

Client-Waiter games on complete and random graphs

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Abstract

For a graph G , a monotone increasing graph property \mathcal{P} and positive integer q , we define the Client-Waiter game to be a two-player game which runs as follows. In each turn Waiter is offering Client a subset of at least one and at most $q + 1$ unclaimed edges of G from which Client claims one, and the rest are claimed by Waiter. The game ends when all the edges have been claimed. If Client's graph has property \mathcal{P} by the end of the game, then he wins the game, otherwise Waiter is the winner. In this paper we study several Client-Waiter games on the edge set of the complete graph, and the so called H -game, in which Client tries to build a copy of some fixed graph H , played on the edge set of the random graph. For the complete graph we consider games where Client tries to build a large star, a long path and a large connected component. We obtain lower and upper bounds on the critical bias for these games and compare them with the corresponding Waiter-Client games and with the probabilistic intuition. For the H -game on the random graph we show that the known results for the corresponding Maker-Breaker game are essentially the same for the Client-Waiter game, and we extend those results for the biased games and for trees.

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1 Introduction

Positional games are games of complete information with no random moves. The inception of the study of positional games goes back to the seminal papers of Hales and Jewett [18], of Lehman [25], and of Erdős and Selfridge [16]. It has developed since to be a recognized area of combinatorics, and the last decade has seen a burst of papers and an increasing interest in this field (see, for example, the monograph of Beck [4] and the recent monograph [19]).

Several variants of positional games have been considered in the literature. In the classical **Maker-Breaker** game the two players ('Maker' and 'Breaker') alternately claim elements from a set X (*the board*). Maker wins the game if he fully claims some set from a predefined family $\mathcal{F} \subseteq \mathcal{P}(X)$, which we call the *winning sets*, and Breaker wins otherwise.

Recently, some attention was turned to another type of game, namely the **Waiter-Client** game. In this variant, initially defined and studied by Beck (e.g. [3]) under the name "Picker-Chooser", in each turn Waiter is picking a subset of $q + 1$ free elements, where q is a fixed positive integer called *the bias*. Client then needs to choose one element from this subset which he claims, while Waiter claims the remaining q elements. If there are less than $q + 1$ remaining elements then in the last turn Waiter will claim all of them. Waiter's goal is to force Client to fully claim a winning set while Client tries to avoid it. It can be easily shown that this game is *bias monotone*, i.e. for two positive integers $q_1 < q_2$, a winning strategy for Waiter when playing with bias q_2 implies a winning strategy when playing with bias q_1 . We can therefore define the *critical bias*, q_c , to be the unique integer for which Waiter has a winning strategy if and only if $q \leq q_c$.

The **Client-Waiter** game runs much the same as the Waiter-Client with two differences. The first is that Client now tries to claim a winning set and Waiter tries to prevent it. The second is that we introduce a special monotonicity rule which states that Waiter may offer any number of elements between 1 and $q + 1$ in a turn. Client still claims one element in each turn (including the last one). The motivation for this rule is the fact that without it the game is not bias monotone. (Consider for example a game with n pairwise disjoint winning sets of size 2. In this game Waiter wins whenever $q + 1$ is even and loses otherwise.) The *critical bias*, q_c , in this game can thus be defined to be the unique integer for which Client has a winning strategy if and only if $q \leq q_c$.

We denote by $WC(X, \mathcal{F}, q)$ and $CW(X, \mathcal{F}, q)$ the Waiter-Client and Client-Waiter games, with board X , winning sets \mathcal{F} and bias q . Usually when considering a Waiter-Client or a Client-Waiter game we are interested in finding, or at least bounding, the critical bias. Another question we might ask is what is the probability threshold for the property "Waiter wins $WC(X_p, \mathcal{F}_p, q)$ " or "Client wins $CW(X_p, \mathcal{F}_p, q)$ ", where X_p is a random subset of X generated by removing each element from X randomly and independently with probability $1 - p$ and \mathcal{F}_p is the subfamily of \mathcal{F} which includes only the sets of \mathcal{F} which are subsets of X_p .

Besides the interest in those games for their own right it was observed that in many cases they exhibit a strong probabilistic intuition. That is, the outcome of many Waiter-Client and Client-Waiter games is roughly the same as what we would expect it to be

when both players just play randomly (although a random strategy for a single player is usually far from being optimal). See for example [3, 4, 10, 14, 8, 9, 20, 21].

In this paper we look into Client-Waiter games played on the edge set of a graph G , where G is either K_n — the complete graph on n vertices, or $G_{n,p}$ — the random graph generated by taking every edge of K_n to be in the graph randomly and independently with probability p . The $G_{n,p}$ model is the most commonly studied probability distribution on graphs (see [12], [23], and the most recent [17]).

For an infinite series of events $\{A_n\}_{n \geq 1}$ we say that A_n happens *with high probability* (*w.h.p.*) if $\lim_{n \rightarrow \infty} \Pr[A_n] = 1$. Let \mathcal{P} be a monotone increasing graph property. We say that $G_{n,p}$ goes through a *phase transition* around p^* with relation to \mathcal{P} if w.h.p. $G_{n,p} \notin \mathcal{P}$ whenever $p \ll p^*$, while w.h.p. $G_{n,p} \in \mathcal{P}$ when $p \gg p^*$.

For games on K_n we bound the critical bias for a few games where Client tries to achieve some monotone graph theoretic property. The interpretation of the probabilistic intuition in this case is that the critical bias should be $q_c \approx 1/p^*$, where p^* is the value around which $G_{n,p}$ goes through a phase transition with relation to this property.

For a fixed graph H we define *the H -game*, denoted $CW(G, H, q)$, to be the Client-Waiter game on the edge-set of G with bias q , where Client’s goal is to build in his graph a copy of H . We investigate the H -game on the edges of $G_{n,p}$ and find the value of p^* for the property “Client wins $CW(G_{n,p}, H, q)$ ”.

Not many biased Client-Waiter games on graphs have been studied to date. Bednarska-Bzdęga, Hefetz and Łuczak ([10]) showed that in a Client-Waiter game on the edges of K_n with bias $q = (1 + o(1))n/\ln n$ Waiter can isolate a vertex in Client’s graph, while if the bias is $q = (1 - o(1))n/\ln n$ then Client can guarantee his graph will be k -vertex connected, or alternatively he can guarantee his graph is Hamiltonian. This fits very well with the probabilistic intuition as $G_{n,p}$ goes through a phase transition with relation to the properties of being k -vertex connected and Hamiltonian around $p^* = \ln n/n$ (see, for example, Chapters 4.2 and 6.2 of [17]). Recently Hefetz, Krivelevich and Tan ([21]) analysed the non-planarity, K_t -minor and non- k -colorability Client-Waiter games. They showed that these games also exhibit some probabilistic intuition (though not as strong as in the former games).

We start by considering the **maximum-degree game**, i.e. the game in which Client tries to claim a star of maximum possible size. For integers n, q let $\mathcal{S}(n, q)$ denote the size (no. of edges) of a largest star graph Client can build when playing a Client-Waiter game with bias q on $E(K_n)$.

Proposition 1. *For any positive integers n, k , if $q \geq \lceil n/k \rceil - 2$ then $\mathcal{S}(n, q) \leq 2k$, while for $k \geq 2$, if $q < \frac{n-1}{k-1} - 1$ then $\mathcal{S}(n, q) \geq k$.*

Notice that the claim in the above proposition is not asymptotic, and k can be fixed or it can be a function of n .

In the **large component game** Client tries to build in his graph a connected component as large as possible, while in the **path game** he tries to build a path as long as possible. Let $\mathcal{C}(n, q), \mathcal{P}(n, q)$ denote the size of a largest component, and the length of a

longest path (no. of edges), respectively, that Client can build in the Client-Waiter game on the edges of K_n with bias q .

Theorem 2.

- (i) For every n and for every $k > 0$, if $q \geq 6n^{2^k/(2^k-1)}$ then $\mathcal{C}(n, q) < 3^k$.
- (ii) For every $\epsilon > 0$ and n large enough, if $q \geq 1.6n$ then $\mathcal{C}(n, q) < (\ln n)^{\log_2 3 + \epsilon}$.
- (iii) For every $0 < \epsilon < 1$ and n large enough, if $q \leq (1 - \epsilon)\frac{n}{2}$ then $\mathcal{C}(n, q) \geq e^{-5/2\epsilon + 3/2}n$.

Theorem 3.

- (i) For every n and for every $k > 0$, if $q \geq 3n^{2^k/(2^k-1)}$ then $\mathcal{P}(n, q) < 2k$.
- (ii) If $q \geq n$ and n is large enough then $\mathcal{P}(n, q) < 3 \ln \ln n$.
- (iii) For every $\epsilon > 0$ small enough there is n_0 such that for every $n > n_0$, if $q \leq (1 - \epsilon)\frac{n}{2}$ then $\mathcal{P}(n, q) \geq e^{-12/\epsilon}n$.
- (iv) For every $\epsilon > 0$ small enough there is n_0 such that for every $n > n_0$, if $q \leq \epsilon n$ then $\mathcal{P}(n, q) \geq (1 - 8\epsilon \ln(1/\epsilon))n$.

