On total positivity of Catalan-Stieltjes matrices

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Submitted: Jul 1, 2016; Accepted: Nov 5, 2016; Published: Nov 25, 2016
Mathematics Subject Classifications: 05A18; 05A15, 05A30

Abstract

Recently Chen-Liang-Wang (Linear Algebra Appl. **471** (2015) 383–393) present some sufficient conditions for the total positivity of Catalan-Stieltjes matrices. Our aim is to provide a combinatorial interpretation of their sufficient conditions. More precisely, for any Catalan-Stieltjes matrix A we construct a digraph with a weight, which is positive under their sufficient conditions, such that every minor of A is equal to the sum of the weights of families of nonintersecting paths of the digraph. We have also an analogous result for the minors of a Hankel matrix associated to the first column of a Catalan-Stieltjes matrix.

1 Introduction

The study of totally positive matrices appears in various areas such as orthogonal polynomials, combinatorics, algebraic geometry, stochastic processes, game theory, differential equations, representation theory, Brownian motion, electrical networks, and chemistry; see [13, 5, 6, 19, 9]. In this paper we shall consider the total positivity of some special lower triangular matrices. Recall that an infinite real matrix M is said to be *totally positive* (TP) if every minor of M is *nonnegative.*

Definition 1.1. An infinite lower triangular matrix of real numbers $A^{\gamma,\sigma,\tau} := (a_{n,k})_{n,k\geqslant 0}$ is called a *Catalan-Stieltjes matrix* if there are three sequences of positive numbers $\gamma := (r_k)_{k\geqslant 0}$, $\sigma := (s_k)_{k\geqslant 0}$ and $\tau := (t_k)_{k\geqslant 1}$ such that

$$\begin{aligned} a_{n,0} &= s_0 a_{n-1,0} + t_1 a_{n-1,1};\\ a_{n,k} &= r_{k-1} a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1} \quad (k \geqslant 1,\ n \geqslant 1), \end{aligned} \tag{1.1}$$

where $a_{0,0} = 1$ and $a_{n,k} = 0$ unless $n \geqslant k \geqslant 0$.

Although a matrix defined by (1.1) is called a *Catalan matrix* in [2, p. 291], we prefer to call it a Catalan-Stieltjes matrix because, when $r_k = 1$, Stieltjes [23] first introduced such a matrix in his step-by-step method to expand a continued J-fraction into a power series; see [24, Section 53] and [10]. Indeed, the matrix (1.1) implies the following continued fraction expansion of the ordinary generating function of the first column of $A^{\gamma,\sigma,\tau}$:

$$\sum_{n=0}^{\infty} a_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{\lambda_1 z^2}{1 - s_1 z - \cfrac{\lambda_2 z^2}{1 - s_2 z - \cdots}}}, \tag{1.2}$$

where $\lambda_{k+1} = r_k t_{k+1}$ for $k \geqslant 0$. The sequence $(a_{n,0})$ is usually called a *moment sequence* in the theory of orthogonal polynomials (see [26]). Conversely, starting from (1.2) with nonnegative s_k and λ_{k+1} ($k \geqslant 0$), and any factorisation $r_k t_{k+1}$ of λ_{k+1} such that $r_k, t_{k+1} \geqslant 0$ ($k \geqslant 0$), we can recover the moment sequence $(a_{n,0})$ using matrix(1.1).

Recently Chen-Liang-Wang [7] proved some sufficient conditions for the total positivity of Catalan-Stieltjes matrices. At the end of their paper they asked for a combinatorial interpretation of their results. The aim of this paper is to present such a combinatorial interpretation using a classical lemma of Lindström [16]. As in [5, 12, 11], our strategy is to first interpret the matrix A as a path matrix of some planar network within two boundary vertex sets, and then apply Lindström's lemma [16] to write every minor of A as a sum of positive weights of families of nonintersecting paths. We first recall some basic definitions of this methodology. Let $G = (V, E)$ be an infinite acyclic digraph, where V is the vertex set and E the edge set. If $S := (A_i)_{i \geqslant 0}$ and $T := (B_i)_{i \geqslant 0}$ are two sequences of vertices in G, we say that the triple (G, S, T) is a *network*. We assume that there is a weight function $w : E \to \mathbb{R}$ and define the *associated path matrix* $M = (m_{i,j})_{i,j \geqslant 0}$ by

$$m_{i,j} = \sum_{\gamma : A_i \to B_j} w(\gamma),$$

where the sum is over all the paths γ from A_i to B_j and the *weight* of a path is the product of its edge weights. By convention we define $m_{i,j} = 1$ if $A_i = B_j$.

Let $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ be two positive increasing integer sequences. The I, J minor of a matrix M is defined by $\det M_{I,J}$, where $M_{I,J}$ is the submatrix of M corresponding to row set I and column set J. Let $A_I = \{A_i : i \in I\}$, $B_J = \{B_j : j \in J\}$ be two n-sets of vertices of $\mathcal{G}$, which need not be disjoint. For any permutation $\sigma \in \mathfrak{S}_n$ denote by $N(\mathcal{G}; A_I, B_{\sigma(J)})$ the set of n-tuples $(p_1, \ldots, p_n)$ where p_i is a path from A_{i_k} to $B_{j_{\sigma(k)}}$ such that any two paths in the tuple are vertex-disjoint. The weight of P is defined by $w(P) = \prod_{i=1}^{n} w(p_i)$. The following result is due to Lindström's [16]. See also[12, 2].

