On the Potts antiferromagnet on random graphs

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Abstract

Extending a prior result of Contucci et al. [Comm. Math. Phys. 2013], we determine the free energy of the Potts antiferromagnet on the Erdős-Rényi random graph at all temperatures for average degrees $d \leqslant (2k-1)\ln k - 2 - k^{-1/2}$. In particular, we show that for this regime of d there does not occur a phase transition.

Keywords: Potts model, random graphs

1 Introduction

1.1 Background and motivation

The Gibbs measure of the *k-spin Potts antiferromagnet at inverse temperature* $\beta \geqslant 0$ on a graph $G = (V, E)$ is the probability measure on the set of all maps $\sigma : V \to [k] = \{1, \ldots, k\}$ defined by

$$\mu_{G,\beta}(\sigma) = \frac{\exp(-\beta \mathcal{H}_G(\sigma))}{Z_\beta(G)}, \quad \text{where} \quad \mathcal{H}_G(\sigma) = |\{e \in E : |\sigma(e)| = 1\}| \qquad (1)$$

$$\text{and} \quad Z_\beta(G) = \sum_{\tau : V \to [k]} \exp(-\beta \mathcal{H}_G(\tau)).$$

Thus, if we think of $[k]$ as a set of colors, then the function $\mathcal{H}_G$, the *Hamiltonian* of G, maps a color assignment σ to the number of monochromatic edges. Moreover, $\beta \in [0, \infty) \mapsto Z_\beta(G)$ is known as the *partition function*. The Potts antiferromagnet is one of

the best-known models of statistical physics. Accordingly, it has been studied extensively on a wide class of graphs, particularly lattices [10, 27, 24]. The aim of the present paper is to study the model on the Erdős-Rényi random graph $\boldsymbol{G} = \boldsymbol{G}(n, m)$. Throughout the paper, we let $m = \lceil dn/2 \rceil$ for a number $d > 0$ that remains fixed as $n \to \infty$. We also assume that the number $k \geqslant 3$ of colors remains fixed as $n \to \infty$.

The Potts model on the random graph $\boldsymbol{G}$ is of interest partly due to the connection to the k-colorability problem. Indeed, the larger β, the more severe the "penalty factor" of $\exp(-\beta)$ that each monochromatic edge induces in (1). Thus, if the underlying graph is k-colorable, then for large β the Gibbs measure will put most of its weight on color assignments that leave few edges monochromatic. Ultimately, one could think of the uniform distribution on k-colorings as the "$\beta = \infty$"-case of the Gibbs measure (1). Now, consider the problem of finding a k-coloring of the random graph by a local search algorithm such as Simulated Annealing. Then most likely the algorithm will start from a color assignment that has quite a few monochromatic edges. As the algorithm proceeds, it will attempt to gradually reduce the number of monochromatic edges by running the Metropolis process for the Gibbs measure (1) with a value of β that increases over time. Specifically, β has to be large enough to make progress but small enough so that the algorithm does not get trapped in a local minimum of the Hamiltonian. Hence, to figure out whether such a local search algorithm will find a proper k-coloring in polynomial time, it is instrumental to study the "shape" of the Hamiltonian.

To this end, it is key to get a handle on the *free energy*, defined as $\mathbb{E}[\ln Z_\beta(\boldsymbol{G})]$. We take the logarithm because $Z_\beta(\boldsymbol{G})$ scales exponentially in the number n of vertices. As a standard application of Azuma's inequality shows that $\ln Z_\beta(\boldsymbol{G})$ is concentrated about its expectation (see Fact 2 below), $\frac{1}{n}|\ln Z_\beta(\boldsymbol{G}) - \mathbb{E}[\ln Z_\beta(\boldsymbol{G})]|$ converges to 0 in probability. Furthermore, if $\mathbb{E}[\ln Z_\beta(\boldsymbol{G})] \sim \ln \mathbb{E}[Z_\beta(\boldsymbol{G})]$ for certain d, β, then the Hamiltonian can be studied via an easily accessible probability distribution called the *planted model*. This trick has been applied to the "proper" graph coloring problem as well as to other random constraint satisfaction problems successfully [2, 28].

1.2 The main result

Because our motivation largely comes from the random graph coloring problem, we are going to confine ourselves to values of d where the random graph $\boldsymbol{G}$ is k-colorable w.h.p. Although the precise k-colorability threshold $d_{k-\mathrm{col}}$ is not currently known, we have [12, 14]

$$(2k - 1)\ln k - 2\ln 2 + o_k(1) \leqslant d_{k-\mathrm{col}} \leqslant (2k - 1)\ln k - 1 + o_k(1), \tag{2}$$

where $o_k(1)$ hides a term that tends to 0 in the limit of large k. The following theorem determines $\frac{1}{n}\mathbb{E}[\ln Z_\beta(\boldsymbol{G})]$ almost up to the lower bound from (2).

Theorem 1. *There is $k_0 > 0$ such that for all $k \geqslant k_0$, $d \leqslant d_\star = (2k-1)\ln k - 2 - k^{-1/2}$, $\beta > 0$ we have*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\ln Z_\beta(\boldsymbol{G})] = \lim_{n\to\infty} \frac{1}{n}\ln \mathbb{E}[Z_\beta(\boldsymbol{G})] = \ln k + \frac{d}{2}\ln(1 - (1 - \exp(-\beta))/k). \tag{3}$$

Clearly, the function on the r.h.s. of (3) is analytic in $\beta \in (0, \infty)$. Thus, in the language of mathematical physics Theorem 1 implies that the Potts antiferromagnet on the random graph does not exhibit a phase transition for any average degree $d < d_\star$.

1.3 Related work

The problem of determining the k-colorability threshold of the random graph was raised in the seminal paper by Erdős and Rényi and is thus the longest-standing open problem in the theory of random graphs [20]. Achlioptas and Friedgut [1] proved the existence of a non-uniform sharp threshold. Moreover, a simple greedy algorithm finds a k-coloring for degrees up to about $k \ln k$, approximately half the k-colorability threshold [3]. Further, Achlioptas and Naor [4] used the second moment method to establish a lower bound of $d_{k-\mathrm{col}} \geqslant 2(k-1)\ln k + o_k(1)$, which matches the first-moment upper bound $d_{k-\mathrm{col}} \leqslant (2k-1)\ln k + o_k(1)$ up to about an additive $\ln k$. Coja-Oghlan and Vilenchik [14] improved the lower bound to $d_{k-\mathrm{col}} \geqslant (2k-1)\ln k - 2\ln 2 + o_k(1)$ via a second moment argument that incorporates insights from non-rigorous physics work [25]. On the other hand, Coja-Oghlan [12] proved $d_{k-\mathrm{col}} \leqslant (2k-1)\ln k - 1 + o_k(1)$. The results from [4, 14] were subsequently generalized to various other models, including random regular graphs and random hypergraphs [5, 13, 17, 22].

The Potts antiferromagnet on the random graph was studied before by Contucci, Dommers, Giardina and Starr [15], who generalized the second moment argument from [4] to the Potts model. In particular, [15] shows that (3) holds for all $\beta \geqslant 0$ if $d \leqslant (2k-2)\ln k - 2$. An analogous result was recently obtained (among other things) by Banks and Moore [6] for a variant of the stochastic block model that resembles the Potts antiferromagnet. Their proof is based on [4] as well. In the present paper we improve the corresponding results of [6, 15] by extending the physics-enhanced second moment argument from [14] to the Potts antiferromagnet.

Physics considerations suggest that for average degrees $d > (2k-1)\ln k - 2\ln 2 + o_k(1)$ a phase transition does occur, i.e., the function $\beta \in (0, \infty) \mapsto \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\ln Z_\beta(\boldsymbol{G})]$ is non-analytic [23, 26, 25]. The existence and location of the condensation phase transition has been established asymptotically in the hypergraph 2-coloring and the hardcore model and precisely in the regular k-SAT model and the k-colorability problem [7, 8, 9, 11]. However, the Potts antiferromagnet is conceptually more challenging than hardcore, k-SAT or hypergraph 2-coloring because the "variables" (viz. vertices) can take more than two values (colors). Potts is also more difficult than k-coloring because of the presence of the inverse temperature parameter β. In fact, the present work is partly motivated by studying condensation in the Potts antiferromagnet, and we hope that Theorem 1 and its proof may pave the way to pinpointing the phase transition precisely, see Section 2.5 below. Additionally, as mentioned above, Theorem 1 implies that for $d \leqslant (2k-1)\ln k - 2 - k^{-1/2}$ the Hamiltonian can be studied by way of the planted model. Finally, the *ferromagnetic* Potts model (where the Gibbs measure favors monochromatic edges) is far better understood than the antiferromagnetic version [16].

1.4 Preliminaries

Throughout the paper we assume that $k \geqslant k_0$ for a large enough constant $k_0 > 0$. Moreover, let

$$c_\beta = 1 - \exp(-\beta).$$

Unless specified otherwise, the standard O-notation refers to the limit $n \to \infty$. We always assume tacitly that n is sufficiently large. Additionally, we use asymptotic notation in the limit of large k with a subscript k.

Fact 2. *For any $\delta > 0$ there is $\varepsilon = \varepsilon(\delta, \beta, d) > 0$ such that $\limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{P}[| \ln Z_\beta(\boldsymbol{G}) - \mathbb{E}[\ln Z_\beta(\boldsymbol{G})]| > \delta n] < -\varepsilon.$*

Proof. If G, G' are multi-graphs such that G' can be obtained from G by adding or deleting a single edge, then $|\ln Z_\beta(G) - \ln Z_\beta(G')| \leqslant 2\beta$. Hence, the assertion follows from Azuma's inequality. $\square$

If s is an integer, we write $[s]$ for the set $\{1, \ldots, s\}$. Further, if v is a vertex of a graph G, then $\partial v = \partial_G(v)$ is the set of neighbors of v in G. If ρ is a matrix, then by ρ_i we denote the ith row of ρ and by ρ_{ij} the jth entry of ρ_i. Further, the Frobenius norm of a $k \times k$-matrix ρ is

$$\|\rho\|_2 = \left[\sum_{i,j \in [k]} \rho_{ij}^2 \right]^{1/2}.$$

For a probability distribution $p : \Omega \to [0, 1]$ on a finite set Ω we denote by

$$H(p) = - \sum_{x \in \Omega} p(x) \ln p(x)$$

the entropy of p (with the convention that $0 \ln 0 = 0$). Additionally, if ρ is a $k \times k$-matrix with non-negative entries, then we let

$$H(\rho) = - \sum_{i,j \in [k]} \rho_{ij} \ln \rho_{ij}.$$

Further, $h : [0, 1] \to \mathbf{R}$ denotes the function

$$h(z) = -z \ln z - (1 - z) \ln(1 - z).$$

We will use the following standard fact about the entropy.

Fact 3. *Let $p \in [0, 1]^k$ be such that $\sum_{i=1}^k p_i = 1$. Let $I \subset [k]$ and suppose that $q = \sum_{i \in I} p_i \in (0, 1)$. Then*

$$H(p) \leqslant h(q) + q \ln |I| + (1 - q) \ln(k - |I|).$$

Lemma 4 (Chernoff bound, e.g. [21]). *Let X be a binomial random variable with mean $\mu > 0$. Then for any $t > 1$, we have $\mathbb{P}[X > t\mu] \leqslant \exp[-t\mu \ln(t/e)]$.*

2 Outline

We prove Theorem 1 by generalizing the second moment argument for k-colorings from [14] to the partition function of the Potts antiferromagnet. In this section we describe the proof strategy. Most of the technical details are left to the subsequent sections.

