Large Deviations for Permutations
Avoiding Monotone Patterns

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Abstract

For a given permutation τ, let P_N^τ be the uniform probability distribution on the set of N-element permutations σ that avoid the pattern τ. For $\tau = \mu_k := 123\ldots k$, we consider $P_N^{\mu_k}(\sigma_I = J)$ where $I \sim \gamma N$ and $J \sim \delta N$ for $\gamma, \delta \in (0, 1)$. If $\gamma + \delta \neq 1$, then we are in the large deviations regime with the probability decaying exponentially, and we calculate the limiting value of $P_N^{\mu_k}(\sigma_I = J)^{1/N}$. We also observe that for $\tau = \lambda_{k,\ell} := 12\ldots\ell k(k-1)\ldots(\ell+1)$ and $\gamma + \delta < 1$, the limit of $P_N^\tau(\sigma_I = J)^{1/N}$ is the same as for $\tau = \mu_k$.

Keywords: random permutation; monotone pattern-avoiding permutation; left-to-right minimum; large deviations

1 Introduction and Statement of Results

This paper concerns an aspect of the probabilistic properties of a class of pattern-avoiding permutations. As surveyed in the books of Bóna [4] and Kitaev [9], pattern avoidance has been of considerable interest in combinatorial theory, interacting with fields ranging

from algebraic combinatorics to the theory of algorithms. In the next few paragraphs, we give a brief description of the context.

For each positive integer N, let $\mathcal{S}_N$ be the set of all permutations of $1, 2, \ldots, N$. We represent a permutation $\sigma \in \mathcal{S}_N$ as a string of numbers using the one-line notation $\sigma = \sigma_1 \ldots \sigma_N$. We also view σ as the function on $\{1, \ldots, N\}$ that maps i to $\sigma(i) = \sigma_i$. The graph of the function σ is the set of N points $\{(i, \sigma_i) : i = 1, \ldots, N\}$ in $\mathbb{Z}^2$. Given $\tau \in \mathcal{S}_k$ (with $k \leqslant N$), we say that a permutation $\sigma \in \mathcal{S}_N$ avoids the pattern τ (or "σ is τ-avoiding") if there is no k-element subsequence of $\sigma_1, \ldots, \sigma_N$ having the same relative order as τ. (See Section 1.1 for a more formal definition.) Let $\mathcal{S}_N(\tau)$ be the set of permutations in $\mathcal{S}_N$ that avoid τ. For example, the permutation 24153 is not in $\mathcal{S}_5(312)$ because it contains the subsequence 413, which has the same relative order as 312. In contrast, the permutation 35421 has no such subsequence, and hence $35421 \in \mathcal{S}_5(312)$.

We write $|\mathcal{A}|$ to denote the number of elements in a set $\mathcal{A}$. Knuth [10] proved that $|\mathcal{S}_N(\tau)|$ is the same for all $\tau \in \mathcal{S}_3$ and is equal to the Nth Catalan number, that is $\binom{2N}{N}/(N+1)$ for every N. For $\tau \in \mathcal{S}_k$ with $k \geqslant 4$, the values of $|\mathcal{S}_N(\tau)|$ depend on the pattern τ and have been computed for only some cases. For example, Gessel [6] used generating functions to show that

$$|\mathcal{S}_N(1234)| \;=\; 2 \sum_{k=0}^{N} \binom{2k}{k} \binom{N}{k}^2 \frac{3k^2 + 2k + 1 - N - 2kN}{(k+1)^2(k+2)(N-k+1)} \,.$$

In 2004 Marcus and Tardos [13] proved that

$$L(\tau) \;:=\; \lim_{N \to \infty} |\mathcal{S}_N(\tau)|^{1/N} \text{ exists and is finite for every } \tau,$$

thereby confirming the Stanley-Wilf conjecture that had been open for more than two decades. For example, for $k \geqslant 3$ and $1 \leqslant \ell \leqslant k-2$, consider the patterns

$$\mu_k \;=\; 123\ldots k \qquad \text{and} \qquad \lambda_{k,\ell} \;=\; 123\ldots(\ell-1)\ell k(k-1)\ldots(\ell+1)\,;$$

that is, μ_k is the increasing pattern of length k, and $\lambda_{k,\ell}$ is obtained by reversing the last $k-\ell$ elements of μ_k. A theorem due to Regev [15] implies that $L(\mu_k) = (k-1)^2$. Backelin, West and Xin [3] prove that μ_k and $\lambda_{k,\ell}$ are Wilf equivalent, i.e. that $|\mathcal{S}_N(\mu_k)| = |\mathcal{S}_N(\lambda_{k,\ell})|$ for every N, which implies that $L(\lambda_{k,\ell}) = (k-1)^2$. More generally, [3] finds a bijection from $\mathcal{S}_N(\tau_1 \ldots \tau_\ell(\ell+1)\ldots(k-1)k)$ to $\mathcal{S}_N(\tau_1 \ldots \tau_\ell k(k-1)\ldots(\ell+1))$ for any $\tau \in \mathcal{S}_\ell$.

Recently, some researchers have taken a probabilistic viewpoint towards investigating pattern-avoiding permutations, especially for patterns in $\mathcal{S}_3$. They have been concerned with the configurational properties of a typical τ-avoiding permutation of length N—more precisely, of a permutation drawn uniformly at random from the set $\mathcal{S}_N(\tau)$. Accordingly, we shall write P_N^τ to denote the uniform probability distribution over the set $\mathcal{S}_N(\tau)$. In this paper several Theorems and Propositions use certain conditions on γ, δ, I_N and J_N and to avoid repetition we write these conditions below and refer to them as needed.

Condition 1. *Let γ and δ be fixed numbers in $(0,1)$ such that $\gamma < 1 - \delta$. For each N, let I_N and J_N be integers in $[1, N]$ such that*

$$\lim_{N \to \infty} \frac{I_N}{N} = \gamma \quad and \quad \lim_{N \to \infty} \frac{J_N}{N} = \delta. \tag{1}$$

The following result, proven independently by Miner and Pak [14] and by Atapour and Madras [2], motivates the present paper.

Theorem 2. *[2, 14] Let γ, δ, I_N and J_N be as specified in Condition 1. Then*

$$\lim_{N \to \infty} P_N^{123}(\sigma_{I_N} = J_N)^{1/N} = \frac{1}{4} G(\gamma, 1 - \delta; 1) \tag{2}$$

$$= \lim_{N \to \infty} P_N^{132}(\sigma_{I_N} = J_N)^{1/N}, \tag{3}$$

where we define

$$G(u, v; 1) := \frac{(u+v)^{(u+v)}(2 - u - v)^{2-u-v}}{u^u v^v (1-u)^{(1-u)}(1-v)^{(1-v)}}. \tag{4}$$

Since $G(u, v; 1) < 4$ whenever $u \neq v$, we see that the probabilities $P_N^{123}(\sigma_{I_N} = J_N)$ and $P_N^{132}(\sigma_{I_N} = J_N)$ decay exponentially in N when $\gamma < 1 - \delta$. Thus, a random 123-avoiding or 132-avoiding permutation is exponentially unlikely to contain any points ϵN below the diagonal $\{(i, N-i+1) : 1 \leqslant i \leqslant N\}$; we refer to this as the "large deviations" regime. In the case that $\gamma > 1 - \delta$, Equation (2) still holds (by symmetry about the diagonal), but for $\tau = 132$ there is no exponential decay—i.e. the limit in Equation (3) is 1. In fact, $P_N^{132}(\sigma_{I_N} = J_N)$ is asymptotically proportional to $N^{-3/2}$ ([12], [14]). Madras and Pehlivan [12] also examined joint probabilities under P_N^{132}, proving for example that the probability that graph of σ has two specified points below the diagonal is of order N^{-3} (under certain conditions on the points). Rizzolo, Hoffman, and Slivken [7] proved that for $\tau \in \mathcal{S}_3$, the shape of a τ-avoiding random permutation can be described by Brownian excursion. Janson [8] studied the number of occurrences of another pattern π inside a random 132-avoiding permutation.