We did not find a matching lower bound for Theorems 2(i) and 3(i). We conjecture the following.

Conjecture 4. For every positive integer k and every constant $C > 0$, Client wins $CW(K_n, P_k, Cn)$, provided n is large enough.

Bednarska-Bzdęga proved in [7] several criteria for Waiter’s win in biased Client-Waiter games and as an application showed the following bounds on the critical bias in the Client-Waiter H -game.

Theorem 5 (6.1 in [7]). *Let H be a graph with at least two edges. For every $0 < \epsilon < 1$ and n large enough the following holds. If $q \geq n^{1/m'(H)+\epsilon}$ then Waiter wins $CW(K_n, H, q)$ while if $q \leq n^{1/m''(H)-\epsilon}$ Client wins $CW(K_n, H, q)$, where*

$$m'(H) = \max_{H' \subseteq H: v_{H'} \geq 1} \frac{e_{H'} - 1}{v_{H'}}$$

$$m''(H) = \max_{H' \subseteq H: v_{H'} \geq 3} \frac{e_{H'} + 1}{v_{H'} - 2}$$

Here we give an improvement of the lower bound using the hypergraph containers result of Saxton and Thomason ([29]). The idea for this proof was suggested to us by Bednarska-Bzdęga.

The *density* of a graph G is defined to be $d(G) = e_G/v_G$, and its *2-density* to be $d_2(G) = (e_G - 1)/(v_G - 2)$ or 0 if $v_G \leq 2$. The *maximum density* of G is $m(G) = \max_{G' \subseteq G} d(G')$,

and its *maximum 2-density* is $m_2(G) = \max_{G' \subseteq G} d_2(G')$. We say that a graph G is *balanced* (resp. *2-balanced*) if $d(G) = m(G)$ (resp. $d_2(G) = m_2(G)$). If for any proper subgraph $G' \subset G$ we have $d(G') < m(G)$ (resp. $d_2(G') < m_2(G)$) then G is *strictly balanced* (*strictly 2-balanced*).

Proposition 6. *For every graph H there is $c > 0$ such that Client wins $CW(K_n, H, q)$ whenever $q \leq cn^{1/m_2(H)} / \ln n$.*

In our treatment of the Client-Waiter games on the random graph we are very much influenced by the proofs of Nenadov, Steger and Stojaković ([28]) for the unbiased Maker-Breaker H -game played on random graphs. We show that the results in [28] are also true in the Client-Waiter game. Moreover, we extend these results for any fixed bias and show that the case of $H = K_3$, which was an exception in the unbiased game, is no longer such when $q \geq 2$.

Theorem 7. *Let H be a graph which is not a forest. If either $q \geq 2$ or there exists $H' \subseteq H$ such that $d_2(H') = m_2(H)$, H' is strictly 2-balanced and it is not a triangle, then there exist constants $c, C > 0$ which depend only on H and q such that in the Client-Waiter game $CW(G_{n,p}, H, q)$*

$$\lim_{n \rightarrow \infty} \Pr[\text{Client wins}] = \begin{cases} 1, & p \geq Cn^{-1/m_2(H)}, \\ 0, & p \leq cn^{-1/m_2(H)}. \end{cases}$$

Note: the same is true for the corresponding Maker-Breaker game.

For the case $H = K_3$ and $q = 1$, which is missing in the above theorem, we have the following theorem, whose proof is much the same as Theorem 1.3 of [26] for the corresponding Maker-Breaker game and is therefore not included here.

Theorem 8. *For every $p = p(n)$,*

$$\lim_{n \rightarrow \infty} \Pr[\text{Client wins } CW(G_{n,p}, K_3, 1)] = \lim_{n \rightarrow \infty} \Pr[G_{n,p} \text{ contains } K_5 - e].$$

Lastly, we prove that for trees Client can win even when $p \ll n^{-1}$.

Proposition 9. *For positive integers k, q Client has a strategy to win $CW(G_{n,p}, T_{k,k}, q)$ w.h.p., where $T_{k,k}$ is the complete k -ary tree of height k , and $p = n^{-1 - (k(q+1))^{-2(k+1)}}$.*

Note: the same is true for the corresponding Maker-Breaker game.

1.1 Related results

In a groundbreaking paper ([15]), Erdős and Rényi showed that the random graph $G_{n,p}$ goes through a phase transition around $p = 1/n$, from typically having only connected components of size at most logarithmic in n to having a linear sized ('giant') component. Later Ajtai, Komlós and Szemerédi ([1]) proved that in the supercritical regime $p =$

$(1 + \epsilon)/n$, $G_{n,p}$ will typically contain not only a giant component, but a linearly-long path (see also [24] for a simple proof).

The Maker-Breaker and Waiter-Client variants of the component game are excellent demonstrations of the probabilistic intuition. Indeed, Bednarska and Łuczak ([6]) showed in particular that when $q = (1 + \epsilon)n$ then Breaker can prevent a component of size larger than $1/\epsilon$, while if $q = (1 - \epsilon)n$ then Maker can build a component of size at least ϵn . Moreover, Krivelevich and Sudakov ([24]) proved that in the latter case Maker can even achieve a linearly long path. A similar sharp phase transition was sketched for the Waiter-Client component game by Bednarska-Bzdęga et al. ([8]). Theorems 2(ii)+(iii) and 3(ii)+(iii) show that in the Client-Waiter component game a phase transition takes place when q is between $n/2$ and $1.6n$, and in the path game when q is between $n/2$ and n . Moreover, when $q = n$ Waiter can even limit Client's longest path to be of order $\ln \ln n$ which indicates some weakness of Client in comparison to the probabilistic intuition. A similar Client's weakness can be observed in Theorem 3(i) where we would expect Client to be able to achieve a path of length 2^k while we prove that he cannot hope to get more than $2k$.

In [9] Bednarska-Bzdęga, Hefetz, and Łuczak showed that for any tree T_k on k vertices the critical bias for $WC(K_n, T_k, q)$ is $\Theta(n^{k/(k-1)})$ and that the critical bias for $WC(K_n, K_k, q)$ is $\Theta(n^{2/(k-1)})$ and conjectured that for any graph H the critical bias for $WC(K_n, H, q)$ is $\Theta(n^{1/m(H)})$. Notice that this is in compliance with the random intuition as the probability threshold for $G_{n,p}$ to contain a copy of H is $n^{-1/m(H)}$. On the other hand, in the Client-Waiter game with bias of order $n^{k/(k-1)}$ Client will not be able to build a star graph with three vertices (by Proposition 1) and he will not be able to build a path of length $3 \log_2 k$ (by Theorem 3(i)). This shows that in these games Client is also weak compared to Waiter in the corresponding Waiter-Client game. It is plausible that the upper bound in Theorem 5 can be improved to $\Theta(n^{1/m(H)})$ as well.

We mention here two results of similar games for the Maker-Breaker variation, though their bounds are not comparable to ours. The first is by Beck ([2]), it states that Maker can build a cycle of length at least $(1 - e^{-1/200\epsilon})n$ when playing on K_n against Breaker with a bias of ϵn (while on the other hand Breaker can isolate at least $\frac{\epsilon}{2}e^{-1/\epsilon}n$ vertices in Maker's graph).

The second is due to Bednarska and Łuczak ([5]) who proved that the critical bias for the Maker-Breaker H -game is $\Theta(n^{1/m_2(H)})$.

1.2 Preliminaries

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in some of our proofs. Our graph-theoretic notation is standard and follows those in [30]. In particular, we use the following.

For a graph G , let $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively, and let $v_G = |V(G)|$ and $e_G = |E(G)|$. For two sets $A, B \subseteq V(G)$, let $E_G(A)$ denote the set of edges of G with both endpoints in A and let $e_G(A) = |E_G(A)|$. Let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and the other endpoint in B (formally, $E_G(A, B) = \{e \in E(A \cup B) : e \cap A \neq \emptyset, e \cap B \neq \emptyset\}$), and let $e_G(A, B) = |E_G(A, B)|$.

Notice that if $A \subseteq B$ then $E_G(A, B) = E_G(A) + E_G(A, B \setminus A)$. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by the set S , and $N_G(S) = \{v \in V(G) \setminus S : \exists u \in S \text{ such that } (uv) \in E(G)\}$ denotes the external neighbourhood of S in G . For a vertex $u \in V(G)$ we abbreviate $N_G(\{u\})$ as $N_G(u)$ and let $d_G(u) = |N_G(u)|$ denote the degree of u in G . Often, when there is no risk of confusion, we omit the subscript G from the notation above. The *maximum degree* of a graph G is $\Delta(G) := \max\{d_G(u) : u \in V(G)\}$ and the *minimum degree* of a graph G is $\delta(G) := \min\{d_G(u) : u \in V(G)\}$.

For a family \mathcal{F} of subsets of X , we define the *transversal family* of \mathcal{F} to be $\mathcal{F}^* = \{A \subseteq X : A \cap B \neq \emptyset \text{ for every } B \in \mathcal{F}\}$.