2.1 The first moment

As a first step we calculate the first moment $\mathbb{E}[Z_\beta(\boldsymbol{G})]$. This is pretty straightforward; in fact, it has been done before [15]. Nonetheless, we go over the calculations to introduce a few concepts that will prove important in the second moment argument as well.

Proposition 5 ([15]). *For all $\beta, d > 0$ we have $\mathbb{E}[Z_\beta(\boldsymbol{G})] = \Theta\left(k^n \left(1 - c_\beta/k\right)^m\right)$.*

To lower-bound $Z_\beta(\boldsymbol{G})$ we follow Achlioptas and Naor [4] and work with "balanced" color assignments whose color classes are all about the same size. Specifically, call $\sigma : [n] \to [k]$ *balanced* if $\left||\sigma^{-1}(i)| - \frac{n}{k}\right| \leqslant \sqrt{n}$ for all $i = 1, \ldots, k$. Of course, by Stirling's formula the set $\mathcal{B} = \mathcal{B}(n, k)$ of all balanced $\sigma : [n] \to [k]$ has size $|\mathcal{B}| = \Theta(k^n)$. Let

$$Z_{\beta,\mathrm{bal}}(\boldsymbol{G}) = \sum_{\sigma \in \mathcal{B}} \exp\left(-\beta \mathcal{H}_{\boldsymbol{G}}(\sigma)\right)$$

be the partition function restricted to balanced maps. Moreover, let

$$\mathcal{H}_{K_n}(\sigma) = \sum_{i=1}^{k} \binom{|\sigma^{-1}(i)|}{2}.$$

be the number of monochromatic edges of the complete graph. Then uniformly for all balanced σ,

$$\mathcal{H}_{K_n}(\sigma) = k \binom{\frac{n}{k} + O(\sqrt{n})}{2} = \binom{n}{2} \frac{1}{k} + O(n). \tag{4}$$

Hence, by Stirling's formula

$$\mathbb{E}\left[\exp\left(-\beta \mathcal{H}_{\boldsymbol{G}}(\sigma)\right)\right] = \sum_{m_1=0}^{m} \exp(-\beta m_1) \binom{\mathcal{H}_{K_n}(\sigma)}{m_1} \binom{\binom{n}{2} - \mathcal{H}_{K_n}(\sigma)}{m - m_1} \binom{\binom{n}{2}}{m}^{-1}$$

$$= \Theta(1) \sum_{m_1=0}^{m} \binom{m}{m_1} \left(\frac{\mathcal{H}_{K_n}(\sigma)}{\exp(\beta)\binom{n}{2}}\right)^{m_1} \left(1 - \frac{\mathcal{H}_{K_n}(\sigma)}{\binom{n}{2}}\right)^{m - m_1}. \tag{5}$$

Combining (4) and (5), we find

$$\mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})] = \sum_{\sigma \in \mathcal{B}} \mathbb{E}\left[\exp\left(-\beta \mathcal{H}_{\boldsymbol{G}}(\sigma)\right)\right] = \Theta\left(k^n \left(1 - c_\beta/k\right)^{\frac{nd}{2}}\right). \tag{6}$$

On the other hand, for all σ we have $\mathcal{H}_{K_n}(\sigma) \geqslant \frac{1}{k}\binom{n}{2} - n$ by convexity. Therefore, (5) yields

$$\mathbb{E}[Z_\beta(\boldsymbol{G})] = \sum_\sigma \mathbb{E}\left[\exp\left(-\beta\mathcal{H}_{\boldsymbol{G}}(\sigma)\right)\right] \leqslant O\left(k^n\left(1 - c_\beta/k\right)^{\frac{nd}{2}}\right). \tag{7}$$

Combining (6) and (7), we obtain Proposition 5. Moreover, comparing (6) and (7), we see that $\mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})]$ and $\mathbb{E}[Z_\beta(\boldsymbol{G})]$ are of the same order of magnitude. Since it is technically more convenient to work with $Z_{\beta,\mathrm{bal}}(\boldsymbol{G})$, we are going to perform the second moment argument for that random variable.

2.2 The second moment

Following [4], we define the *overlap matrix* $\rho(\sigma,\tau) = (\rho_{ij}(\sigma,\tau))_{i,j\in[k]}$ of $\sigma,\tau : [n] \to [k]$ by letting

$$\rho_{ij}(\sigma,\tau) = \frac{k}{n}|\sigma^{-1}(i) \cap \tau^{-1}(j)|. \tag{8}$$

Thus, $k^{-1}\rho_{ij}(\sigma,\tau)$ is the fraction of vertices with color i under σ and color j under τ. Let $\mathcal{R} = \mathcal{R}(n,k) = \{\rho(\sigma,\tau) : \sigma,\tau \in \mathcal{B}\}$ be the set of all possible overlap matrices and set

$$Z_{\rho,\mathrm{bal}}(\boldsymbol{G}) = \sum_{\substack{(\sigma,\tau)\in\mathcal{B}^2 \\ \rho(\sigma,\tau)=\rho}} \exp\left(-\beta\left(\mathcal{H}_{\boldsymbol{G}}(\sigma) + \mathcal{H}_{\boldsymbol{G}}(\tau)\right)\right).$$

Then

$$\mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})^2] = \sum_{(\sigma,\tau)\in\mathcal{B}^2} \mathbb{E}\left[\exp\left(-\beta\left(\mathcal{H}_{\boldsymbol{G}}(\sigma) + \mathcal{H}_{\boldsymbol{G}}(\tau)\right)\right)\right] = \sum_{\rho\in\mathcal{R}} \mathbb{E}[Z_{\rho,\mathrm{bal}}(\boldsymbol{G})]. \tag{9}$$

Further, define

$$f_{d,\beta}(\rho) = H(k^{-1}\rho) + \frac{d}{2}\ln\left[1 - \frac{2}{k}c_\beta + \frac{\|\rho\|_2^2}{k^2}c_\beta^2\right]. \tag{10}$$

Then an elementary argument similar to the proof of Proposition 5 yields

Proposition 6 ([15]). *Uniformly for all $\rho \in \mathcal{R}$ we have $\mathbb{E}[Z_{\rho,\mathrm{bal}}(\boldsymbol{G})] = \exp(nf_{d,\beta}(\rho) + o(n))$.*

The function $f_{d,\beta}$ is a sum of an entropy term $H(k^{-1}\rho)$ and an "energy term"

$$E(\rho) = E_{d,\beta}(\rho) = \frac{d}{2}\ln\left[1 - \frac{2}{k}c_\beta + \frac{\|\rho\|_2^2}{k^2}c_\beta^2\right].$$

For future reference we note that

$$\frac{\partial}{\partial\rho_{ij}}H(k^{-1}\rho) = \frac{1}{k}\left(-1 - \ln(\rho_{ij})\right), \tag{11}$$

$$\frac{\partial}{\partial \rho_{ij}} E(\rho) = \frac{d}{k^2} \frac{c_\beta^2 \rho_{ij}}{1 - \frac{2}{k} c_\beta + \|\rho\|_2^2 c_\beta / k^2}. \tag{12}$$

The number $|\mathcal{R}|$ of summands on the right hand side of (9) is easily bounded by n^{k^2}. Therefore,

$$\frac{1}{n} \ln \mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})^2] = \frac{1}{n} \ln \sum_{\rho \in \mathcal{R}} \mathbb{E}[Z_{\rho,\mathrm{bal}}(\boldsymbol{G})] \sim \max_{\rho \in \mathcal{R}} \frac{1}{n} \ln \mathbb{E}[Z_{\rho,\mathrm{bal}}(\boldsymbol{G})] \sim \max_{\rho \in \mathcal{R}} f_{d,\beta}(\rho). \tag{13}$$

Denote by $\mathcal{S}$ the set of all singly-stochastic matrices and by $\mathcal{D}$ the set of all doubly-stochastic $k \times k$ matrices, respectively. Then $\bigcup_{n \geqslant 1} \mathcal{R}(n,k) \cap \mathcal{D}$ is a dense subset of $\mathcal{D}$. Together with (13) the continuity of f therefore implies

$$\frac{1}{n} \ln \mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})^2] \sim \max_{\rho \in \mathcal{D}} f_{d,\beta}(\rho). \tag{14}$$

Setting $\bar{\rho} = k^{-1}\mathbf{1}$ to be the barycenter of $\mathcal{D}$, we obtain from Proposition 6 that

$$f_{d,\beta}(\bar{\rho}) \sim \frac{2}{n} \ln \mathbb{E}\left[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})\right]. \tag{15}$$

Hence, just as in the case of proper k-colorings [4, 15], a *necessary* condition for the success of the second moment method is that the function $f_{d,\beta}$ attains its maximum on $\mathcal{D}$ at the point $\bar{\rho}$.

2.3 Small average degree or high temperature

Contucci, Dommers, Giardina and Starr [15] proved that the maximum in (14) is indeed attained at $\bar{\rho}$ if the average degree is a fair bit below the k-colorability threshold.

Theorem 7 ([15]). *Assume that $d < 2(k-1)\ln(k-1)$. Then (3) holds for all $\beta > 0$.*

Comparing this result with (2), we see that Theorem 7 applies to degrees about an additive $\ln k$ below the k-colorability threshold. The proof of Theorem 7 builds upon ideas of Achlioptas and Naor [4]. More precisely, solving the maximization problem from (14) directly emerges to be surprisingly difficult. Hence, Achlioptas and Naor suggested to enlarge the domain to the set of *singly* stochastic matrices. Clearly, the maximum over the larger space is an upper bound on the maximum over the set of doubly-stochastic matrices. Further, because the set of singly-stochastic matrices is a product of simplices, the relaxed optimization problem can be tackled with a fair bit of technical work. Crucially, for $d < 2(k-1)\ln(k-1)$ the maximum of the relaxed problem is attained at $\bar{\rho}$. However, for only slightly larger values of d the maximum is attained at a different point, and thus the relaxed second moment argument fails.

Apart from the case of small d, the second case that is relatively straightforward is that of small β (the "high temperature" case in physics jargon). More precisely, in Section 3 we will prove the following.

Proposition 8. *If $d \in [2(k-1)\ln(k-1), (2k-1)\ln k - 2]$ and $\beta \leqslant \ln k$, then (3) holds.*

For $d \in [2(k-1)\ln(k-1), (2k-1)\ln k - 2]$ Proposition 8 improves upon the result from [15], which yields (3) merely for $\beta \leqslant \beta_0$ for an absolute constant β_0 (independent of k). The proof of Proposition 8 is by way of relaxing (13) to singly-stochastic matrices as well and builds upon arguments developed in [14] for k-colorability.

2.4 Large degree and low temperature

The most challenging constellation is that of d beyond $2(k-1)\ln(k-1)$ and β large. In this regime we do not know how to solve the maximization problem (13). In particular, the trick of relaxing the problem to the set of all singly-stochastic matrices does not work. Instead, following [14] we are going *add* further constraints to the problem. That is, we are going to apply the second moment method to a modified random variable that is constructed so as to ensure that certain parts of the domain $\mathcal{D}$ cannot contribute to (13) significantly.

The construction is guided by the physics prediction [23] that for large d and β the Gibbs measure $\mu_{\boldsymbol{G}}$ "decomposes" into an exponential number of well-separated clusters. Of course, it would be non-trivial to turn this notion into a precise mathematical statement because the support of $\mu_{\boldsymbol{G}}$ is the entire cube $[k]^n$. However, the probability mass is expected to be distributed very unevenly, with large swathes of the cube carrying very little mass.

