q + 1$.*

Since the matrices $M_G^{\uparrow n}$ are related to inclusion matrices, knowing the ranks of inclusion matrices over $\mathbb{F}_q$ will be crucial to verifying the condition in Corollary 1.5.

Theorem 1.6 (Frankl [6], Wilson [20]) *For fixed integers $0 \leqslant b \leqslant a \leqslant r - b$ and a prime $p = \operatorname{char}(\mathbb{F}_q)$, we have*

$$\operatorname{rank}_{\mathbb{F}_q} I_r(a, b) = \sum_{\substack{0\leqslant i\leqslant b \\ p\nmid \binom{a-i}{b-i}}} \binom{r}{i} - \binom{r}{i-1}. \tag{1.8}$$

For example, when $n = 0$ and $G \subset \mathbb{F}_q^k$ is an arc of size $2k-3$, the matrix $M_G^{\uparrow 0}$ is the inclusion matrix $I_{2k-3}(k-1, k-2)$. Theorem 1.6 thus implies the first assertion of Theorem 1.7.

Theorem 1.7 *If $G \subset \mathbb{F}_q^k$ is an arc of size $2k-3$, then the matrix $M_G^{\uparrow 0}$ has full row rank exactly when $k \leqslant p$. Hence $g(k,q) = q+1$ when $k \leqslant p$.*

The second assertion of Theorem 1.7 follows from Corollary 1.5 when q is not prime. If q is prime, then Corollary 1.5 implies that $g(k,q) = q+1$ when $k \leqslant (q+4)/2$ and hence the second assertion of Theorem 1.7 follows from duality. The second assertion of Theorem 1.7 was first proved by Ball [1].

In Section 7, we again use Theorem 1.6 to verify the condition in Corollary 1.5 when $n = 1$ and $k \leqslant 2p-2 \leqslant q$.

Theorem 1.8 *If $k \leqslant 2p-2 \leqslant q$ and $G \subset \mathbb{F}_q^k$ is an arc of size $2k-2$, then the matrix $M_G^{\uparrow 1}$ has full row rank. Hence, if q is not prime, then $g(k,q) = q+1$ when $k \leqslant 2p-2$.*

The bound $k \leqslant 2p-2$ in the first assertion of Theorem 1.8 cannot be improved because one can check using a computer that if $G \subset \mathbb{F}_9^5$ is a subarc of size 8 of the normal rational curve $\mathcal{R}_5 \subset \mathbb{F}_9^5$, then the matrix $M_G^{\uparrow 1}$ does not have full row rank. The second assertion of Theorem 1.8 follows from Corollary 1.5 and was first proved by Ball and De Buele [4].

Recalling that $p = \mathrm{char}\,(\mathbb{F}_q)$, we conjecture that if $0 \leqslant n \leqslant q$ and

$$2 \leqslant k \leqslant \min\left\{p + n(p-2), \frac{q+4-n}{2}\right\}, \tag{1.9}$$

then the condition in Corollary 1.5 holds.

Conjecture 1.9 If $0 \leqslant n \leqslant q$, k satisfies (1.9), and $G \subset \mathbb{F}_q^k$ is an arc of size $2k-3+n$, then the matrix $M_G^{\uparrow n}$ has full row rank.

Observe that Theorem 1.7 and Theorem 1.8 prove Conjecture 1.9 when $n = 0$ and $n = 1$. For larger values of n, we have computational evidence to support Conjecture 1.9.

If Conjecture 1.9 is true then, by Corollary 1.5, $g(k,q) = q+1$ when (1.4) holds.

Corollary 1.10 *If Conjecture 1.9 is true for any particular n satisfying*

$$0 \leqslant n \leqslant \left|\frac{q-2p+4}{2p-3}\right|, \tag{1.10}$$

then $g(k,q) = q+1$ when $k \leqslant p+n(p-2)$. If Conjecture 1.9 is true, then $g(k,q) = q+1$ when (1.4) holds.

1.1.1 Classification

The matrices $M_G^{\uparrow n}$ are also useful for determining when the normal rational curve $\mathcal{R}_k \subset \mathbb{F}_q^k$ is the unique arc of size $q+1$ up to linear equivalence. The second main result of this article is that if $0 \leqslant n \leqslant q-2k$ and for any arc $G \subset \mathbb{F}_q^k$ of size $2k-2+n$, the matrix $M_G^{\uparrow n}$ contains a certain vector in its column space, then the normal rational curve is the unique arc of size $q+1$ up to linear equivalence.

To state our theorem precisely, we define a matrix $H_G^{\uparrow n}$ that is equivalent to the matrix $M_G^{\uparrow n}$ so that the vector we require in the column space has a nice form. Recall that we have defined arcs to be ordered sets and that if $(X, <)$ is an ordered set then $A \subset X$ is smaller than $B \subset X$ in colex order if the largest element of the symmetric difference $A \triangle B$ lies in B.

Definition 1.11 Let $G \subset \mathbb{F}_q^k$ be an arc, let $0 \leqslant n \leqslant |G| - k + 1$, and let B be the basis of $\mathbb{F}_q^k$ fixed in Definition 1.3. For each $C \in \binom{G}{k-1}$, let $L_C \in \binom{G\setminus C}{n}$ be the last n-subset of $\binom{G\setminus C}{n}$ in colex order. Let $J_G^{\uparrow n}$ be a diagonal matrix with rows and columns indexed by $\binom{G}{k-1}$ and (C, C)-entry

$$J_G^{\uparrow n}(C, C) = \prod_{y \in L_C} \det(y, C)_B^{-1}. \tag{1.11}$$

Define the matrix $H_G^{\uparrow n} = J_G^{\uparrow n} M_G^{\uparrow n}$ and put the rows of the matrix $H_G^{\uparrow n}$ in colex order.

Observe that the entries of the matrix $H_G^{\uparrow n}$ are independent of the basis B. We restate our second main result precisely using the matrices $H_G^{\uparrow n}$.

Theorem 1.12 *If $0 \leqslant n \leqslant q-2k$ and for every arc $G \subset \mathbb{F}_q^k$ of size $2k-2+n$, the column space of the matrix $H_G^{\uparrow n}$ contains a vector $v \in \mathbb{F}_q^{\binom{2k-2+n}{k-1}}$ such that $v_i = 1$ if $i \in \{1, \ldots, k\}$ and $v_i = 0$ otherwise, then the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$ up to linear equivalence.*

When $n = 0$ and $G \subset \mathbb{F}_q^k$ is an arc of size $2k-2$, the matrix $H_G^{\uparrow 0}$ equals the inclusion matrix $I_{2k-2}(k-1, k-2)$, so we can easily verify that the column space of the matrix $H_G^{\uparrow 0}$ contains the required vector when $k \leqslant p = \operatorname{char}(\mathbb{F}_q)$.

Theorem 1.13 *If $k \leqslant p = \operatorname{char}(\mathbb{F}_q)$ and $G \subset \mathbb{F}_q^k$ is an arc of size $2k-2$, then the column space of the matrix $H_G^{\uparrow 0}$ contains a vector $v \in \mathbb{F}_q^{\binom{2k-2}{k-1}}$ such that $v_i = 1$ if $i \in \{1, \ldots, k\}$ and $v_i = 0$ otherwise. Hence, if $k \leqslant p$ and $k \neq (q+1)/2$, then the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$ up to linear equivalence.*

It is easy to see that the bound $k \leqslant p$ in the first assertion of Theorem 1.13 cannot be improved. The second assertion of Theorem 1.13 was first proved by Ball in [1], although the condition $k \neq (q+1)/2$ was missing there.

We conjecture in Conjecture 1.14 that if $k \leqslant 2p-2 \leqslant q$ and $G \subset \mathbb{F}_q^k$ is an arc of size $2k$, then the column space of the matrix $H_G^{\uparrow 2}$ contains the required vector in

Theorem 1.12. We have computational evidence to support Conjecture 1.14, and we note that if Conjecture 1.14 is true, then the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$ up to linear equivalence when $k \leqslant 2p-2 \leqslant q$.

Conjecture 1.14 When $k \leqslant 2p-2 \leqslant q$, for every arc $G \subset \mathbb{F}_q^k$ of size $2k$, the column space of the matrix $H_G^{\uparrow 2}$ contains a vector $v \in \mathbb{F}_q^{\binom{2k}{k-1}}$ such that $v_i = 1$ if $i \in \{1, \ldots, k\}$ and $v_i = 0$ otherwise.

1.1.2 Verifying Conjecture 1.2 and Classifying Large Arcs Computationally

An important benefit of the conditions in Corollary 1.5 and Theorem 1.12 is that they can be checked with a computer. Corollary 1.5 and Theorem 1.12 may consequently be of use in verifying Conjecture 1.2 and classifying large arcs computationally. For example, if one could classify arcs in $\mathbb{F}_q^k$ of size $2k-2$ up to linear equivalence, then one could test the rank of the matrix $M_G^{\uparrow 1}$ for a representative G from each linear equivalence class. If the matrix $M_G^{\uparrow 1}$ has full row rank, then Corollary 1.5 would rule out the possibility that any arc in the linear equivalence class of G could be extended to an arc of size $q+2$. If the matrix $M_G^{\uparrow 1}$ does not have full row rank, then one could extend G to an arc H of size $2k-1$ and check if the matrix $M_H^{\uparrow 2}$ has full row rank. This should dramatically reduce the space of possible subarcs of arcs of size $q+2$. In the same way, Theorem 1.12 can be used to check if the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$ up to linear equivalence. These algorithms should be possible to implement because the question of classifying arcs up to linear equivalence has already been considered in [8] and [13].

1.2 Important Remarks and Outline of Paper

The results in this paper are joint work with Simeon Ball, but he has elected to write a separate exposition of some of these results in [3]. A straightforward consequence of the proof of Theorem 1.4 is Theorem 1.15, which shows that that the conclusion of Theorem 1.4 holds if the matrix $M_G^{\uparrow n}$ satisfies the slightly weaker condition of having a vector of weight one in its column space. Theorem 1.15 is the main result of [3].