Although patterns of length 3 are amenable to precise probabilistic results, analogues for longer patterns seem to be much harder. One reason for this is that for $\tau \in \mathcal{S}_3$, there are nice bijections from $\mathcal{S}_N(\tau)$ to the set of Dyck paths of length $2N$, and these bijections translate various configurational properties of τ-avoiding permutations into tractable properties of Dyck paths (e.g. [7],[12]). (At a more metaphysical level: when the Catalan numbers appear in a problem, nice things happen.) However, nice bijections are much harder to find for patterns of length 4. Although exact formulas for $|\mathcal{S}_N(\tau)|$ are known for some patterns τ of length 4, their proofs are much more complicated than for length 3 and do not seem to be useful for investigating properties of P_N^τ. In this paper our goal is to extend the large deviation result of Theorem 2 to the patterns μ_k for $k \geqslant 4$. In contrast to the proof for μ_3, our derivation of the precise large deviations results does not require exact formulas for finite values of N.

We shall examine the cardinalities of sets of the form

$$\mathcal{F}_N(I, J; \tau) := \{\sigma \in \mathcal{S}_N(\tau) : \sigma_I = J\}. \tag{5}$$

Then in terms of the uniform distribution over $\mathcal{S}_N(\tau)$, we have

$$P_N^\tau(\sigma_I = J) = \frac{|\mathcal{F}(I, J; \tau)|}{|\mathcal{S}_N(\tau)|}.$$

Monte Carlo simulations by Gökhan Yıldırım (as seen in Figure 1) suggests as N gets larger the number of points well below the $x + y = 1$ line decreases.
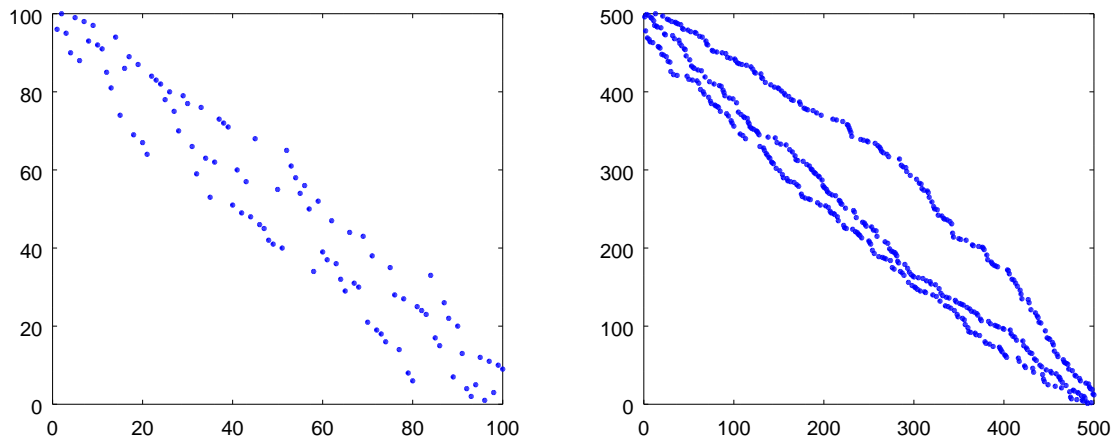

Figure 1: Randomly generated 1234-avoiding permutation with N = 100 on the left and N= 500 on the right figure

We shall typically consider the case $J \ll N - I$ (i.e., points "below the diagonal"); when $\tau = \mu_k$, the case $J \gg N - I$ follows from symmetry considerations. Since we know the asymptotics of the denominator $|\mathcal{S}_N(\tau)|$ for our patterns of interest, and since our methods are essentially combinatorial, we shall henceforth discuss only the numerator, dealing directly with $|\mathcal{F}_N(I, J; \tau)|$ and related combinatorial quantities.

Theorem 3. *Fix $k \geqslant 4$ and $1 \leqslant \ell \leqslant k - 2$. Let γ, δ, I_N and J_N be as specified in Condition 1. Then*

$$\lim_{N \to \infty} |\mathcal{F}_N(I_N, J_N; \mu_k)|^{1/N} = G(\gamma, 1 - \delta; (k-2)^2) \tag{6}$$

$$= \lim_{N \to \infty} |\mathcal{F}_N(I_N, J_N; \lambda_{k,\ell})|^{1/N}, \tag{7}$$

where we define

$$G(u, v; c) := 4c\, g(u, v; c)g(v, u; c)g(1 - u, 1 - v; c)g(1 - v, 1 - u; c) \tag{8}$$

$$and \qquad g(x, y; c) := \left(\frac{2cx + (y - x) - \sqrt{(y - x)^2 + 4cxy}}{x(c - 1)}\right)^{-x}. \tag{9}$$

Figure 2 gives an example of the level curves of $G(u, v; c)$ for $(u, v) \in [0, 1]^2$ and $c = 4$.
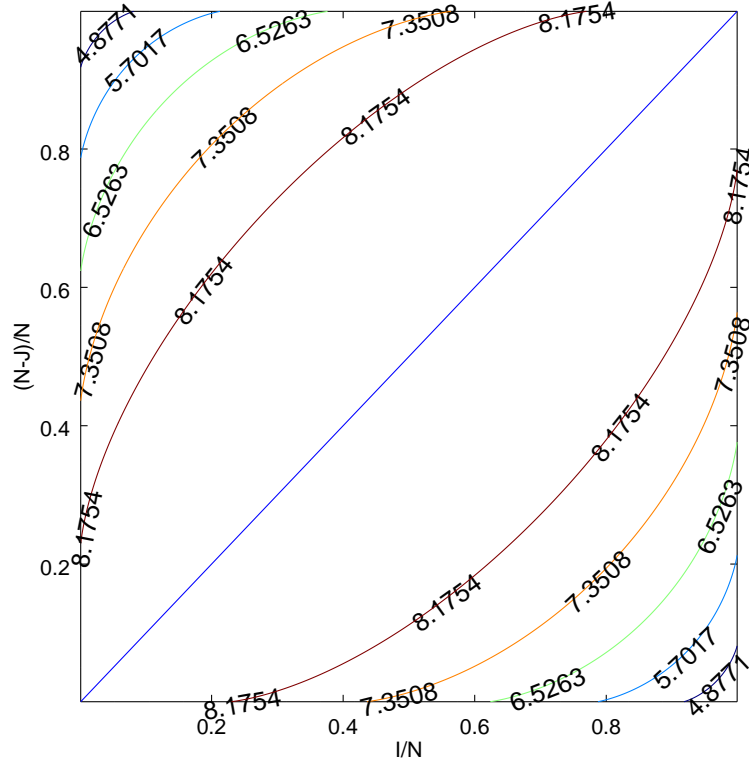

Figure 2: Level curves in $[0, 1]^2$ for $G(u, v; 4)$, showing limit values of $|\mathcal{F}_N(I, J; 1234)|^{1/N}$. To obtain limit values of $P_N^{1234}(\sigma_{I_N} = J_N)^{1/N}$, divide the displayed values by 9.

Remark 4. When $J_N \approx N - I_N$ (i.e., when we are close to the diagonal), then we are in the (limiting) case $\gamma = 1 - \delta$. This is not a "large deviation," since $G(u, u; (k-2)^2) = L(\mu_k)$; indeed,

$$g(x, x; c) = \left(\frac{2cx - \sqrt{4cx^2}}{x(c-1)} \right)^{-x} = \left(\frac{2\sqrt{c}(\sqrt{c} - 1)}{c - 1} \right)^{-x} = \left(\frac{2\sqrt{c}}{\sqrt{c} + 1} \right)^{-x},$$

and it follows that

$$G(u, u; c) = 4c \left(\frac{\sqrt{c} + 1}{2\sqrt{c}} \right)^2 = (\sqrt{c} + 1)^2,$$

which equals $(k-1)^2$ when we substitute $c = (k-2)^2$. The regime $|N - I_N - J_N| = o(N)$ is examined by Fineman, Slivken, Rizzolo, and Hoffman (in preparation).

Remark 5. The numerator and denominator inside the parentheses in Equation (9) are both 0 when we set $c = 1$. Therefore we define $g(x, y; 1)$ by taking the limit of $g(x, y; c)$ as $c \to 1^+$. We then obtain

$$g(x, y; 1) = \left(\frac{2x}{x + y} \right)^{-x}$$

which in turn implies that $G(u, v; 1)$ is given by Equation (4). Thus our Theorem 3 formally recovers Theorem 2.