Assume that some Client-Waiter game, played on the edge set of a graph $G = (V, E)$, is in progress. At any given moment during this game, let E_C, E_W, E_F denote the set of edges that were claimed by Client, resp. Waiter, resp. unclaimed (free) up to that moment. We denote their respective sizes by $e_C = |E_C|$, $e_W = |E_W|$, $e_F = |E_F|$. If $A, B \subseteq V$ are two sets then $E_C(A)$ is the set of Client's edges inside A , $E_C(A, B)$ is the set of Client's edges with one end in A and the other in B , and $e_C(A) = |E_C(A)|$ and $e_C(A, B) = |E_C(A, B)|$. Similarly we define $E_W(A), e_W(A), E_F(A)$, etc.

The rest of this paper is organized as follows: in Section 2 we quote two useful criteria for Client's win in Client-Waiter and Waiter-Client games, and we state and prove a result of our own which is of independent interest. In Section 3 we discuss games on the complete graphs and prove Proposition 1, Theorems 2 and 3, and Proposition 6. In Section 4 we discuss games on $G_{n,p}$ and give the proofs of Theorem 7 and Proposition 9. Section 5 is devoted to concluding remarks.

2 Game-Theoretic Tools

In this section we present several general criteria for the existence of a winning strategy for Client in a Client-Waiter or Waiter-Client game.

Theorem 10 (implicit in [4]). *Let q be a positive integer, let X be a finite set, let \mathcal{F} be a family of subsets of X . If*

$$\sum_{A \in \mathcal{F}} (q+1)^{-|A|} < 1$$

then Client has a winning strategy in the $WC(X, \mathcal{F}, q)$ game.

Theorem 10 gives a criterion for Client to *avoid* the family \mathcal{F} , which can sometimes be helpful in showing that his claimed subset has some desirable property (see for example the proofs of Theorems 2(iii) and 3(iii) in Section 3.2.4). However, recall that in the Client-Waiter game we introduced a monotonicity rule which allows Waiter to offer less than $q+1$ elements in a turn. This difference between the games prevents us from using Theorem 10. The next theorem provides a workaround.

Theorem 11. *Let q be a positive integer, let X be a finite set, let \mathcal{F} be a family of subsets of X and let $\Phi(\mathcal{F}) = \sum_{A \in \mathcal{F}} (q+1)^{-|A|}$. Then, playing a Client-Waiter game on X with bias q , Client has a strategy to claim a set $X_C \subseteq X$ of size $|X_C| \geq \lfloor |X|/(q+1) \rfloor$ which fully contains at most $2\Phi(\mathcal{F})$ sets of \mathcal{F} .*

Proof. Denote by W_i the set of elements offered by Waiter at the i -th turn, and let $\alpha_i = |W_i|/(q+1)$. Suppose Client plays the following random strategy: he picks an element from W_i uniformly at random, and then with probability α_i puts it in X_C . If element $x \in X$ is offered in the i -th turn, then $\Pr[x \in X_C] = \alpha_i/|W_i| = 1/(q+1)$. Fix some $A \in \mathcal{F}$. If, in some turn, Waiter offered at least two elements of A , then surely Client will not fully claim A . Since any element must be offered at some point we get that $\Pr[A \subseteq X_C] \leq (q+1)^{-|A|}$, and thus $\mathbb{E}(|\{A \in \mathcal{F} : A \subseteq X_C\}|) \leq \Phi(\mathcal{F})$. It follows by Markov's inequality that

$$\Pr[|\{A \in \mathcal{F} : A \subseteq X_C\}| > 2\Phi] < 1/2. \quad (1)$$

Let m denote the total number of turns played in the game. Note that $|X_C| = \sum_{i=1}^m Z_i$ where Z_1, \dots, Z_m are independent Bernoulli random variables with $\Pr[Z_i = 1] = \alpha_i$. Hence

$$\Pr[|X_C| \geq \lfloor |X|/(q+1) \rfloor] \geq \Pr[\text{Bin}(|X|, 1/(q+1)) \geq \lfloor |X|/(q+1) \rfloor] \geq 1/2, \quad (2)$$

where the first inequality holds by Theorem 5 from [22]. Combining (1) and (2) we conclude that with positive probability both $|X_C| \geq \lfloor |X|/(q+1) \rfloor$ and $|\{A \in \mathcal{F} : A \subseteq X_C\}| \leq 2\Phi$, and therefore there is a strategy for Client which will ensure a subset with these properties. \square

For a set X and a family of subsets \mathcal{F} we defined \mathcal{F}^* to be the transversal family of \mathcal{F} . If Client wins the $CW(X, \mathcal{F}^*, q)$ game, then he has claimed at least one element from every set of \mathcal{F} . The next theorem is therefore very useful in those situations where we want to show that Client can prevent Waiter from fully claiming a set of \mathcal{F} .

Theorem 12 (implicit in Theorem 3.2 of [20]). *Let q be a positive integer, let X be a finite set and let \mathcal{F} be a family of subsets of X . If*

$$\sum_{A \in \mathcal{F}} e^{-|A|/(q+1)} < 1,$$

then Client has a winning strategy for the $CW(X, \mathcal{F}^, q)$ game.*

3 Games on K_n

3.1 Star game

Proof of Proposition 1. First notice that the lower bound is trivial since if $q+1 < (n-1)/(k-1)$ then at the end of the game $e_C > n(k-1)/2$ which means that the average degree in Client's graph will be higher than $k-1$. We turn to the upper bound.

Consider the following strategy for Waiter. In the first turn Waiter chooses some vertex v_0 and offers $q+1$ edges incident to that vertex. Suppose Client picks (v_0, v_1) . In the second turn Waiter will offer some $q+1$ arbitrary edges incident to v_1 . In general, if on the i -th turn Client chose the edge (v_{i-1}, v_i) , and there are free edges incident to v_i ,

then in the next turn Waiter will offer an arbitrary subset of those edges of size $q + 1$, or all of them if there are less than $q + 1$. If there are no free edges incident to v_i then Waiter will choose some other vertex with free edges incident to it, and offer a subset of size $q + 1$ (or all) of those edges. Consider some vertex v . According to the above strategy, we can pair Client's edges which are incident to v (with exception, perhaps, of the last edge) such that each pair was claimed in consecutive turns, and when Client claimed the later of the two Waiter claimed q edges incident to v (again with exception of the last edge). This observation leads to the upper bound on Client's maximum degree: $\Delta_C \leq 2\lceil(n - 1)/(q + 2)\rceil$. Assuming $q \geq \lceil n/k \rceil - 2$ we get

$$\Delta_C \leq 2 \left\lceil \frac{n - 1}{q + 2} \right\rceil \leq 2 \left\lceil \frac{k(n - 1)}{n} \right\rceil = 2k. \quad \square$$

3.2 Large component and long path games

3.2.1 Waiter's strategies

We start by presenting Waiter's strategies in these games. These strategies will be promptly analysed to get the upper bounds of Theorems 2 and 3.

Strategy S_C for Waiter in the Component game

Let n, q be integers with $q + 1 \geq n - 1$. We describe strategy S_C for Waiter in a Client-Waiter game on $E(K_n)$ with bias q . Waiter will maintain 3 subsets $X, Y, U \subseteq V$. Initiate $X = Y = \emptyset$; $U = V$. We describe the strategy in 3 stages.

Stage I

In the i -th turn Waiter picks a maximal set of vertices $T_i \subseteq U$ such that $e_F(T_i, U) \leq q + 1$. Denote $t_i = |T_i|$, to be used later in the proof.

Waiter will offer all edges $E_F(T_i, U)$. Suppose Client picks an edge (x, y) with $x \in T_i$, then we add x to X and y to Y , and remove $T_i \cup \{y\}$ from U .

We enumerate the vertices in $X = \{x_1, x_2, \dots\}$, $Y = \{y_1, y_2, \dots\}$ by the order of their addition. It is not hard to verify the following properties right after the i -th turn:

1. Client's graph is a perfect matching between X and Y . In particular $|X| = |Y| = i$.
2. For any $1 \leq j \leq i$, x_j has free edges only to (some or all) y_k with $k < j$.

This stage ends when $U = \emptyset$. We denote by s the number of turns played in the first stage.

Stage II

This stage will last for at most s turns. In the i -th turn Waiter will offer all free edges between y_i and all isolated vertices in Client's graph. By the end of this stage Client's graph is made of s components, each of size at most 3. We claim that between any pair of components there are at most 3 free edges, and those are all the remaining free edges in the game.

Indeed, let C_i, C_j be two components. We may assume that $C_i = \{x_i, y_i, z_i\}$, $C_j = \{x_j, y_j, z_j\}$ where (y_i, z_i) and (y_j, z_j) are the edges claimed in this stage, and that $i < j$. By property (2) of the previous stage, x_i does not have a free edge to any vertex in C_j

and (x_j, z_i) is not available as well. By Waiter's play on this stage the edge (y_i, z_j) is not free and the edge (z_i, z_j) was offered sometime during the previous stage, so the free edges between C_i and C_j satisfy $E_F(C_i, C_j) \subseteq \{(y_i, y_j), (y_i, x_j), (z_i, y_j)\}$.