Theorem 1.15 *Let $G \subset \mathbb{F}_q^k$ be an arc and let $n \in \mathbb{N}$ be a natural number such that*

$$n + k - 1 \leqslant |G| \leqslant \frac{q + 2k - 2 + n}{2}. \tag{1.12}$$

If the matrix $M_G^{\uparrow n}$ has a vector of weight one in its column space, then the arc G cannot be extended to an arc of size $q + 2k - 1 + n - |G|$.

For the most interesting application of Theorem 1.4, namely Corollary 1.5, we do not believe that Theorem 1.15 offers any benefit over Theorem 1.4. In other words, we believe that if $0 \leqslant n \leqslant q - 2k + 4$ and if for every arc $G \subset \mathbb{F}_q^k$ of size $2k - 3 + n$ the matrix $M_G^{\uparrow n}$ has a vector of weight one in its column space, then for every such arc G the matrix $M_G^{\uparrow n}$

has full row rank. Indeed, the bound on k in our stronger Conjecture 1.9 matches exactly the bound on k in Ball's weaker Conjecture 1 in [3].

This paper builds on the methods initiated in [1], [2], and [4]. An important change in the proof approach of [1], [2], and [4] lies in the definition of certain parameters α_A in Lemma 6.1. In [2, Chapter 7], the analogue of the parameter α_A in Lemma 6.1 is referred to as $Q(A, F)$ and its definition is dependent on a smaller subarc of a larger arc. In Lemma 6.1 and in [3, Section 3], the parameters α_A are defined so that they no longer depend on the smaller subarc and only depend on the larger arc. This change is crucial to the proof of Theorem 1.4.

The three main ingredients in the proofs of Theorem 1.4 and Theorem 1.12 are duality, polynomial interpolation, and Segre's Lemma of Tangents. Section 2 discusses the properties of polynomial interpolation that we use. Section 3 explains the concept of tangent functions. In Section 4, we reduce our first main result Theorem 1.4 to Theorem 4.2. In Section 5, we reduce Theorem 4.2 to Lemma 5.3. In Section 6, we state and prove Segre's Lemma of Tangents and use it to prove Lemma 5.3, thus completing the proofs of Theorem 1.4 and Theorem 4.2. In Section 7, we prove Theorem 1.8 and thus prove Conjecture 1.9 when $n = 1$. In Section 8, we prove Theorem 1.12 and Theorem 1.13.

2 Polynomial Interpolation

That one can uniquely determine a polynomial $f \in \mathbb{F}[X]$ in one variable of degree at most t over any field $\mathbb{F}$ from $t+1$ of its values is well-known. Similarly, one can recover a homogeneous polynomial in two variables $f \in \mathbb{F}(X, Y)$ of degree t by knowing values of f on the points of an arc $\{(x_i, y_i) : i \in \{1, \ldots, t+1\}\}$ of size $t+1$ in $\mathbb{F}^2$.

Suppose $f(X, Y) = \sum_{i=0}^{t} c_i X^i Y^{t-i}$ is a homogeneous polynomial in two variables of degree t and we know its values $f(x_i, y_i)$ on the points of an arc $\{(x_i, y_i) : i \in \{1, \ldots, t+2\}\}$ of size $t+2$ in $\mathbb{F}^2$. Let $P \in M_{t+1,t+2}(\mathbb{F})$ be a matrix with (i, j)-entry $P(i, j) = x_j^{i-1} y_j^{t-i+1}$ and let $\vec{c} = [c_0, \ldots, c_t]$ and $\vec{z} = [f(x_1, y_1), \ldots, f(x_{t+2}, y_{t+2})]$. As P has more columns than rows, its columns are linearly dependent. Hence, there is a solution $\vec{w} = [w_1, \ldots, w_{t+2}]^T$ to $P\vec{w} = \vec{0}$ and thus $\vec{z}\vec{w} = \vec{0}$ because $\vec{c}P = \vec{z}$. We now show in Theorem 2.1 that a solution $\vec{w}$ to $P\vec{w} = \vec{0}$ and $\vec{z}\vec{w} = \vec{0}$ is given by

$$w_i = \prod_{\substack{j=1 \\ j \neq i}}^{t+2} (x_i y_j - x_j y_i)^{-1}, \quad i \in \{1, \ldots, t+2\}. \tag{2.13}$$

Theorem 2.1 is a key ingredient in the proof of Theorem 1.4.

Theorem 2.1 *Suppose $f(X, Y) \in \mathbb{F}[X, Y]$ is a homogeneous polynomial in two variables of degree t and $\{(x_i, y_i) : i \in \{1, \ldots, t+2\}\}$ is an arc of size $t+2$ in $\mathbb{F}^2$. We then have*

$$\sum_{i=1}^{t+2} f(x_i, y_i) \prod_{\substack{j=1 \\ j \neq i}}^{t+2} (x_i y_j - x_j y_i)^{-1} = 0. \tag{2.14}$$

Proof. Using the definitions of P, $\vec{w}$, and $\vec{z}$ from the preceding paragraph, let B be a square matrix whose columns are the first $t+1$ columns of the matrix P. Let $\vec{b}$ be the last column of the matrix P. Note that a solution $\vec{r} = [r_1, \ldots, r_{t+1}]^T$ to $B\vec{r} = \vec{b}$ gives a solution $\vec{w}$ to $P\vec{w} = \vec{0}$ with $w_i = r_i$ for $i \in \{1, \ldots, t+1\}$ and $w_{t+2} = -1$.

Since $\{(x_i, y_i) : i \in \{1, \ldots, t+2\}\}$ is an arc of size $t+2$ in $\mathbb{F}^2$, we may assume that $y_1, \ldots, y_{t+1}$ are nonzero. Hence the matrix B is nonsingular, so by Cramer's Rule, a solution $\vec{r}$ to $B\vec{r} = \vec{b}$ is given by $r_i = \det(B_i)/\det(B)$ where B_i is the matrix formed by replacing the i^{th} column of B with $\vec{b}$. Using the formula for the determinant of a Vandermonde matrix,

$$\det(B) = (y_1 \cdots y_{t+1})^t \prod_{1 \leqslant l < m \leqslant t+1} \left(\frac{x_m}{y_m} - \frac{x_l}{y_l} \right) = \prod_{1 \leqslant l < m \leqslant t+1} (x_m y_l - x_l y_m),$$

$$\det(B_i) = \prod_{\substack{1 \leqslant l < m \leqslant t+1 \\ l \neq i, m \neq i}} (x_m y_l - x_l y_m) \prod_{1 \leqslant l < i} (x_{t+2} y_l - x_l y_{t+2}) \prod_{i < m \leqslant t+1} (x_m y_{t+2} - x_{t+2} y_m).$$

Hence, after a little algebraic manipulation,

$$r_i = \frac{\det(B_i)}{\det(B)} = -\frac{\prod_{1 \leqslant l \leqslant t+1} (x_{t+2} y_l - x_l y_{t+2})}{\prod_{\substack{1 \leqslant l \leqslant t+2 \\ l \neq i}} (x_i y_l - x_l y_i)}. \tag{2.15}$$

Multiplying the corresponding solution $\vec{w}$ to $P\vec{w} = \vec{0}$ by $-\prod_{1 \leqslant l \leqslant t+1} (x_{t+2} y_l - x_l y_{t+2})^{-1}$ yields the solution $\vec{w}$ to $\vec{z}\vec{w} = \vec{0}$ given by (2.13). □

3 Tangent Functions

Let $S \subset \mathbb{F}_q^k$ be an arc and let $A \subset S$ have size $k-2$. We first count in Lemma 3.1 the number of $(k-1)$-dimensional subspaces of $\mathbb{F}_q^k$ that intersect S precisely in A.

Lemma 3.1 (Ball [1, 2]) *Let $S \subset \mathbb{F}_q^k$ be an arc and let $A \subset S$ have size $k-2$. Let $H_A^1, \ldots, H_A^t$ be the $(k-1)$-dimensional subspaces of $\mathbb{F}_q^k$ whose intersection with S is A. We have*

$$t := q + k - 1 - |S|. \tag{3.16}$$

Proof. Since A is a linearly independent set of size $k-2$, the number of $(k-1)$-dimensional subspaces of $\mathbb{F}_q^k$ that contain A is $q+1$. Since S is an arc, a $(k-1)$-dimensional subspace of $\mathbb{F}_q^k$ that contains A can contain at most one other vector of $S \setminus A$. □

Given an arc $S \subset \mathbb{F}_q^k$ and a subset $A \subset S$ of size $k-2$, we now define the tangent function at A, which can be viewed as a homogeneous polynomial in two variables with respect to certain bases of $\mathbb{F}_q^k$. We will then apply Theorem 2.1 to the tangent functions $f_{A,S}$ for various $A \subset S$ to prove Theorem 1.4.

Definition 3.2 (Ball [1, 2]) Let $S \subset \mathbb{F}_q^k$ be an arc and let $A \subset S$ have size $k-2$. Let $H_A^1, \ldots, H_A^t$ be the $(k-1)$-dimensional subspaces of $\mathbb{F}_q^k$ defined in Lemma 3.1, where t is given by (3.16). Let $\beta_A^i : \mathbb{F}_q^k \to \mathbb{F}_q$ be a linear functional whose kernel is H_A^i. We define the tangent function at A, denoted $f_{A,S} : \mathbb{F}_q^k \to \mathbb{F}_q$, by

$$f_{A,S}(x) = \prod_{i=1}^{t} \beta_A^i(x). \tag{3.17}$$

Observe that $f_{A,S}(x) = 0$ precisely when $x \in \bigcup_{i=1}^{t} H_A^i$ and that $f_{A,S}$ is defined up to a scalar factor.

Notation: Recall that an arc $S \subset \mathbb{F}_q^k$ is ordered. If $R_1, \ldots, R_l$ are subsets of S we use $(R_1, \ldots, R_l)$ to mean write the vectors in R_1 in order first, and then the vectors in R_2 etc. When R_i is a singleton set, we simply write the vector. For example, if $x, y \in S \setminus A$ and B is a basis of $\mathbb{F}_q^k$, we write $\det(x, y, A)_B$ for the determinant of the matrix whose rows are the vectors x, y, and the elements of A in order written with respect to B.