Fortunately, we do not need to define clusters etc. precisely. Instead, adapting the construction from [14], we just define a new random variable $Z_{\beta,\mathrm{sep}}(\boldsymbol{G})$ that comes with a "hard-wired" notion of well-separated clusters. To be precise, for a graph G denote by $\Sigma_{G,\beta}$ the set of all $\tau \in \mathcal{B}$ that enjoy the following property.

SEP1 For every $i \in [k]$ the set $\tau^{-1}(i)$ spans at most $2n \exp(-\beta) k^{-1} \ln k$ edges.

Further, let $\kappa = \ln^{20} k / k$. We call $\sigma \in \mathcal{B}$ *separable* if $\sigma \in \Sigma_{G,\beta}$ and if

SEP2 for every $\tau \in \Sigma_{G,\beta}$ and all $i, j \in [k]$ such that $\rho_{ij}(\sigma, \tau) \geqslant 0.51$ we have $\rho_{ij}(\sigma, \tau) \geqslant 1 - \kappa$.

Let $\mathcal{B}_{sep} = \mathcal{B}_{sep}(G, \beta) \subset \mathcal{B}$ denote the set of all separable maps and define

$$Z_{\beta,\mathrm{sep}}(\boldsymbol{G}) = \sum_{\sigma \in \mathcal{B}_{sep}(\boldsymbol{G},\beta)} \exp(-\beta \mathcal{H}_{\boldsymbol{G}}(\sigma)).$$

To elaborate, condition **SEP1** provides that the subgraphs induced on the individual color classes are quite sparse. Indeed, recalling that each monochromatic edge incurs a "penalty factor" of $\exp(-\beta)$, we expect that in a typical sample from the Gibbs measure the total number of monochromatic edges is about $nd \exp(-\beta)/(2k)$. Moreover, suppose that $\sigma \in \Sigma_{G,\beta}$ satisfies **SEP2** and $\tau \in \Sigma_{G,\beta}$ is another color assignment. Let $i, j \in [k]$. Then **SEP2** provides that there are only two possible scenarios.

(i) If $\rho_{ij}(\sigma, \tau) < 0.51$, then the color classes $\sigma^{-1}(i)$, $\tau^{-1}(j)$ are "quite distinct" and we may think of σ, τ as belonging to different "clusters".

(ii) If $\rho_{ij}(\sigma, \tau) \geqslant 0.51$, then in fact $\rho_{ij}(\sigma, \tau) \geqslant 1 - \kappa$. Thus, the color classes $\sigma^{-1}(i)$, $\tau^{-1}(j)$ are nearly identical. Hence, if there is a permutation $\pi : [k] \to [k]$ such that $\rho_{i\pi(i)}(\sigma, \tau) \geqslant 0.51$ for all $i \in [k]$, then we may think of σ, τ as belonging to the same "cluster".

The upshot is that separability rules out the existence of any "middle ground", i.e., we do not have to consider overlaps ρ with entries $\rho_{ij} \in (0, 51, 1 - \kappa)$.

The following proposition, which we prove in Section 4, shows that imposing separability has no discernible effect on the first moment.

Proposition 9. *Assume that $d \in [2(k-1)\ln(k-1), (2k-1)\ln k - 2]$ and $\beta \geqslant \ln k$. Then*

$$\mathbb{E}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G})] \sim \mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})].$$

The point of working with separable color assignments is that the maximization problem that arises in the second moment computation of $Z_{\beta,\mathrm{sep}}(\boldsymbol{G})$ comes with further constraints that are not present in (13). Specifically, we only need to optimize over $\rho \in \mathcal{D}$ such that $\rho_{ij} \notin (0.51, 1 - \kappa)$ for all $i, j \in [k]$. In Section 5 we will use these constraints to derive the following.

Proposition 10. *Let $d \in [2(k-1)\ln(k-1), d_\star]$ and $\beta \geqslant \ln k$. Then $\frac{1}{n} \ln \mathbb{E}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G})^2] \sim \frac{2}{n} \ln \mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})]$.*

Corollary 11. *If $d \in [2(k-1)\ln(k-1), d_\star]$ and $\beta \geqslant \ln k$, then (3) holds.*

Proof. On the one hand, Jensen's inequality gives

$$\mathbb{E}[\ln Z_\beta(\boldsymbol{G})] \leqslant \ln \mathbb{E}[Z_\beta(\boldsymbol{G})]. \tag{16}$$

On the other hand, by Propositions 9 and 10 and the Paley-Zigmund inequality,

$$\mathbb{P}[Z_\beta(\boldsymbol{G}) \geqslant \mathbb{E}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G})]/2] \geqslant \mathbb{P}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G}) \geqslant \mathbb{E}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G})]/2]$$
$$\geqslant \frac{\mathbb{E}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G})]^2}{4\mathbb{E}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G})^2]} = \exp(o(n)). \tag{17}$$

Combining (17) with Proposition 5, (6) and Proposition 9, we obtain

$$\mathbb{P}[\ln Z_\beta(\boldsymbol{G}) \geqslant \ln \mathbb{E}[Z_\beta(\boldsymbol{G})] - \ln \ln n] \geqslant \exp(o(n)). \tag{18}$$

Further, (18) and Fact 2 yield $n^{-1}\mathbb{E}[\ln Z_\beta(\boldsymbol{G})] \geqslant n^{-1} \ln \mathbb{E}[Z_\beta(\boldsymbol{G})] + o(1)$. Finally, combining this lower bound with the upper bound (16) completes the proof. $\square$

Finally, Theorem 1 follows from Theorem 7, Proposition 8 and Corollary 11.

2.5 Outlook: the condensation phase transition

According to non-rigorous physics methods [23, 26] for d only slightly above the bound from Theorem 1 the formula (3) does not hold for all $\beta > 0$ anymore. While the exact formula is quite complicated (e.g., it involves the solution to a distributional fixed point problem), the critical degree satisfies $d_{k,\text{cond}} = (2k-1)\ln k - 2\ln 2 + o_k(1)$. Thus, for $d > d_{k,\text{cond}}$ there occurs a phase transition at a certain critical inverse temperature $\beta_{k,\text{cond}}(d)$. The *existence* of a critical $\beta_{k,\text{cond}}(d)$ follows from prior results on the random graph coloring problem [8]. However, the value of $\beta_{k,\text{cond}}(d)$ is not (rigorously) known.

The physics intuition of how this phase transition comes about is as follows. For $\beta < \beta_{k,\text{cond}}(d)$ the Gibbs measure decomposes into an exponential number of clusters that each have probability mass $\exp(-\Omega(n))$. Hence, if we sample $\boldsymbol{\sigma}, \boldsymbol{\tau}$ independently from the Gibbs measure, then most likely they belong to different clusters, in which case their overlap should be very close to $\bar{\rho}$. By contrast, for $\beta > \beta_{k,\text{cond}}(d)$ a bounded number of clusters dominate the Gibbs measure, i.e., there are individual clusters whose probability mass is $\Omega(1)$. In effect, for $\beta > \beta_{k,\text{cond}}(d)$ the overlap of two randomly chosen color assignments is *not* concentrated on the single value $\bar{\rho}$ anymore, because there is a non-vanishing probability that both belong to the same cluster. In effect, the second moment method fails. In fact, we expect that $\mathbb{E}[\ln Z_\beta(\boldsymbol{G})] < \ln \mathbb{E}[Z_\beta(\boldsymbol{G})] - \Omega(n)$ for all $\beta > \beta_{k,\text{cond}}(d)$.

But even the second moment argument for separable color assignments does not quite reach the expected critical degree $d_{k,\text{cond}}$. Indeed, for $d > (2k-1)\ln k - 2 + o_k(1)$ the maximum over the set of separable overlaps is attained at $\rho_{ij} = \alpha \mathbf{1}\{i = j\} + \frac{1-\alpha}{k-1}\mathbf{1}\{i \neq j\}$ with $\alpha = 1 - 1/k + o_k(1/k)$. In terms of the physics intuition, this overlap matrix corresponds to pairs of color assignments that belong to the same cluster. In other words, the second moment method fails because the *expected* cluster size blows up. A similar problem occurs in the k-colorability problem [14]. There the issue was resolved by explicitly controlling the *median* cluster size, which is by an exponential factor smaller than the expected cluster size [8]. We expect that a similar remedy applies to the Potts model, although the fact that monochromatic edges are allowed entails that the proof method from [8] does not apply. In any case, Theorem 1 reduces the task of determining the phase transition to the problem of controlling the median cluster size.

Furthermore, also in the case of degrees above $d_{k-\text{col}}$ at least the existence of a phase transition has been established rigorously [15]. It would be most interesting to see if the present methods can be extended to $d > d_{k-\text{col}}$ in order to obtain a more precise estimate of $\beta_{k,\text{cond}}(d)$.

3 Singly stochastic analysis

We prove Proposition 8 by way of the following proposition regarding the maximum of $f_{d,\beta}$ over the set of singly-stochastic matrices.

Proposition 12. *If $d \in [2(k-1)\ln(k-1), (2k-1)\ln k - 2]$ and $\beta \leqslant \ln k$, then $f_{d,\beta}(\bar{\rho}) > f_{d,\beta}(\rho)$ for all $\rho \in \mathcal{S} \setminus \{\bar{\rho}\}$.*

To prove Proposition 12 we will closely follow the proof strategy developed for the graph coloring problem in [14, Section 4]. Basically, that argument dealt with optimizing the function $f_{d,\infty}$ (i.e., c_β is replaced by 1) over $\mathcal{S}$ and we extend that argument to finite values of β. In fact, the following monotonicity statement shows that it suffices to prove Proposition 12 for $\beta = \ln k$; related monotonicity statements were used in [9] for hypergraph 2-coloring and in [7] for regular k-SAT.

Lemma 13. *For all $d > 0$, $\beta \geqslant 0$, $\rho \in \mathcal{S}$ we have*

$$\frac{\partial}{\partial \beta} f_{d,\beta}(\bar{\rho}) \leqslant \frac{\partial}{\partial \beta} f_{d,\beta}(\rho) < 0.$$

Hence, if $f_{d,\beta'}(\bar{\rho}) \geqslant f_{d,\beta'}(\rho)$ for $\beta' \in [0,\infty]$, then $f_{d,\beta}(\bar{\rho}) \geqslant f_{d,\beta}(\rho)$ for all $\beta < \beta'$.

Proof. Differentiating by β reveals that $\beta \mapsto f_{d,\beta}(\rho)$ is monotonous.

$$\frac{\partial}{\partial \beta} f_{d,\beta}(\rho) = -\frac{d}{2} \frac{\frac{2}{k} - \|\rho\|_2^2 \frac{2c_\beta}{k^2} e^{-\beta}}{1 - \frac{2}{k}c_\beta + \frac{\|\rho\|_2^2}{k^2}c_\beta^2} < 0. \tag{19}$$

Setting $y = \|\rho\|_2^2$ and construing $\frac{\partial}{\partial \beta} f_{d,\beta}(\rho)$ as a map of y,

$$\phi : [1,k] \to \mathbf{R}, \quad \phi(y) \mapsto -\frac{d}{2} \frac{\frac{2}{k} - y \frac{2c_\beta}{k^2} e^{-\beta}}{1 - \frac{2}{k}c_\beta + \frac{y}{k^2}c_\beta^2},$$

differentiating $\frac{\partial}{\partial \beta} f_{d,\beta}(\rho)$ by y, we obtain

$$\frac{\partial}{\partial y} \phi(y) = \frac{\frac{1}{k^2} 2c_\beta e^{-\beta} \left(1 - \frac{2}{k}c_\beta + \frac{y}{k^2}c_\beta^2\right) - \left(-\frac{2}{k}e^{-\beta} + \frac{y}{k^2} 2c_\beta e^{-\beta}\right) \frac{c_\beta^2}{k^2}}{\left(1 - \frac{2}{k}c_\beta + \frac{y}{k^2}c_\beta^2\right)^2}$$

$$= \frac{\frac{2c_\beta e^{-\beta}}{k^2}\left(1 - \frac{c_\beta}{k}\right) + y \frac{2c_\beta^3 e^{-\beta}}{k^3}\left(1 - \frac{1}{k}\right)}{\left(1 - \frac{2}{k}c_\beta + \frac{y}{k^2}c_\beta^2\right)^2} \geqslant 0 \qquad \text{for } y \in [1,k]. \tag{20}$$

Hence, $y \mapsto \frac{\partial}{\partial \beta} f_{d,\beta}(\rho)$ has a global minimum at $y = 1$. Because $y = \|\rho\|_2^2 = 1$ is only the case for $\rho = \bar{\rho}$ the combination of (19) and (20) yields the assertion. $\square$

The following basic observation concerning the partial derivatives of $f_{d,\beta}$ is reminiscent of [14, Lemma 4.11].