Remark 6. Assume that γ, δ, I_N and J_N are as in Condition 1 except that $\gamma > 1 - \delta$. Then Equation (6) still holds (by symmetry), while $\lim_{N \to \infty} |\mathcal{F}_N(I_N, J_N; \lambda_{k,\ell})|^{1/N} = (k-1)^2$ by Proposition 3.1 of [2] (i.e., $\lim_{N \to \infty} P_N^{\lambda_{k,\ell}}(\sigma_{I_N} = J_N)^{1/N} = 1$).

The term $(k-2)^2$ appears in Equations (6) and (7) because it is the value of $L(\mu_{k-1})$. This is highlighted and generalized in Theorem 9 below.

Definition 7. Let N and A be positive integers, and let τ be a fixed permutation. Define

$$\mathcal{S}_N^{*A}(\tau) := \{ \sigma \in \mathcal{S}_N(\tau) : \sigma_i > N - i - A \text{ for every } i = 1, \dots, N \}.$$

Thus, the graph of a permutation in $\mathcal{S}_N^{*A}(\tau)$ has no point that is more than A units below $\{(i, N + 1 - i) : 1 \leqslant i \leqslant N\}$, the decreasing diagonal of $[1, N]^2$.

Then Theorem 1.2 of [2] implies that for every $\epsilon > 0$, $|\mathcal{S}_N^{*N\epsilon}(123)|/|\mathcal{S}_N(123)|$ and $|\mathcal{S}_N^{*N\epsilon}(132)|/|\mathcal{S}_N(132)|$ converge to 1 exponentially rapidly as $N \to \infty$.

Definition 8. For $\omega \in \mathcal{S}_m$, let $1 \oslash \omega$ be the permutation $1(\omega_1 + 1)(\omega_2 + 1) \dots (\omega_m + 1)$ in $\mathcal{S}_{m+1}$.

For example, $1 \oslash 3124 = 14235$. Observe that $1 \oslash \mu_{k-1} = \mu_k$ and $1 \oslash \lambda_{k-1,\ell-1} = \lambda_{k,\ell}$.

Most of the present paper will focus on the proof of the following theorem.

Theorem 9. *Let $\hat{\tau}$ be a pattern of length 3 or more, and assume that*

$$\lim_{N \to \infty} |\mathcal{S}_N^{*N\epsilon}(\hat{\tau})|^{1/N} = L(\hat{\tau}) \quad \text{for every } \epsilon > 0. \tag{10}$$

Let $\tau = 1 \oslash \hat{\tau}$. Let γ, δ, I_N and J_N be as specified in Condition 1. Then

$$\lim_{N \to \infty} |\mathcal{F}_N(I_N, J_N; \tau)|^{1/N} = G(\gamma, 1 - \delta; L(\hat{\tau})). \tag{11}$$

Remark 10. (a) Theorem 1.2 of [2] implies that Equation (10) holds for μ_3 and $\lambda_{3,1}$.
(b) Theorem 1.3(b) of [2] implies that if Equation (10) holds, then $\hat{\tau}_1$ must equal 1. The converse of this statement has neither been proved nor disproved; however, simulations in [2] and [11] suggest that (10) is false for $\hat{\tau} = 1324$.

As we shall see in Section 4, Theorem 3 follows from Theorem 9 by induction on k, with Remark 10(a) leading to the base case $k = 4$. The idea behind the proof of Theorem 9 consists of three main steps. An important role is played by the set $\mathcal{F}_N^*(I, J; \tau)$ of permutations in $\mathcal{F}_N(I, J; \tau)$ for which (I, J) is a left-to-right minimum (i.e., $\sigma_i > J$ for all $i < I$). The first step is to derive an explicit upper bound to show that $|\mathcal{F}_N^*(I, J; \tau)|^{1/N}$ is less than or equal to $G(\gamma, 1 - \delta; L(\hat{\tau}))$ in the limit. The second step is to use monotonicity of G to show that we can replace $\mathcal{F}^*$ by $\mathcal{F}$ in the preceding assertion. The third step uses the dominant terms from the upper bound of the first step to construct a lower bound on $|\mathcal{F}_N(I, J; \tau)|^{1/N}$ that is arbitrarily close to the upper bound. Section 2 carries out the first two steps, while Section 3 performs the third step. Section 4 ties the pieces together to complete the proofs of the two theorems. Section 1.1 presents some basic definitions and a useful lemma.

We close this section with a physical analogy to help visualize our results about μ_k. It is well known that an N-element permutation σ is in $\mathcal{S}_N(\mu_k)$ if and only if σ can be partitioned into $k - 1$ decreasing subsequences. It is not hard to see that these decreasing subsequences are all likely to stay close to the decreasing diagonal of $[1, N]^2$. Think of the subsequences as $k - 1$ elastic strings, each with one end tied to the point $(1, N)$ and the other end tied to $(N, 1)$, and each string tight. Requiring σ_I to equal J is like forcing one of the strings to pass through the point (I, J). With this constraint, the rest of the string deforms into two line segments, one from $(1, N)$ to (I, J) and the other from (I, J) to $(N, 1)$. Tension in the string dictates how the mass of the string is balanced among the two segments, and the mass is evenly distributed within each segment. This physical picture parallels our lower bound construction in Section 3.

1.1 Some Formalities and Preliminaries

For a string ω of length k whose entries are all distinct numbers, let $\mathbf{Patt}(\omega)$ be the permutation in $\mathcal{S}_k$ that has the same relative order as ω. E.g., $\mathbf{Patt}(91734) = 51423$. More precisely, $\mathbf{Patt}(\omega_1 \omega_2 \ldots \omega_k)$ is the unique permutation π in $\mathcal{S}_k$ with the property that for all $i, j \in \{1, \ldots, k\}$, $\omega_i < \omega_j$ if and only if $\pi_i < \pi_j$.

Assume $\tau \in \mathcal{S}_k$ and $\sigma \in \mathcal{S}_N$. We say that σ contains the pattern τ if there exists $1 \leqslant I_1 < I_2 < \cdots < I_k \leqslant N$ such that $\mathbf{Patt}(\sigma_{I_1} \sigma_{I_2} \ldots \sigma_{I_k}) = \tau$. We say that σ avoids the pattern τ if σ does not contain τ. We write $\mathcal{S}_N(\tau)$ for the set of all permutations in $\mathcal{S}_N$ that avoid τ.

For functions f and g, we write $f \sim g$ to mean $\lim_{N \to \infty} f(N)/g(N) = 1$.

Definition 11. A finite subset of $\mathbb{Z}^2$ is said to be *decreasing* if it can be written in the form $\{(x(m), y(m)) : m = 1, \ldots, w\}$ with $x(1) < x(2) < \cdots < x(w)$ and $y(1) > y(2) > \cdots > y(w)$ for some $w \geqslant 0$.

We shall also use the following well-known results.

Lemma 12. *(i) Let s and t be integers satisfying $0 \leqslant s \leqslant t$. Then*

$$\binom{t}{s} \leqslant \frac{t^t}{s^s(t - s)^{t-s}} \, .$$

(ii) Let $\{s_N\}$ and $\{t_N\}$ be sequences of integers with $0 \leqslant s_N \leqslant t_N$ such that $\lim_{N\to\infty} s_N/N = S$ and $\lim_{N\to\infty} t_N/N = T$. Then

$$\lim_{N\to\infty} \binom{t_N}{s_N}^{1/N} = \frac{T^T}{S^S(T-S)^{T-S}} \,.$$

In this lemma, we interpret 0^0 to be 1.