Let $S \subset \mathbb{F}_q^k$ be an arc and let $A \subset S$ have size $k-2$. Let $B = (b_1, b_2, A)$ be a basis of $\mathbb{F}_q^k$. Also let $T \subset S \setminus A$ have size $t+2$, where t is defined by (3.16). In Lemma 3.3, we use Theorem 2.1 to obtain an equation for a pair (A, T) where $A \in \binom{S}{k-2}$ and $T \in \binom{S \setminus A}{t+2}$.

Lemma 3.3 (Ball [1, 2]) *Let $S \subset \mathbb{F}_q^k$ be an arc and let $A \subset S$ have size $k-2$. Let $B = (b_1, b_2, A)$ be a basis of $\mathbb{F}_q^k$. If $T \subset S \setminus A$ has size $t+2$, where t is given by* (3.16), *then*

$$\sum_{x \in T} f_{A,S}(x) \prod_{y \in T \setminus \{x\}} \det(x, y, A)_B^{-1} = 0. \tag{3.18}$$

Proof. With respect to the basis B, the linear functional β_A^i in (3.17) is linear in just the first two coordinates since its kernel contains A. Hence, the tangent function $f_{A,S}$ is a homogeneous polynomial in two variables of degree t, where t is given by (3.16). Since S is an arc, when we write the vectors in T in terms of the basis B, their first two coordinates form an arc of size $t+2$ in $\mathbb{F}_q^2$. Hence, we can apply Theorem 2.1 to $f_{A,S}$ and T, and note that with respect to B, we have $\det(x, y, A)_B = x_1 y_2 - y_1 x_2$. □

Theorem 2.1 and Lemma 3.3 explain how the entries of the matrix $M_G^{\uparrow n}$ arise because in Lemma 5.5 we show how the product of determinants in (3.18) is related to the product of determinants in the entries of the matrix $M_G^{\uparrow n}$.

4 Proof that Theorem 4.2 Implies Theorem 1.4

Let $S \subset \mathbb{F}_q^k$ be an arc and let $G \subset S$ have size $t+k+n$, where t is defined by (3.16) and $n \geqslant 0$. For each $U \in \binom{G}{n}$, we can analyze the system of equations obtained from applying Lemma 3.3 to pairs (A, T) where $A \in \binom{G \setminus U}{k-2}$ and $T = G \setminus (A \cup U)$.

Corollary 4.1 *Let $S \subset \mathbb{F}_q^k$ be an arc and let $G \subset S$ have size $t+k+n$, where t is defined by (3.16) and $n \geqslant 0$. For $A \in \binom{G}{k-2}$, define a basis $B(A) = (b_1, b_2, A)$ of $\mathbb{F}_q^k$. If $P_G^{\uparrow n}$ is a matrix whose rows are indexed by $\binom{G}{k-1}$, whose columns are indexed by ordered pairs (U, A) where $U \in \binom{G}{n}$ and $A \in \binom{G\setminus U}{k-2}$, and whose $(C,(U,A))$-entry is*

$$P_G^{\uparrow n}(C,(U,A)) = \begin{cases} f_{A,S}(C \setminus A) \prod_{y \in G\setminus(C\cup U)} \det(C \setminus A, y, A)^{-1}_{B(A)} & \text{if } A \subset C \subset G \setminus U \\ 0 & \text{otherwise,} \end{cases}$$

then $\vec{1} P_G^{\uparrow n} = \vec{0}$.

Proof. For $U \in \binom{G}{n}$ and $A \in \binom{G\setminus U}{k-2}$, let $T = G \setminus (A \cup U)$. Since $|G| = t+k+n$, we have that $T \in \binom{S\setminus A}{t+2}$. Applying Lemma 3.3, we have that

$$\sum_{x \in G\setminus(A\cup U)} f_{A,S}(x) \prod_{y \in (G\setminus(A\cup U))\setminus\{x\}} \det(x, y, A)^{-1}_{B(A)} = 0. \tag{4.19}$$

Let us rewrite (4.19) so that it will be easier to express the system of equations given by (4.19) in matrix form. For any fixed $U \in \binom{G}{n}$ and $A \in \binom{G\setminus U}{k-2}$, we have

$$\sum_{A \subset C \in \binom{G\setminus U}{k-1}} f_{A,S}(C \setminus A) \prod_{y \in G\setminus(C\cup U)} \det(C \setminus A, y, A)^{-1}_{B(A)} = 0. \tag{4.20}$$

Consequently, letting $P_G^{\uparrow n}$ be the matrix defined in Corollary 4.1, we see that we can write the system of equations given by (4.20) in matrix form as $\vec{1} P_G^{\uparrow n} = \vec{0}$. □

The equation $\vec{1} P_G^{\uparrow n} = \vec{0}$ contains a wealth of information about the arc $S \subset \mathbb{F}_q^k$ and is crucial to the proof of Theorem 1.4. At the moment, the matrix $P_G^{\uparrow n}$ defined in Corollary 4.1 may seem ugly and difficult to analyze, but we claim that $P_G^{\uparrow n}$ is equivalent to the much simpler matrix $M_G^{\uparrow n}$ defined in (1.5), which depends only on the arc G.

Theorem 4.2 *Let $S \subset \mathbb{F}_q^k$ be an arc and let $G \subset S$ have size $t+k+n$, where t is defined by (3.16) and $n \geqslant 0$. If $P_G^{\uparrow n}$ is the matrix defined in Corollary 4.1, then there exist invertible diagonal matrices D_1 and D_2 so that $D_1 P_G^{\uparrow n} D_2 = M_G^{\uparrow n}$, where $M_G^{\uparrow n}$ is defined by (1.5).*

We now reduce Theorem 1.4 and Theorem 1.15 to Theorem 4.2.

Proof of Theorem 1.4 and Theorem 1.15. We prove the contrapositive: namely that if $G \subset \mathbb{F}_q^k$ can be extended to an arc $S \subset \mathbb{F}_q^k$ of size $q+2k-1+n-|G|$, then the matrix $M_G^{\uparrow n}$ cannot have full row rank or a vector of weight one in its column space. First we show the arc G satisfies the hypotheses of Corollary 4.1. As S has size $q+2k-1+n-|G|$, the arc G has size $t+k+n$, where t is defined by (3.16). By Corollary 4.1, we have $\vec{1} P_G^{\uparrow n} = \vec{0}$ and so by Theorem 4.2, we have $\vec{0} = (\vec{1} D_1^{-1}) M_G^{\uparrow n}$. Since D_1 is an invertible matrix, all entries of $\vec{1} D_1^{-1}$ are nonzero. Hence, the matrix $M_G^{\uparrow n}$ cannot have full row rank or a vector of weight one in its column space. □

5 Proof that Lemma 5.3 Implies Theorem 4.2

To prove Theorem 4.2, we first define matrices $Q_G^{\uparrow n}$, $R_G^{\uparrow n}$ and $I_G^{\uparrow n}$ given a subset $G \subseteq S$ where $S \subset \mathbb{F}_q^k$ is an arc. The matrix $I_G^{\uparrow n}$ is a signed inclusion matrix and to define the signing we need the following notation.

Definition 5.1 Let X be an ordered set, let A be an ordered subset of X, and let C be an ordered subset of X that contains A and has size $|A|+1$. We define $\tau(A, C)$ to be the minimum number of transpositions needed to order $(A, C \setminus A)$ as C.

Definition 5.2 Let $G \subseteq S \subset \mathbb{F}_q^k$, where S is an arc, and let $0 \leqslant n \leqslant |G| - k + 1$. Let $Q_G^{\uparrow n}$, $R_G^{\uparrow n}$, and $I_G^{\uparrow n}$ respectively be matrices whose rows are indexed by $\binom{G}{k-1}$, whose columns are indexed by ordered pairs (U, A) where $U \in \binom{G}{n}$ and $A \in \binom{G\setminus U}{k-2}$, and whose $(C, (U, A))$-entries respectively are

$$Q_G^{\uparrow n}(C, (U, A)) = \begin{cases} f_{A,S}(C \setminus A) & \text{if } A \subset C \subset G \setminus U \\ 0 & \text{otherwise,} \end{cases} \tag{5.21}$$

$$R_G^{\uparrow n}(C, (U, A)) = \begin{cases} \prod_{y \in G\setminus(C\cup U)} \det(C \setminus A, y, A)^{-1}_{B(A)} & \text{if } A \subset C \subset G \setminus U \\ 0 & \text{otherwise,} \end{cases} \tag{5.22}$$

$$I_G^{\uparrow n}(C, (U, A)) = \begin{cases} (-1)^{\tau(A,C)(t+1)} & \text{if } A \subset C \subset G \setminus U \\ 0 & \text{otherwise,} \end{cases} \tag{5.23}$$

where $f_{A,S}$ is defined by Definition 3.2, $\tau(A, C)$ is defined by Definition 5.1, and t is defined by (3.16).

We will prove in Section 6 that if $S \subset \mathbb{F}_q^k$ is an arc, then the matrix $Q_S^{\uparrow 0}$ defined in (5.21) is equivalent to the matrix $I_S^{\uparrow 0}$ defined in (5.23).

Lemma 5.3 *Let $S \subset \mathbb{F}_q^k$ be an arc. If $Q_S^{\uparrow 0}$ is the matrix defined in* (5.21) *and $I_S^{\uparrow 0}$ is the matrix defined in* (5.23)*, then there exist invertible diagonal matrices E_1 and E_2 such that $E_1 Q_S^{\uparrow 0} E_2 = I_S^{\uparrow 0}$.*

We now use Lemma 5.3 to prove that if $S \subset \mathbb{F}_q^k$ is an arc and $G \subset S$ satisfies the constraints of Corollary 4.1 then the matrix $Q_G^{\uparrow n}$ defined in (5.21) is equivalent to the matrix $I_G^{\uparrow n}$ defined in (5.23).