Claim 14. *Let $\rho \in \mathcal{S}$. With $i, j, l \in [k]$ such that $\rho_{il}, \rho_{ij} > 0$ set $\delta = \rho_{il} - \rho_{ij}$.*

i) Then

$$\text{sign}\left(\frac{\partial}{\partial \rho_{ij}} f_{d,\beta}(\rho) - \frac{\partial}{\partial \rho_{il}} f_{d,\beta}(\rho)\right) = \text{sign}\left(1 + \frac{\delta}{\rho_{ij}} - \exp\left(\frac{dc_\beta \delta}{k - 2c_\beta + c_\beta^2 \|\rho\|_2^2/k}\right)\right).$$

ii) If $\partial E(\rho)/\partial \rho_{ij} < 1/k$ then there is $\delta^* > 0$ such that for all $0 < \delta < \delta^*$

$$1 + \frac{\delta}{\rho_{ij}} - \exp\left(\frac{dc_\beta \delta}{k - 2c_\beta + c_\beta^2 \|\rho\|_2^2/k}\right) > 0. \tag{21}$$

If $\partial E(\rho)/\partial \rho_{ij} \geqslant 1/k$, the left hand side of (21) is negative for all $\delta > 0$.

Proof. By (11), (12) and the choice of δ,

$$\frac{\partial}{\partial \rho_{ij}} f_{d,\beta}(\rho) - \frac{\partial}{\partial \rho_{il}} f_{d,\beta}(\rho) = \frac{1}{k}\left[\ln\left(1 + \frac{\delta}{\rho_{ij}}\right) - \frac{dc_\beta^2 \delta}{k - 2c_\beta + c_\beta \|\rho\|_2^2/k}\right]. \tag{22}$$

The first part of the claim follows because the signs of the terms in (22) are invariant under exponentiation of the minuend $\phi(\delta) = \ln(1 + \delta/\rho_{ij})$ and subtrahend $\psi(\delta) = dc_\beta^2 \delta/(k - 2c_\beta + c_\beta \|\rho\|_2^2/k)$. The second part follows from the observation that the linear function $\exp(\phi) : \mathbf{R}^+ \to \mathbf{R}$ intersects at most once with the strictly convex function $\exp(\psi) : \mathbf{R}^+ \to \mathbf{R}$. This is only the case if the derivative of $\exp(\phi)$ in $\delta = 0$ is strictly greater than that of $\exp(\psi)$. $\square$

The following lemma provides a general "maximum entropy" principle that we will use repeatedly (cf. [14, Proposition 4.7]).

Lemma 15. *Let $d \leqslant (2k - 1)\ln k$ and $\beta > 0$. For $\rho \in \mathcal{S}$, a fixed row i and a set of columns $J \subset [k]$, set $\hat{\rho}_{ab} = \sum_{j \in J} \rho_{ij}/|J|$ for all $(a,b) \in \{i\} \times J$ and $\hat{\rho}_{ab} = \rho_{ab}$ for all $(a,b) \notin \{i\} \times J$. Let $\lambda \geqslant 3 \ln \ln k/\ln k$. If $|J| \geqslant k^\lambda$ and $\max_{j \in J} \rho_{ij} < \lambda/2 - \ln \ln k/\ln k$, then $f_{d,\beta}(\hat{\rho}) > f_{d,\beta}(\rho)$ if $\rho \neq \hat{\rho}$.*

Proof. We may assume that $0 \leqslant \min_{j \in J} \rho_{ij} < \max_{j \in J} \rho_{ij}$. Otherwise, we would have $\hat{\rho} = \rho$ and there is nothing to prove. Now let

$$\mathcal{S}_\rho = \left\{\tilde{\rho} : \tilde{\rho}_{ab} = \rho_{ab} \text{ for all } (a,b) \notin \{i\} \times J \text{ and } \max_{j \in J} \tilde{\rho}_{ij} \leqslant \max_{j \in J} \rho_{ij}\right\}$$

denote the set of all possible overlaps that vary in entries from $\{i\} \times J$. $\mathcal{S}_\rho$ is a closed subset of $\mathcal{S}$ and therefore contains a maximal overlap $\check{\rho} \in \arg\max_{\tilde{\rho} \in \mathcal{S}} f_{d,\beta}(\tilde{\rho})$. Evidently the derivative of H tends to infinity as ρ_{ij} tends to zero, while the derivative of E remains bounded. Therefore in a maximal overlap each entry $\check{\rho}_{ij}$, $j \in J$ is positive. As a whole, we know that $0 < \min_{j \in J} \check{\rho}_{ij} \leqslant \max_{j \in J} \check{\rho}_{ij} \leqslant 1$. By means of Claim 14 it remains to show that $\check{\delta} = \max_{j \in J} \check{\rho}_{ij} - \min_{j \in J} \check{\rho}_{ij} = 0$.

Let $a \in J$ denote the index of $\check{\rho}_{ia} = \min_{j \in J} \check{\rho}_{ij}$. Because $|J|\check{\rho}_{ia} \leqslant \sum_{j \in J} \check{\rho}_{ij}$ and $d \leqslant 2k \ln k - \ln k$, we have

$$\frac{1}{\check{\rho}_{ia}} \geqslant |J| \geqslant k^\lambda \geqslant 3 \ln k > 2 \ln k\left(\frac{kc_\beta^2}{k - 2c_\beta + c_\beta^2/k}\right) \geqslant \frac{k}{\check{\rho}_{ia}} \frac{\partial}{\partial \check{\rho}_{ia}} E(\check{\rho}),$$

where the final inequality is an immediate consequence of (12). As $\hat{\delta} = \lambda/2 - \ln\ln k / \ln k$, $\|\check{\rho}\|_2^2 \geqslant 1$ and $d \leqslant 2k\ln k - \ln k$,

$$\exp\left(\frac{dc_\beta^2 \hat{\delta}}{k - 2c_\beta + c_\beta^2\|\check{\rho}\|_2^2/k}\right) \leqslant \exp\left(\frac{d\hat{\delta}}{k(1 - c_\beta/k)^2}c_\beta^2\right) \leqslant \exp(2\hat{\delta}\ln k)$$

$$\leqslant k^\lambda \ln^{-2} k \leqslant |J| \ln^{-2} k \leqslant \frac{1}{\check{\rho}_{ia}}\frac{1}{\ln^2 k} < \frac{1}{\check{\rho}_{ia}}\frac{\ln\ln k}{2\ln k} \leqslant \frac{1}{\check{\rho}_{ia}}\left(\frac{\lambda}{2} - \frac{\ln\ln k}{\ln k}\right) \leqslant \frac{\hat{\delta}}{\check{\rho}_{ia}}$$

confirms that

$$\mathrm{sign}\left(1 + \frac{\delta}{\check{\rho}_{ia}} - \exp\left(\frac{dc_\beta\delta}{k - 2c_\beta + c_\beta^2\|\check{\rho}\|_2^2/k}\right)\right) = 1$$

holds for any $\delta < \hat{\delta}$. Suppose that $\check{\delta} > 0$. Then $0 < \delta \leqslant \max_{j\in J} \check{\rho}_{ij} \leqslant \hat{\delta}$ and Claim 14 imply that a matrix $\check{\rho}'$ obtained from $\check{\rho}'$ by decreasing $\max_{j\in J}\check{\rho}_{ij}$ by a sufficiently small $\xi > 0$ and increasing $\check{\rho}_{ia}$ by the same value ξ results in $f_{d,\beta}(\check{\rho}') > f_{d,\beta}(\check{\rho})$, which contradicts the maximality of $\check{\rho}$. Hence, a maximal overlap ρ satisfies $\check{\delta} = \max_{j\in J}\check{\rho}_{ij} - \min_{j\in J}\check{\rho}_{ij} = 0$ for any i, J chosen according to our assumption. $\qquad\square$

In order to achieve a global bound on $\max_{\rho\in\mathcal{S}} f_{d,\beta}(\rho)$ we need to pin down the structure of a maximizing matrix ρ. To this end, the following elementary fact is going to be useful.

Fact 16 ([14, Lemma 4.15]). *Let $\xi : \varepsilon \in (0, k/2) \mapsto k^{2\varepsilon/k}(\varepsilon^{-1} - k^{-1})$. Let $\mu = \frac{k}{2}(1 - \sqrt{1 - 2/\ln k})$. Then ξ is decreasing on the interval $(0, \mu)$ and increasing on $(\mu, k/2)$. Furthermore, we have $-1/2 \leqslant \xi'(\varepsilon) \leqslant -3/2$ for $\varepsilon \in (0.99, 1.01)$.*

The following lemma rules out the possibility that the maximizer of $f_{d,\beta}$ has an entry close to $1/2$ (cf. [14, Lemma 4.13]).

Lemma 17. *Let $\beta > 0$ and $d = 2k\ln k - c$, where $c = O_k(\ln k)$. If $\rho \in \mathcal{S}$ has an entry $\rho_{ij} \in [0.49, 0.51]$, then there is $\rho' \in \mathcal{S}$ such that $f_{d,\beta}(\rho') \geqslant f_{d,\beta}(\rho) + \frac{\ln k}{5k}$.*

Proof. By means of Lemma 15 we will specify ρ' and provide above bound for $f_{d,\beta}(\rho) - f_{d,\beta}(\rho')$ in a distinction of two cases. Without loss of generality we may assume that the entry in the interval $[0.49, 0.51]$ is ρ_{11}. Suppose ρ maximizes $f_{d,\beta}$ subject to the condition that $\rho_{11} \in [0.49, 0.51]$.