Stage III

In the last stage Waiter creates an auxiliary board of K_s and identifies each vertex of this board with one of Client's components of the original game. Given Waiter's strategy for the Client-Waiter game on $E(K_s)$ with a bias of $\lfloor q/3 \rfloor$, he can use this strategy to play in the original game by offering all free edges between two components each time this strategy requires him to offer the edge between the corresponding vertices on the auxiliary board, and if Client chooses some edge which connects two of his components on the original board then Waiter will translate it to the appropriate edge between the corresponding vertices in the auxiliary board. This gives us the recursion

$$\mathcal{C}(n, q) \leq 3 \cdot \mathcal{C}(s, \lfloor q/3 \rfloor). \quad (3)$$

Strategy S_P for Waiter in the Path game

We keep all the notations of the previous strategy. This strategy is very similar, we note only the differences. Waiter plays in 2 stages.

Stage I

In the i -th turn Waiter will pick a maximal set of vertices $T_i \subseteq U$ such that $e_F(T_i, V) \leq q + 1$. Denote: $t_i = |T_i|$.

Waiter will offer all edges $E_F(T_i, V)$. Suppose Client picks an edge (x, y) with $x \in T_i$, then we add x to X and y to Y , and remove $T_i \cup \{y\}$ from U .

We will have the following properties after the i th turn:

1. Client's graph is a union of at most i disjoint stars with all the center vertices in Y and the leaves in X .
2. All the free edges have both endpoints in $Y \cup U$.

This stage will end when $U = \emptyset$. By the end of this stage the vertices in Y are centers of disjoint stars in Client's graph, and the edges with both endpoints in Y are all the remaining free edges in the game. We denote $s = |Y|$. Clearly, s is at most the number of turns played at the first stage.

Stage II

As in the last stage of strategy S_C Waiter translates the game to an auxiliary game on $E(K_s)$, but this time with the same bias q . This leads to the following recursion

$$\mathcal{P}(n, q) \leq \mathcal{P}(s, q) + 2. \quad (4)$$

Strategies S_C and S_P will be used twice each in the proofs of (i) and (ii) of Theorem 2 and Theorem 3, respectively. The difference between proving (i) and (ii) will be in the degree of precision required in the analysis of the strategies.

3.2.2 k -sized component and k -path

The first part of Theorem 2 states that if $q \geq 6n^{2^k/(2^k-1)}$ then Waiter can prevent a component of size 3^k .

Proof of Theorem 2(i). When $q \geq n^2/2$, Client's graph will be a single edge, so the theorem is true for $k = 1$. We assume $q < n^2/2$ and proceed by induction on k .

Let n be arbitrary and set $q = 6n^{2^k/(2^k-1)}$. Waiter will play strategy S_C . Recall that for each $1 \leq i \leq s$, t_i is the size of a largest subset $T_i \subset U$ such that Waiter can offer all edges $E_F(T_i, U)$. Certainly $t_i \geq \lfloor q/n \rfloor$ for any i . So the number s of turns at the first stage is at most

$$\begin{aligned} s &\leq \left\lceil \frac{n}{\lfloor q/n \rfloor} \right\rceil \leq \frac{n^2}{q-n} + 1 \leq \frac{n^2}{q-q/6} + 1 = \frac{1}{5} \cdot \frac{n^2}{q/6} + 1 < \frac{17}{60} \cdot \frac{n^2}{q/6} < \frac{1}{3} \left(\frac{q}{6}\right)^{(2^{k-1}-1)/(2^{k-1})} \\ &\implies \left\lfloor \frac{q}{3} \right\rfloor \geq 6s^{2^{k-1}/(2^{k-1}-1)}. \end{aligned}$$

We can therefore use the induction hypothesis with recursion (3) and get that

$$\mathcal{C}(n, q) \leq 3 \cdot \mathcal{C}(s, \lfloor q/3 \rfloor) < 3^k. \quad \square$$

The first part of Theorem 3 states that if $q \geq 3n^{2^k/(2^k-1)}$ then Waiter can prevent a path of length $2k$.

Proof of Theorem 3(i). The proof is very similar to the above proof of Theorem 2(i). One easily checks that the claim is true for $k = 1$. In the induction step we take $q = 3n^{2^k/(2^k-1)}$, and get

$$\begin{aligned} s &\leq \left\lceil \frac{n}{\lfloor q/n \rfloor} \right\rceil \leq \frac{n^2}{q-n} + 1 \leq \frac{n^2}{q-q/3} + 1 = \frac{1}{2} \cdot \frac{n^2}{q/3} + 1 < \frac{2}{3} \cdot \frac{n^2}{q/3} = \frac{2}{3} \left(\frac{q}{3}\right)^{(2^{k-1}-1)/(2^{k-1})} \\ &\implies q \geq 3s^{2^{k-1}/(2^{k-1}-1)}, \end{aligned}$$

which together with recursion (4) implies

$$\mathcal{P}(n, q) \leq \mathcal{P}(s, q) + 2 < 2k. \quad \square$$

3.2.3 Polylogarithmic component and polylogarithmic path

In the second part of Theorem 2 we claim that if $q \geq 1.6n$ then Waiter can prevent a component of size $(\ln n)^{\log_2 3 + \epsilon}$. To show that, need a finer analysis of Waiter's strategy. This is done in the next two lemmas.

Lemma 13. For any $\gamma > \log_2 3$ Waiter can prevent a component of size $2 \left(\frac{\ln n}{\ln(q/n)}\right)^\gamma$ in Client's graph when playing a Client-Waiter game on $E(K_n)$ with bias $q \geq \lceil 6^{1/(2-3^{1/\gamma})} \rceil n$.

Remark 14. notice that when $q = 6n^{2^k/(2^k-1)}$, then for large n, k this lemma roughly gives us $\mathcal{C}(n, q) < 2 \cdot 3^k$ which is only slightly worse than the statement in Theorem 2(i).

Proof. Fix $\gamma > \log_2 3$ and set $q = cn$ with $c = \lceil 6^{1/(2-3^{1/\gamma})} \rceil$. Notice that $c \geq 7$. The claim is true for $n \leq 2c + 1$ since then if $q = cn$ then $q \geq \binom{n}{2}$. We proceed by induction on n . Waiter will play strategy S_C . In each turn i at the first stage we have $t_i \geq \lfloor q/n \rfloor$, therefore the first stage lasts for at most

$$s \leq \left\lceil \frac{n}{\lfloor q/n \rfloor} \right\rceil \leq \frac{n}{q/n - 1} + 1 < \frac{n^2 + q}{q - n}$$

turns, which is also an upper bound on the number of components in Client's graph at the end of the second stage. We have

$$\begin{aligned} \frac{\lfloor q/3 \rfloor}{s} &\geq \frac{(q/3 - 1)(q - n)}{n^2 + q} = \left(\frac{q}{n}\right)^2 \cdot \frac{(1/3 - 1/q)(1 - 1/c)}{1 + q/n^2} \\ &\geq \left(\frac{q}{n}\right)^2 \cdot \left(\frac{1}{3} - \frac{1}{100}\right) \cdot \frac{6}{7} \cdot \frac{2}{3} \geq 0.18 \cdot \left(\frac{q}{n}\right)^2, \end{aligned} \quad (5)$$

(we used the assumptions that $100 < 2c(c + 1) \leq q \leq n^2/2$). We get that

$$\frac{\lfloor q/3 \rfloor}{s} \geq 0.18 \cdot \left(\frac{q}{n}\right)^2 \geq \frac{c^{2-3^{1/\gamma}}}{6} \cdot \left(\frac{q}{n}\right)^{3^{1/\gamma}} > \left(\frac{q}{n}\right)^{3^{1/\gamma}}.$$

Using recursion (3) and our induction hypothesis we derive

$$\mathcal{C}(n, q) \leq 3 \cdot \mathcal{C}(s, \lfloor q/3 \rfloor) < 6 \left(\frac{\ln n}{\ln(\lfloor q/3 \rfloor/s)}\right)^\gamma < 6 \left(\frac{\ln n}{3^{1/\gamma} \ln(q/n)}\right)^\gamma = 2 \left(\frac{\ln n}{\ln(q/n)}\right)^\gamma. \quad \square$$

Lemma 15. *There is n_0 such that for all integers $n > n_0$ and $q \geq 1.6n$, $\mathcal{C}(n, q) \leq 3\mathcal{C}(n_1, q_1)$ with $q_1 = \lfloor q/3 \rfloor$ and $\frac{q_1}{n_1} \geq 1.001 \frac{q}{n}$.*

Proof. In order to prove this lemma we need to get a better bound on s than that we used in Lemma 13. Denote by s_k the number of moves in the first stage for which $t_i = k$. For simplicity and clarity of the calculations we make the following sub-optimal assumptions.

- While $n \geq |U| > q/2$ we assume $t_i = 1$. During this time U gets decreased by at least 2 vertices a turn.
- For every $2 \leq k \leq \sqrt{q}$, while $q/k \geq |U| > q/(k + 1)$ we assume $t_i = k$. During this time U gets decreased by at least k vertices a turn.
- When $|U| \leq \sqrt{q}$ Waiter can take $T = U$, hence we can assume that $s_k = 0$ for $k > \sqrt{q}$.