Lemma 1.2 (Lindström)**.** *We have*

$$\det M_{I,J} = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) \sum_{P \in N(\mathcal{G}; A_I, B_{\sigma(J)})} w(P). \tag{1.3}$$

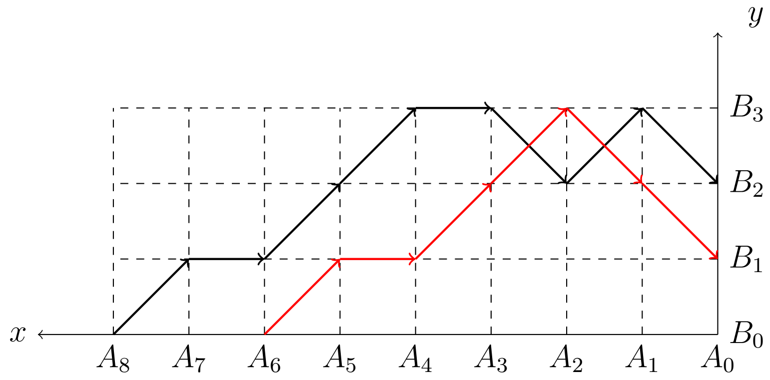

Figure 1.1: Two nonintersecting paths from $(8,0)$ to $(0,2)$ and from $(6,0)$ to $(0,1)$ in the Motzkin network $(\mathcal{M},(A_i),(B_i))$, where only the edges in the paths are depicted.

The basic idea is to find a planar graph G along with two sequences of vertices S and T so that $N(G;A_I,B_{\sigma(J)})$ is empty except when σ is identity. As there is no general method for constructing a simple planar network in order to prove that a given matrix is totally positive, to motivate our approach, we will start with the Motzkin path description of the matrix coefficients $a_{n,k}$ in (1.1). Consider the digraph $\mathcal{M}=(V,E)$, where $V=\mathbb{Z}\times\mathbb{N}$ and

$$E=\{(i,j)\to(i+1,j+1),\ (i,j)\to(i+1,j),\ (i,j)\to(i+1,j-1)|i,j\geqslant 0\}.$$

A path in $\mathcal{M}$ is called a *Motzkin path*. An example of two nonintersecting Motzkin paths in $\mathcal{M}$ is depicted in Figure 1.1. It is well-known and easy to verify (see [10]) that the coefficient $a_{n,k}$ is equal to the sum of weights of Motzkin paths from $A_n:=(n,0)$ to $B_k:=(0,k)$, where the arrows are weighted as follows:

$$\begin{aligned} w((i+1,j)\to(i,j+1))&=r_i,\\ w((i+1,j)\to(i,j))&=s_j,\\ w((i+1,j+1)\to(i,j))&=t_{j+1}, \end{aligned}$$

for $i\geqslant 1, j\geqslant 0$. It follows that the Catalan-Stieltjes matrix A is a path matrix of the Motzkin network $(\mathcal{M},(A_i),(B_i))$. Unfortunately, the signed expression (1.3) does not manifest any positivity for $\det A_{I,J}$ in general, except for the special case $t_k=0$; see Proposition 3.1. Actually it is easy to see that $\mathcal{M}$ is not a planar graph.

In the next section we will suitably modify the Motzkin network along with its weight in order to make a planar network and recover the total positivity conditions in [7]. We also show how to use our path model to carry these positivity conditions over to the Hankel matrix associated to the first column of $A^{\gamma,\sigma,\tau}$. In Section 3 we sepcialize our general results to some well-known combinatorial matrices as well as coefficientwise total positivity of their polynomial analogous. We conclude this paper with two open problems in Section 4 and an appendix about the computation of some Catalan-Stieltjes matrices in Section 5.

2 Minors of Catalan-Stieltjes and Hankel matrices

We consider the graph $\mathcal{G} = (V, E)$ where

- the vertex set V is equal to $V = \{(i,j), (i+\frac{1}{2}, j+\frac{1}{2}) \mid (i,j) \in \mathbb{Z}\times\mathbb{N}\}$;
- the edge set E is equal to

$$\{(i,j) \to (i-1,j); (i,j) \to (i-\frac{1}{2}, j+\frac{1}{2}); (i,j) \to (i-\frac{1}{2}, j-\frac{1}{2});$$
$$(i-\frac{1}{2}, j+\frac{1}{2}) \to (i-1, j+1); (i-\frac{1}{2}, j+\frac{1}{2}) \to (i-1,j) \mid i \in \mathbb{Z}, j \geqslant 0\}.$$

Comparing with the Motzkin graph $\mathcal{M}$ we see that each crossing point $(i+\frac{1}{2}, j+\frac{1}{2})$ of two edges is transformed to a vertex in $\mathcal{G}$ as shown below:

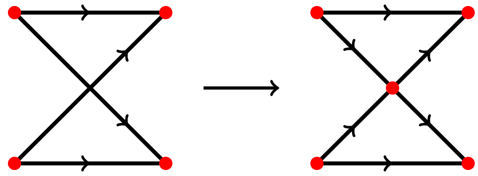

Figure 2.1: Planar embedding of the Motzkin graph

Moreover, instead of three types of arrows (or edges) in $\mathcal{M}$ there are now five types of arrows (or edges) in $\mathcal{G}$ that are summarized in the following diagram:

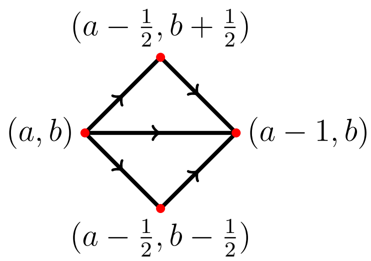

Figure 2.2: Five types of arrows where $(a,b) \in \mathbb{Z}\times\mathbb{N}$.

2.1 Catalan-Stieltjes network

Definition 2.1. Let $A_i := (i, 0)$ and $B_i := (0, i)$ for $i \geqslant 0$. The Catalan-Stieltjes network is defined to be the triple $(\mathcal{G}, (A_i), (B_i))$. See Figure 2.3.