For the first case, suppose that $\rho_{1j} < 0.49$ for all $j \geqslant 2$. By setting $J = \{2, \ldots, k\}$ and $\lambda = \ln(k-1)/\ln k$ in Lemma 15, we have $\rho_{1j} = (1 - \rho_{11})/(k-1)$ for all $j \geqslant 2$. Let ρ' denote the matrix obtained from ρ by setting $\rho_1' = (1/k, \ldots, 1/k)$ and $\rho_i' = \rho_i$ for $i \geqslant 2$. In the following assume that k is sufficiently large. By Fact 3 we have

$$H(\rho_1) \leqslant h(\rho_{11}) + (1 - \rho_{11})\ln(k-1) \leqslant \ln 2 + 0.51\ln k.$$

Consequently

$$H(k^{-1}\rho_1') - H(k^{-1}\rho_1) \geqslant \frac{0.48\ln k}{k}. \tag{23}$$

In comparison, the Frobenius norm of ρ_1 is bounded by

$$\|\rho_1\|_2^2 \leqslant 0.51^2 + (k-1)\left(\frac{0.51}{k-1}\right)^2 \leqslant 0.261,$$

while

$$\frac{\partial}{\partial\|\rho\|_2^2}E(\rho) = \frac{d}{2k^2}\frac{c_\beta^2}{1 - 2/kc_\beta + \|\rho\|_2^2/k^2c_\beta^2} = \frac{2k\ln k + O_k(\ln k)}{2k}O_k\left(\frac{1}{k}\right)$$

$$\leqslant \frac{\ln k}{k}\left(1 + O_k\left(\frac{1}{k}\right)\right). \tag{24}$$

Therefore

$$E(\rho) - E(\rho') \leqslant \frac{0.262\ln k}{k}. \tag{25}$$

The combination of (23) and (25) verifies

$$f_{d,\beta}(\rho') \geqslant f_{d,\beta}(\rho) + 0.218\frac{\ln k}{k} \geqslant f_{d,\beta}(\rho) + \frac{\ln k}{5k}$$

for $\beta \geqslant \ln k$. By Lemma 13

$$f_{d,\beta}(\rho') \geqslant f_{d,\beta}(\rho) + \frac{\ln k}{5k} \tag{26}$$

holds for any $0 \leqslant \beta \leqslant \ln k$. Finally we show (26) for the case that a row consists of two entries greater than 0.49. Without loss of generality we may assume that $\rho_{11} \geqslant \rho_{12} \geqslant 0.49$ and $\rho_{1j} < 0.02$ for $j \geqslant 3$. Lemma 15 with parameters $J = \{2,\ldots,k\}$ and $\lambda = \ln(k-1)/\ln k$ gives $\rho_{1j} = (1 - \rho_{11} - \rho_{12})/(k-2)$ for all $j \geqslant 3$. Hence, for sufficiently large k

$$H(\rho_1) \leqslant h(\rho_{11}) + h(\rho_{12}) + (0.02)\ln(k-2) \leqslant 2\ln 2 + 0.02\ln k \leqslant 0.03\ln k.$$

Moreover the norm is bounded by

$$\|\rho_1\|_2^2 = \rho_{11}^2 + \rho_{12}^2 + (k-2)\left(\frac{1 - \rho_{11} - \rho_{12}}{k-2}\right)^2 \leqslant 0.501.$$

Consequently

$$E(\rho) - E(\rho') \leqslant \frac{0.51\ln k}{k}, \tag{27}$$

$$H(k^{-1}\rho_1') - H(k^{-1}\rho_1) \geqslant 0.97\frac{\ln k}{k}. \tag{28}$$

The combination of (27) and (28) yields (26) for $\beta \geqslant \ln k$. By Lemma 13 the assertion follows for $0 \leqslant \beta \leqslant \ln k$. $\qquad\square$

Generalizing [14, Lemma 4.16], as a next step we characterize the structure of the local maxima of $f_{d,\beta}$ on $\mathcal{S}$.

Lemma 18. *Let $\beta > 0$ and $d = 2k \ln k - c$, where $c = O_k(\ln k)$. Let $\rho \in \mathcal{S}$.*

(1) Suppose that row $i \in [k]$ has no entries in $[0.49, 0.51]$ and $\rho_{ij} \leqslant 0.49$ for all $j \in [k]$. Let ρ' be the stochastic matrix with entries

$$\rho'_{hj} = \rho_{hj} \text{ and } \rho'_{ij} = \frac{1}{k} \quad \text{ for all } j \in [k], h \in [k] \setminus \{i\}. \tag{29}$$

Then $f_{d,\beta}(\rho) \leqslant f_{d,\beta}(\rho')$.

(2) Suppose that row $i \in [k]$ has no entries in $[0.49, 0.51]$ and $\rho_{ij} \geqslant 0.51$ for some $j \in [k]$. Then there is a number $\alpha = \frac{1}{k} + \tilde{O}_k(1/k^2)$ such that for the stochastic matrix ρ'' with entries

$$\rho''_{hj} = \rho_{hj} \text{ and } \rho''_{ii} = 1 - \alpha, \rho''_{ih} = \frac{\alpha}{k-1} \quad \text{ for all } j \in [k], h \in [k] \setminus \{i\} \tag{30}$$

we have $f_{d,\beta}(\rho) \leqslant f_{d,\beta}(\rho'')$.

(3) Let $\beta \leqslant \ln k$. Suppose that row $i \in [k]$ has an entry $\rho_{ij} \in [0.49, 0.51]$. Then the matrix ρ' with (29) satisfies $f_{d,\beta}(\rho) \leqslant f_{d,\beta}(\rho')$.

Proof. Claim (1) is an immediate consequence of Lemma 15 when setting $J = [k]$, $\lambda = 1$ and applying the $\rho \mapsto \hat{\rho}$ operation on the i-th row.

For Claim (2) we may again assume that $i = j = 1$ and therefore $\rho_{11} \geqslant 0.51$. Let $\hat{\rho} \in \mathcal{S}$ maximize $f_{d,\beta}$ subject to the conditions that $\hat{\rho}$ coincides with ρ everywhere but in the first row and $\hat{\rho}_{11} \geqslant 0.51$. A necessary condition for $\hat{\rho}$ to be maximal is that the mass in the remaining open entries is equally distributed. $\hat{\rho}_{11} \geqslant 0.51$ implies that for all $j \geqslant 2$ the entries $\hat{\rho}_{1j}$ are bounded by 0.49. Setting $\lambda = \ln(k-1)/\ln k$, Lemma 15 applies to row $i = 1$ and $J = \{2, \ldots, k\}$ confirming that for all $j \geqslant 2$ we have $\hat{\rho}_{1j} = (1 - \hat{\rho}_{11})/(k-1)$.

Let $0 \leqslant \varepsilon \leqslant 0.49k$ be such that $\hat{\rho}_{11} = 1 - \varepsilon/k$. To prove the assertion we need to show that $\varepsilon = 1 + \tilde{O}_k(1/k)$. Set $\delta = \hat{\rho}_{11} - \hat{\rho}_{12}$. Then because $\hat{\rho}$ maximizes $f_{d,\beta}$ Claim 14 implies that

$$\text{either } \varepsilon \in \{0, 0.49k\}, \text{ or } 1 + \frac{\delta}{\hat{\rho}_{12}} = \exp\left(\frac{dc_\beta^2 \delta}{k - 2c_\beta + c_\beta^2 \frac{\|\hat{\rho}\|_2^2}{k}}\right). \tag{31}$$

Equations (22) and (11) show that $\partial/\partial\rho_{11} H(\rho_1)$ tends to $-\infty$ as ρ_{11} tends to 1, while $\partial/\partial\rho_{11} E(\rho_1)$ remains bounded. Hence, a maximal $\hat{\rho}$ is bound to satisfy $\varepsilon > 0$.

By $\|\hat{\rho}\|_2^2 \geqslant 1$ we have $k - 2c_\beta + c_\beta^2 \frac{\|\hat{\rho}\|_2^2}{k} \geqslant k(1 - c_\beta/k)^2$. Moreover we have $\delta = \hat{\rho}_{11} - O_k(1/k)$ due to all entries in the first row being $(1 - \hat{\rho}_{11})/(k-1)$. With $d = 2k \ln k + O_k(\ln k)$ and $\beta \geqslant \ln k$ we obtain

$$\exp\left(\frac{dc_\beta^2 \delta}{k - 2c_\beta + c_\beta^2 \frac{\|\hat{\rho}\|_2^2}{k}}\right) = k^{2\hat{\rho}_{11}}\left(1 + \tilde{O}_k(1/k)\right) = k^{2(1-\varepsilon/k)}\left(1 + O_k(1/k)\right)$$

and

$$1 + \frac{\delta}{\hat{\rho}_{12}} = \frac{\hat{\rho}_{11}}{\hat{\rho}_{12}} = \frac{(k-1)\hat{\rho}_{11}}{1 - \hat{\rho}_{11}} = k^2(1/\varepsilon - 1/k)(1 + O_k(1/k)).$$

Thus, setting $\xi : \varepsilon \mapsto k^{2\varepsilon/k}(1/\varepsilon - 1/k)$ there is $\eta = O_k(\ln k/k)$ such that

$$(1 - \eta)\xi(\varepsilon) \leqslant \left(1 + \frac{\delta}{\hat{\rho}_{12}}\right) \exp\left(\frac{dc_\beta^2 \delta}{k - 2c_\beta + c_\beta \frac{\|\hat{\rho}\|_2^2}{k}}\right) \leqslant (1 + \eta)\xi(\varepsilon). \tag{32}$$

Fact 16 reveals that ξ has a unique local minimum in $\mu = \frac{k}{2}(1 - \sqrt{1 - 2/\ln k})$ while ξ is decreasing on $(0, \mu)$ and increasing on $(\mu, k/2)$. Furthermore we have $\xi(\varepsilon) \in [-3/2, -1/2]$ for $\varepsilon \in (0.99, 1.01)$. Therefore, setting $\gamma = \ln^2 k/k$, we have

$$\xi(\varepsilon) \leqslant \begin{cases} \xi(0.49k) \leqslant k^{0.98}\left(\frac{1}{0.49k} - \frac{1}{k}\right) < \frac{1}{1+\eta} & \text{for } \varepsilon \in [\mu, 0.49k] \\ \xi(1 + \gamma) < \frac{1}{1+\eta} & \text{for } \varepsilon \in [1 + \gamma, \mu] \end{cases}$$

and

$$\xi(\varepsilon) \geqslant \xi(1 - \gamma) > \frac{1}{1 - \eta}, \qquad \text{for } \varepsilon \in (0, 1 - \gamma).$$

These bounds applied to (32) yield

$$1 + \frac{\delta}{\hat{\rho}_{12}} - \exp\left(\frac{dc_\beta^2 \delta}{k - 2c_\beta + c_\beta \frac{\|\hat{\rho}\|_2^2}{k}}\right) \begin{cases} > 0 & \text{for } \varepsilon \in (0, 1 - \gamma), \\ < 0 & \text{for } \varepsilon \in [1 + \gamma, 0.49k]. \end{cases} \tag{33}$$

Altogether (31) and (33) with $\varepsilon > 0$ imply $\varepsilon = 1 + \tilde{O}_k(1/k)$ and therefore $\hat{\rho}_{11} = 1 - 1/k + \tilde{O}_k(1/k^2)$ by Claim 14. Hence $\hat{\rho}$ satisfies (30) and $f_{d,\beta}(\hat{\rho}) \geqslant f_{d,\beta}(\rho)$ for any $\beta \geqslant \ln k$. By Lemma 13 $f_{d,\beta}(\hat{\rho}) \geqslant f_{d,\beta}(\rho)$ holds for any $0 \leqslant \beta \leqslant \ln k$ as well.

By definition of ρ' Claim (3) is a Corollary of Lemma 17. $\qquad \square$

The following Lemma, which extends [14, Lemma 4.14] to finite β, estimates the function values attained at points near the "candidate maxima" from Lemma 18.