Proof: Part (*ii*) follows from Stirling's formula, and part (*i*) is proven for example in Lemma 2.1(b) in [2]. $\qquad\square$

2   The Upper Bound

We begin with some definitions. For a given permutation σ, define

$$\mathcal{M} \equiv \mathcal{M}(\sigma) := \{(i, \sigma_i) : \sigma_i < \sigma_t \text{ for every } t < i\}. \tag{12}$$

That is, $\mathcal{M}$ is the set of points of the graph of σ corresponding to left-to-right minima. Next, let $\sigma \setminus \mathcal{M}$ be the string consisting of those σ_t such that $(t, \sigma_t) \notin \mathcal{M}(\sigma)$. Figure 3 shows an example. More generally, if $\mathcal{A}$ is a subset of $\mathbb{Z}^2$, let $\sigma \setminus \mathcal{A}$ denote the string consisting of those σ_t such that $(t, \sigma_t) \notin \mathcal{A}$.
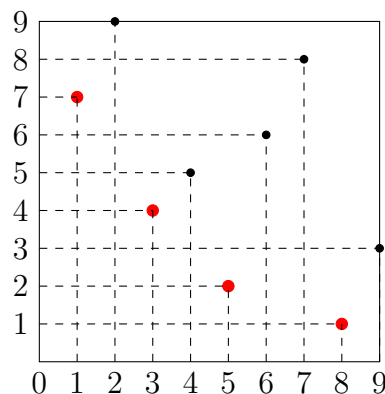

Figure 3: The graph of $\sigma = 794526813 \in S_9$. Here, $\mathcal{M} = \{(1,7), (3,4), (5,2), (8,1)\}$ and $\sigma \setminus \mathcal{M} = 95683$.

The following observations are useful. We omit the straightforward proof.

Lemma 13. *(i) A permutation σ is uniquely determined by the set $\mathcal{M}$ and the permutation $\mathbf{Patt}(\sigma \setminus \mathcal{M})$.*
(ii) Let $\hat{\tau}$ be a pattern with $\hat{\tau}_1 = 1$. The permutation σ avoids $1 \oslash \hat{\tau}$ if and only if $\mathbf{Patt}(\sigma \setminus \mathcal{M})$ avoids $\hat{\tau}$.

Recall from Section 1 that

$$\mathcal{F}_N^*(I, J; \tau) \;=\; \{\sigma \in \mathcal{F}_N(I, J; \tau) \,:\, \sigma_i > J \text{ for all } i < I\,\}.$$

We shall now perform the first step in the proof of our main theorem.

Proposition 14. *Let $\hat{\tau}$ be a pattern of length 3 or more such that $\hat{\tau}_1 = 1$, and let $\tau = 1 \oslash \hat{\tau}$. Let γ, δ, I_N and J_N be as specified in Condition 1. Then*

$$\limsup_{N \to \infty} |\mathcal{F}_N^*(I_N, J_N; \tau)|^{1/N} \;\leqslant\; G(\gamma, 1 - \delta; L(\hat{\tau})). \tag{13}$$

Proof: For $I \in [1, N]$ and $\sigma \in \mathcal{S}_N$, we define

$$\mathcal{M}^{<I} \;=\; \{(i, \sigma_i) \in \mathcal{M}(\sigma) : i < I\} \quad \text{and} \quad \mathcal{M}^{>I} \;=\; \{(i, \sigma_i) \in \mathcal{M}(\sigma) : i > I\}\,.$$

Fix I and J in $[1, N]$ with $J < N - I$. Suppose we know that $\sigma \in \mathcal{F}_N^*(I, J; \tau)$, $l = |\mathcal{M}^{<I}|$ and $m = |\mathcal{M}^{>I}|$. Then $\mathcal{M}^{<I}$ is a set of l integral points in $[1, I) \times (J, N]$, and this set must be decreasing (recall Definition 11). Therefore there are at most $\binom{I-1}{l}\binom{N-J}{l}$ possible realizations of $\mathcal{M}^{<I}$. Similarly, there are at most $\binom{N-I}{m}\binom{J-1}{m}$ possibilities for $\mathcal{M}^{>I}$. Recalling Lemma 13, we obtain the following bound:

$$\begin{aligned}
|\mathcal{F}_N^*(I, J; \tau)| & \\
&\leqslant \sum_{l=0}^{I-1} \sum_{m=0}^{J-1} \binom{I-1}{l}\binom{N-J}{l}\binom{N-I}{m}\binom{J-1}{m} |\mathcal{S}_{N-l-m-1}(\hat{\tau})| \\
&\leqslant H(I-1, N-J; L(\hat{\tau}))\, H(J-1, N-I; L(\hat{\tau}))\, L(\hat{\tau})^{N-1} \tag{14}
\end{aligned}$$

where we define

$$H(a, b; c) \;:=\; \sum_{n=0}^{a} \binom{a}{n}\binom{b}{n} c^{-n}\,. \tag{15}$$

In the last step, the bound $|\mathcal{S}_{N-l-m-1}(\hat{\tau})| \leqslant L(\hat{\tau})^{N-l-m-1}$ is proven in Theorem 1 in [1].

We now wish to bound $H(a, b; c)$ for $a \leqslant b$ and $c > 1$. By Lemma 12(i), we have

$$H(a, b; c) \;\leqslant\; (a + 1) \sup\{f(y; a, b, c) : 0 \leqslant y \leqslant a\} \tag{16}$$

where

$$f(y; a, b, c) \;=\; \left(\frac{y}{a}\right)^{-y} \left(1 - \frac{y}{a}\right)^{y-a} \left(\frac{y}{b}\right)^{-y} \left(1 - \frac{y}{b}\right)^{y-b} c^{-y}\,. \tag{17}$$

We now pause to state and prove a lemma, which will also be useful later.

Lemma 15. *Fix real numbers $a, b > 0$ and $c > 1$. Define the function f as in Equation (17) for real y in the interval $[0, a \wedge b]$ (where $a \wedge b$ is the minimum of a and b). We interpret $0^0 = 1$, which makes f continuous on this interval. Then there is a unique point $y^* \equiv y^*[a, b, c]$ that maximizes f in this interval, and $0 < y^* < a \wedge b$. Furthermore,*

$$y^*[a, b, c] \;=\; \frac{\sqrt{(a - b)^2 + 4cab} - (a + b)}{2(c - 1)} \tag{18}$$

and the maximum value of f is

$$f(y^*[a, b, c]; a, b, c) = 2^{a+b} g(a, b; c) g(b, a; c), \tag{19}$$

where g was defined in Equation (9).

Proof: By calculus, $\log f$ is a strictly concave function of y on $[0, a \wedge b]$, and is maximized at the (unique) point $y^* \equiv y^*[a, b, c]$ in $(0, a \wedge b)$ that satisfies the equation

$$(a - y^*)(b - y^*) = c(y^*)^2. \tag{20}$$

Thus Equation (17) becomes

$$
\begin{aligned}
f(y^*; a, b, c) &= \frac{a^a \, b^b}{(y^*)^{2y^*} (a - y^*)^{a-y^*} (b - y^*)^{b-y^*} c^{y^*}} \\
&= \left(1 - \frac{y^*}{a}\right)^{-a} \left(1 - \frac{y^*}{b}\right)^{-b} \quad \text{(using (20)).}
\end{aligned}
\tag{21}
$$

Solving the quadratic equation (20) for the positive root gives

$$y^*[a, b, c] = \frac{\sqrt{(a + b)^2 + 4(c - 1)ab} - (a + b)}{2(c - 1)}, \tag{22}$$

which leads to Equation (18). Finally, inserting (18) into (21) gives (19). $\qquad\square$

We now return to the proof of Proposition 14. By Equation (16) and Lemma 15, we have

$$H(I - 1, N - J; c) \leqslant I \, 2^{N+I-J-1} g(I - 1, N - J; c) \, g(N - J, I - 1; c). \tag{23}$$

By Equation (1) and the explicit form of g, we can take the limit in Equation (23) to get

$$\limsup_{N \to \infty} H(I_N - 1, N - J_N; c)^{1/N} \leqslant 2^{1+\gamma-\delta} g(\gamma, 1 - \delta; c) \, g(1 - \delta, \gamma; c).$$

Similarly, we have

$$\limsup_{N \to \infty} H(J_N - 1, N - I_N; c)^{1/N} \leqslant 2^{1+\delta-\gamma} g(\delta, 1 - \gamma; c) \, g(1 - \gamma, \delta; c).$$

Proposition 14 follows directly from the above (with $c = L(\hat{\tau})$) and Equation (14). $\qquad\square$

Our next task is to replace $\mathcal{F}_N^*$ by $\mathcal{F}_N$ in the statement of Proposition 14. We shall do this by proving a monotonicity property of G (Lemma 17) and then using a compactness argument.

Proposition 16. *Under the hypotheses of Proposition 14, we have*

$$\limsup_{N \to \infty} |\mathcal{F}_N(I_N, J_N; \tau)|^{1/N} \leqslant G(\gamma, 1 - \delta; L(\hat{\tau})). \tag{24}$$

We begin by showing that G decreases as we move away from the diagonal. We emphasize that in this lemma, "increasing" and "decreasing" are used in their strict sense.