Lemma 5.4 *Let $G \subset S \subset \mathbb{F}_q^k$ where S is an arc. If $Q_G^{\uparrow n}$ is the matrix defined in* (5.21)*, then there exist invertible diagonal matrices F_1 and F_2 such that $F_1 Q_G^{\uparrow n} F_2 = I_G^{\uparrow n}$.*

Proof. Recall that, by Lemma 5.3, there exist invertible diagonal matrices E_1 and E_2 such that $E_1 Q_S^{\uparrow 0} E_2 = I_S^{\uparrow 0}$. Let F_1 be the submatrix of E_1 whose rows and columns are indexed by $\binom{G}{k-1}$. Let F_2 be the submatrix of E_2 whose rows and columns are indexed by ordered pairs (U, A), where $U \in \binom{G}{n}$ and $A \in \binom{G\setminus U}{k-2}$. As the entries of $Q_G^{\uparrow n}$ and $I_G^{\uparrow n}$ don't depend on $U \in \binom{G}{n}$, we have $F_1 Q_G^{\uparrow n} F_2 = I_G^{\uparrow n}$. □

We now prove that if $S \subset \mathbb{F}_q^k$ is an arc and $G \subset S$ satisfies the constraints of Corollary 4.1 then the matrix $R_G^{\uparrow n}$ defined in (5.22) is equivalent to the Hadamard product $I_G^{\uparrow n} \circ M_G^{\uparrow n}$.

Lemma 5.5 *Let $S \subset \mathbb{F}_q^k$ be an arc and let $G \subset S$ satisfy the hypotheses of Corollary 4.1. If $R_G^{\uparrow n}$ is the matrix defined in* (5.22)*, then there exist invertible diagonal matrices F_3 and F_4 such that $F_3 R_G^{\uparrow n} F_4 = I_G^{\uparrow n} \circ M_G^{\uparrow n}$.*

Proof. Since $|G| = t+k+n$, for $C \in \binom{G}{k-1}$ and $U \in \binom{G}{n}$, we have $|G \setminus (C \cup U)| = t+1$. Hence, for $A \subset C$, we have

$$(-1)^{(t+1)(k-1)} \prod_{y \in G \setminus (C \cup U)} \det(C \setminus A, y, A)_{B(A)}^{-1} = \prod_{y \in G \setminus (C \cup U)} \det(y, A, C \setminus A)_{B(A)}^{-1} \tag{5.24}$$

because there are $k-1$ transpositions needed to make $C \setminus A$ the last row.

Recall that for each $A \in \binom{G}{k-2}$, we defined a basis $B(A) = (b_1, b_2, A)$ of $\mathbb{F}_q^k$. Now let B be the basis of $\mathbb{F}_q^k$ fixed in Definition 1.3 and let $M(B(A), B)$ be the change-of-basis matrix from $B(A)$ to B. Observe that

$$\det(y, A, C \setminus A)_{B(A)}^{-1} \det(M(B(A), B))^{-1} = \det(y, A, C \setminus A)_B^{-1}. \tag{5.25}$$

Define F_4 to be a diagonal matrix with rows and columns indexed by ordered pairs (U, A) where $U \in \binom{G}{n}$ and $A \in \binom{G \setminus U}{k-2}$, and $((U, A), (U, A))$-entry

$$F_4((U, A), (U, A)) = (-1)^{(t+1)(k-1)} \det(M(B(A), B))^{-(t+1)}. \tag{5.26}$$

By (5.24) and (5.25), the $(C, (U, A))$-entry of $R_G^{\uparrow n} F_4$ is

$$R_G^{\uparrow n} F_4(C, (U, A)) = \begin{cases} \prod_{y \in G \setminus (C \cup U)} \det(y, A, C \setminus A)_B^{-1} & \text{if } A \subset C \subset G \setminus U \\ 0 & \text{otherwise.} \end{cases} \tag{5.27}$$

Observe that

$$\prod_{y \in G \setminus (C \cup U)} \det(y, A, C \setminus A)_B^{-1} = (-1)^{\tau(A,C)(t+1)} \prod_{y \in G \setminus (C \cup U)} \det(y, C)_B^{-1} \tag{5.28}$$

because moving $C \setminus A$ from the end to its proper place in the ordering of C requires $\tau(A, C)$ transpositions for each of the $t+1$ determinants in the product.

Hence, defining F_3 to be a diagonal matrix with rows and columns indexed by $\binom{G}{k-1}$ and (C, C)-entry

$$F_3(C, C) = \prod_{y \in G \setminus C} \det(y, C)_B, \tag{5.29}$$

we see that $F_3 R_G^{\uparrow n} F_4 = I_G^{\uparrow n} \circ M_G^{\uparrow n}$. □

Finally we reduce Theorem 4.2 to Lemma 5.3.

Proof of Theorem 4.2 We express the matrix $P_G^{\uparrow n}$ defined in Corollary 4.1 as the Hadamard product $P_G^{\uparrow n} = Q_G^{\uparrow n} \circ R_G^{\uparrow n}$, where the matrices $Q_G^{\uparrow n}$ and $R_G^{\uparrow n}$ are defined in (5.21) and (5.22) respectively.

Using Lemma 5.3, we show in Lemma 5.4 that there exist invertible diagonal matrices F_1 and F_2 such that $F_1 Q_G^{\uparrow n} F_2 = I_G^{\uparrow n}$. By Lemma 5.5, there exist invertible diagonal matrices F_3 and F_4 such that $F_3 R_G^{\uparrow n} F_4 = I_G^{\uparrow n} \circ M_G^{\uparrow n}$. Setting $D_1 = F_1 F_3$ and $D_2 = F_2 F_4$, Theorem 4.2 follows. □

6 Proof of Lemma 5.3

In this section, we prove Lemma 5.3 and hence complete the proofs of Theorem 1.4 and Theorem 4.2. In Lemma 6.1, we show that Lemma 5.3 holds if we can find a vector in the nullspace of a certain matrix L all of whose coordinates are nonzero. In Lemma 6.2, we state and prove Segre's Lemma of Tangents, which we use in Lemma 6.3 to show that the matrix L does not have full column rank. Consequently, L has nonzero vectors in its nullspace and we prove Lemma 5.3 by showing that any nonzero vector in the nullspace of L must have all coordinates nonzero.

Recall that an arc $S \subset \mathbb{F}_q^k$ is ordered and that if $(X, <)$ is an ordered set then $A \subset X$ is smaller than $B \subset X$ in lex order if the smallest element of the symmetric difference $A \triangle B$ lies in A.

Lemma 6.1 *Let $S \subset \mathbb{F}_q^k$ be an arc. Let L be a matrix whose columns are indexed by $\binom{S}{k-2}$ and whose rows are indexed by ordered pairs (A, A') where $A, A' \in \binom{S}{k-2}$, $A \cup A' \in \binom{S}{k-1}$, and $A < A'$ in lex order. Let the $((A, A'), A'')$ entry of L be*

$$L((A, A')), A'') = \begin{cases} (-1)^{\tau(A, A \cup A')(t+1)} f_{A,S}(A' \setminus A) & \text{if } A'' = A \\ (-1)^{\tau(A', A \cup A')(t+1)+1} f_{A',S}(A \setminus A') & \text{if } A'' = A' \\ 0 & \text{otherwise,} \end{cases} \tag{6.30}$$

where t is defined by (3.16). If there exists a vector $\vec{\alpha} \in \mathbb{F}_q^{\binom{|S|}{k-2}}$ in the nullspace of L all of whose coordinates are nonzero, then Lemma 5.3 holds.

Proof. We write the coordinates of $\vec{\alpha}$ as α_A where $A \in \binom{S}{k-2}$. Since $\vec{\alpha}$ is in the nullspace of L, if $Q_S^{\uparrow 0}$ is the matrix defined in (5.21), $C \in \binom{S}{k-1}$, $A, A' \in \binom{C}{k-2}$, and t is defined by (3.16), then

$$(-1)^{\tau(A,C)(t+1)} \alpha_A Q_S^{\uparrow 0}(C, A) = (-1)^{\tau(A',C)(t+1)} \alpha_{A'} Q_S^{\uparrow 0}(C, A'). \tag{6.31}$$

Define E_2 to be a diagonal matrix with rows and columns indexed by $\binom{S}{k-2}$ and (A, A) entry $E_2(A, A) = \alpha_A$. Since the coordinates of $\vec{\alpha}$ are nonzero and $Q_S^{\uparrow 0}(C, A) \neq 0$ when

$A \subset C$, there exist nonzero constants $\alpha_C \in \mathbb{F}_q$ for $C \in \binom{S}{k-1}$ such that the (C, A) entry of $Q_S^{\uparrow 0} E_2$ is

$$Q_S^{\uparrow 0} E_2(C, A) = \begin{cases} (-1)^{\tau(A,C)(t+1)} \alpha_C & \text{if } A \subset C \\ 0 & \text{otherwise,} \end{cases} \tag{6.32}$$

by (6.31). Consequently, defining E_1 to be a diagonal matrix with rows and columns indexed by $\binom{S}{k-1}$ and (C, C)-entry $E_1(C, C) = \alpha_C^{-1}$, we see that Lemma 5.3 holds. □

To prove the existence of a vector $\vec{\alpha} \in \mathbb{F}_q^{\binom{|S|}{k-2}}$ satisfying the hypotheses of Lemma 6.1, we first show in Lemma 6.3 that the matrix L defined in (6.30) does not have full column rank over $\mathbb{F}_q$. For this, we need Lemma 6.2, which is called Segre's Lemma of Tangents and gives a relationship between values of different tangent functions.