Lemma 19. *Let ρ_s denote the matrix whose the top s rows coincide with the identity matrix and whose last $k - s$ rows coincide with $\bar{\rho}$. If $\beta = \ln k$ and $d \leqslant (2k - 1) \ln k$ then $f_{d,\beta}(\bar{\rho}) > f_{d,\beta}(\rho_s)$ for all $s = 1, \ldots, k$.*

Proof. We have

$$H(k^{-1}\bar{\rho}) = \ln k + \frac{1}{k} \sum_{i=1}^{k} H(\rho_i) = 2\ln k, \quad E(\bar{\rho}) = \frac{d}{2} \ln\left[1 - \frac{2}{k}c_\beta + \frac{1}{k^2}c_\beta^2\right] = d \ln\left[1 - \frac{c_\beta}{k}\right].$$

Further,

$$H(k^{-1}\rho_s) = \ln k + \frac{1}{k} \sum_{i=1}^{k} H(\rho_i) = \ln k + \frac{k-s}{k} \ln k, \tag{34}$$

$$E(\rho_s) = \frac{d}{2} \ln \left[1 - \frac{2}{k} c_\beta + \left(\frac{k-s}{k} + s \right) \frac{c_\beta^2}{k^2} \right]. \tag{35}$$

Hence,

$$f_{d,\beta}(\bar{\rho}) = 2 \ln k + d \ln[1 - c_\beta/k],$$

$$f_{d,\beta}(\rho_s) = \frac{2k-s}{k} \ln k + \frac{d}{2} \ln \left[1 - \frac{2}{k} c_\beta + \left(\frac{k-s}{k} + s \right) \frac{c_\beta^2}{k^2} \right].$$

The assertion $f_{d,\beta}(\bar{\rho}) > f_{d,\beta}(\rho_s)$ holds iff $H(k^{-1}\bar{\rho}) - H(k^{-1}\rho_s) = \frac{s}{k} \ln k > E(\rho_s) - E(\bar{\rho})$, i.e.

$$E(\rho_s) - E(\bar{\rho}) = \frac{d}{2} \ln \left[\frac{1 - \frac{2}{k} c_\beta + \left(\frac{k-s}{k} + s \right) \frac{c_\beta^2}{k^2}}{(1 - \frac{c_\beta}{k})^2} \right] = \frac{d}{2} \ln \left[1 + \frac{\left(s - \frac{s}{k} \right) \frac{c_\beta^2}{k^2}}{\left(1 - \frac{c_\beta}{k} \right)^2} \right] < \frac{s}{k} \ln k. \tag{36}$$

Setting $x = (s - s/k) c_\beta^2/k^2 (1 - c_\beta/k)^{-2}$ a mercator series expansion

$$\frac{d}{2} \ln(1 + x) = \frac{d}{2} \left[x - \frac{x^2}{2} + O_k(x^3) \right]$$

$$\leqslant \frac{2k \ln k - \ln k}{2} \left[x - \frac{x^2}{2} \right] = \ln k \left[kx - k\frac{x^2}{2} - \frac{x}{2} + \frac{x^2}{4} \right].$$

along with the representation

$$\frac{\left(s - \frac{s}{k} \right) \frac{c_\beta^2}{k^2}}{\left(1 - \frac{c_\beta}{k} \right)^2} = \frac{c_\beta^2}{k} \frac{s}{k} \left(1 - \frac{1}{k} \right) \frac{1}{(1 - c_\beta/k)^2} = \frac{c_\beta^2}{k} \frac{s}{k} \left(1 - \frac{1}{k} \right) \left(1 + 2c_\beta/k + O_k(1/k^2) \right)$$

$$= \frac{1}{k} \frac{s}{k} \left(1 - \frac{2}{k} + O_k(k^{-2}) \right) \left(1 + \frac{1}{k} + O_k(k^{-2}) \right) \qquad \text{[as } \beta = \ln k]$$

reduces the proof to validating the inequality

$$1 > \frac{(k-1/2)}{k} \left(1 - \frac{2}{k} + O_k(k^{-2}) \right) \left(1 + \frac{1}{k} + O_k(k^{-2}) \right)$$

$$+ \frac{(1/4 - k/2)}{k^2} \left[\left(1 - \frac{2}{k} + O_k(k^{-2}) \right) \left(1 + \frac{1}{k} + O_k(k^{-2}) \right) \right]^2 \frac{s}{k}. \tag{37}$$

This is indeed true, since the first summand is bounded by $1 - k^{-2}$ and the second summand is negative. $\square$

Corollary 20. *With ρ_s defined as in Lemma 19 the inequality $f_{d,\beta}(\bar{\rho}) > f_{d,\beta}(\rho_s)$ holds for all $0 < s < k$ and $0 < \beta \leqslant \ln k$.*

Proof of Proposition 12. In the case $\beta = 0$ we have $f_{d,\beta}(\rho) = H(k^{-1}\rho)$. On $[0,1]^{k \times k} \supset \mathcal{S}$ the entropy function is maximized by the uniform distribution on $[k]^2$, i.e. the matrix

$\bar{\rho}$. Consider the case $0 < \beta \leqslant \ln k$. Because ρ is stochastic each row of ρ has at most one entry greater than 0.51. We call ρ s-stable if there are precisely s rows with entries greater than 0.51. For any $s \in \{1, \ldots, k\}$ and any s-stable matrix ρ, using Lemma 18 we obtain a matrix ρ' such that $f_{d,\beta}(\rho') \geqslant f_{d,\beta}(\rho)$ where ρ' is achieved by moving from ρ in direction ρ_s. Together with Corollary 20 this yields the assertion. $\qquad\square$

Proof of Proposition 8. For any choice of n, β or d Jensen's inequality shows

$$\frac{1}{n} \ln \mathbb{E}[Z_\beta(\boldsymbol{G})] \geqslant \frac{1}{n} \mathbb{E}[\ln Z_\beta(\boldsymbol{G})]. \tag{38}$$

We claim that $d \in [2(k-1)\ln(k-1), (2k-1)\ln k - 2]$ and $\beta \leqslant \ln k$ allows for

$$\frac{1}{n} \ln \mathbb{E}[Z_\beta(\boldsymbol{G})] \leqslant \frac{1}{n} \mathbb{E}[\ln Z_\beta(\boldsymbol{G})] + o(1). \tag{39}$$

By (6), there is $C_b >$ such that

$$\mathbb{E}[Z_\beta(\boldsymbol{G})] \leqslant C_b \mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})]. \tag{40}$$

Hence, combining Propositions 6 and 12 we have

$$\mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})^2] = \sum_{\rho \in \mathcal{R}} \exp(n f_{d,\beta}(\rho) + o(n))$$

$$\leqslant \exp(o(n)) \exp(n f_{d,\beta}(\bar{\rho})/2) \leqslant \exp(o(n)) \mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})]^2.$$

Analogously to the proof of Corollary 11 we apply the Paley-Zigmund inequality and obtain

$$\liminf_{n \to \infty} \mathbb{P}[n^{-1} \ln(Z_\beta(\boldsymbol{G})) \geqslant n^{-1} \ln \mathbb{E}[Z_\beta(\boldsymbol{G})] - o(1)] \geqslant \exp(o(n)).$$

The concentration result in Fact 2 therefore yields $\frac{1}{n} \mathbb{E}[\ln Z_\beta(\boldsymbol{G})] \geqslant \frac{1}{n} \ln \mathbb{E}[Z_\beta(\boldsymbol{G})] - o(1)$.
$\qquad\square$

4 High degree, low temperature: the first moment

Throughout this section we assume that $d \in [2(k-1)\ln(k-1), (2k-1)\ln k - 2]$ and $\beta \geqslant \ln k$. In this section we prove Proposition 9. The principal tool is going to be the following experiment called the *planted model*; similar constructions for hypergraph 2-coloring or k-SAT played an important role in [7, 9].

PM1 Choose a map $\hat{\boldsymbol{\sigma}} : [n] \to [k]$ uniformly at random.

PM2 Letting

$$p_1 = \frac{dk \exp(-\beta)}{n(k - c_\beta)}, \qquad p_2 = \frac{dk}{n(k - c_\beta)},$$

obtain a random graph $\hat{\boldsymbol{G}}$ on $[n]$ by independently including every edge $\{v, w\}$ of the complete graph such that $\hat{\boldsymbol{\sigma}}(v) \neq \hat{\boldsymbol{\sigma}}(w)$ with probability p_2 and every edge $\{v, w\}$ such that $\hat{\boldsymbol{\sigma}}(v) = \hat{\boldsymbol{\sigma}}(w)$ with probability p_1.

The following lemma sets out the connection between the planted model and the first moment.

Lemma 21. *If $\mathcal{A}$ is a set of graph/color assignment pairs (G, σ) such that*

$$\mathbb{P}\left[(\hat{G}, \hat{\sigma}) \in \mathcal{A} | \hat{\sigma} \in \mathcal{B}\right] = o(n^{-1/2}),$$

then

$$\mathbb{E} \sum_{\sigma \in \mathcal{B}} \exp(-\beta \mathcal{H}_{G}(\sigma)) \mathbf{1}_{\{(G,\sigma) \in \mathcal{A}\}} = o(\mathbb{E}[Z_{\beta,\mathrm{bal}}(G)]).$$

Proof. Because $k^{-1} p_1 + (1 - 1/k) p_2 = d/n$, the expected number of edges of $\hat{G}$ is $m + O(\sqrt{n})$. Hence, the assumption $\mathbb{P}\left[(\hat{G}, \hat{\sigma}) \in \mathcal{A} | \hat{\sigma} \in \mathcal{B}\right] = o(n^{-1/2})$ implies that

$$\mathbb{P}\left[(\hat{G}, \hat{\sigma}) \in \mathcal{A} | \hat{\sigma} \in \mathcal{B}, |E(\hat{G})| = m\right] = o(1). \tag{41}$$

Writing out the l.h.s. of (41), we obtain

$$\mathbb{P}\left[(\hat{G}, \hat{\sigma}) \in \mathcal{A} | \hat{\sigma} \in \mathcal{B}, |E(\hat{G})| = m\right]$$

$$=\Theta(k^{-n}) \sum_{\substack{(G,\sigma) \in \mathcal{A} \\ \sigma \in \mathcal{B}, |E(G)|=m}} \frac{p_1^{\mathcal{H}_G(\sigma)} p_2^{m-\mathcal{H}_G(\sigma)} (1-p_1)^{\mathcal{H}_{K_n}(\sigma)-\mathcal{H}_G(\sigma)} (1-p_2)^{\binom{n}{2}-\mathcal{H}_{K_n}(\sigma)-m+\mathcal{H}_G(\sigma)}}{\mathbb{P}\left[|E(\hat{G}) = m|\right]}$$

$$=\frac{\Theta(k^{-n})}{(1-c_\beta/k)^m} \left(\frac{d}{n}\right)^m \cdot$$

$$\sum_{\substack{(G,\sigma) \in \mathcal{A} \\ \sigma \in \mathcal{B}, |E(G)|=m}} \frac{\exp(-\beta \mathcal{H}_G(\sigma))(1-p_1)^{k^{-1}\binom{n}{2}-\mathcal{H}_G(\sigma)} (1-p_2)^{(1-k^{-1})\binom{n}{2}-m+\mathcal{H}_G(\sigma)}}{\mathbb{P}\left[\mathrm{Bin}(\binom{n}{2}, d/n) = m\right]};$$

in the last step we used (4) and the observation that $k^{-1} p_1 + (1 - 1/k) p_2 = d/n$. Further, combining the above with (6), we get

$$\mathbb{P}\left[(\hat{G}, \hat{\sigma}) \in \mathcal{A} | \hat{\sigma} \in \mathcal{B}, |E(\hat{G})| = m\right] \mathbb{E}[Z_{\beta,\mathrm{bal}}(G)]$$

$$=\Theta(1) \binom{\binom{n}{2}}{m}^{-1} \cdot$$

$$\sum_{(G,\sigma) \in \mathcal{A}, \sigma \in \mathcal{B}, |E(G)|=m} \exp(-\beta \mathcal{H}_G(\sigma)) \left(\frac{1-p_1}{1-p}\right)^{k^{-1}\binom{n}{2}-\mathcal{H}_G(\sigma)} \left(\frac{1-p_2}{1-p}\right)^{(1-k^{-1})\binom{n}{2}-m+\mathcal{H}_G(\sigma)}$$

$$=\Theta(1) \binom{\binom{n}{2}}{m}^{-1} \cdot$$

$$\sum_{(G,\sigma) \in \mathcal{A}, \sigma \in \mathcal{B}, |E(G)|=m} \exp(-\beta \mathcal{H}_G(\sigma)) = \Theta(1) \sum_{\sigma \in \mathcal{B}} \mathbb{E}\left[\exp(-\beta \mathcal{H}_G(\sigma)) \mathbf{1}_{\{(G,\sigma) \in \mathcal{A}\}}\right].$$

Thus, the assertion follows from (41). $\qquad \square$

We are going to combine Lemma 21 with the following proposition, which shows that separability is a likely event in the planted model.