Lemma 17. *Fix $c > 1$. The function $G(u, v; c)$ defined in Equation (8) is increasing in u and decreasing in v for $0 < u < v < 1$. By symmetry, it is also increasing in v and decreasing in u for $0 < v < u < 1$. In particular, G is maximized when $u = v$, where we have*

$$G(u, u; c) = (\sqrt{c} + 1)^2 \quad \text{for every } u \in (0, 1). \tag{25}$$

Proof: Recall that Equation (25) was proved in Remark 4.

Since c is fixed, we shall suppress it in the following notation. Let $r(u, v) = \sqrt{(v - u)^2 + 4cuv}$ and $h(u, v) = [2cu + (v - u) - r(u, v)]/u$. Then

$$G(u, v; c) = 4c(c - 1)^2 h(u, v)^{-u} h(v, u)^{-v} h(1{-}u, 1{-}v)^{1-u} h(1{-}v, 1{-}u)^{1-v}$$

and hence

$$\begin{aligned} \ln G(u, v; c) = {} & \ln(4c(c - 1)^2) - u \ln(h(u, v)) - v \ln(h(v, u)) \\ & - (1 - u) \ln(h(1{-}u, 1{-}v)) - (1 - v) \ln(h(1{-}v, 1{-}u)) \, . \end{aligned} \tag{26}$$

By routine calculus and some algebraic manipulation, we obtain

$$\frac{\partial}{\partial u} \ln(h(u, v)) = \frac{v}{u \, r(u, v)} \quad \text{and} \quad \frac{\partial}{\partial u} \ln(h(v, u)) = -\frac{1}{r(u, v)} \, . \tag{27}$$

Using this and Equation (26), we can show that

$$\frac{\partial}{\partial u} \ln G(u, v; c) = -\ln(h(u, v)) + \ln(h(1{-}u, 1{-}v)) \, . \tag{28}$$

From this and Equation (27), we also obtain

$$\frac{\partial^2}{\partial u^2} \ln G(u, v; c) = -\frac{v}{u \, r(u, v)} - \frac{(1 - v)}{(1 - u) \, r(1{-}u, 1{-}v)} < 0$$

for every u and v in $(0, 1)$. Therefore $G(u, v; c)$ is strictly concave in u for fixed v (and, by symmetry, it is strictly concave in v for fixed u).

Since $h(u, u) = 2c - 2\sqrt{c}$ for every u, it follows that the partial derivative in Equation (28) is zero whenever $u = v$. By symmetry, the same is true for the partial derivative with respect to v. Combining this with the concavity result of the previous paragraph completes the proof of the lemma. $\square$

Proof of Proposition 16: It is easy to see that

$$\mathcal{F}_N(I_N, J_N; \tau) \subseteq \bigcup_{1 \leqslant u \leqslant I_N, 1 \leqslant t \leqslant J_N} \mathcal{F}_N^*(u, t; \tau).$$

Let $u(N)$ and $t(N)$ be the values of u and t that maximize $|\mathcal{F}_N^*(u, t; \tau)|$ over u in $[1, I_N]$ and t in $[1, J_N]$. Then we have

$$|\mathcal{F}_N(I_N, J_N; \tau)| \leqslant N^2 |\mathcal{F}_N^*(u(N), t(N); \tau)|. \tag{29}$$

Let $LS = \limsup_{N \to \infty} |\mathcal{F}_N^*(u(N), t(N); \tau)|^{1/N}$. There exists a subsequence N' such that $|\mathcal{F}_{N'}^*(u(N'), t(N'); \tau)|^{1/N'}$ converges to LS. By compactness of $[0, 1]^2$, this subsequence has a sub-subsequence N'' for which
$(u(N'')/N'', t(N'')/N'')$ converges to a point $(\tilde{u}, \tilde{t})$ in $[0, \gamma] \times [0, \delta]$. Thus Proposition 14 tells us that $LS \leqslant G(\tilde{u}, 1 - \tilde{t}; L(\hat{\tau}))$. The monotonicity of G (in Lemma 17) implies that $G(\tilde{u}, 1 - \tilde{t}; L(\hat{\tau})) \leqslant G(\gamma, 1 - \delta; L(\hat{\tau}))$. Therefore $LS \leqslant G(\gamma, 1 - \delta; L(\hat{\tau}))$. Hence, using Equation (29), we obtain Equation (24). $\qquad \square$

3 The Lower Bound

To get the lower bound on $|\mathcal{F}_N(I, J; \tau)|$, we shall perform an explicit construction of some permutations in $\mathcal{F}_N^*(I, J; \tau)$ (this is done in the proof of Proposition 20 below). The construction is motivated by examining the dominant terms in our proof of the upper bound, and showing that they are approximately achieved.

The main result of this section is the following.

Proposition 18. *Under the hypotheses of Theorem 9, we have*

$$\liminf_{N \to \infty} |\mathcal{F}_N^*(I_N, J_N; \tau)|^{1/N} \geqslant G(\gamma, 1 - \delta; L(\hat{\tau})). \tag{30}$$

The proof of Proposition 18 relies on Proposition 20 and Lemma 21. We shall first state these two auxiliary results, then prove Proposition 18, and conclude the section by proving the two auxiliary results.

The construction of Proposition 20 uses a positive parameter A, which will afterwards be of the order $N\epsilon$ for fixed small ϵ. We start with a definition.

Definition 19. Let w, M_1, and M_2 be positive integers, with $w \leqslant M_1 \wedge M_2$.
• Let $\mathbf{Dec}(w; M_1, M_2)$ be the collection of all w-element decreasing subsets of $\{1, \ldots, M_1\} \times \{1, \ldots, M_2\}$. (Recall Definition 11.)
• For given $A > 0$, let $\mathbf{Dec}^{*A}(w; M_1, M_2)$ be the collections of all w-element sets $\mathcal{B} \in \mathbf{Dec}(w; M_1, M_2)$ such that

$$y < M_2 - x \frac{M_2}{M_1} + A \quad \text{for all } (x, y) \in \mathcal{B}. \tag{31}$$

The collections $\mathbf{Dec}(0; M_1, M_2)$ and $\mathbf{Dec}^{*A}(0; M_1, M_2)$ each contain one member: the empty set.

Observe that the line $y = M_2 - xM_2/M_1$ is the decreasing diagonal of the rectangle $[0, M_1] \times [0, M_2]$. Thus, $\mathbf{Dec}^{*A}(w; M_1, M_2)$ is the collection of sets in $\mathbf{Dec}(w; M_1, M_2)$ that rise less than A above the diagonal.

Proposition 20. *Let $\hat{\tau}$ be a pattern of length 3 or more such that $\hat{\tau}_1 = 1$, and let $\tau = 1 \oslash \hat{\tau}$. Let N, I, J, and A be positive integers with $J < N - I - 2A$. Let w_1 and w_2 be integers with*

$$0 \leqslant w_1 \leqslant I - 1 \quad and \quad 0 \leqslant w_2 \leqslant J - 1. \tag{32}$$

Then (recall Definitions 7 and 19)

$$
\begin{aligned}
|\mathcal{F}_N^*(I, J; \tau)| \geqslant \ & |\mathbf{Dec}^{*A}(w_1; I - 1, N - 2A - J)| \\
& \times \ |\mathbf{Dec}^{*A}(w_2; N - 2A - I, J - 1)| \\
& \times \ |\mathcal{S}_{N-w_1-w_2-1}^{*A}(\hat{\tau})|.
\end{aligned} \tag{33}
$$

Lemma 21. *Consider sequences of positive integers $w(N)$, $M_1(N)$, $M_2(N)$, and A_N such that*

$$\lim_{N\to\infty} \frac{w(N)}{N} = \theta, \quad \lim_{N\to\infty} \frac{M_1(N)}{N} = \alpha, \quad \lim_{N\to\infty} \frac{M_2(N)}{N} = \beta, \quad \lim_{N\to\infty} \frac{A_N}{N} = \epsilon,$$

with $0 < \theta < \alpha \wedge \beta$ and $\epsilon > 0$. Then

$$\lim_{N\to\infty} \frac{|\mathbf{Dec}^{*A_N}(w(N); M_1(N), M_2(N))|}{|\mathbf{Dec}(w(N); M_1(N), M_2(N))|} = 1 \tag{34}$$

and (for f defined by Equation (17))

$$f(\theta; \alpha, \beta, c) \, c^\theta = \lim_{N\to\infty} |\mathbf{Dec}(w(N); M_1(N), M_2(N))|^{1/N} \tag{35}$$

$$= \lim_{N\to\infty} |\mathbf{Dec}^{*A_N}(w(N); M_1(N), M_2(N))|^{1/N} \tag{36}$$

for any c. (Notice that $f(\theta; \alpha, \beta, c)c^\theta$ is independent of c by definition.)