Lemma 6.2 (Ball [1, 2]) *Let $S \subset \mathbb{F}_q^k$ be an arc and let t be defined by* (3.16)*. For a subset $D \subset S$ of size $k-3$ and a subset $\{u, v, w\} \in S \setminus D$, we have*

$$f_{D\cup\{u\},S}(v) f_{D\cup\{v\},S}(w) f_{D\cup\{w\},S}(u) = (-1)^{t+1} f_{D\cup\{u\},S}(w) f_{D\cup\{v\},S}(u) f_{D\cup\{w\},S}(v). \tag{6.33}$$

Proof. Observe that $B = (u, v, w, D)$ is a basis of $\mathbb{F}_q^k$ because S is an arc. For $x \in \mathbb{F}_q^k$, let $x = (x_1, \ldots, x_k)$ be the coordinates of x with respect to B. By (3.17),

$$f_{D\cup\{w\},S}(x) = \prod_{i=1}^{t} (\beta^i_{D\cup\{w\}}(u) x_1 + \beta^i_{D\cup\{w\}}(v) x_2). \tag{6.34}$$

Our first goal is to show that

$$\left\{ -\frac{\beta^i_{D\cup\{w\}}(u)}{\beta^i_{D\cup\{w\}}(v)} : i \in \{1, \ldots, t\} \right\} \cup \left\{ \frac{x_2}{x_1} : x \in S \setminus B \right\} = \mathbb{F}_q \setminus \{0\}. \tag{6.35}$$

To accomplish this, observe that the first set on the left hand side of (6.35) contains t nonzero elements of $\mathbb{F}_q$ because for $i \in \{1, \ldots, t\}$, the $(k-1)$-dimensional subspaces $H^i_{D\cup\{w\}}$ defined in Lemma 3.1 are all distinct and intersect S only in $D \cup \{w\}$. Now observe that the second set on the left hand side of (6.35) is disjoint from the first set and contains $|S| - k$ nonzero elements of $\mathbb{F}_q$ because S is an arc and because for $i \in \{1, \ldots, t\}$, the $(k-1)$-dimensional subspaces $H^i_{D\cup\{w\}}$ defined in Lemma 3.1 intersect S only in $D \cup \{w\}$. Since $t + |S| - k = q - 1$, (6.35) is established.

Since the product of the nonzero elements of a finite field $\mathbb{F}_q$ equals -1, (6.35) implies

$$\prod_{i=1}^{t} \left(-\frac{\beta^i_{D\cup\{w\}}(u)}{\beta^i_{D\cup\{w\}}(v)} \right) \prod_{x \in S\setminus B} \frac{x_2}{x_1} = -1. \tag{6.36}$$

By (6.34), we can rewrite (6.36) as

$$f_{D\cup\{w\},S}(u) \prod_{x\in S\setminus B} x_2 = (-1)^{t+1} f_{D\cup\{w\},S}(v) \prod_{x\in S\setminus B} x_1. \tag{6.37}$$

Repeating the argument above with the $(k-2)$-subsets $D \cup \{u\}$ and $D \cup \{v\}$, we have

$$f_{D\cup\{u\},S}(v) \prod_{x\in S\setminus B} x_3 = (-1)^{t+1} f_{D\cup\{u\},S}(w) \prod_{x\in S\setminus B} x_2 \tag{6.38}$$

$$f_{D\cup\{v\},S}(w) \prod_{x\in S\setminus B} x_1 = (-1)^{t+1} f_{D\cup\{v\},S}(u) \prod_{x\in S\setminus B} x_3. \tag{6.39}$$

Multiplying (6.37), (6.38), and (6.39), and canceling $\prod_{x\in S\setminus B} x_1x_2x_3$ from both sides, we see that (6.33) holds. □

Now we use Lemma 6.2 to show that the matrix L defined in (6.30) does not have full column rank over $\mathbb{F}_q$.

Lemma 6.3 *Let $S \subset \mathbb{F}_q^k$ be an arc. If L is the matrix defined in* (6.30)*, then L does not have full column rank over $\mathbb{F}_q$.*

Proof. Write $S = \{s_1, \ldots, s_{|S|}\}$ in order. We use the ordering of S to write the elements of $A \in \binom{S}{k-2}$, and the elements of $S \setminus A \in \binom{S}{|S|-k+2}$ in order as $A = \{a_1, \ldots, a_{k-2}\}$ and $S \setminus A = \{\bar{a}_1, \ldots, \bar{a}_{|S|-k+2}\}$.

Let $L_{(A,A')}$ denote the row of L that is indexed by (A, A'). To prove that L does not have full column rank over $\mathbb{F}_q$, we will show that the rows in L are spanned by

$$\mathcal{R} = \{L_{(\{\bar{a}_1\}\cup\{a_1,\ldots,a_{k-3}\},A)} : A \neq \{s_1, \ldots, s_{k-2}\}\}. \tag{6.40}$$

To accomplish this, we must order the rows of L. First, list the rows of $\mathcal{R}$ and then list the remaining rows in lex order. We will show that each row of L that is not in $\mathcal{R}$ can be written as a linear combination of two rows of L that precede it. Hence, by induction, every row of L can be written as a linear combination of rows in $\mathcal{R}$.

Let $L_{(A,A')}$ be a row of L that is not in $\mathcal{R}$. We distinguish two cases.

Case 1: There exists $s \in S \setminus (A \cap A')$ such that s precedes $A \setminus A'$ and $A' \setminus A$ in the ordering of S.

Let $\hat{A} = \{s\} \cup (A \cap A')$ and note that $\hat{A} < A < A'$ in lex order. Also observe that $\tau(A, \hat{A}\cup A) = \tau(A', \hat{A}\cup A')$, $\tau(\hat{A}, \hat{A}\cup A') = \tau(A, A\cup A')$, and $\tau(A', A\cup A') = \tau(\hat{A}, \hat{A}\cup A)+1$. Let t be defined by (3.16) and define $w_1 = (\tau(A, \hat{A}\cup A) + \tau(A, A\cup A') + 1)(t+1) + (t+2)$ and $w_2 = (\tau(A', \hat{A} \cup A') + \tau(A', A \cup A'))(t+1)$. Applying Lemma 6.2 with $D = A \cap A'$, $u = A \setminus A'$, $v = A' \setminus A$ and $w = s$ implies that $L_{(A,A')}$ is a linear combination of $L_{(\hat{A},A)}$ and $L_{(\hat{A},A')}$:

$$L_{(A,A')} = (-1)^{w_1} \frac{f_{A,S}(A' \setminus A)}{f_{A,S}(s)} L_{(\hat{A},A)} + (-1)^{w_2} \frac{f_{A',S}(A \setminus A')}{f_{A',S}(s)} L_{(\hat{A},A')}. \tag{6.41}$$

If $A \cap A' = \{a_1, \ldots, a_{k-3}\}$ and $s = \bar{a}_1$, then $L_{(\hat{A},A)} \in \mathcal{R}$; otherwise the rows $L_{(\hat{A},A)}$ and $L_{(\hat{A},A')}$ precede $L_{(A,A')}$.

Case 2: There does not exist $s \in S \setminus (A \cap A')$ such that s precedes $A \setminus A'$ and $A' \setminus A$ in the ordering of S.

Write $D = A \cap A' = \{d_1, \ldots, d_{k-3}\}$ using the ordering of S. Observe that $A \setminus A'$ precedes d_{k-3} in the ordering of S; otherwise $D = \{s_1, \ldots, s_{k-3}\}$ and $A \setminus A' = s_{k-2}$, which would imply $L_{(A,A')} \in \mathcal{R}$. Let $\hat{A} = D \setminus \{d_{k-3}\} \cup \{A \setminus A'\} \cup \{A' \setminus A\}$ and note that $\hat{A} < A < A'$ in lex order because $A \setminus A'$ precedes d_{k-3} in the ordering of S. Since the union of any two of A, A', and $\hat{A}$ equals the union of all three, we have $L_{(A,A')} = -L_{(\hat{A},A)} + L_{(\hat{A},A')}$. Also, the rows $L_{(\hat{A},A)}$ and $L_{(\hat{A},A')}$ precede $L_{(A,A')}$. □

We now prove Lemma 5.3 and thus complete the proofs of Theorem 1.4 and Theorem 4.2.

Proof of Lemma 5.3 By Lemma 6.3, the matrix L defined in (6.30) does not have full column rank over $\mathbb{F}_q$, so there exists a nonzero vector $\vec{\alpha}$ in the nullspace of L. The coordinates α_A of $\vec{\alpha}$ satisfy (6.31) and we now show that they are all nonzero. Suppose, for a contradiction, that there exists $\hat{A} \in \binom{S}{k-2}$ such that $\alpha_{\hat{A}} = 0$. By (6.31), $\alpha_{A'} = 0$ for all $A' \in \binom{S}{k-2}$ such that $\hat{A} \cup A' \in \binom{S}{k-1}$. Repeating this argument, we see that $\alpha_A = 0$ for all $A \in \binom{S}{k-2}$, which contradicts that $\vec{\alpha} \neq 0$. Therefore, all coordinates of $\vec{\alpha}$ are nonzero so Lemma 5.3 holds by Lemma 6.1. □

Let F be the subset consisting of the first $k-2$ elements of S. For a subset $A \in \binom{S}{k-2}$, let $D = A \cap F$, let $A \setminus F = \{x_1, \ldots, x_r\}$, let $F \setminus A = \{z_1, \ldots, z_r\}$, and let s be the minimum number of transpositions required to order $(F \cap A, F \setminus A)$ as F. Let t be defined by (3.16). One can show that an explicit solution for a nonzero vector $\vec{\alpha} \in \mathbb{F}_q^{\binom{|S|}{k-2}}$ in the nullspace of L is given by

$$\alpha_A = (-1)^{(r+s)(t+1)} \prod_{i=1}^{r} \frac{f_{D \cup \{z_r, \ldots, z_i, x_{i-1}, \ldots, x_1\}, S}(x_i)}{f_{D \cup \{z_r, \ldots, z_{i+1}, x_i, \ldots, x_1\}, S}(z_i)}, \tag{6.42}$$

which motivates Ball's definition of α_A in [3, Section 3].

7 Proof of Theorem 1.8

Let $G \subset \mathbb{F}_q^k$ be an arc of size $2k-2$. For $C \in \binom{G}{k-1}$, let $e(C) \in \mathbb{F}_q^{\binom{2k-2}{k-1}}$ be the C-coordinate vector; that is $e(C)_{C'} = 1$ if $C = C'$ and $e(C)_{C'} = 0$ otherwise. To prove that the matrix $M_G^{\uparrow 1}$ defined in (1.5) has full row rank over $\mathbb{F}_q$ when $k \leqslant 2p - 2 \leqslant q$, we will show that for each $C \in \binom{G}{k-1}$, the C-coordinate vector $e(C)$ lies in the column space of $M_G^{\uparrow 1}$.