We can therefore bound $s_1 \leq \lceil (n - q/2)/2 \rceil$ and $s_k \leq \lceil (q/k - q/(k + 1))/k \rceil = \lceil q/k^2(k + 1) \rceil$ for $2 \leq k \leq \sqrt{q}$, which leads to the following bound on s for n large enough :

$$s \leq \sum_{k=1}^{\lfloor \sqrt{q} \rfloor} s_k \leq \frac{n - q/2}{2} + \sum_{k=2}^{\infty} \frac{q}{k^2(k + 1)} + \sqrt{q} \leq \frac{n}{2} - \frac{q}{4} + \frac{q(\pi^2 - 9)}{6} + o(n) < 0.999 \frac{n}{3}.$$

Notice that the necessity to achieve the last inequality is the reason for the constant 1.6 in our upper bound. Set $n_1 = s$, $q_1 = \lfloor q/3 \rfloor$. Then

$$q_1 \geq \frac{q}{3} - 1 \geq \frac{qn_1}{0.999n} - 1 > 1.001 \frac{q}{n} n_1,$$

where the last inequality is for large enough n . By our recursion:

$$\mathcal{C}(n, q) \leq 3 \cdot \mathcal{C}(n_1, q_1). \quad \square$$

Proof of Theorem 2(ii). Let n be an integer. Due to the monotonicity of the game it is enough to prove the Theorem for $q = 1.6n$. Let $\gamma = \log_2 3 + \epsilon/2$ and let $c = c(\gamma)$ be such that Waiter can prevent a component of size $2(\ln n / \ln c)^\gamma$ when playing with bias cn (which we get from Lemma 13). Set $m = \ln c / \ln 1.001$. Let n_1, q_1 be the integers guaranteed by Lemma 15. We can recurrently use this lemma to get a sequence $\{(n_i, q_i)\}_{i=1}^\ell$ of pairs with $\frac{q_i}{n_i} \geq 1.001 \frac{q_{i-1}}{n_{i-1}}$; with ℓ the first such that $q_\ell \geq cn_\ell$. Notice that necessarily $\ell \leq m$ and that since for any i , $q_i = \lfloor q_{i-1}/3 \rfloor$,

$$n_{\ell-1} > \frac{q_{\ell-1}}{c} \geq \frac{q}{c \cdot 4^\ell} = \frac{1.6n}{c \cdot 4^\ell},$$

and so if $n > n_0 4^\ell c / 1.6$, where n_0 is from Lemma 15, then our use of the lemma was valid. Finally we get from Lemma 13

$$\mathcal{C}(n, q) \leq 3 \cdot \mathcal{C}(n_1, q_1) \leq \dots \leq 3^\ell \cdot \mathcal{C}(n_\ell, q_\ell) < 2 \cdot 3^m \left(\frac{\ln n_\ell}{\ln c} \right)^\gamma \leq (\ln n)^{\log_2 3 + \epsilon},$$

for n large enough. □

We turn to the proof of the second part of Theorem 3 where we claimed that if $q \geq n$ and n is large enough then Waiter can prevent a path of length $3 \ln \ln n$.

Proposition 16. For all integers $n, q = cn$ with $c \geq 2$, $\mathcal{P}(n, q) < 2 \log_2 \log_{2c/3} q + 1$.

Proof. Fix c . The claim is true for every $1 \leq n \leq 2c + 1$ since then $q \geq \binom{n}{2}$. We proceed by induction on n . Playing strategy S_P we have that in every turn at the first stage

$$t_i \geq \left\lfloor \frac{q}{n} \right\rfloor.$$

In each turn Client chooses an edge (x_i, y_i) with $x_i \in T_i$ and $y_i \in V$. We consider three cases:

- If $y_i \in V \setminus U$ then $y_i \in Y$ and the edge (x_i, y_i) is just an additional edge to an existing star in Client's graph.
- If $y_i \in T_i$ then y_i has no more free edges and Waiter can ignore the edge (x_i, y_i) in the next stage of his strategy.

- If $y_i \in U \setminus T_i$ then Client has just created a new star and U got decreased by $t_i + 1$ vertices.

Therefore, by the end of the first stage Client's graph is a union of at most

$$s \leq \left\lceil \frac{n}{\lfloor q/n \rfloor + 1} \right\rceil \leq \frac{q}{c^2} + 1 \leq \frac{3q}{2c^2}$$

disjoint stars (we used in the last inequality that $n > 2c \implies q > 2c^2$).

Hence $q \geq \frac{2c^2}{3} \cdot s$, and we get from our recursion and the induction hypothesis that

$$\mathcal{P}(n, q) \leq 2 + \mathcal{P}(s, q) < 3 + 2 \log_2 \log_{(2c/3)^2} q = 2 \log_2 \log_{2c/3} q + 1. \quad \square$$

Proof of Theorem 3(ii). Let n be an integer. Due to monotonicity of the game we may assume $q = n$. When Waiter plays strategy S_P Client's graph after the first stage contains at most $s \leq n/2$ disjoint stars, hence $q \geq 2s$ and by the recursion and Proposition 16

$$\mathcal{P}(n, q) \leq 2 + \mathcal{P}(s, q) < 3 + 2 \log_2 \log_{4/3} q \leq 3 \ln \ln n,$$

for n large enough. □

3.2.4 Client's side: linear-sized component and linear-sized path

Part three of Theorem 2 states that if $q \leq (1 - \epsilon) \frac{n}{2}$ then Client can build a component of size at least $e^{-5/2\epsilon+3/2}n$.

Proof of Theorem 2(iii). Set $\delta = \frac{\epsilon}{1 - \epsilon}$, $\theta = e^{-2.5/\delta-1}$ and let

$$\mathcal{F} := \{E(H) : H \subseteq K_n, v_H \leq \theta n, e_H = (1 + \delta)v_H\}.$$

Then

$$\begin{aligned} \Phi(\mathcal{F}) &:= \sum_{i=4}^{\lfloor \theta n \rfloor} \binom{n}{i} \binom{\binom{i}{2}}{(1 + \delta)i} (q + 1)^{-(1 + \delta)i} < \sum_{i=4}^{\lfloor \theta n \rfloor} \left[\frac{en}{i} \left(\frac{(1 - \epsilon)ei}{2} \right)^{1 + \delta} \left(\frac{(1 - \epsilon)n}{2} \right)^{-(1 + \delta)} \right]^i \\ &= \sum_{i=4}^{\lfloor \theta n \rfloor} \left[e^{2 + \delta} \left(\frac{i}{n} \right)^\delta \right]^i \leq \sum_{i=4}^{\lfloor \theta n \rfloor} (e^{2 + \delta} \theta^\delta)^i < \sum_{i=4}^{\infty} e^{-i/2} < \frac{1}{2}. \end{aligned}$$

By Theorem 11 Client has a strategy such that by the end of the game his graph contains a subgraph G_C , with $e(G_C) = \frac{n(n - 1)}{2(q + 1)} \geq \frac{n}{1 - \epsilon}$ edges such that every connected component U with size at most θn has less than $(1 + \delta)|U|$ edges in G_C . Suppose that all the components in G_C are of size less than θn . Then

$$|E(G_C)| = \sum_{U \in \text{comp}(G_C)} e_{G_C}(U) < (1 + \delta) \sum_{U \in \text{comp}(G_C)} |U| \leq (1 + \delta)n = \frac{n}{1 - \epsilon},$$

which is a contradiction. Therefore Client has a connected component of size at least θn . □

Part three of Theorem 3 states that if $q \leq (1 - \epsilon)\frac{n}{2}$ then Client can build a path of length at least $e^{-12/\epsilon}n$.

Proof of Theorem 3(iii).

Lemma 17. *Let $\epsilon, \gamma > 0$ and $\delta_1 > \delta_2 > 0$ be constants. Let G be a graph on n vertices with the following properties:*

1. G has $e_G \geq (1 + \epsilon)n$ edges,
2. every set S of size $|S| \leq \delta_1 n$ spans $e(S) < (1 + \epsilon)|S|$ edges,
3. every set S of size $|S| \leq \delta_2 n$ spans $e(S) < (1 + \epsilon/2)|S|$ edges,
4. for every set S of size $|S| \leq \gamma n$, we have $e(S, V \setminus S) < \frac{\epsilon \delta_2}{2}n$ edges.

Then G contains a path of length at least γn .

Proof. By the first property G must contain some connected component C with $e(C) \geq (1 + \epsilon)|C|$. By the second property $|C| > \delta_1 n$. Consider the DFS algorithm as defined, for example, in [24]. As a quick reminder: we take an arbitrary ordering of the vertices of G and run a DFS exploration on G by maintaining three sets of vertices: S — the vertices we have finished exploring, U — a LIFO stack with the vertices we are currently exploring, and T — the unvisited vertices. Consider an execution of the DFS algorithm on C , starting with $S = U = \emptyset$ and $T = C$, and completing when $S = C$ and $U = T = \emptyset$. We will use the following properties of this algorithm:

- at any given moment there are no edges of G between S and T , and
- at any given moment U spans a path in C .