Proof of Proposition 18: Let $c = L(\hat{\tau})$. Choose $\epsilon > 0$ such that $\gamma < 1 - \delta - 2\epsilon$. Let $\{A_N\}$ be a sequence of positive integers such that $\lim_{N\to\infty} A_N/N = \epsilon$. Therefore $J_N < N - I_N - 2A_N$ holds for all sufficiently large N.

Let $\{w_1(N)\}$ and $\{w_2(N)\}$ be sequences of positive integers such that

$$\lim_{N\to\infty} \frac{w_1(N)}{N} = y^*[\gamma, 1 - \delta - 2\epsilon, c] =: y_1^* \quad \text{and}$$

$$\lim_{N\to\infty} \frac{w_2(N)}{N} = y^*[1 - \gamma - 2\epsilon, \delta, c] =: y_2^*.$$

Lemma 15 assures us that $y_1^* < \gamma \wedge (1 - \delta - 2\epsilon)$ and $y_2^* < (1 - \gamma - 2\epsilon) \wedge \delta$, and therefore Equation (32) holds for all sufficiently large N (where I is interpreted to be I_N, etc.). Using these sequences in Proposition 20 and invoking Lemma 21 and Equations (19) and (10), we see that the Nth root of the right hand side of Equation (33) converges to

$$
\begin{aligned}
& 2^{\gamma + 1 - \delta - 2\epsilon} g(\gamma, 1 - \delta - 2\epsilon; c) \, g(1 - \delta - 2\epsilon, \gamma; c) \, c^{y_1^*} \\
& \times \ 2^{1 - \gamma + \delta - 2\epsilon} g(1 - \gamma - 2\epsilon, \delta; c) \, g(\delta, 1 - \gamma - 2\epsilon; c) \, c^{y_2^*} \times c^{1 - y_1^* - y_2^*}.
\end{aligned} \tag{37}
$$

Thus Equation (37) is a lower bound for $\liminf_{N\to\infty} |\mathcal{F}_N^*(I_N, J_N; \tau)|^{1/N}$ for all sufficiently small positive ϵ. Now let ϵ decrease to 0. By the continuity of g, the expression of Equation (37) converges to $G(\gamma, 1 - \delta; c)$. This proves the proposition. $\qquad\square$

Proof of Proposition 20: Fix N, I, J, A, w_1 and w_2 as specified. We shall prove the proposition by constructing an injection from $\mathcal{D}$ into $\mathcal{F}_N^*(I, J; \tau)$, where

$$\begin{aligned}
\mathcal{D} \;=\; &\mathbf{Dec}^{*A}(w_1; I - 1, N - 2A - J) \times \mathbf{Dec}^{*A}(w_2; N - 2A - I, J - 1) \\
&\times \; \mathcal{S}_{N - w_1 - w_2 - 1}^{*A}(\hat{\tau}).
\end{aligned}$$

Consider $(\mathcal{B}_1, \mathcal{B}_2, \phi) \in \mathcal{D}$ (that is, $\mathcal{B}_1$ is one of the w_1-element sets in $\mathbf{Dec}^{*A}(w_1; I - 1, N - 2A - J)$, and so on). Let $\Psi \equiv \Psi(\mathcal{B}_1, \mathcal{B}_2)$ be the $(w_1 + w_2 + 1)$-element decreasing set defined by

$$(\mathcal{B}_1 + (0, J)) \cup \{(I, J)\} \cup (\mathcal{B}_2 + (I, 0))$$

(where $\mathcal{B} + (x, y)$ denotes translation of the set $\mathcal{B}$ by the vector (x, y)). Thus Ψ is a decreasing subset of $[1, N - 2A] \times [1, N - 2A]$ that contains (I, J).

We claim that

$$y \;<\; N - x - A \quad \text{for every } (x, y) \in \Psi. \tag{38}$$

For $(x, y) = (I, J)$, this follows from our assumption $J < N - I - 2A$. For (x, y) in $\mathcal{B}_1 + (0, J)$, we have $(x, y - J) \in \mathcal{B}_1$ and hence

$$y - J \;<\; (N - 2A - J) - x \,\frac{N - 2A - J}{I - 1} + A \;<\; N - A - J - x$$

(using $I < N - 2A - J$), which verifies the claim in this case. A similar argument works if $(x, y) \in \mathcal{B}_2 + (I, 0)$. Therefore the claim (38) is true.

Given Ψ and a permutation $\phi \in \mathcal{S}_{N - w_1 - w_2 - 1}^{*A}(\hat{\tau})$, we shall define a permutation $\sigma \in \mathcal{S}_N$ such that Ψ is contained in the graph of σ (i.e., $y = \sigma_x$ whenever $(x, y) \in \Psi$) and $\mathbf{Patt}(\sigma \setminus \Psi) = \phi$. Let $w = w_1 + w_2 + 1$, and write the elements of Ψ as $(x(\ell), y(\ell))$ ($\ell = 1, \ldots, w$) with $x(\ell)$ increasing in ℓ and $y(\ell)$ decreasing in ℓ. Define the functions Γ_x and Γ_y from $\{1, \ldots, N - w\}$ into $\{1, \ldots, N\}$ as follows. Writing $x(0) = 0$ and $x(w + 1) = N + 1$, and observing that $x(\ell) - \ell$ is decreasing in ℓ, we define

$$\begin{aligned}
\Gamma_x(i) \;=\; i + m \quad &\text{where } m \text{ satisfies } x(m) - m < i \leqslant x(m + 1) - (m + 1); \\
&\text{i.e., where } m \text{ satisfies } x(m) < i + m < x(m + 1).
\end{aligned}$$

The possible values for m are $0, 1, \ldots, w$. Analogously, writing $y(0) = N + 1$ and $y(w + 1) = 0$, we define

$$\begin{aligned}
\Gamma_y(i) \;=\; i + n \quad &\text{where } n \text{ satisfies } y(w - n + 1) - n + 1 \leqslant i < y(w - n) - n; \\
&\text{i.e., where } n \text{ satisfies } y(w - n + 1) < i + n < y(w - n).
\end{aligned}$$

Again, the possible values for n range from 0 to w. Observe that Γ_x (respectively, Γ_y) is the unique strictly increasing function from $\{1, \ldots, N - w\}$ to $\{1, \ldots, N\} \setminus \{x(1), \ldots, x(w)\}$

(respectively, $\{1, \ldots, N\} \setminus \{y(1), \ldots, y(w)\}$). Now define $\sigma_1, \ldots, \sigma_N$ by

$$
\begin{aligned}
\sigma_{x(\ell)} &= y(\ell) \quad \text{for } \ell = 1, \ldots, w, \\
\sigma_{\Gamma_x(i)} &= \Gamma_y(\phi_i) \quad \text{for } i = 1, \ldots, N - w.
\end{aligned}
$$