Lemma 2.2. *For any of the following four weight functions on the edges of* $\mathcal{G}$*:*

- $w((i,j) \to (i-1,j)) = s_j - t_j - r_j$ *(resp.* $s_j - t_{j+1} - r_{j-1}, s_j - t_j r_{j-1} - 1, s_j - t_{j+1} r_j - 1$*);*
- $w((i,j) \to (i-\frac{1}{2}, j+\frac{1}{2})) = r_j$ *(resp.* $1, 1, r_j$*);*

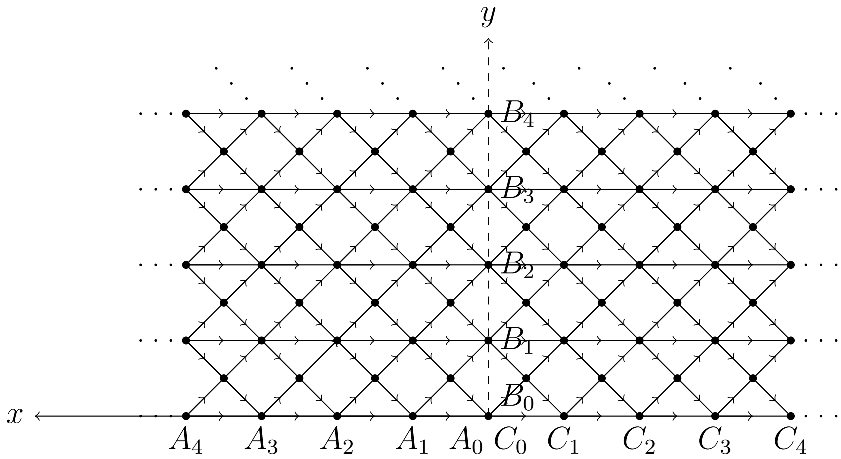

Figure 2.3: Catalan-Stieltjes network $(\mathcal{G}, (A_i),\ (B_i))$ and Hankel network $(\mathcal{G}, (A_i),\ (C_i))$

- $w((i, j+1) \to (i-\frac{1}{2}, j+\frac{1}{2})) = t_{j+1}$ *(resp.* $1, t_{j+1}, 1$*);*
- $w((i-\frac{1}{2}, j+\frac{1}{2}) \to (i-1, j+1)) = 1$ *(resp.* $r_j, r_j, 1$*);*
- $w((i-\frac{1}{2}, j+\frac{1}{2}) \to (i-1, j)) = 1$ *(resp.* $t_{j+1}, 1, t_{j+1}$*);*

the matrix (1.1) *is a path matrix from* (A_i) *to* (B_j) *of the network* $(\mathcal{G}, (A_i),\ (B_i))$*, namely,*

$$a_{i,j} = \sum_{\gamma: A_i \to B_j} w(\gamma). \tag{2.1}$$

Proof. Let $w_{i,j}$ be the right-hand side of (2.1). It suffices to prove that $w_{i,j}$ satisfy the recurrence (1.1). Among the four weight functions, we just prove the first one because the other cases can be verified in the same manner. Firstly, by definition $w_{0,0} = 1$. We can classify the paths from A_{n+1} to B_k according to their intersecting points with the line $x = 1$ as follows:

- All the paths from A_{n+1} to $(1, k)$ plus the last step $(1, k) \to B_k$;
- All the paths from A_{n+1} to $(1, k)$ plus the last two steps $(1, k) \to (\frac{1}{2}, k-\frac{1}{2})$ and $(\frac{1}{2}, k-\frac{1}{2}) \to B_k$;
- All the paths from A_{n+1} to $(1, k)$ plus the last two steps $(1, k) \to (\frac{1}{2}, k+\frac{1}{2})$ and $(\frac{1}{2}, k+\frac{1}{2}) \to B_k$;
- All the paths from A_{n+1} to $(1, k-1)$ plus the last two steps $(1, k-1) \to (\frac{1}{2}, k-\frac{1}{2})$ and $(\frac{1}{2}, k-\frac{1}{2}) \to B_k$;
- All the paths from A_{n+1} to $(1, k+1)$ plus the last two steps $(1, k+1) \to (\frac{1}{2}, k+\frac{1}{2})$ and $(\frac{1}{2}, k+\frac{1}{2}) \to (0, k)$.

It is clear that the sum of the weights of the paths from A_{n+1} to $(1,k)$ (resp. $(1,k-1)$, $(1,k+1)$) is $w_{n,k}$ (resp. $w_{n,k-1}$, $w_{n,k+1}$). So,

$$\begin{aligned} w_{n+1,k} &= r_{k-1}w_{n,k-1} + (s_k - t_k - r_k)w_{n,k} + t_{k+1}w_{n,k+1} + t_k w_{n,k} + r_k w_{n,k} \\ &= r_{k-1}w_{n,k-1} + s_k w_{n,k} + t_{k+1}w_{n,k+1}, \end{aligned}$$

which is the recurrence (1.1). □

Theorem 2.3. *Let $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_n)$ be two positive increasing integer sequences. For any of the four weight functions in Lemma 2.2, we have*

$$\det A_{I,J} = \sum_{P \in N(\mathcal{G}; A_I, B_J)} w(P). \tag{2.2}$$

Proof. By Lemma 2.2 and Lindström's lemma, we can write $\det A_{I,J}$ as a double sum as (1.3) except that the nonintersection condition forces the path p_k to go from A_{i_k} to B_{j_k} for all $k = 1, \dots, n$, namely $\sigma \in \mathfrak{S}_n$ must be identity. □

From Theorem 2.3 we derive immediately the main results of Chen-Liang-Wang [7, Theorems 2.10 and 2.11, Corollary 2.12].