Set $t_0 = 0$ and $t_f = \lceil |C|/(\delta_2 n) \rceil$. For any $1 \leq i \leq t_f - 1$ let t_i be the moment in which $|S| = \delta_2 n i$ and let S_i , $1 \leq i \leq t_f$, be the vertices which were added to S between time t_{i-1} and t_i . Since $|S_i| \leq \delta_2 n$ for all i , we get from the third property that

$$\sum_{1 \leq i < j \leq t_f} e(S_i, S_j) = e(C) - \sum_{i=1}^{t_f} e(S_i) \geq (1 + \epsilon)|C| - (1 + \epsilon/2) \sum_{i=1}^{t_f} |S_i| = \frac{\epsilon}{2}|C|.$$

Thus there is $1 \leq i_0 \leq t_f - 1$ such that $\sum_{i_0 < j \leq t_f} e(S_{i_0}, S_j) \geq \frac{\epsilon|C|}{2(t_f - 1)} \geq \frac{\epsilon \delta_2 n}{2}$. Since at time t_{i_0} all these edges are between S_{i_0} and U , we get from property 4 that at that moment U spans a path of length at least γn . \square

We now return to the proof of Theorem 3(iii). Since the game is bias monotone, we may assume $q = \frac{n}{2(1+\epsilon)} > (1-\epsilon)\frac{n}{2}$. Set $\delta_1 = e^{-3/\epsilon-1}$, $\delta_2 = \delta_1^2$, and $\gamma = (\epsilon\delta_2)^2$. Define

$$\begin{aligned}\mathcal{F}_1 &:= \{E(H) : H \subseteq K_n, v_H \leq \delta_1 n, e_H = (1+\epsilon)v_H\}, \\ \mathcal{F}_2 &:= \{E(H) : H \subseteq K_n, v_H \leq \delta_2 n, e_H = (1+\epsilon/2)v_H\}, \\ \mathcal{F}_3 &:= \{E(H) : H \subseteq K_n, H = (S \cup (V \setminus S), E), |S| \leq \gamma n, e_H = \frac{\epsilon\delta_2}{2}n\}.\end{aligned}$$

The notation $(S \cup (V \setminus S), E)$ means a bipartite graph between S and its complement. Note that when bounding $\Phi(\mathcal{F}_3)$ it is enough to consider those subsets of \mathcal{F}_3 with $|S| = \gamma n$. This is because γ and $\epsilon\delta_2/2$ are small and so if there is a subset S such that $|S| < \gamma n$ and with $\frac{\epsilon\delta_2}{2}n$ edges to its complement then we can always add to S vertices not in its neighbourhood until it reaches a size of γn . We calculate

$$\begin{aligned}\Phi(\mathcal{F}_1) &= \sum_{i=4}^{\lfloor \delta_1 n \rfloor} \binom{n}{i} \binom{\binom{i}{2}}{(1+\epsilon)i} (q+1)^{-(1+\epsilon)i} < \sum_{i=4}^{\lfloor \delta_1 n \rfloor} \left[\frac{en}{i} \left(\frac{ei}{2(1+\epsilon)} \right)^{1+\epsilon} \left(\frac{n}{2(1+\epsilon)} \right)^{-(1+\epsilon)} \right]^i \\ &= \sum_{i=4}^{\lfloor \delta_1 n \rfloor} \left[e^{2+\epsilon} \left(\frac{i}{n} \right)^\epsilon \right]^i \leq \sum_{i=4}^{\infty} (e^{2+\epsilon}\delta_1^\epsilon)^i = \sum_{i=4}^{\infty} e^{-i} < 0.1;\end{aligned}$$

$$\begin{aligned}\Phi(\mathcal{F}_2) &= \sum_{i=4}^{\lfloor \delta_2 n \rfloor} \binom{n}{i} \binom{\binom{i}{2}}{(1+\epsilon/2)i} (q+1)^{-(1+\epsilon/2)i} \\ &< \sum_{i=4}^{\lfloor \delta_2 n \rfloor} \left[\frac{en}{i} \left(\frac{ei}{2(1+\epsilon/2)} \right)^{1+\epsilon/2} \left(\frac{n}{2(1+\epsilon)} \right)^{-(1+\epsilon/2)} \right]^i \\ &= \sum_{i=4}^{\lfloor \delta_2 n \rfloor} \left[\left(\frac{1+\epsilon}{1+\epsilon/2} \right)^{1+\epsilon/2} e^{2+\epsilon/2} \left(\frac{i}{n} \right)^{\epsilon/2} \right]^i \leq \sum_{i=4}^{\lfloor \delta_2 n \rfloor} \left[e^{2+\epsilon} \left(\frac{i}{n} \right)^{\epsilon/2} \right]^i \leq \sum_{i=4}^{\infty} (e^{2+\epsilon}\delta_2^{\epsilon/2})^i \\ &= \sum_{i=4}^{\infty} e^{-i} < 0.1;\end{aligned}$$

$$\begin{aligned}\Phi(\mathcal{F}_3) &= \binom{n}{\gamma n} \binom{\gamma(1-\gamma)n^2}{\frac{\epsilon\delta_2}{2}n} (q+1)^{-\frac{\epsilon\delta_2}{2}n} \\ &\leq \left(\frac{e}{\gamma} \right)^{\gamma n} \left(\frac{2\gamma(1-\gamma)n}{\epsilon\delta_2} \right)^{\epsilon\delta_2 n/2} \left(\frac{n}{2(1+\epsilon)} \right)^{-\epsilon\delta_2 n/2} \\ &\leq \left[\left(\frac{e}{\epsilon^2\delta_2^2} \right)^{2\epsilon\delta_2} 4(1+\epsilon)\epsilon\delta_2 \right]^{\epsilon\delta_2 n/2} = o(1),\end{aligned}$$

where the last equality is for ϵ small enough. Note that we get that for n large enough

$$\Phi(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3) \leq \Phi(\mathcal{F}_1) + \Phi(\mathcal{F}_2) + \Phi(\mathcal{F}_3) < 1/2,$$

and by Theorem 11 Client has a strategy to claim a subgraph G_C which has all the properties of Lemma 17 and therefore contains a path of length at least γn . \square

The purpose of part four of Theorem 3 is to show that when $q \leq \epsilon n$ and $\epsilon \rightarrow 0$, then Client can build a graph which is “almost” Hamiltonian. More precisely, he can build a path of length at least $(1 - 8\epsilon \ln(1/\epsilon))n$.

Proof of Theorem 3(iv). We will use the following lemma.

Lemma 18 (Lemma 4.4 in [11]). *Let G be a graph on n vertices. Suppose that for every two disjoint sets $A, B \subset V(G)$ such that $|A|, |B| \geq k$, there is at least one edge between A and B . Then G contains a path of length $n - 2k + 1$.*

Let $q = \epsilon n$, and let $\delta > 0$ to be determined. Define

$$\mathcal{F} := \{E_{K_n}(A, B) : A \cap B = \emptyset, |A| = |B| = \delta n\}.$$

If Client wins the $CW(K_n, \mathcal{F}^*, q)$ game then he has an edge between any two disjoint subsets of size δn , and by Lemma 18 he has a path of length $(1 - 2\delta)n$. So it is enough to verify the condition of Theorem 12:

$$\sum_{A \in \mathcal{F}} e^{-|A|/(q+1)} \leq \binom{n}{\delta n}^2 e^{-\delta^2 n^2 / 2q} \leq \left[\left(\frac{e}{\delta} \right)^2 e^{-\delta/2\epsilon} \right]^{\delta n}.$$

The last expression will be $o(1)$ when

$$\frac{\delta}{2\epsilon} > 2(1 + \ln(1/\delta)),$$

and this is true for $\delta = 4\epsilon \ln(1/\epsilon)$ and ϵ small enough. □

3.3 The H -game

Proof of Proposition 6. The idea of the proof (suggested by Bednarska-Bzdęga) is to use the following theorem of hypergraph containers.

Theorem 19 (implicit in Theorem 2.3 in [29]). *Let H be a graph with at least three vertices. Then there are $n_0, \delta > 0$ such that for every $n \geq n_0$ there is a collection \mathcal{C} of subgraphs of K_n such that*

1. Every H -free subgraph of K_n is contained in some $C \in \mathcal{C}$.
2. For every $C \in \mathcal{C}$, $e_C \leq (1 - \delta) \binom{n}{2}$.
3. $|\mathcal{C}| \leq n^{n_0 n^2 - 1/m_2(H)}$.

The lower bound for the $CW(K_n, H, q)$ game is an easy application of the above theorem and of the criterion for Client’s win in Theorem 12. Let \mathcal{C} be the collection of subgraphs for the graph H guaranteed by Theorem 19. It is enough to show that Client can claim at least one edge in the complement of every graph $C \in \mathcal{C}$. Since the number

of edges in any such complement is at least $\delta \binom{n}{2}$, recalling the bound on $|\mathcal{C}|$ we can verify the condition of Theorem 12.

$$\sum_{A \in \mathcal{F}} e^{-|A|/(q+1)} \leq n^{n_0 n^{2-1/m_2(H)}} e^{-\delta \binom{n}{2}/(q+1)} = o(1),$$

provided $q \leq cn^{1/m_2(H)}/\ln n$ for some $c = c(H) > 0$. □

4 The H-Game on $G_{n,p}$

We note two well known facts about random graphs which will be used in this section without reference.