For a fixed $U \in \binom{G}{1}$, recall that we noted in Section 1.1 that the submatrix $M_G^{\uparrow 1}(U)$ of $M_G^{\uparrow 1}$ equals $D_U I_{2k-3}(k-1, k-2)$. Hence, to understand the column space of $M_G^{\uparrow 1}$, we must understand the column space of $I_{2k-3}(k-1, k-2)$. By Theorem 1.6, the inclusion matrix $I_{2k-3}(k-1, k-2)$ is invertible over $\mathbb{F}_q$ exactly when $k \leqslant p = \text{char}(\mathbb{F}_q)$ so our first goal is to determine a spanning set for the orthogonal space of the column space of $I_{2k-3}(k-1, k-2)$ over $\mathbb{F}_q$ when $k > p$. This will allow us to prove that a vector $\vec{y}$ lies in

the column space of $I_{2k-3}(k-1,k-2)$ over $\mathbb{F}_q$ by showing that $\vec{y}$ is orthogonal to every vector in the spanning set.

Lemma 7.1 *If $k > p = \mathrm{char}\,(\mathbb{F}_q)$ then, over $\mathbb{F}_q$, the nullspace of the inclusion matrix $I_{2k-3}(k+p-2,k-1)$ is the column space of $I_{2k-3}(k-1,k-2)$.*

Proof. Over $\mathbb{F}_q$, the column space of $I_{2k-3}(k-1,k-2)$ clearly lies in the nullspace of the inclusion matrix $I_{2k-3}(k+p-2,k-1)$ so it suffices to show that the nullity of $I_{2k-3}(k+p-2,k-1)$ equals the rank of $I_{2k-3}(k-1,k-2)$. Observe that the inclusion matrix $I_{2k-3}(k-2,k-p-1)$ equals the inclusion matrix $I_{2k-3}(k+p-2,k-1)^\top$ so by Theorem 1.6 and Lucas' Theorem [14],

$$\mathrm{nullity}_{\,\mathbb{F}_q} I_{2k-3}(k+p-2,k-1) = \binom{2k-3}{k-2} - \sum_{i\in J}\binom{2k-3}{i} - \binom{2k-3}{i-1}, \tag{7.43}$$

where $J = \{0 \leqslant i \leqslant k-2 : i = k-1 \pmod p\}$. On the other hand, by Theorem 1.6 and Lucas' Theorem,

$$\mathrm{rank}_{\,\mathbb{F}_q} I_{2k-3}(k-1,k-2) = \sum_{i\in L}\binom{2k-3}{i} - \binom{2k-3}{i-1}, \tag{7.44}$$

where $L = \{0 \leqslant i \leqslant k-2 : i \neq k-1 \pmod p\}$. Since (7.43) equals (7.44), the lemma follows. □

We now define some special vectors in $\mathbb{F}_q^{\binom{2k-3}{k-1}}$. We will show in the proof of Theorem 1.8 that variants of these vectors lie in the column space of $M_G^{\uparrow 1}$.

Definition 7.2 For $0 \leqslant i \leqslant k-2$, suppose that $X = \{x_1,\ldots,x_i\}$, $Y = \{y_1,\ldots,y_i\}$, and $\Delta = \{y_{i+1},\ldots,y_{k-1}\}$ are disjoint subsets of $\{1,\ldots,2k-3\}$. For $\tau \subseteq \{1,\ldots,i\}$, let $X_\tau = \{x_j : j \in \tau\}$ and let $Y_\tau = \{y_j : j \in \tau\}$. Define the vector $\vec{v}_i(X,Y,\Delta) \in \mathbb{F}_q^{\binom{2k-3}{k-1}}$ with coordinates indexed by $\binom{\{1,\ldots,2k-3\}}{k-1}$ as

$$\vec{v}_i(X,Y,\Delta)_C = \begin{cases} (-1)^{|\tau|} & \text{if } C = X_\tau \cup (Y \setminus Y_\tau) \cup \Delta \text{ for } \tau \subseteq \{1,\ldots,i\} \\ 0 & \text{otherwise.} \end{cases} \tag{7.45}$$

We now show that if $k > p = \mathrm{char}\,(\mathbb{F}_q)$, the vector $\vec{v}_{k-p}(X,Y,\Delta)$ defined in (7.45) lies in the column space of $I_{2k-3}(k-1,k-2)$ over $\mathbb{F}_q$.

Lemma 7.3 *If $k > p = \mathrm{char}\,(\mathbb{F}_q)$, then for any choice of X, Y, and Δ satisfying the constraints in Definition 7.2, the vector $\vec{v}_{k-p}(X,Y,\Delta)$ defined in* (7.45) *lies in the column space of $I_{2k-3}(k-1,k-2)$ over $\mathbb{F}_q$.*

Proof. By Lemma 7.1, it suffices to show that the vector $\vec{v}_{k-p}(X,Y,\Delta)$ lies in the nullspace of the inclusion matrix $I_{2k-3}(k+p-2,k-1)$ for any choice of X, Y, and Δ satisfying the constraints in Definition 7.2. For $H \in \binom{\{1,\ldots,2k-3\}}{k+p-2}$, let $I_{2k-3}(k+p-2,k-1)_H$

be the row of the inclusion matrix $I_{2k-3}(k+p-2,k-1)$ corresponding to H. We want to show that

$$I_{2k-3}(k+p-2,k-1)_H \vec{v}_{k-p}(X,Y,\Delta) = 0. \tag{7.46}$$

Define $\overline{H} = \{1,\ldots,2k-3\} \setminus H$ and define $R(X) = \{1 \leqslant i \leqslant k-p : x_i \in \overline{H}\}$ and $R(Y) = \{1 \leqslant i \leqslant k-p : y_i \in \overline{H}\}$. Moreover, define $\mathcal{F}$ to be the family of $(k-1)$-subsets C of $\{1,\ldots,2k-3\}$ such that $I_{2k-3}(k+p-2,k-1)_{(H,C)} \neq 0$ and $\vec{v}_{k-p}(X,Y,\Delta)_C \neq 0$. If H does not contain Δ or if $R(X)$ and $R(Y)$ have nonempty intersection, then $\mathcal{F} = \emptyset$ and thus (7.46) holds. Otherwise, the elements of $\mathcal{F}$ are of the form $C = \Delta \cup X_\tau \cup (Y \setminus Y_\tau)$ where $\tau = R(Y) \cup U$ and $U \subseteq \{1,\ldots,k-p\} \setminus (R(X) \cup R(Y))$. Let $W = \{1,\ldots,k-p\} \setminus (R(X) \cup R(Y))$ and observe that $W \neq \emptyset$ because $R(X)$ and $R(Y)$ are disjoint and because $|\overline{H}| = k-p-1$. Since the left hand side of (7.46) equals

$$\sum_{\substack{\tau = R(Y) \cup U \\ U \subseteq W}} (-1)^{|\tau|} = (-1)^{|R(Y)|} \sum_{j=0}^{|W|} \binom{|W|}{j} (-1)^j = (-1)^{|R(Y)|}(1-1)^{|W|} = 0, \tag{7.47}$$

the lemma follows. □

For a fixed $U \in \binom{G}{1}$, recall that we noted in Section 1.1 that the submatrix $M_G^{\uparrow 1}(U)$ of $M_G^{\uparrow 1}$ equals $D_U I_{2k-3}(k-1,k-2)$. Since Lemma 7.3 shows that the vector $\vec{v}_{k-p}(X,Y,\Delta)$ lies in the column space of the inclusion matrix $I_{2k-3}(k-1,k-2)$ when $k > p$, we have that for $U \in \binom{G}{1}$, the vector $D_U \vec{v}_{k-p}(X,Y,\Delta)$ lies in the column space of the submatrix $M_G^{\uparrow 1}(U)$. Padding the vector $D_U \vec{v}_{k-p}(X,Y,\Delta)$ with zeroes in the appropriate places thus gives a vector in the column space of $M_G^{\uparrow 1}$. To make this precise, we now define variants of the vectors $\vec{v}_i(X,Y,\Delta)$ defined in Definition 7.2.

Definition 7.4 Let $G \subset \mathbb{F}_q^k$ be an arc of size $2k-2$ and let $U \in \binom{G}{1}$. Let B be the basis of $\mathbb{F}_q^k$ from Definition 1.3. For $0 \leqslant i \leqslant k-2$, let $X = \{x_1,\ldots,x_i\}$, $Y = \{y_1,\ldots,y_i\}$, and $\Delta = \{y_{i+1},\ldots,y_{k-1}\}$ be disjoint subsets of $G \setminus U$. For $\tau \subseteq \{1,\ldots,i\}$, let X_τ and Y_τ be defined as in Definition 7.2. Define the vector $\vec{v}_i(U,X,Y,\Delta) \in \mathbb{F}_q^{\binom{2k-2}{k-1}}$ with coordinates indexed by $\binom{G}{k-1}$ as

$$\vec{v}_i(U,X,Y,\Delta)_C = \begin{cases} (-1)^{|\tau|} \det(U,C)_B & \text{if } C = X_\tau \cup (Y \setminus Y_\tau) \cup \Delta \text{ for } \tau \subseteq \{1,\ldots,i\} \\ 0 & \text{otherwise.} \end{cases} \tag{7.48}$$

Observe that the vector $\vec{v}_{k-p}(U,X,Y,\Delta)$ is the vector $D_U \vec{v}_{k-p}(X,Y,\Delta)$ padded with zeroes in all coordinates $C \in \binom{G}{k-1}$ that have nonempty intersection with U. Consequently, when $k > p$, the vector $\vec{v}_{k-p}(U,X,Y,\Delta)$ lies in the column space of $M_G^{\uparrow 1}$ for any choice of $U \in \binom{G}{1}$ and any choice of X, Y, and Δ satisfying the constraints in Definition 7.4.