Corollary 2.4. *If the three sequences of nonnegative numbers $(r_k)_{k\geqslant 0}, (s_k)_{k\geqslant 0}$ and $(t_k)_{k\geqslant 1}$ satisfy one of the following conditions:*

(*i*) $s_0 \geqslant r_0$ *and* $s_k \geqslant r_k + t_k$ *for* $k \geqslant 1$;

(*ii*) $s_0 \geqslant t_1$ *and* $s_k \geqslant t_{k+1} + r_{k-1}$ *for* $k \geqslant 1$;

(*iii*) $s_0 \geqslant 1$ *and* $s_k \geqslant r_{k-1} \cdot t_k + 1$ *for* $k \geqslant 1$;

(*iv*) $s_0 \geqslant r_0 \cdot t_1$ *and* $s_k \geqslant r_k \cdot t_{k+1} + 1$ *for* $k \geqslant 1$;

then the Catalan-Stieltjes matrix A defined by (1.1) *is totally positive.*

Remark 2.5. The above conditions (i)-(iii) are exactly Theorems 2.10 and 2.11 of Chen-Liang-Wang [7], while the special $r_k = 1$ case of (iv) is Corollary 2.12 of [7].

We will give several examples of Catalan-Stieltjes networks in Section 3. For the reader's convenience, in Figure 2.4 we present a rotated version of the Catalan-Stieltjes network in Figure 2.3.

2.2 Hankel network

A sequence $\alpha = (a_n)_{n\geqslant 0}$ of real numbers is *Hankel-totally positive* if the associated Hankel matrix $H := H(\alpha) = (a_{i+j})_{i,j\geqslant 0}$ is totally positive. For brevity, we use *H-TP* to denote Hankel-totally positive or Hankel-totally positivity in what follows. It is known (see [19,

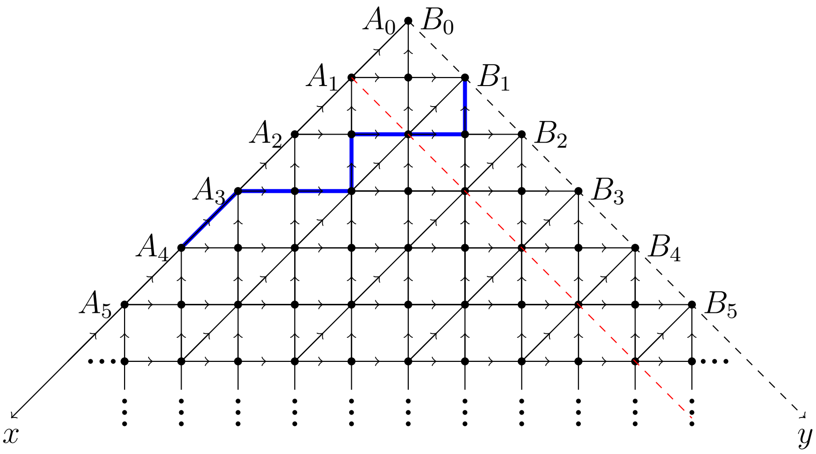

Figure 2.4: Catalan-Stieltjes network $(\mathcal{G}, (A_i),\ (B_i))$

Theorem 4.4]) that the H-TP condition on α is equivalent to sat that α is a *Stieltjes moment sequence*, i.e., there is an integral representation of the form

$$a_n = \int_0^{+\infty} x^n d\mu(x), \quad \text{for} \quad n \geqslant 0,$$

where μ is a non-negative measure μ on $[0, +\infty)$.
A sequence $\alpha = (a_n)_{n \geqslant 0}$ is said to be generated by a Catalan-Stieltjes matrix (1.1) if it coincides with its first column, namely, $a_n = a_{n,0}$ for all $n \in \mathbb{N}$. In [15] Liang-Mu-Wang gave some sufficient conditions on the H-TP of a sequence generated by a Catalan-Stieltjes matrix. Actually, for such a sequence α, we can derive from Lemma 2.2 a lattice path interpretation for each minor of the associated Hankel matrix $H(\alpha)$, which implies sufficient conditions on the H-TP of α. Recall that a sequence $\alpha = (a_n)_{n \geqslant 0}$ is *strongly log-convex* if $a_n a_{m+1} \geqslant a_m a_{n+1}$ for all $m \geqslant n \geqslant 0$. Clearly the H-TP of α implies that it is strongly log-convex.

Definition 2.6. Let $A_i := (i, 0)$ and $C_i := (-i, 0)$ for $i \geqslant 0$. The Hankel network is defined to be the triple $\mathcal{H} := (\mathcal{G}, (A_i),\ (C_i))$. See Figure 2.3.

Theorem 2.7. *Let $\alpha = (a_n)_{n \geqslant 0}$ be a sequence generated by a Catalan-Stieltjes matrix* (1.1) *and $H = (a_{i+j})$ the associated Hankel matrix. Then, for any two positive increasing integer sequences $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ we have*

$$\det H_{I,J} = \sum_{P \in N(\mathcal{G}; A_I, C_J)} w(P),$$

where w is any of the four weight functions in Lemma 2.2.

Proof. By Lemma 2.2, the coefficient $a_{i+j,0}$ is the sum of weights of paths from A_i to C_j in Hankel network $\mathcal{H}$; see Figure 2.3. In other words, the matrix H is the path matrix of $\mathcal{H}$ from (A_i) to (C_j). It is clear that the only possible permutation in Lindström's lemma is identity, so each minor of H reduces to the sum of positive weights of nonintersecting paths families from $(A_i)_{i \in I}$ to $(C_j)_{j \in J}$ in $\mathcal{H}$. $\square$

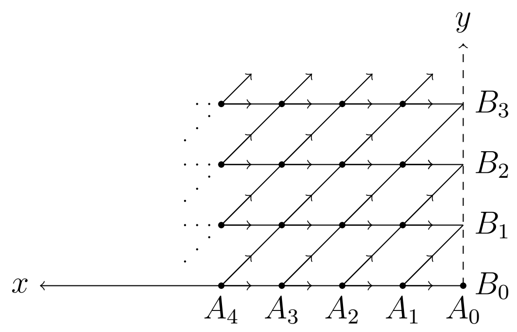

Figure 3.1: Stirling network

Corollary 2.8. *The sequence* $(a_{n,0})_{n\geqslant 0}$ *of* (1.1) *is Hankel totally positive if the three sequences* $(r_k), (s_k)$ *and* (t_{k+1}) *satisfy one of the four conditions (i)–(iv) of Corollary 2.4.*

Remark 2.9. 1. The first condition (i) of Corollary 2.4 is Corollary 2.4 of [15].