Proposition 22. *We have* $\mathbb{P}[\hat{\boldsymbol{\sigma}} \text{ is separable in } \hat{\boldsymbol{G}} | \hat{\boldsymbol{\sigma}} \in \mathcal{B}] = 1 - o(n^{-1/2})$.

To prove Proposition 22 we generalize the argument for proper k-colorings from [14, Section 3] to the Potts antiferromagnet. In the following we let $V_i = \hat{\boldsymbol{\sigma}}^{-1}(i)$ for $i \in [k]$.

Lemma 23. *Let* $i \in [k]$. *For* $S \subset V_i$ *let* $X_{S,i} = |\{v \in V \setminus V_i : \partial_{\hat{G}} v \cap S = \emptyset\}|$. *Given* $\hat{\boldsymbol{\sigma}} \in \mathcal{B}$ *the following statement holds with probability* $1 - \exp(-\Omega(n))$.

$$\text{Let } i \in [k]. \text{ Then for all } S \subset V_i \text{ of size } \tfrac{k}{n}|S| \in [0, 501, 1 - k^{-0.499}] \text{ we have } X_{S,i} \leqslant \tfrac{n}{k}(1 - \alpha - \kappa) - n^{2/3}. \tag{42}$$

Proof. It suffices to prove the statement for $i = 1$ and we set $X_S = X_{S,1}$. Moreover, let $\alpha \in [0.501, 1 - k^{-0.499}]$. For a fixed $S \subset V_1$ and $v \in V \setminus V_1$ the number $|\partial_{\hat{G}} v \cap S|$ is a binomial random variable with parameters $|S| = \frac{\alpha n}{k}$ and p_2. Hence, $\mathbb{P}[\partial v \cap S = \emptyset] = (1 - p_2)^{|S|}$. Consequently, X_S itself is a binomial variable with mean $|V \setminus V_1|(1 - p_2)^{|S|}$. Because σ is balanced, we have $|V \setminus V_1| \sim n(1 - 1/k)$. Further, our assumptions on d, β entail

$$(1 - p_2)^{|S|} \leqslant \exp(-p_2|S|) \leqslant \exp\left(-\alpha \frac{2k \ln k}{k - c_\beta} + \alpha \frac{3 \ln k}{k - c_\beta}\right) \leqslant (1 + o_k(1))k^{-2\alpha}.$$

Therefore, $\mathbb{E}[X_S] \leqslant n(1 + o_k(1))(1 - 1/k)k^{-2\alpha}$. Thus, Lemma 4 yields

$$\mathbb{P}[X_S > (1 - \alpha - \kappa)\frac{n}{k} - n^{2/3}] \leqslant \exp\left[-(1 - \alpha - \kappa + o(1))\frac{n}{k} \ln\left(\frac{1 - \alpha - \kappa}{ek^{1-2\alpha}}\right)\right].$$

The total number of sets S of size $\alpha n/k$ is $\binom{|V_1|}{\alpha \frac{n}{k}} \leqslant \exp\left[\frac{n}{k} h(\alpha)\right]$. Hence, by the union bound

$$\mathbb{P}\left[\exists S : X_S \geqslant (1 - \alpha - \kappa)\frac{n}{k} - n^{2/3}\right]$$
$$\leqslant \exp\left[\frac{n}{k}\left(2h(\alpha) + (1 - \alpha) + (1 - 2\alpha)(1 - \alpha - \kappa)\ln k + o(1)\right)\right]$$
$$\leqslant \exp\left[\frac{n}{k}\left((1 - \alpha)(3 - 2\ln(1 - \alpha)) + (2(1 - \alpha)^2 - (1 - 2\kappa)(1 - \alpha) + \kappa)\ln k + o(1)\right)\right]. \tag{43}$$

Substituting $y = 1 - \alpha$ and differentiating, we obtain

$$\frac{\partial}{\partial y} y(3 - 2\ln y) + (2y^2 - (1 - 2\kappa)y + \kappa)\ln k = 1 - 2\ln y + 4y \ln k - (1 - 2\kappa)\ln k,$$
$$\frac{\partial^2}{\partial y^2} y(3 - 2\ln y) + (2y^2 - (1 - 2\kappa)y + \kappa)\ln k = -\frac{2}{y} + 4\ln k,$$
$$\frac{\partial^3}{\partial y^3} y(3 - 2\ln y) + (2y^2 - (1 - 2\kappa)y + \kappa)\ln k = 2.$$

Hence, the first derivative is negative at the left boundary point $y = k^{-0.499}$, positive at the right boundary point $y = 0.499$ and convex on the entire interval. Furthermore, we check that $y(3 - 2\ln y) + (2y^2 - (1 - 2\kappa)y + \kappa)\ln k < 0$ for $y \in \{0.499, k^{-0.499}\}$. Therefore, the assertion follows from (43). $\qquad\square$

Lemma 24. *Given $\hat{\boldsymbol{\sigma}} \in \mathcal{B}$ the random graph $\hat{\boldsymbol{G}}$ has the following property with probability $1 - \exp(-\Omega(n))$.*

> *Let $i \in [k]$ and let $Y = Y(\hat{\boldsymbol{G}}, \hat{\boldsymbol{\sigma}})$ be the number of vertices $v \notin V_i$ with fewer than 15 neighbors in V_i. Then $Y \leqslant \frac{\kappa n}{3k \ln k}$.* $\qquad(44)$

Proof. Suppose $i = 1$. Given $\hat{\boldsymbol{\sigma}} \in \mathcal{B}$ for $v \notin V_1$ the number $|\partial_{\hat{\boldsymbol{G}}} v \cap V_1|$ of neighbors in V_1 is a binomial variable with mean $\lambda = |V_1| p_2 \sim d/(k - c_\beta) > 2\ln k + O_k(\ln k/k)$. Hence, the probability of a vertex having at most 14 neighbors in V_1 is upper bounded by $2\lambda^{14} \exp(-\lambda) \leqslant 3k^{-2}\ln^{14} k$. Therefore, Y is dominated by a binomial variable with mean $\mu \leqslant 3nk^{-2}\ln^{14} k$. Finally, the assertion follows from Lemma 4 and the choice of κ. $\qquad\square$

Claim 25. *Given $\hat{\boldsymbol{\sigma}} \in \mathcal{B}$ the random graph $\hat{\boldsymbol{G}}$ has the following property with probability $1 - O(n^{-1})$.*

> *If $W \subset V$ has size $W \leqslant k^{-4/3}n$, then W spans no more than $5|W|$ edges.* $\qquad(45)$

Proof. Given $\hat{\boldsymbol{\sigma}}$ for any edge of the complete graph the probability of being present in $\hat{\boldsymbol{G}}$ is bounded by p_2. Therefore, by the union bound and with room to spare, for any $0 < \gamma \leqslant k^{-4/3}$ we find

$$\mathbb{P}\left[\exists W \subset V, |W| = \gamma n : W \text{ spans } 5|W| \text{ edges} \mid \hat{\boldsymbol{\sigma}} \in \mathcal{B}\right]$$
$$\leqslant \binom{n}{\gamma n}\binom{\binom{\gamma n}{2}}{5\gamma n}p_2^{5\gamma n} \leqslant \left[\frac{e}{\gamma}\left(\frac{e\gamma d}{5}\right)^5\right]^{\gamma n} \leqslant \left(\gamma^4 d^5\right)^{\gamma n}.$$

Summing over $1/n \leqslant \gamma \leqslant k^{-4/3}$ completes the proof. $\qquad\square$

Proof of Proposition 22. Suppose that $\hat{\boldsymbol{\sigma}}$ is balanced. By our assumptions on d, β for each i the number of edges spanned by $\hat{\boldsymbol{\sigma}}^{-1}(i)$ in $\hat{\boldsymbol{G}}$ is a binomial random variable with mean

$$(1 + o(1))\binom{n/k}{2}p_1 \leqslant (1 + o(1))\frac{dn}{2k(k - c_\beta)}\exp(-\beta) \leqslant (1 + o_k(1))nk^{-1}\exp(-\beta)\ln k.$$

Hence, Lemma 4 shows that $(\hat{\boldsymbol{G}}, \hat{\boldsymbol{\sigma}})$ satisfies **SEP1** with probability $1 - \exp(-\Omega(n))$.

With respect to **SEP2**, we continue to condition on $\hat{\boldsymbol{\sigma}} \in \mathcal{B}$. By Lemma 23, Lemma 24 and Claim 25 we may assume that $\hat{\boldsymbol{G}}$ has the properties (42), (44) and (45). In order to show separability we may without loss of generality restrict ourselves to the case of $i = j = 1$. Thus, suppose that $\tau \in \Sigma_{\boldsymbol{G}, \beta}$ satisfies $\rho_{11}(\hat{\boldsymbol{\sigma}}, \tau) \geqslant 0.51\frac{n}{k}$ and assume for contradiction that $\alpha = \frac{k}{n}|S| = \rho_{11}(\hat{\boldsymbol{\sigma}}, \tau) < 1 - \kappa$. Let

$$S = \hat{\boldsymbol{\sigma}}^{-1}(1) \cap \tau^{-1}(1), \quad R = \hat{\boldsymbol{\sigma}}^{-1}(1) \setminus \tau^{-1}(1), \quad T = \tau^{-1}(1) \setminus \hat{\boldsymbol{\sigma}}^{-1}(1).$$

Because σ and τ are balanced, we have

$$|T \cup S| \sim \frac{n}{k} \sim |R \cup S|. \tag{46}$$

Let $T_0 = \{v \in T : \partial_{\hat{G}} v \cap S = \emptyset\}$ and let $T_1 = T \setminus T_0$. Then **SEP1** and our assumptions on d and β ensure that $|T_1| \leqslant \frac{4n \ln k}{k \exp(\beta)}$. Consequently, the assumption $\beta \geqslant \ln k$ yields

$$|T_0| \geqslant \frac{n}{k}(1 - \alpha - O_k(\ln k/k)).$$

Since the vertices in T_0 do not have neighbors in S, (42) implies that

$$\alpha > 1 - k^{-0.49}. \tag{47}$$

Further, let $U = \{v \in T : |\partial v \cap \hat{\sigma}^{-1}(1)| \geqslant 15\}$. Then (44) implies that $|T| \leqslant |U| + \kappa n/(k \ln k)$. Therefore, (46) and our assumption $\alpha < 1 - \kappa$ yield

$$|U| \geqslant (1 - o_k(1))\frac{\kappa n}{k} \qquad \text{and} \qquad |R| - o_k(\kappa)\frac{n}{k} \leqslant |U| \leqslant |R| + o(n). \tag{48}$$