Then the string $\sigma := \sigma_1 \sigma_2 \ldots \sigma_N$ is well defined and σ is a permutation in $\mathcal{S}_N$ whose graph contains Ψ, and that $\mathbf{Patt}(\sigma \setminus \Psi) = \phi$. See Figure 4.
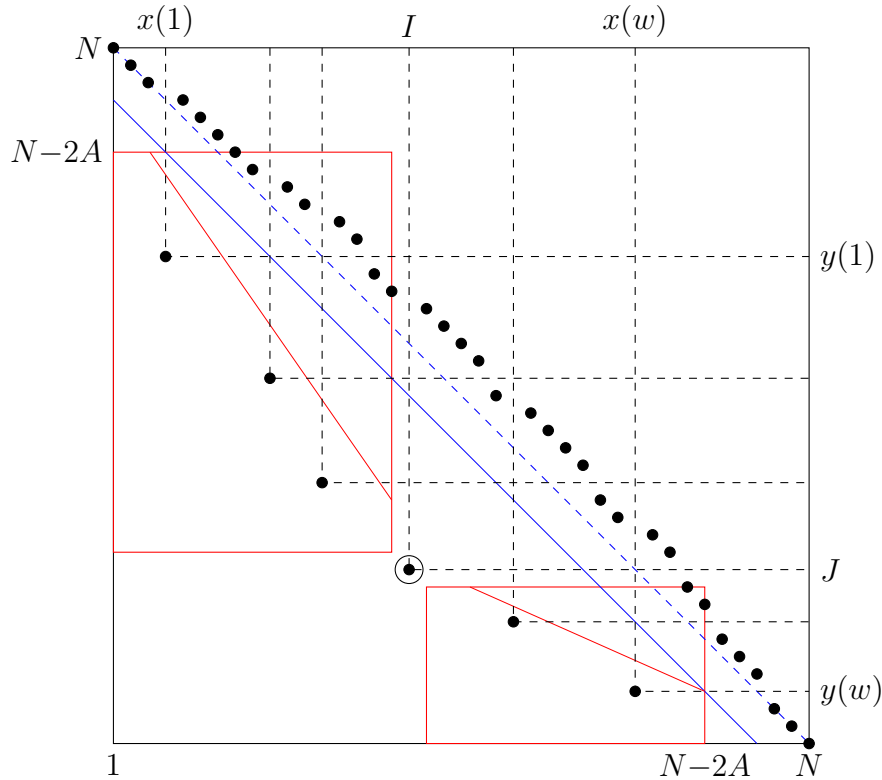

Figure 4: An example of the permutation σ constructed in the proof of Proposition 20, in which $N = 41$, $w_1 = 3$, $w_2 = 2$, $w = 6$, and $A = 3$, and the permutation ϕ is the decreasing permutation of length $N - w$. The circled black dot is at (I, J). The dashed blue line is the diagonal of $[1, N]^2$. The two red rectangles enclose $\mathcal{B}_1 + (0, J)$ and $\mathcal{B}_2 + (I, 0)$. The sloped red line segment within each red rectangle is drawn A units above the diagonal of the rectangle. No point of Ψ is above a sloped red line segment. The solid blue line is the line $y = N - x - A$, which partitions the graph of σ as described in the Key Claim in the proof. The two sloped red line segments lie below the solid blue line. Observe that $I = x(w_1 + 1)$ and $J = y(w_1 + 1)$.

The proof of the proposition is based on the following claim. Let $\Psi_x = \{x(1), x(2), \ldots, x(w)\}$.

Key Claim: *We have $\sigma_j < N - j - A$ for every $j \in \Psi_x$, and $\sigma_j > N - j - A$ for every $j \notin \Psi_x$.*

Once the Key Claim is proven, we proceed as follows. The Key Claim implies that $\Psi \subset \mathcal{M}(\sigma)$ (recall Equation (12)). Therefore, since $Patt(\sigma \setminus \Psi)$ avoids $\hat{\tau}$, so does $Patt(\sigma \setminus \mathcal{M}(\sigma))$. Hence, by Lemma 13$(ii)$, σ avoids τ. It follows that $\sigma \in \mathcal{F}_N^*(I, J; \tau)$. Consequently, writing $Q(\mathcal{B}_1, \mathcal{B}_2, \phi) = \sigma$, we have defined a function $Q : \mathcal{D} \to \mathcal{F}_N^*(I, J; \tau)$. To see that the function Q is one-to-one, suppose $Q(\mathcal{B}_1, \mathcal{B}_2, \phi) = \sigma$. Since Ψ is contained in the graph of σ, the Key Claim shows that $\Psi(\mathcal{B}_1, \mathcal{B}_2)$ is uniquely determined by σ, as is ϕ. Finally, since (I, J) is specified, $\mathcal{B}_1$ and $\mathcal{B}_2$ are determined by $\Psi(\mathcal{B}_1, \mathcal{B}_2)$. Hence Q is one-to-one, and the proposition follows.

It only remains to prove the Key Claim. For $j \in \Psi_x$, say $j = x(\ell)$, we have $\sigma_j = y(\ell)$, and the assertion of the Key Claim follows from Equation (38). Now suppose $j \notin \Psi_x$. Then for some $i \in [1, N - w]$ we have $j = \Gamma_x(i)$ and $\sigma_j = \Gamma_y(\phi_i)$. Since $\phi \in \mathcal{S}_{N-w}^{*A}(\hat{\tau})$, we know that $\phi_i > (N - w) - i - A$. Following the notation in the definitions of Γ_x and Γ_y, let $m = \Gamma_x(i) - i$ and $n = \Gamma_y(\phi_i) - \phi_i$. Then $x(m) < i + m < x(m + 1)$ and $y(w - n + 1) < \phi_i + n < y(w - n)$. Also, we have

$$
\begin{aligned}
\sigma_j &= \phi_i + n \\
&> (N - w) - i - A + n \\
&= N - w - (j - m) - A + n \, .
\end{aligned}
$$

Thus, to show $\sigma_j > N - j - A$, as required for proving the Key Claim, we need to show that $m \geqslant w - n$.

Assume that $m \geqslant w - n$ is false, i.e. that $m + 1 \leqslant w - n$. Since $y(\ell) \geqslant y(\ell + 1) + 1$ for every ℓ, we see that

$$
y(m + 1) \geqslant y(w - n) + (w - n) - (m + 1) \, .
$$

Using this inequality and those of the preceding paragraph, we obtain

$$
\begin{aligned}
N - w - A &< \phi_i + i \\
&\leqslant y(w - n) - n - 1 + x(m + 1) - m - 1 \\
&\leqslant [y(m + 1) - w + n + m + 1] - n + x(m + 1) - m - 2 \\
&\leqslant N - A - w - 1 \qquad \text{(by (38))}
\end{aligned}
$$

which is a contradiction. Therefore $m \geqslant w - n$. This proves the Key Claim, and hence the proposition. $\qquad\square$

Proof of Lemma 21: For positive integers w and M, let $\mathbf{Seq}(w; M)$ be the set of all w-element subsets of $\{1, 2, \ldots, M\}$. We shall write a member of $\mathbf{Seq}(w; M)$ as a w-element vector with the entries in increasing order: $\vec{x} = (x(1), x(2), \ldots, x(w))$, with $x(1) < \cdots < x(w)$. Then there is a natural bijection $\Theta : \mathbf{Seq}(w; M_1) \times \mathbf{Seq}(w; M_2) \to \mathbf{Dec}(w; M_1, M_2)$ via

$$
\Theta(\vec{x}, \vec{z}) = \{(x(1), z(w)), (x(2), z(w - 1)), \ldots, (x(w), z(1))\} \, .
$$

In particular, we have

$$|\mathbf{Dec}(w; M_1, M_2)| \;=\; |\mathbf{Seq}(w; M_1)|\,|\mathbf{Seq}(w; M_2)| \;=\; \binom{M_1}{w}\binom{M_2}{w}. \qquad (39)$$

Applying Lemma 12 to Equation (39) proves Equation (35). Equation (36) will follow immediately once we have proven Equation (34).

For positive integers A, we now define

$$\mathbf{Seq}^{*A}(w; M) \;=\; \left\{ \vec{x} \in \mathbf{Seq}(w; M) : \left| x(\ell) - \ell\,\frac{M}{w+1} \right| < A \text{ for } \ell = 1, \dots, w \right\}.$$

Roughly speaking, a w-element subset of $\{1, \dots, M\}$ is in $\mathbf{Seq}^{*A}(w; M)$ if its elements are within distance A of a uniform spacing configuration over the interval. We shall now show the following.