2. If $(a_{n,0})$ is the first column of a Catalan-Stieltjes matrix (1.1), then it is also generated by the Catalan-Stieltjes matrix $(\tilde{a}_{n,k})$ defined by

$$\tilde{a}_{n,k} = t_k\tilde{a}_{n-1,k-1} + s_k\tilde{a}_{n-1,k} + r_k\tilde{a}_{n-1,k+1} \quad (k \geqslant 0,\ n \geqslant 1), \tag{2.3}$$

where $\tilde{a}_{0,0} = 1$ and $\tilde{a}_{n,k} = 0$ unless $n \geqslant k \geqslant 0$. Applying the condition (i) (resp. (iii)) of Corollary 2.4 to the matrix (2.3) we get the condition (ii) (resp. (iv)) of Corollary 2.4, so we need only to verify conditions (i) and (iii) of Corollary 2.8.

3 Applications to some combinatorial matrices

It is known [28, 7] that instead of the total order of real numbers we can consider the *partial order* of the commutative ring $\mathbb{R}[x]$ of polynomials with real coefficients as follows: a polynomial in $\mathcal{R}[x]$ is *coefficientwise nonnegative* if it has nonnegative coefficients and $p(x) \succcurlyeq q(x)$ if $p(x) - q(x)$ is coefficientwise nonnegative. Thus we can generalize the previous notions to coefficentwise log-convexity and coefficentwise-Hankel total positivity. For example, a sequence in $\mathcal{R}[x]$ is called *coefficientwise-Hankel totally positive* if the associated Hankel matrix is coefficientwise totally positive. Clearly the totally positive results in the previous sections can be restated in terms of coefficientwise totally positive sequence. In what follows we consider some special cases of Catalan-Stieltjes network and Hankel network in connection with some classical combinatorial sequences. One souce of such examples can be found in Viennot's Lecture Note [26] because almost all the moment sequences of classical orthogonal polynomials have interesting combinatorial interpretations.

3.1 Stirling network

If $t_k = 0$ for all $k \geqslant 1$, the recurrence (1.1) reduces to

$$a_{n,k} = r_{k-1}a_{n-1,k-1} + s_k a_{n-1,k} \quad (k \geqslant 0,\ n \geqslant 1). \tag{3.1}$$

Since the Stirling numbers of second kind $S(n,k)$ satisfy (3.1) with $r_k = 1$ and $t_k = k$ we call the corresponding graph *Stirling network*. On the other hand, as there is no *down* step in the Motzkin paths, the corresponding network (see Figure 1.1) reduces to Figure 3.1.

Proposition 3.1. *Let $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_n)$ be two positive increasing integer sequences. The minors of the matrix $A = (a_{i,j})$ satisfying* (3.1) *has the following combinatorial interpretation*

$$\det A_{I,J} = \sum_{P \in N(\mathcal{G}; A_I, B_J)} w(P). \tag{3.2}$$

In particular A is coefficientwise totally positive if r_k and s_k are polynomials in x with nonnegative coefficients for all $k \geqslant 0$.

Remark 3.2. Mongelli [18, Theorem 5] gave the above combinatorial interpretation in the special case $r_k = 1$ and $s_k = k(z+1)$. Generalizing the positivity part of Mongelli's result Zhu [29] proved the above total positivity result in the special case where r_k and s_k are quadratic polynomials of k.

3.2 Narayana network of type A

The *Narayana polynomials* $N_n(x)$ (see [26, 28]) are defined by

$$N_n(x) = \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} x^k.$$

It is known (see Appendix) that $(N_n(x))$ is the moment sequence generated by the Catalan-Stieltjes matrix $A^{\gamma,\sigma,\tau}$ (see (1.1)) with

$$\gamma = (0, 1, 1, \dots), \quad \sigma = (x, 1+x, 1+x, \dots), \quad \tau = (x, x, \dots).$$

Since these three sequences satisfy all the conditions in Corollary 2.4, the matrix $A^{\gamma,\sigma,\tau}$ is TP and the sequence $(N_n(x))$ is coefficientwise-H-TP. When $x = 1$ the matrix reduces to the Catalan triangle of Aigner [1]:

$$C = (C_{n,k}) = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 9 & 5 & 1 & & \\ 14 & 28 & 20 & 7 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $C_{n+1,0} = C_{n,0} + C_{n,1}$, $C_{n+1,k} = C_{n,k-1} + 2C_{n,k} + C_{n,k+1}$. The corresponding *Narayana network of type A* is depicted in Figure 3.2. As the weight of arrows $(i,j) \to (i-1,j)$

is $s_j - t_j - r_j = 0$ for all $i \geqslant 1$ and $j \geqslant 0$, so there is no such arrows in Figure 3.2. For example, if we choose $I = \{2, 3\}$ and $J = \{0, 1\}$, then:

$$\det \begin{pmatrix} 2 & 3 \\ 5 & 9 \end{pmatrix} = 3$$

and the three pairs of nonintersecting paths from $\{A_2, A_3\}$ to $\{B_0, B_1\}$ are drawn as red, green and blue pairs of paths in Figure 3.2.