Hence, **SEP1** implies that $S \cup U$ spans no more than $2nk^{-1}\exp(-\beta)\ln k \leqslant |U|$ edges. Consequently, $U \cup R$ spans at least $14|U|$ edges. Thus, combining (45) and (48), we conclude that $|U \cup R| > nk^{-4/3}$. But then (46) and (48) show that $1 - \alpha + o(1) \geqslant \frac{k}{n}|R| > \frac{k}{3n}|U \cup R| \geqslant \frac{1}{3}k^{-1/3}$, in contradiction to (47). $\qquad \square$

Proof of Proposition 9. By linearity of expectation, applying Lemma 21 to Proposition 22 yields $\mathbb{E}[Z_{\beta,\mathrm{bal}}(\boldsymbol{G})] \sim \mathbb{E}[Z_{\beta,\mathrm{sep}}(\boldsymbol{G})]$. $\qquad \square$

5 High degree, low temperature: the second moment

To prove Proposition 10 we call a doubly-stochastic $k \times k$-matrix ρ *separable* if $\rho_{ij} \notin (0.51, 1 - \kappa)$ for all $i, j \in [k]$. Moreover, ρ is *s-stable* if $s = |\{(i,j) \in [k]^2 : \rho_{ij} > 0.51\}|$. Let $\mathcal{D}_{\mathrm{sep}} \subset \mathcal{D}$ be the set of all separable matrices and let $\mathcal{D}_{s,\mathrm{sep}} \subset \mathcal{D}_{\mathrm{sep}}$ be the set of all s-stable matrices so that $\mathcal{D}_{\mathrm{sep}} = \bigcup_{s=0}^{k} \mathcal{D}_{s,\mathrm{sep}}$. The key step is to optimize the function $f_{d,\beta}$ over $\mathcal{D}_{\mathrm{sep}}$.

Proposition 26. *If $2(k-1)\ln(k-1) \leqslant d \leqslant d_\star$ and $\beta \geqslant \ln k$, then $f_{d,\beta}(\rho) < f_{d,\beta}(\bar{\rho})$ for all $\rho \in \mathcal{D}_{\mathrm{sep}} \setminus \{\bar{\rho}\}$.*

A similar statement for the function

$$f_{d,\infty}(\rho) = H(k^{-1}\rho) + \frac{d}{2}\ln\left[1 - \frac{2}{k} + \frac{\|\rho\|_2^2}{k^2}\right], \tag{49}$$

the limit of $f_{d,\beta}(\rho)$ as $\beta \to \infty$, played a key role in [14]. Specifically, we have

Proposition 27 ([14, Propositions 4.4–4.6, 4.8]). *Assume that $d = (2k-1)\ln k - 2\ln 2$. Then $f_{d,\infty}(\rho) < f_{d,\infty}(\bar{\rho})$ for all $0 \leqslant s < k$, $\rho \in \mathcal{D}_{s,\mathrm{sep}} \setminus \{\bar{\rho}\}$.*

We prove Proposition 26 by combining Proposition 27 with monotonicity in both d and β. In fact, Lemma 13 readily provided monotonicity in β. Further, with respect to d we have the following.

Lemma 28. *For every $d > 0$, $\rho \in \mathcal{S}$ we have*

$$\frac{\partial}{\partial d} f_{d,\infty}(\bar{\rho}) \leqslant \frac{\partial}{\partial d} f_{d,\infty}(\rho) < 0.$$

Hence, if $f_{d',\beta}(\bar{\rho}) \geqslant f_{d',\beta}(\rho)$, then $f_{d,\beta}(\bar{\rho}) \geqslant f_{d,\beta}(\rho)$ for all $0 \leqslant d < d'$.

Proof. Recalling that $1 \leqslant \|\rho\|_2^2 \leqslant k$, we find

$$\frac{\partial}{\partial d} f_{d,\infty}(\rho) = \frac{1}{2} \ln \left(1 - \frac{2}{k} + \frac{\|\rho\|_2^2}{k^2} \right) < 0.$$

The assertion follows because $\bar{\rho}$ minimizes the Frobenius norm on $\mathcal{S}$. $\square$

Corollary 29. *Let $\beta \geqslant 0$ and $d \leqslant d_\star$. For all $1 \leqslant s \leqslant k - 1$ and $\rho \in \bigcup_{s<k} \mathcal{D}_{s,\mathrm{sep}} \setminus \{\bar{\rho}\}$ we have $f_{d,\beta}(\rho) < f_{d,\beta}(\bar{\rho})$.*

Proof. Suppose that $\rho \in \bigcup_{s<k} \mathcal{D}_{s,\mathrm{sep}} \setminus \{\bar{\rho}\}$. Combining Proposition 27 with Lemma 28, we see that

$$\lim_{\gamma \to \infty} f_{d,\gamma}(\rho) = f_{d,\infty}(\rho) < f_{d,\infty}(\bar{\rho}) = \lim_{\gamma \to \infty} f_{d,\gamma}(\bar{\rho}).$$

Hence, $f_{d,\gamma}(\rho) < f_{d,\gamma}(\bar{\rho})$ for $\gamma > \beta$ sufficiently large. Therefore, Lemma 13 entails that $f_{d,\beta}(\rho) < f_{d,\beta}(\bar{\rho})$. $\square$

Observe that Proposition 27 (and hence Corollary 29) does not cover the k-stable case.

Lemma 30. *Let $\beta \geqslant 0$ and $d \leqslant d_\star$. For all $\rho \in \mathcal{D}_{k,\mathrm{sep}}$ we have $f_{d,\beta}(\rho) < f_{d,\beta}(\bar{\rho})$.*

Proof. Because $\rho \in \mathcal{D}_{k,\mathrm{sep}}$ for each $i \in [k]$ there is precisely one entry greater than 0.51. Without loss of generality we may assume that $\rho_{ii} \geqslant 0.51$. By Lemma 18 there is $\alpha = k^{-1} + \tilde{O}_k(k^{-2})$ such that the matrix ρ' obtained from ρ by substituting any row ρ_i for a row ρ'_i with $\rho'_{ii} = 1 - \alpha$ and $\rho'_{ij} = \alpha/(k-1)$ for $j \neq i$ satisfies $f_{d,\beta}(\rho') \geqslant f_{d,\beta}(\rho)$. Hence, a maximizer of $\rho \in \mathcal{D}_{k,\mathrm{sep}}$ is of the form $\rho_{\mathrm{stable}} = (1 - 1/k)\mathrm{id} + 1/k^2 \mathbf{1}$. Because the matrix ρ_{stable} does not further improve from applying the transformation in Lemma 18, it remains to show that

$$f_{d,\beta}(\bar{\rho}) > f_{d,\beta}(\rho_{\mathrm{stable}}). \tag{50}$$

In the zero temperature case with $d = (2k - 1) \ln k - c$, we have

$$f_{d,\infty}(\bar{\rho}) = \ln k + \frac{1}{k} \sum_{i \leqslant k} H(\bar{\rho}_i) + \frac{d}{2} \ln \left[1 - \frac{2}{k}\mathbf{1} + \|\bar{\rho}\|_2^2 \frac{1}{k^2} \right] = 2 \ln k + d \ln \left[1 - \frac{1}{k} \right]$$

$$[\text{as } \|\bar{\rho}\|_2^2 = 1]$$

$$= 2\ln k - d\left(\frac{1}{k} + \frac{1}{2k^2} + O_k\left(k^{-3}\right)\right) \tag{51}$$

$$= 2\ln k - (2k\ln k - \ln k - c)\left(\frac{1}{k} + \frac{1}{2k^2} + O_k\left(k^{-3}\right)\right)$$

$$= \frac{c}{k} + O_k\left(\frac{\ln k}{k^2}\right). \tag{52}$$

On the other hand the matrix ρ_{stable} satisfies

$$H(k^{-1}\rho_{\text{stable}}) = \ln k + \frac{1}{k}\sum_{i \leqslant k} H\left((1 - 1/k + 1/k^2, 1/k^2, \ldots, 1/k^2)\right)$$

$$= \ln k - \left(1 - \frac{1}{k} + \frac{1}{k^2}\right)\ln\left(1 - \frac{1}{k} + \frac{1}{k^2}\right) + \frac{(k-1)}{k^2}\ln k^2. \tag{53}$$

Because $\|\rho_{\text{stable}}\|_2^2 = \frac{k(k-1)}{k^4} + k(1 - \frac{1}{k} + \frac{1}{k^2})^2$ and $\beta \geqslant \ln k$, setting $d = (2k-1)\ln k - c$ we obtain

$$E(\rho_{\text{stable}}) = \frac{d}{2}\ln\left[1 - \frac{2}{k} + \frac{1}{k^2}\left(\frac{k(k-1)}{k^4} + k\left(1 - \frac{1}{k} + \frac{1}{k^2}\right)^2\right)\right]$$

$$= \frac{d}{2}\ln\left[1 - \left(\frac{1}{k} + \frac{2}{k^2} + O_k\left(k^{-3}\right)\right)\right] \tag{54}$$

$$= -\frac{d}{2}\left(\frac{1}{k} + \frac{1}{k^2} + \frac{1}{2}\left(\frac{1}{k} + \frac{2}{k^2}\right)^2 + O_k\left(k^{-3}\right)\right)$$

$$= -\left(k\ln k - \frac{\ln k}{2} - \frac{c}{2}\right)\left(\frac{1}{k} + \frac{5}{2k^2} + O_k\left(k^{-3}\right)\right)$$

$$= -\ln k - \frac{2\ln k}{k} + \frac{c}{2k} + O_k\left(\frac{\ln k}{k^2}\right). \tag{55}$$

Consequently

$$f_{d,\infty}(\rho_{\text{stable}}) = \frac{1}{k} + \frac{c}{2k} + O_k\left(\frac{\ln k}{k^2}\right). \tag{56}$$

From (52) and (56) we see that $f_{d,\beta}(\bar{\rho}) > f_{d,\beta}(\rho_{\text{stable}})$ holds for any $d \leqslant (2k-1)\ln k - 2 - \omega_k(\ln k/k)$ and $\beta = \infty$. Lemma 13 concludes the proof by extending (50) to $\beta \geqslant \ln k$. $\square$

Proof of Proposition 26. Because $\mathcal{D}_{\text{sep}}$ decomposes into disjoint subsets $\mathcal{D}_{s,\text{sep}}$ and $s = 0, 1, \ldots, k$ Proposition 26 is immediate from Corollary 29 and Lemma 30. $\square$

Proof of Proposition 10. By definition of $\mathcal{B}_{sep}$ and Proposition 22 we have

$$\mathbb{E}[Z_{\beta,\text{sep}}(\boldsymbol{G})^2] \sim \sum_{\rho \in \mathcal{R} \cap \mathcal{B}_{sep}} \mathbb{E}[Z_{\rho,\text{bal}}(\boldsymbol{G})]. \tag{57}$$

By Propositions 6 and 26,

$$\sum_{\rho \in \mathcal{R} \cap \mathcal{B}_{sep}} \mathbb{E}[Z_{\rho,\mathrm{bal}}(\boldsymbol{G})] = \sum_{\rho \in \mathcal{R} \cap \mathcal{B}_{sep}} \exp(n f_{d,\beta}(\rho) + o(n)) \tag{58}$$

$$= \exp\left(2n \ln k + nd \ln\left(1 - c_\beta/k\right) + o(n)\right). \tag{59}$$

Combining (57)–(59) with (6) and taking logarithms yields the assertion. $\square$

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