- When $n^{-k/(k-1)} \ll p \ll n^{-(k+1)/k}$ w.h.p. $G_{n,p}$ is a forest with copies of all trees with at most k vertices, and no tree with more than k vertices.
- Let α, c be positive constants. If $p \leq cn^{-1/\alpha}$ then for any fixed graph G with $m(G) > \alpha$ w.h.p. $G \not\subseteq G_{n,p}$. Put another way, for c, α and p as above and any fixed $k > 0$, w.h.p. any subgraph G of $G_{n,p}$ on at most k vertices has density $m(G) \leq \alpha$.

4.1 Client's side

The proof of Maker's side (the 1-statement) in [28] relayed on hypergraph containers as an auxiliary tool. That method would have worked here as well. However, we give an alternative proof using another tool — a variant of the famous KLR conjecture which was proved in ([13]). We start with a few definitions.

Definition 20. A bipartite graph between sets U and V is (ϵ, d) -lower-regular if, for every $U' \subseteq U$ and $V' \subseteq V$ with $|U'| \geq \epsilon|U|$ and $|V'| \geq \epsilon|V|$, the density $d(U', V')$ of edges between U' and V' satisfies $d(U', V') \geq d$.

Given a graph H with vertex set $[k]$, we denote by $\mathcal{G}(H, n, d, \epsilon)$ the collection of all graphs G with vertex set $V_1 \cup \dots \cup V_k$, where V_1, \dots, V_k are pairwise disjoint sets of size n each, whose edge sets consist of e_H (ϵ, d) -lower-regular bipartite graphs, one graph between V_i and V_j for each $ij \in E(H)$.

For an arbitrary graph G and $p \in [0, 1]$, we denote by G_p the random subgraph of G , where each edge of G is included with probability p independently of all other edges.

Theorem 21 (implied by Theorem 2.1 in [13]). *Let H be an arbitrary graph. For every $d > 0$, there exist $\epsilon, C > 0$ such that if $p \geq Cn^{-1/m_2(H)}$, then the following holds. For every $G \in \mathcal{G}(H, n, d, \epsilon)$, w.h.p. the random graph G_p has the following property: Every subgraph G' of G_p in $\mathcal{G}(H, n, dp, \epsilon)$ contains a copy of H .*

We are ready to prove Client's side in Theorem 7.

Proof of the 1-statement of Theorem 7. Choose d such that $d < 1/(q+1)$ and

$$\frac{d}{1/2-d}(1 + \ln 2 - \ln d) < \frac{1}{q+1}$$

(this is possible since the LHS goes to 0 when d goes to 0), and let $\epsilon = \epsilon(H, d)$ be that of Theorem 21. Let $k = |V(H)|$. At the beginning of the game Client will fix an equipartition of the vertices of $G_{n,p}$ to k parts V_1, \dots, V_k . He will then follow a strategy which guarantees that by the end of the game his graph, G_C , when restricted to any pair of parts, is (ϵ, dp) -lower-regular. To see that this means that w.h.p. he will claim a copy of H , take G in Theorem 21 to be the complete k -partite graph on V_1, \dots, V_k . Clearly $G \in \mathcal{G}(H, n/k, d, \epsilon)$ and $G_C \cap G \in \mathcal{G}(H, n/k, dp, \epsilon)$, and we can consider G_p as $G \cap G_{n,p}$. It remains to show that Client indeed has such a strategy. To this end we will use Theorem 12, and define

$$\mathcal{F} := \{F \subseteq E_{G_{n,p}}(U_1, U_2) : U_1 \cap U_2 = \emptyset, |U_1| = |U_2| = \frac{\epsilon n}{k}, |F| \geq e_{G_{n,p}}(U_1, U_2) - dp\epsilon^2 n^2/k^2\}.$$

Clearly, if Client wins the $CW(G_{n,p}, \mathcal{F}^*, q)$ game then he has achieved his goal. It remains to verify the condition of Theorem 12. Indeed, since w.h.p. the number of edges between any two disjoint subsets of size $\epsilon n/k$ will satisfy

$$\frac{1}{2}p\epsilon^2 n^2/k^2 \leq e_{G_{n,p}}(U_1, U_2) \leq 2p\epsilon^2 n^2/k^2,$$

we get

$$\begin{aligned} \Phi(\mathcal{F}) &\leq \binom{n}{2\epsilon n/k} \binom{2\epsilon n/k}{\epsilon n/k} \binom{2p\epsilon^2 n^2/k^2}{dp\epsilon^2 n^2/k^2} e^{-(\frac{1}{2}-d)p\epsilon^2 n^2/(q+1)k^2} \\ &\leq \left(\frac{ek}{2\epsilon}\right)^{2\epsilon n/k} 2^{2\epsilon n/k} \left(\frac{2e}{d}\right)^{dp\epsilon^2 n^2/k^2} e^{-(\frac{1}{2}-d)p\epsilon^2 n^2/(q+1)k^2} \\ &\leq C_{\epsilon,k}^n \exp\left(\frac{p\epsilon^2 n^2}{k^2}(d(1 + \ln 2 - \ln d) - (1/2 - d)/(q+1))\right) \rightarrow 0, \end{aligned}$$

by our assumption on d . □

4.2 Waiter's side

We start with the case of a graph H for which there exists $H' \subseteq H$ such that $d_2(H') = m_2(H)$, H' is strictly 2-balanced and it is not a tree or a triangle. Due to the monotonicity of the Client-Waiter game, it is enough to consider the unbiased ($q = 1$) case. Moreover, it is enough to show that Waiter can prevent Client from claiming a copy of H' , and so we may assume that $H' = H$. Our proof follows very closely that of Theorem 2 in [28]. We start with a general sufficient condition for Waiter's win.

Proposition 22. *Let H be a strictly 2-balanced graph which is neither a tree nor a triangle. If G is a graph such that $m(G) \leq m_2(H)$, then Waiter has a winning strategy for the $CW(G, H, 1)$ game.*

Proof. The *arboricity* of a graph G is defined by

$$ar(G) = \max_{G' \subseteq G} \frac{e(G')}{v(G') - 1}.$$

The Nash-Williams arboricity theorem ([27]) states that any graph G can be decomposed into $\lceil ar(G) \rceil$ edge-disjoint forests.

Lemma 23. *Let G, H be graphs such that*

$$\left\lceil \frac{ar(G)}{2} \right\rceil < ar(H),$$

then Waiter has a winning strategy for the $CW(G, H, 1)$ game.

Proof. Set $k = \left\lceil \frac{ar(G)}{2} \right\rceil$, and partition $E(G)$ into $2k$ edge-disjoint forests. Divide these forests into pairs. By Theorem 2 in [14] Waiter can force Client's graph to be a forest when playing on the edges of a union of two edge-disjoint forests. Thus, when playing on G Waiter has a strategy to force Client's graph to be a union of k edge disjoint forests. For any subset $S \subseteq V(G)$, the number of Client's edges spanned by S will be at most $k(|S| - 1)$, which means that Client's graph has arboricity at most k , hence it cannot contain H . \square

Lemma 24. *Let G, H be graphs such that*

$$\left\lceil \frac{m(G)}{2} \right\rceil < m(H),$$

then Waiter has a winning strategy for the $CW(G, H, 1)$ game.

Proof. We first orient the edges of G in the following manner. Set $k = \lceil m(G) \rceil$. Construct a bipartite graph between $E(G)$ and k copies of $V(G)$ and connect each edge to all the copies of the vertices which are incident to it. Since any subset of edges $A \subseteq E(G)$ spans at least $|A|/m(G)$ vertices we get that this graph satisfies Hall's condition with respect to $E(G)$. Hence we have a matching which covers $E(G)$. For every $v \in V$ orient the edges of v such that v is the source of e if and only if e is connected to a copy of v in this matching. Since there are k copies of v , its out-degree will be at most k . Now Waiter can then play on each vertex at a time, offering only pairs of edges for which the current vertex is a source. The maximum out-degree in Client's graph will then be at most $\left\lceil \frac{m(G)}{2} \right\rceil$, which means that its maximal density is lower than $m(H)$, and it certainly does not contain H . \square

To prove the proposition we consider several cases of maximal 2-density of H and use the two lemmas to show that in any case Waiter has a winning strategy. Since this is practically the same as in Theorem 18 in [28] we omit the details. The inquisitive reader can find them in Appendix A. \square

Consider the game $CW(G, H, 1)$ on the edges of some arbitrary graph G . Certainly, any edge in G which does not take part in any copy of H is irrelevant to the outcome of the game. Moreover, if some copy of H in G has two edges neither of which takes part in another copy of H , then Waiter can offer these two edges in a single turn, thus preventing Client from claiming this copy of H while not risking any other copy. This leads to the following definition.