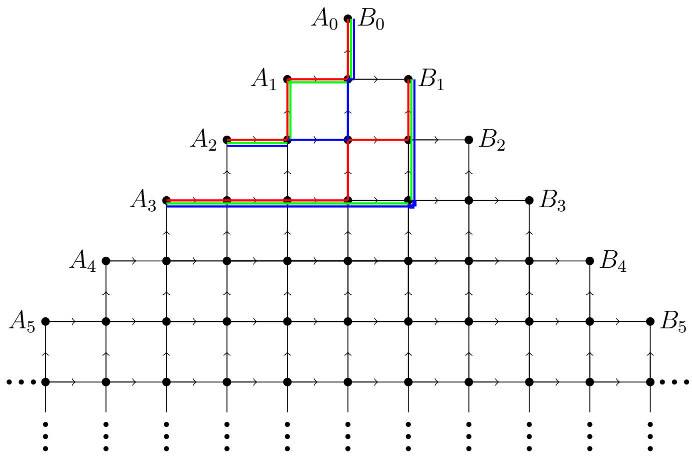

Figure 3.2: Narayana network of type A

3.3 Catalan-Shapiro network

Shapiro [20] proved that the ballot numbers $B_{n,k} = \frac{k}{n}\binom{2n}{n+k}$ $(n, k \geqslant 1)$ satisfy the recurrence

$$B_{n+1,k} = B_{n,k-1} + 2B_{n,k} + B_{n,k+1}.$$

Thus the sequence $(B_{n,1})_{n\geqslant 1})$ of Catalan numbers is the moment sequence generated by the Catalan-Stieltjes matrix $\left(B_{n+1,k+1}\right)_{n,k\geqslant 0}$:

$$B = \left(B_{n+1,k+1}\right)_{n,k\geqslant 0} = \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly all the four conditions of Corollary 2.4 are satisfied, so the matrix B is TP and the sequence $(B_{n,1})_{n\geqslant 1})$ is H-TP. Note that the total positivity of B was first proved in [25]. The corresponding *Catalan-Shapiro network* is depicted in Figure 3.3, where the edge $(i, j) \to (i - 1, j)$ has weight 1 if $j = 0$, and 0 otherwise.

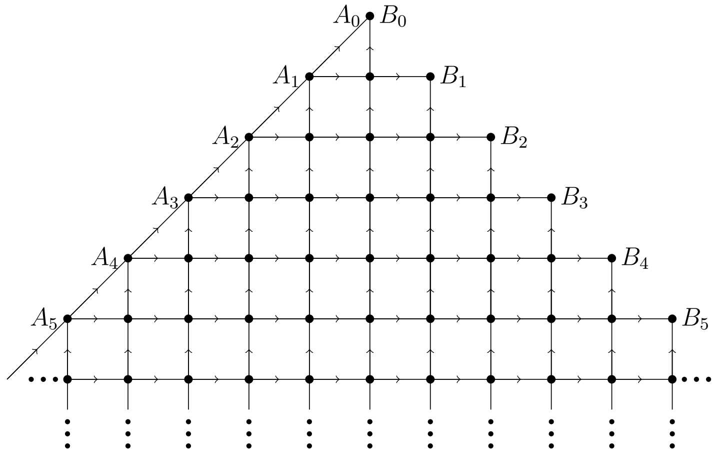

Figure 3.3: Catalan-Shapiro network

3.4 Bell network

The *Bell polynomials* are defined by

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k.$$

It is known [26, 10] that $B_n(x)$'s are generated by the Catalan-Stieltjes matrix $(a_{n,k})$:

$$a_{n,k} = xa_{n-1,k-1} + (k+x)a_{n-1,k} + (k+1)a_{n-1,k+1}, \tag{3.3}$$

where $a_{n,0} = B_n(x)$ for $n \geqslant 0$. Since recurrence (3.3) satisfies just the second condition (ii) of Corollary 2.8, the sequence $(B_n(x))$ is coefficientwise-H-TP. When $x = 1$ it reduces to the Bell triangle [3]

$$X = (X_{n,k}) = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 10 & 6 & 1 & & \\ 15 & 37 & 31 & 10 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $X_{n+1,k} = X_{n,k-1} + (k+1)X_{n,k} + (k+1)X_{n,k+1}$. It satisfies the first and the third conditions in Corollary 2.4. The corresponding *Bell network* is depicted in Figure 3.4.

3.5 Restricted hexagonal network

A *hex tree* is an ordred tree of which each vertex has updegree 0, 1, or 2, and an edge from a vertex of updegree 1 is either left, median, or right. The so-called *restricted hexagonal number* (see [15]) h_n is also the number of hex trees with n edges (see [17, A002212]).

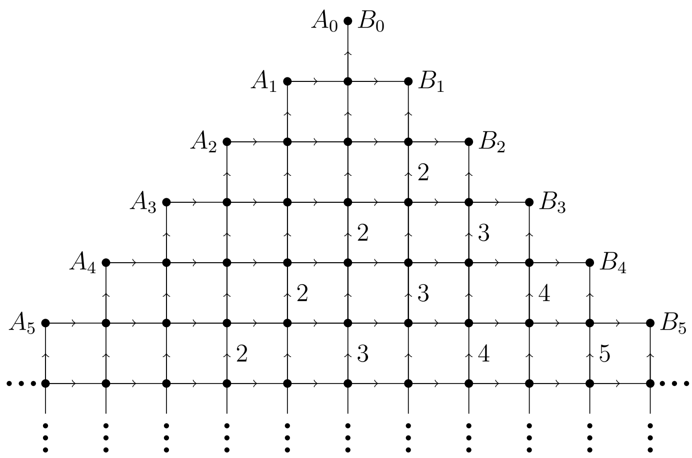

Figure 3.4: Bell network

The numbers h_n have the explicit formula $h_n = \sum_{j=0}^{n} C_{j+1}\binom{n}{j}$ where $C_j = \frac{1}{j+1}\binom{2j}{j}$ are the Catalan numbers. The sequence (h_n) is generated by the Catalan-Stieltjes matrix (see Appendix):

$$H = (h_{n,k}) = \begin{pmatrix} 1 & & & & & \\ 3 & 1 & & & & \\ 10 & 6 & 1 & & & \\ 36 & 29 & 9 & 1 & & \\ 137 & 132 & 57 & 12 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $h_{n+1,k} = h_{n,k-1} + 3h_{n,k} + h_{n,k+1}$. It is easy to see that all the four conditons of Corollary 2.4 are satisfied. Thus the matrix H is TP and the sequence (h_n) is H-TP. The corresponding *restricted hexagonal network* is depicted in Figure 3.5.