In the proof of Theorem 1.8, we show that each of the C-coordinate vectors $e(C)$ are linear combinations of the vectors $\vec{v}_{k-p}(U,X,Y,\Delta)$, and hence lie in the column space of $M_G^{\uparrow 1}$. To specify the linear combination, we need the following lemma.

Lemma 7.5 *Let $G \subset \mathbb{F}_q^k$ be an arc of size $2k-2$ and let B be the basis of $\mathbb{F}_q^k$ fixed in Definition 1.3. Let $C \in \binom{G}{k-1}$ and suppose that $\Delta \subset C$. Let $W \subset G \setminus \Delta$ have size $k - |\Delta|$. For any $u \in G$, we have*

$$\sum_{w \in W} \frac{\det(u, W \setminus w, \Delta)_B}{\det(w, W \setminus w, \Delta)_B} \det(w, C)_B = \det(u, C)_B. \tag{7.49}$$

Proof. Since $W \cup \Delta$ is a basis of $\mathbb{F}_q^k$, we can write $u \in G$ as a unique linear combination of the elements of $W \cup \Delta$. It is easy to see that the coefficient of w in this linear combination is $\det(u, W \setminus w, \Delta)_B / \det(w, W \setminus w, \Delta)_B$. Since $\Delta \subset C$, (7.49) holds. □

The vectors $v_i(U, X, Y, \Delta)$ have three nice properties: For fixed X, Y, and Δ and $U \in \binom{G \setminus (X \cup Y \cup \Delta)}{1}$, the support of the vector $v_i(U, X, Y, \Delta)$ is always the same. Moreover, all the $(k-1)$-subsets C in the support of $v_i(U, X, Y, \Delta)$ contain the same fixed set Δ and have empty intersection with $G \setminus (X \cup Y \cup \Delta)$. Consequently, we can add vectors $v_i(U, X, Y, \Delta)$ for different $U \in \binom{G \setminus (X \cup Y \cup \Delta)}{1}$ using Lemma 7.5 to yield vectors with smaller weight in the column space of $M_G^{\uparrow 1}$. Eventually, we conclude that the C-coordinate vectors $e(C)$, which have weight one, lie in the column space of $M_G^{\uparrow 1}$.

Proof of Theorem 1.8 The matrix $M_G^{\uparrow 1}$ defined in (1.5) has full row rank if and only if its column space contains the C-coordinate vector $e(C)$ for each $C \in \binom{G}{k-1}$. If $k \leqslant p$, then the inclusion matrix $I_{2k-3}(k-1, k-2)$ is invertible by Theorem 1.6. For a fixed $U \in \binom{G}{1}$, recall that we noted in Section 1.1 that the submatrix $M_G^{\uparrow 1}(U)$ of $M_G^{\uparrow 1}$ equals $D_U I_{2k-3}(k-1, k-2)$. Hence, for each $U \in \binom{G}{1}$, the submatrix $M_G^{\uparrow 1}(U)$ is invertible. Thus, the column space of $M_G^{\uparrow 1}$ contains the C-coordinate vector $e(C)$ for each $C \in \binom{G}{k-1}$.

Now suppose that $p < k \leqslant 2p-2 \leqslant q$. Observe that for any $U \in \binom{G}{1}$ and $C \in \binom{G \setminus U}{k-1}$, the C-coordinate vector $e(C)$ is a nonzero scalar multiple of the vector $\vec{v}_0(U, \emptyset, \emptyset, C)$. We show that the vectors $\vec{v}_0(U, \emptyset, \emptyset, C)$ lie in the column space of $M_G^{\uparrow 1}$ by proving that for any $0 \leqslant i \leqslant k-p$, $U \in \binom{G}{1}$, and X, Y, and Δ satisfying the constraints in Definition 7.4, the vector $\vec{v}_i(U, X, Y, \Delta)$ defined in (7.48) lies in the column space of $M_G^{\uparrow 1}$.

The proof is by induction on i. By Lemma 7.3 and the remarks preceding and following Definition 7.4, the statement is true for the base case $i = k-p$. We assume the statement is true for $i \in \{1, \ldots, k-p\}$ and prove the statement for $i-1 \in \{0, \ldots, k-p-1\}$. Let $U \in \binom{G}{1}$ and let $X' = \{x_1, \ldots, x_{i-1}\}$, $Y' = \{y_1, \ldots, y_{i-1}\}$, and $\Delta' = \{y_i, \ldots, y_{k-1}\}$ be disjoint subsets of $G \setminus U$. We will show that the vector $\vec{v}_{i-1}(U, X', Y', \Delta')$ lies in the column space of $M_G^{\uparrow 1}$.

Define $x_i = U$ and let $X = X' \cup \{x_i\}$, $Y = Y' \cup \{y_i\}$, and $\Delta = \Delta' \setminus \{y_i\}$. Write G as the disjoint union $G = X \cup Y \cup \Delta \cup W \cup \Omega$, where $|W| = i+1$ and $|\Omega| = k-2-2i$. Note that $i \leqslant k-p$ and $k \leqslant 2p-2$ imply that $|\Omega| \geqslant 0$. For each $w \in W$, we have that X, Y, and Δ are disjoint subsets of $G \setminus w$ satisfying the constraints of Definition 7.4. Consequently, by the induction hypothesis, for each $w \in W$, the vector $\vec{v}_i(w, X, Y, \Delta)$ lies in the column space of $M_G^{\uparrow 1}$. Moreover, the support of $\vec{v}_i(w, X, Y, \Delta)$, denoted $\mathcal{S}_w$, is the

same for all $w \in W$,

$$\mathcal{S}_w = \left\{ C \in \binom{G}{k-1} : C = \Delta \cup X_\tau \cup (Y \setminus Y_\tau) \text{ for } \tau \subseteq \{1, \ldots, i\} \right\}. \tag{7.50}$$

We now show that the vector $\vec{v}_{i-1}(U, X', Y', \Delta')$ is a linear combination of the vectors $\vec{v}_i(w, X, Y, \Delta)$,

$$\vec{v}_{i-1}(U, X', Y', \Delta') = \sum_{w \in W} \frac{\det(U, W \setminus w, \Delta)_B}{\det(w, W \setminus w, \Delta)_B} \vec{v}_i(w, X, Y, \Delta). \tag{7.51}$$

Observe that the support of $\vec{v}_{i-1}(U, X', Y', \Delta')$, denoted $\mathcal{S}_U$, is a subset of the support $\mathcal{S}_w$ from (7.50),

$$\mathcal{S}_U = \left\{ C \in \binom{G}{k-1} : C = \Delta \cup X_\tau \cup (Y \setminus Y_\tau) \text{ for } \tau \subseteq \{1, \ldots, i-1\} \right\}. \tag{7.52}$$

Consequently, to prove (7.51), we must show that the C-coordinates of the left and right hand sides of (7.51) are equal,

$$\sum_{w \in W} \frac{\det(U, W \setminus w, \Delta)_B}{\det(w, W \setminus w, \Delta)_B} \vec{v}_i(w, X, Y, \Delta)_C = \begin{cases} (-1)^{|\tau|} \det(U, C)_B & \text{if } C \in \mathcal{S}_w \cap \mathcal{S}_U \\ 0 & \text{if } C \in \mathcal{S}_w \setminus \mathcal{S}_U. \end{cases} \tag{7.53}$$

We see that (7.53) follows from Lemma 7.5 because if $C \in \mathcal{S}_w$ then $\Delta \subset C$ and $W \subset G \setminus \Delta$ has size $k - |\Delta|$ so the left hand side of (7.53) equals

$$\sum_{w \in W} \frac{\det(U, W \setminus w, \Delta)_B}{\det(w, W \setminus w, \Delta)_B} (-1)^{|\tau|} \det(w, C)_B = (-1)^{|\tau|} \det(U, C)_B. \tag{7.54}$$

If $C \in \mathcal{S}_w \setminus \mathcal{S}_U$, then $i \in \tau$ which implies that $U \in C$ and hence $\det(U, C)_B = 0$. □

8 Classification

To prove Theorem 1.12, we first state a sufficient condition for an arc $S \subset \mathbb{F}_q^k$ of size $q+1$ to be linearly equivalent to the normal rational curve $\mathcal{R}_k$.

Lemma 8.1 (Roth–Lempel [15]) *Suppose that $S \subset \mathbb{F}_q^k$ is an arc of size $q+1$ and let $B = (e_1, \ldots, e_k) \subset S$ be a basis of $\mathbb{F}_q^k$. For $x \in S \setminus B$, let $x = (x_1, \ldots, x_k)$ be the coordinates of x when written with respect to B. Let $W_{S,B}$ be a matrix whose columns are the vectors $(x_1^{-1}, \ldots, x_k^{-1})^\top$ for $x \in S \setminus B$. If $\operatorname{rank} W_{S,B} = 2$, then S is linearly equivalent to the normal rational curve $\mathcal{R}_k \subset \mathbb{F}_q^k$.*

Suppose that $S \subset \mathbb{F}_q^k$ is an arc of size $q+1$ and that there exists a nonnegative integer n for which the hypothesis of Theorem 1.12 is satisfied. Moreover, let $B = (e_1, \ldots, e_k) \subset S$ be a basis of $\mathbb{F}_q^k$. To prove that the matrix $W_{S,B}$ defined in Lemma 8.1 has $\operatorname{rank} W_{S,B} = 2$,

we will show that any three columns of $W_{S,B}$ are linearly dependent. Given three columns of $W_{S,B}$, we will show they are dependent by constructing a $(k-2)\times k$ matrix Z with $\operatorname{rank} Z = k-2$ so that the three columns of $W_{S,B}$ lie in the nullspace of Z. In other words, we want to find $k-2$ independent vectors in $\mathbb{F}_q^k$ that are orthogonal to each of the three given columns of $W_{S,B}$. Using the notation of Lemma 8.1, observe that for $x \in S \setminus B$ and $1 \leqslant j \leqslant k$, we have $x_j^{-1} = (-1)^{j+1} \det(x, B\setminus\{e_j\})_B^{-1}$. The expression $\det(x, B\setminus\{e_j\})_B$ has appeared before, for example in (5.29), which suggests how to find the required vectors. The following lemma makes this intuition precise.