Property I: If $\vec{x} \in \mathbf{Seq}^{*A}(w; M_1)$ and $\vec{z} \in \mathbf{Seq}^{*A}(w; M_2)$, then

$$\Theta(\vec{x}, \vec{z}) \in \mathbf{Dec}^{*B}(w; M_1, M_2), \text{ where } B = A\left(1 + \frac{M_2}{M_1}\right).$$

Property I says that if $\vec{x}$ and $\vec{z}$ are close to being uniformly spaced on their intervals, then $\Theta(\vec{x}, \vec{z})$ is close to the diagonal of its rectangle. To prove Property I, consider $\vec{x}$ and $\vec{z}$ as specified. Then a generic point of $\Theta(\vec{x}, \vec{z})$, $(x(\ell), z(w+1-\ell))$, satisfies

$$\left| z(w+1-\ell) - \left(M_2 - x(\ell)\frac{M_2}{M_1} \right) \right|$$
$$\leqslant \left| z(w+1-\ell) - (w+1-\ell)\frac{M_2}{w+1} \right| + \frac{M_2}{M_1}\left| x(\ell) - \ell\,\frac{M_1}{w+1} \right|$$
$$< A + \frac{M_2}{M_1}\,A\,.$$

This proves Property I. Now, Property I implies that $|\mathbf{Dec}^{*B}(w; M_1, M_2)| \geqslant |\mathbf{Seq}^{*A}(w; M_1)|\,|\mathbf{Seq}^{*A}(w; M_2)|$. Recalling Equation (39), we see that Equation (34) will follow if we can prove

Property II: $\displaystyle \lim_{N\to\infty} \frac{|\mathbf{Seq}^{*A_N}(w(N); N)|}{\binom{N}{w(N)}} = 1$ whenever

$$\lim_{N\to\infty} \frac{w(N)}{N} =: \theta \in (0,1) \quad \text{and} \quad \lim_{N\to\infty} \frac{A_N}{N} =: \epsilon > 0.$$

We shall prove Property II by converting it into a probabilistic statement. Let $p \in (0,1)$. Let $G_1, G_2, \dots$ be a sequence of independent random variables having the geometric distribution with parameter p; that is, $\Pr(G_i = \ell) = p(1-p)^{\ell-1}$ for $\ell = 1, 2, \dots$. Next, let $T_i = G_1 + G_2 + \cdots + G_i$ for each i. These random variables have negative binomial distributions

$$\Pr(T_{j+1} = \ell + 1) \;=\; \binom{\ell}{j} p^{j+1}(1-p)^{\ell-j} \quad \text{for } \ell \geqslant j. \qquad (40)$$

Moreover, for any $\vec{x} \in \mathbf{Seq}(w; N)$ (writing $x(0) = 0$ and $x(w + 1) = N + 1$),

$$
\begin{aligned}
\Pr(T_\ell = x(\ell) \text{ for } \ell = 1, \ldots, w \,|\, T_{w+1} = N+1) &= \frac{\prod_{\ell=1}^{w+1} p(1-p)^{x(\ell)-x(\ell-1)-1}}{\binom{N}{w} p^{w+1}(1-p)^{N-w}} \\
&= \binom{N}{w}^{-1}. \quad (41)
\end{aligned}
$$

Equation (41) says that *the conditional distribution of $(T_1, \ldots, T_w)$ given that $T_{w+1} = N+1$ is precisely the uniform distribution on $\mathbf{Seq}(w; N)$*. This assertion is true for any p. Let us now fix $p = (w + 1)/N$; we shall soon see why this is a convenient choice.

By Equation (41),

$$
\frac{|\mathbf{Seq}^{*A}(w; N)|}{\binom{N}{w}} = \Pr\left(|T_\ell - \ell/p| < A \text{ for } l = 1, \ldots, w \,|\, T_{w+1} = N + 1\right).
$$

and therefore

$$
0 \leqslant 1 - \frac{|\mathbf{Seq}^{*A}(w; N)|}{\binom{N}{w}} \leqslant \frac{\Pr(\max_{\ell=1,\ldots,w} |T_\ell - \ell/p| \geqslant A)}{\Pr(T_{w+1} = N + 1)}. \quad (42)
$$

Next, we obtain the asymptotic behaviour $\Pr(T_{w+1} = N + 1)$ using Stirling's Formula $m! \sim \sqrt{2\pi m}(m/e)^m$ and $p = (w + 1)/N$, with $w = w(N) \sim \theta N$, as follows.

$$
\begin{aligned}
\Pr(T_{w+1} = N + 1) &= \frac{N!}{w!(N - w)!} \frac{(w + 1)^{w+1}(N - w - 1)^{N-w}}{N^{N+1}} \\
&\sim \frac{\sqrt{2\pi N}}{\sqrt{2\pi w}\sqrt{2\pi(N - w)}} \left(\frac{w+1}{w}\right)^{w+1} \frac{w}{N} \left(\frac{N-w-1}{N-w}\right)^{N-w} \\
&\sim \frac{\sqrt{\theta}}{\sqrt{2\pi(1 - \theta)N}}. \quad (43)
\end{aligned}
$$

For the numerator of the right-hand side of Equation (42), we use Kolmogorov's Inequality [5], along with the property that the random variables G_i have mean $1/p$ and variance $(1 - p)/p^2$:

$$
\begin{aligned}
\Pr\left(\max_{\ell=1,\ldots,w} |T_\ell - \ell/p| \geqslant A\right) &\leqslant \frac{\mathrm{Var}(T_w)}{A^2} \\
&\sim \frac{N(1 - \theta)/\theta^2}{N^2 \epsilon^2} \quad (44)
\end{aligned}
$$

Applying Equations (43) and (44) to Equation (42) proves Property II. This completes the proof of Lemma 21. $\qquad\square$

4 Conclusion

Recalling Remark 10(b), we see that Theorem 9 follows immediately from Propositions 16 and 18.

We now show that Equation (6) of Theorem 3 follows from Theorem 9 by induction. Remark 10(a) tells us that we can apply Theorem 9 when τ is $1 \oslash \mu_3$, which shows that Equation (6) holds for $k = 4$. Now assume that Equation (6) is true for a given $k \geqslant 4$. Lemma 17 and Remark 4 prove that $G(\gamma, 1 - \delta; (k-2)^2) < (k-1)^2$ whenever $\gamma < 1 - \delta$. This means that Equation (6) implies Equation (10) when $\hat{\tau}$ is μ_k, using

$$\mathcal{S}_N(\mu_k) \setminus \mathcal{S}_N^{*N\epsilon}(\mu_k) \quad \subset \quad \bigcup_{i,j \,:\, j \leqslant N - i - N\epsilon} \mathcal{F}_N(i, j; \mu_k)$$

and a compactness argument as in the proof of Proposition 16. Hence Equation (11) holds when τ is μ_{k+1}, in which case $L(\hat{\tau})$ equals $(k-1)^2$. This says that Equation (6) holds with k replaced by $k+1$. This completes the induction, showing that Equation (6) holds for every $k \geqslant 4$.

Finally we shall prove Equation (7) for $k \geqslant 4$ and $1 \leqslant \ell \leqslant k - 2$. The proof of Proposition 2.3 in [3] shows that there is a bijection from $\mathcal{S}_N(1 \ldots \ell(\ell+1) \ldots (k-1)k)$ to $\mathcal{S}_N(1 \ldots \ell k(k-1) \ldots (\ell+1))$ that preserves all the left-to-right minima of each permutation. (To see this, observe that when $A = J_\ell$ in the proof of [3], each right-to-left minimum and everything below it and to its right are all coloured blue, and hence are unchanged by the bijection α.) It follows that

$$\mathcal{F}_N^*(I, J; \lambda_{k,\ell}) \;=\; \mathcal{F}_N^*(I, J; \mu_k)$$

always holds. Using this and our Proposition 18 with $\tau = \mu_k$, we obtain

$$\liminf_{N \to \infty} |\mathcal{F}_N^*(I_N, J_N; \lambda_{k,\ell})|^{1/N} \;\geqslant\; G(\gamma, 1 - \delta; (k-2)^2). \tag{45}$$

Next, by Proposition 16 with $\tau = \lambda_{k,\ell}$, we obtain

$$\begin{aligned}
\limsup_{N \to \infty} |\mathcal{F}_N(I_N, J_N; \lambda_{k,\ell})|^{1/N} \;&\leqslant\; G(\gamma, 1 - \delta; L(\lambda_{k-1,\ell-1})) \\
&=\; G(\gamma, 1 - \delta; (k-2)^2). \tag{46}
\end{aligned}$$

Equations (45) and (46) together imply Equation (7). This completes the proof of Theorem 3.

Acknowledgements

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