In a recent paper [14] Kim and Stanley studied a related polynomial sequence $\big(p_n(x)\big)$ where $p_n(x) = \sum_{j=0}^{n} \frac{1}{j+1}\binom{2j}{j}\binom{n}{j}x^{n-j}$ for $n \geqslant 0$. In particular they proved that this sequence is the moment sequence associated to the Catalan-Stieltjes matrix $(p_{n,k}(x))$:

$$\begin{aligned} p_{n+1,0}(x) &= (x+1)p_{n,0}(x) + p_{n,1}(x), \\ p_{n+1,k+1}(x) &= p_{n,k}(x) + (x+2)p_{n,k+1} + p_{n,k+2} \quad (k \geqslant 1). \end{aligned}$$

where $p_{n,0} = p_n(x)$ for $n \geqslant 0$. Clearly all the conditions of (2.8) are satisfied, so the sequence $(p_n(x))$ is coefficientwise-H-TP.

3.6 Eulerian polynomials

The *Eulerian polynomials* $(A_n(x))$ can be defined by

$$\sum_{k \geqslant 0} (k+1)^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$

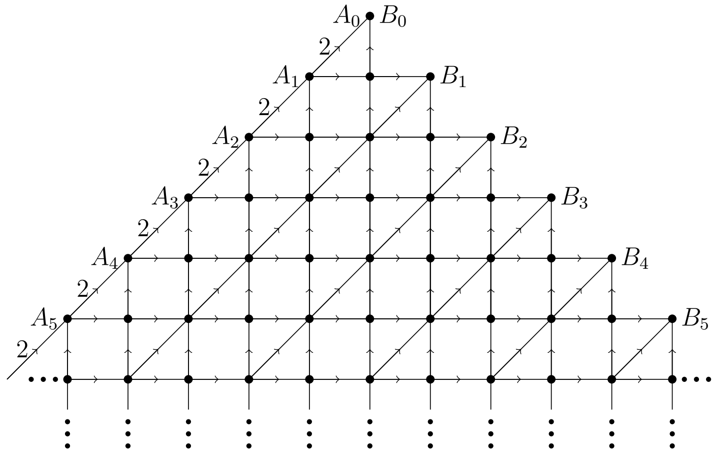

Figure 3.5: Restricted Hexagonal network

They have many combinatorial interpretations and can be generated by the following Catalan-Stieltjes matrix [26, 10]:

$$a_{n,k} = ka_{n-1,k-1} + (k(x+1)+1)a_{n-1,k} + (k+1)xa_{n-1,k+1},$$

where $a_{n,0} = A_n(x)$ for $n \geqslant 0$. By the condition (i) of Corollary 2.8, the sequence $(A_n(x))$ is coefficientwise-H-TP.

3.7 Rising factorials

The rising factorial $\mu_n := (x)_n$ is defined by

$$\mu_0 = 1, \quad \mu_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \geqslant 1.$$

They can be generated by the Catalan-Stieltjes matrix (see [26])

$$a_{n,k} = ka_{n-1,k-1} + (x+2k)a_{n-1,k} + (x-1+k)a_{n-1,k+1} \tag{3.4}$$

where $a_{n,0} = \mu_n$. Since recurrence (3.4) satisfies only the second point of Corollary 2.8, the sequence (μ_n) is coefficientwise-H-TP in $\mathbb{R}[x]$.

3.8 Schröder polynomials

The *Schröder polynomials* $r_n(x)$ (see [4, 28]) are defined by $r_n(x) = N_n(x+1)$, i.e.,

$$r_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} x^{n-k}$$

and generated by the Catalan-Stieltjes matrix (see Appendix):

$$\begin{aligned} a_{n,0} &= (x+1)a_{n-1,0} + (x+1)a_{n-1,1} \\ a_{n,k} &= a_{n-1,k-1} + (x+2)a_{n-1,k} + (x+1)a_{n-1,k+1} \quad (k \geqslant 1). \end{aligned}$$

Since the recurrence satisfies only the condition (i) of Corollary 2.8, so the sequence $(r_n(x))$ is coefficientwise-H-TP.

3.9 Central Delannoy polynomials

The *central Delannoy numbers* [21, 28] are defined by

$$D_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} x^k \tag{3.5}$$

and generated by the Catalan-Stieltjes matrix (see Appendix):

$$\begin{aligned} a_{n,0} &= (2x+1)a_{n-1,0} + (x+1)a_{n-1,1}, \\ a_{n,1} &= 2xa_{n-1,0} + (2x+1)a_{n-1,1} + (x+1)a_{n-1,2}, \\ a_{n,k} &= xa_{n-1,k-1} + (2x+1)a_{n-1,k} + (x+1)a_{n-1,k+1} \quad (k \geqslant 2). \end{aligned}$$

Since the recurrence satisfies only the condition (i) of Corollary 2.8, so the sequence $(D_n(x))$ is coefficientwise-H-TP.

3.10 Narayana polynomials of type B

The *Narayana polynomials of type B* [28] are defined by

$$W_n(x) = \sum_{k \geqslant 0} \binom{n}{k}^2 x^k$$

and generated by the Catalan-Stieltjes matrix (see Appendix)

$$\begin{aligned} a_{n,0} &= 2a_{n-1,0} + xa_{n-1,1}, \\ a_{n,k} &= a_{n-1,k-1} + (x+1)a_{n-1,k} + xa_{n-1,k+1} \quad (k \geqslant 1). \end{aligned}$$

Since the recurrence satisfies only the condition (i) of Corollary 2.8, so the sequence $(W_n(x))$ is Hankel totally positive if $x \geqslant 1$.