Lemma 8.2 *Suppose that $0 \leqslant n \leqslant q-2k$ and that for every arc $G \subset \mathbb{F}_q^k$ of size $2k-2+n$, the column space of the matrix $H_G^{\uparrow n}$ defined in Definition 1.11 contains a vector $v \in \mathbb{F}_q^{\binom{2k-2+n}{k-1}}$ such that $v_i = 1$ if $i \in \{1,\dots,k\}$ and $v_i = 0$ otherwise. If $S \subset \mathbb{F}_q^k$ is an arc of size $q+1$ and $B = (e_1,\dots,e_k) \subset S$ is a basis of $\mathbb{F}_q^k$, then there exist nonzero constants $c_1,\dots,c_k \in \mathbb{F}_q$ such that for any $(k-2)$-subset $A \subset S\setminus B$, we have*

$$\sum_{j=1}^{k} (-1)^{(j+1)(k-1)} c_j \prod_{y\in A} y_j^{-1} = 0, \tag{8.55}$$

where $y = (y_1,\dots,y_k)$ is written with respect to the basis B.

Proof. Let $A \subset S\setminus B$ be a subset of size $k-2$ and let $\hat{L} \subset S\setminus(B\cup A)$ be a subset of size $|\hat{L}| = n$. Define an arc G and its ordering by $G = (B, A, \hat{L})$. Reorder the arc S so that G is the first $2k-2+n$ vectors of S.

Since $|S| = q+1$, we have $t = k-2$, where t is defined by (3.16). Observe that $|G| = t+k+n$ and that $|S\setminus G| \geqslant 1$ since $0 \leqslant n \leqslant q-2k$. Since the arc $G \subset S$ satisfies the hypotheses of Corollary 4.1, we have that $\vec{1}P_G^{\uparrow n} = \vec{0}$. By Theorem 4.2, there exist invertible diagonal matrices D_1 and D_2 such that $D_1 P_G^{\uparrow n} D_2 = M_G^{\uparrow n}$. Recalling Definition 1.11, we have

$$\vec{0} = \vec{1}P_G^{\uparrow n} = \vec{1}(J_G^{\uparrow n} D_1)^{-1}(J_G^{\uparrow n} M_G^{\uparrow n}) D_2^{-1} \quad \text{so} \quad \vec{0} = \vec{1}(J_G^{\uparrow n} D_1)^{-1} H_G^{\uparrow n}. \tag{8.56}$$

Recalling that $D_1 = F_1 F_3$ where F_1 from Lemma 5.4 is defined by the matrix E_1 in Lemma 6.1 and F_3 is defined by (5.29), we see that the C-coordinate of $\vec{1}(J_G^{\uparrow n} D_1)^{-1}$ is

$$(\vec{1}(J_G^{\uparrow n} D_1)^{-1})_C = \alpha_C \prod_{y \in G\setminus(C\cup L_C)} \det(y, C)_B^{-1}, \tag{8.57}$$

where L_C is the last n-subset of $\binom{G\setminus C}{n}$ in colex order.

Note that $L_C = \hat{L}$ for all $C \in \binom{B}{k-1}$ since B is the first k elements of G so

$$(\vec{1}(J_G^{\uparrow n} D_1)^{-1})_{B\setminus\{e_j\}} = \alpha_{B\setminus\{e_j\}} (-1)^{(j+1)(k-1)} \prod_{y\in A} y_j^{-1} \tag{8.58}$$

since $y_j^{-1} = (-1)^{j+1} \det(y, B\setminus\{e_j\})_B^{-1}$.

By assumption, the column space of the matrix $H_G^{\uparrow n}$ contains a vector $v \in \mathbb{F}_q^{\binom{2k-2+n}{k-1}}$ such that $v_i = 1$ if $i \in \{1, \ldots, k\}$ and $v_i = 0$ otherwise. Since the rows of the matrix $H_G^{\uparrow n}$ are in colex order, by (8.56) and (8.58),

$$0 = \langle \vec{1}(J_G^{\uparrow n} D_1)^{-1}, v \rangle = \sum_{j=1}^{k} (-1)^{(j+1)(k-1)} \alpha_{B\setminus\{e_j\}} \prod_{y \in A} y_j^{-1}. \tag{8.59}$$

Hence, Lemma 8.2 follows by setting $c_j = \alpha_{B\setminus\{e_j\}}$. □

We now prove Theorem 1.12.

Proof of Theorem 1.12 Let $S \subset \mathbb{F}_q^k$ be an arc of size $q+1$ and let $B = (e_1, \ldots, e_k) \subset S$ be a basis of $\mathbb{F}_q^k$. Since S is an arc, the matrix $W_{S,B}$ defined in Lemma 8.1 has $\operatorname{rank} W_{S,B} \geqslant 2$ because if a column of $W_{S,B}$ is a multiple of another column of $W_{S,B}$ then two vectors in S are linearly dependent. To prove $\operatorname{rank} W_{S,B} \leqslant 2$, we will show that any three columns of $W_{S,B}$ are linearly dependent. Let $w, x, z \in S \setminus B$. We will show that there exists a $(k-2) \times k$ matrix Z with $\operatorname{rank} Z = k-2$ such that the columns of $W_{S,B}$ corresponding to $w, x, z \in S \setminus B$ are in the nullspace of Z. As $\operatorname{nullity} Z = 2$, this proves that the columns of $W_{S,B}$ corresponding to $w, x, z \in S \setminus B$ are linearly dependent.

To construct Z, first choose a $(k-2)$-subset $A \subseteq S \setminus (B \cup \{w, x, z\})$, which is possible since $0 \leqslant n \leqslant q - 2k$. Write $A = \{a_1, \ldots, a_{k-2}\}$ and define $A_i = A \setminus \{a_i\} \cup \{w\}$. By Lemma 8.2 applied to A_i for $1 \leqslant i \leqslant k$ we have

$$\sum_{j=1}^{k} \left((-1)^{(j+1)(k-1)} c_j \prod_{y \in A\setminus\{a_i\}} y_j^{-1} \right) w_j^{-1} = 0. \tag{8.60}$$

Defining Z to be the $(k-2) \times k$ matrix with (i,j)-entry

$$Z(i,j) = (-1)^{(j+1)(k-1)} c_j \prod_{y \in A\setminus\{a_i\}} y_j^{-1}, \tag{8.61}$$

we see that (8.60) implies that the column of $W_{S,B}$ corresponding to w lies in the nullspace of Z. Repeating the argument above, we similarly have that the columns of $W_{S,B}$ corresponding to x and z lie in the nullspace of Z as well.

To complete the proof, we must show that $\operatorname{rank} Z = k-2$. Multiplying the j^{th} column of Z by $(-1)^{(j+1)(k-1)} c_j^{-1} \prod_{y \in A} y_j$ gives a $(k-2) \times k$ matrix $\overline{Z}$ whose rows are $a_1, \ldots, a_{k-2}$. Since $a_1, \ldots, a_{k-2}$ are linearly independent vectors, $k - 2 = \operatorname{rank} \overline{Z} = \operatorname{rank} Z$. □

Finally we prove Theorem 1.13.

Proof of Theorem 1.13 We first show that if $k \leqslant p = \operatorname{char}(\mathbb{F}_q)$ and $G \subset \mathbb{F}_q^k$ is an arc of size $2k-2$, then the column space of the matrix $H_G^{\uparrow 0}$ contains a vector $v \in \mathbb{F}_q^{\binom{2k-2}{k-1}}$ such that $v_i = 1$ if $i \in \{1, \ldots, k\}$ and $v_i = 0$ otherwise. First note that the matrix $H_G^{\uparrow 0}$ equals the inclusion matrix $I_{2k-2}(k-1, k-2)$.

Let $B = \{1, \ldots, k\}$. For each subset $A \in \binom{\{1,\ldots,2k-2\}}{k-2}$, define $l_A = |A \cap B|$ and $\beta_A = (-1)^{l_A} l_A!(k-2-l_A)!$. For each subset $C \in \binom{\{1,\ldots,2k-2\}}{k-1}$, define $r_C = |C \cap B|$. Define $\vec{\beta} \in \mathbb{F}_q^{\binom{2k-2}{k-2}}$ to be a vector with coordinates indexed by $\binom{\{1,\ldots,2k-2\}}{k-2}$ and entries $\vec{\beta}_A = \beta_A$. Let $\vec{w} = I_{2k-2}(k-1, k-2)\vec{\beta}$.

Consider the C-coordinate of $\vec{w}$. If $C \not\subset B$, then there are $k-1-r_C$ subsets $A \in \binom{C}{k-2}$ such that $l_A = r_C$. The remaining r_C subsets $A \in \binom{C}{k-2}$ satisfy $l_A = r_C - 1$. Consequently,

$$\vec{w}_C = \sum_{A \in \binom{C}{k-2}} \beta_A = (k-1-r_C)\cdot(-1)^{r_C} r_C!(k-2-r_C)! + r_C \cdot (-1)^{r_C-1}(r_C-1)!(k-1-r_C)! = 0. \tag{8.62}$$

On the other hand, if $C \subset B$, then all $A \in \binom{C}{k-2}$ satisfy $l_A = k-2$ so

$$\vec{w}_C = \sum_{A \in \binom{C}{k-2}} \beta_A = (k-1) \cdot (-1)^{k-2}(k-2)!0! = (-1)^{k-2}(k-1)!, \tag{8.63}$$

which is nonzero since $k \leqslant p$. The first part of Theorem 1.13 is proved by setting $v = ((-1)^{k-2}/(k-1)!)\vec{w}$ since the rows of $H_G^{\uparrow 0}$ are in colex order.

By Theorem 1.12 if $k \leqslant \min\{p, q/2\}$, the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$. By the well-known principle of duality, this implies that if $k \leqslant p$ and $k \neq (q+1)/2$, then the normal rational curve $\mathcal{R}_k$ is the unique arc in $\mathbb{F}_q^k$ of size $q+1$. □

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