4 Two open problems

Sokal [22] and Wang-Zhu [27] actually independently proved that the polynomial sequence $(W_n(x))_{n\geqslant 0}$ is coefficientwise-Hankel totally positive. Can one find a planar network proof of this result ? A toy example of Lindström-Gessel-Viennot's methodology is a lattice path model for the total positivity of the Pascal matrix $P := \left(\binom{n}{k}\right)_{n,k\geqslant 0}$. As the Hadamard product of two totally positive matrices is not totally positive in general (see [8]), we speculate that the Hadamard product $P \circ P = \left(\binom{n}{k}^2\right)_{n,k\geqslant 0}$ is totally positive and have checked this until $n = 9$.

Conjecture 4.1. The matrix

$$\left(\binom{n}{k}^2\right)_{n,k\geqslant 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 9 & 9 & 1 & 0 & 0 & \cdots \\ 1 & 16 & 36 & 16 & 1 & 0 & \cdots \\ 1 & 25 & 100 & 100 & 25 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally positive.

5 Appendix

For the reader's convenience, we indicate a quick path to the last four Catalan-Stieltjes matrices of Section 4. Our derivation of the Catalan-Stieltjes matrices relies on the correspondance between (1.1) and (1.2). Introduce two formal power series F and V given by

$$F := F(z,a,b) := \frac{1-az-\sqrt{1-2az+(a^2-4b)z^2}}{2bz^2}, \tag{5.1}$$

$$V := V(z,a,b) := \frac{1}{\sqrt{1-2az+(a^2-4b)z^2}}. \tag{5.2}$$

As $F(1-az-bz^2F)=1$ we have

$$F = \frac{1}{1-az-bz^2F} = \cfrac{1}{1-az-\cfrac{bz^2}{1-az-\cfrac{bz^2}{1-az-\cfrac{bz^2}{\ddots}}}}. \tag{5.3}$$

From $1-az-2bz^2F=1/V$ we derive that

$$V = \frac{1}{1-az-2bz^2F} = \cfrac{1}{1-az-\cfrac{2bz^2}{1-az-\cfrac{bz^2}{1-az-\cfrac{bz^2}{\ddots}}}}. \tag{5.4}$$

1. (*Nayanara polynomials of type A*) Let $G(z,x)=1+\sum_{n\geqslant 1} N_n(x)z^n$ be the generating function of Narayana polynomials $N_n(x)$. It is known [2] that

$$G(z,x) = \frac{1-(x-1)z-\sqrt{1-2(1+x)z+(1-x)^2z^2}}{2z}.$$

Setting $a = x+1$ and $b = x$ in (5.1) we see that $G(z,x) = 1 + xzF(z, x+1, x)$. Note that

$$1 + xzF(z, x+1, x) = \frac{1}{1 - xz - xz^2 F(z, x+1, x)}.$$

We derive then from (5.3) the continued fraction expansion of $G(z,x)$ with $r_k = 1$ ($k \geqslant 0$), $s_0 = x$, $s_k = x+1$ and $t_k = x$ for $k \geqslant 1$.

2. (*the restricted hexagonal numbers*) The numbers h_n has the generating function $H(z) = \sum_{n\geqslant 0} h_n z^n$ is given by

$$H(z) = \frac{1 - 3z - \sqrt{1 - 6z + 5z^2}}{2z^2} = 1 + 3z + 10z^2 + 36z^3 + 137z^4 + \cdots; \qquad (5.5)$$

see [17, A002212]. Hence $H(z) = F(z, 3, 1)$. We derive from (5.3) the continued fraction (1.2) with $s_k = 3$ and $r_k = t_{k+1} = 1$ for $k \geqslant 0$.

3. (*Schröder polynomials*) Since the Schröder polynomials $d_n(x)$ are shifted Narayana polynomials $d_n(x) = N_n(x+1)$ (see [4]), from the previous result we derive immediately the corresponding Catalan-Stieltjes matrix with

$$r_k = 1 \ (k \geqslant 0), \quad s_0 = x+1, \ s_k = x+2, \ t_k = x+1 \quad \text{for} \quad k \geqslant 1.$$

4. (*Central Delannoy polynomials*) The Legendre polynomials $P_n(x)$ are defined by

$$P_n(x) = \sum_{j=0}^{n} \binom{n+j}{j} \binom{n}{j} \left(\frac{x-1}{2}\right)^j$$

and have the generating function

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1}{\sqrt{1 - 2xz + z^2}}.$$

It follows from (3.5) that $D_n(x) = P_n(2x+1)$. Thus

$$\sum_{n\geqslant 0} D_n(x) z^n = \frac{1}{\sqrt{1 - 2(2x+1)z + z^2}}.$$

This is the special case of (5.2) with $a = 2x+1$ and $b = x^2 + x$. We derive then from (5.4) the continued fraction expansion with $r_k = 1$ ($k \geqslant 0$), $s_k = 2x+1$, $t_1 = 2x(x+1)$ and $t_k = x(x+1)$ for $k \geqslant 1$.

5. (*Narayana polynomials of type B*) Using another formula for Legendre polynomials

$$P_n(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k,$$

we see that the polynomials $W_n(x)$ are related to $P_n(x)$ by

$$W_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k = (x-1)^n P_n\left(\frac{x+1}{x-1}\right).$$

It follows from (5.4) that

$$\sum_{n=0}^{\infty} W_n(x) z^n = \frac{1}{\sqrt{1-2(1+x)z+(x-1)^2 z^2}}.$$

This is (5.2) with $a = x+1$ and $b = x$. We derive from (5.4) the continued fraction expansion (1.2) with $r_0 = 2, r_k = 1$ ($k \geqslant 1$), $s_k = 1+x$ for $k \geqslant 0$ and $t_k = x$ for $k \geqslant 1$.

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