Definition 25. A H -core of G is a maximal subgraph $G' \subseteq G$ such that

- every edge of G' is contained in at least one copy of H in G' , and
- every H -copy in G' has at most one edge which does not take part in another H -copy.

A proof of the existence and uniqueness of the H -core can be found in [28]. By the discussion above Waiter wins the $CW(G, H, 1)$ game if and only if he wins $CW(G', H, 1)$ where G' is the H -core of G . Furthermore, it is enough to show that Waiter has a winning strategy for bi-connected (2-connected) components of G' , since these do not share edges (and in particular they do not share H -copies).

Lemma 26 (Lemma 23 in [28]). *Let H be a strictly 2-balanced graph which is not a tree or a triangle. Then there exist constants $c > 0$ and $L > 0$ such that w.h.p. every bi-connected component of the H -core of $G_{n,p}$ has size at most L , provided that $p \leq cn^{-1/m_2(H)}$.*

We can now finish the proof of the 0-statement of Theorem 7 for $H \neq K_3$. Set $G = G_{n,p}$ with $p = cn^{-1/m_2(H)}$. By the above lemma all the bi-connected components of the H -core of G will be of size at most L . By a well known property of $G_{n,p}$ (as was mentioned in the beginning of this section), w.h.p. all the bi-connected components will have maximal density at most $m_2(H)$ and by Proposition 22 Waiter has a winning strategy while playing on each bi-connected component, and thus when playing on all of the H -core, and indeed on all of G .

Next we turn to the case where $H = K_3$ and $q \geq 2$. The Client's side was covered in the proof of the 1-statement, it remains to show the following:

Proposition 27. *There is some constant $c > 0$ such that Waiter wins the $CW(G_{n,p}, K_3, 2)$ game w.h.p. when $p = cn^{-1/2}$.*

Proof. We will prove two lemmas — the first will show that Client can only win on graphs with maximum density higher than 2, and the second will show that when $p = cn^{-1/2}$, if Client wins on $G_{n,p}$ then w.h.p. he wins on some subgraph of bounded order.

Lemma 28. *Let G be a graph with $m(G) \leq 2$, then Waiter has a winning strategy in the $CW(G, K_3, 2)$ game.*

Proof. Suppose to the contrary that G is a minimal graph such that $m(G) \leq 2$ and Client wins the game on G . By Theorem 1.3 in [8] Waiter can force Client's graph to be acyclic when playing on K_6 with bias 2. We may therefore assume that $v_G > 6$. Let $A \subset V(G)$ be a proper subset, and define $H = G[A]$ and $\hat{H} = G[V \setminus A]$. We must have

that $e(A, V \setminus A) \geq 4$, otherwise Waiter can play on H and then on \hat{H} (winning on both by the minimality of G), and then offer all the edges $E(A, V \setminus A)$ and Client will not claim a triangle. In particular $\delta(G) \geq 4$. But since $m(G) \leq 2$, it must be that $m(G) = 2$ and G is 4-regular. This leads to

$$e(A, V \setminus A) = 4v_H - 2e_H \geq 4 \implies e_H \leq 2(v_H - 1). \quad (*)$$

Let v_0 be an arbitrary vertex, and let $N(v_0) = \{v_1, v_2, v_3, v_4\}$. Denote $H = G[\{v_0\} \cup N(v_0)]$ and $\hat{H} = G[V \setminus (\{v_0\} \cup N(v_0))]$. We claim that $G[N(v_0)]$ must be a connected graph, for otherwise we can partition $N(v_0)$ into two parts, A, B , each with size at most 3, and $E(A, B) = \emptyset$. Waiter can then play his winning strategy on $G \setminus \{v_0\}$, then offer $E(v_0, A)$, and on the last turn he will offer $E(v_0, B)$. It is easy to see that in this case Client will not claim a triangle. From this reasoning together with (*) we deduce that $e_H \in \{7, 8\}$. We now consider several cases. In each case we show that Waiter, after playing his winning strategies on H and on \hat{H} , has a strategy to offer the remaining free edges such that Client will not claim a triangle.

1. Suppose $e_H = 8$ and $G[N(v_0)]$ is isomorphic to C_4 ($G[N(v_0)] \cong C_4$). Then every vertex in H has at most one edge connecting it to \hat{H} . Suppose there is $u \in \hat{H}$ such that $e(u, H) = 4$. Then $G[H \cup \{u\}]$ has 12 edges which violates (*). This means that Waiter (after having played and won on H and \hat{H}) can just offer in each turn all the free edges incident to some vertex in \hat{H} . Client will not claim a triangle since he will not have a vertex with degree higher than one in the cut between H and \hat{H} .
2. Suppose $e_H = 8$ and $G[N(v_0)] \cong K_3 + e$. Then there is only one vertex in H with degree 2, let it be v_1 . Waiter will offer all free edges incident to v_1 , and then the remaining (two) edges. Since the two vertices in H which are connected to v_1 have no neighbours in \hat{H} , Client will not be able to claim a triangle.
3. Suppose $e_H = 7$ and $G[N(v_0)] \cong P_4$. Suppose the path is $\{v_1, v_2, v_3, v_4\}$. As in case 1, there is no $u \in \hat{H}$ such that $e(u, H) = 4$. We have 3 sub-cases.
 - (a) There is $u \in \hat{H}$ such that u is connected to v_1, v_2, v_3 . Then the graph $G[\{v_0, v_1, v_2, v_3, u\}]$ is isomorphic to H of case 1. The case where u is connected to v_2, v_3, v_4 is treated similarly.
 - (b) There is $u \in \hat{H}$ such that u is connected to v_1, v_3, v_4 . Waiter will offer $(u, v_3), (u, v_4)$ and the other free edge incident to v_4 , and in the next turn he can safely offer the remaining free edges. The case where u is connected to v_1, v_2, v_4 is treated similarly.
 - (c) We can assume that for any vertex $u \in \hat{H}$, $e(u, H) \leq 2$. If there is $u \in \hat{H}$ which is connected to both v_1, v_2 then Waiter will offer these edges together with the other free edge of v_1 , otherwise he will offer only the two free edges of v_1 . Likewise if there is $u \in \hat{H}$ which is connected to both v_3, v_4 then Waiter will offer these edges together with the other free edge of v_4 , otherwise he will offer only the two free edges of v_4 . Finally, if there is $u \in \hat{H}$ which is connected

to both v_2, v_3 then Waiter will offer these two edges together. The rest of the edges (if any) can be offered in arbitrary order.

4. Suppose $e_H = 7$ and $G[N(v_0)] \cong S_3$. Let v_1 be the centre vertex. In the last three turns Waiter can offer in each turn the two free edges of v_i for $i = 2, 3, 4$.

We have shown that in every case Waiter has a winning strategy, hence there is no such G . \square

Continuing with the proof of the proposition, the next definition and the lemma that follows are influenced by the ideas of Nenadov, Steger and Stojaković in [28], but we need to make some necessary changes, since (as mentioned there) their proof will not go through for $H = K_3$ as that would be a contradiction to the result of [26].

Let G be a graph. An edge in G is *loose* if it does not take part in any triangle, it is *open* if it takes part in precisely one triangle, and it is *half-open* if it takes part in precisely two triangles. Otherwise, it is *closed*.

Definition 29. A K_3 -core of G is a maximal subgraph $G' \subseteq G$ such that

- there are no loose edges in G' ,
- every triangle in G' has at most one open edge, and
- every half-open edge is in at least one triangle which has no open edges.

Consider the following process for generating a K_3 -core of graph G . We set T to be the set of all triangles of G , and define the subgraph $G_T := \bigcup_{t \in T} t$. Iteratively we remove from T all triangles with more than one open edge in G_T , and all pairs of triangles which share a half-open edge and both have an open edge in G_T , updating G_T after each step. When the process ends G_T is a K_3 -core of G .

Claim 30. *Let G be a graph, and let \mathcal{G} be the family of bi-connected (2-connected) components of a K_3 -core of G . Suppose that for any $G' \in \mathcal{G}$, Waiter has a winning strategy in the $CW(G', K_3, 2)$ game. Then Waiter has a winning strategy in the $CW(G, K_3, 2)$ game.*

Proof. Since the bi-connected components are pairwise edge-disjoint, a winning strategy for each separate component yields a winning strategy for the K_3 -core. After winning on the K_3 -core Waiter will add the removed triangles in the K_3 -core generating process described above, but in reverse order. Each time he adds a triangle which has more than one open edge he will offer Client two of those open edges, and each time he adds a pair of triangles which share an half-open edge he will offer this edge and another open edge from each of those triangles. Finally, when there are no more triangles to add he can just play arbitrarily. It is not hard to verify that this is a winning strategy for Waiter. \square

The next Lemma is rather technical. Its proof can be found in the Appendix.

Lemma 31. *There are constants $c, L > 0$ such that w.h.p. every bi-connected component of any K_3 -core of $G_{n,p}$ is of size at most L , when $p \leq cn^{-1/2}$.*

We can now finish the proof of the proposition. From Lemma 31 we get that when $p = cn^{-1/2}$ then w.h.p. Client wins if and only if he wins on some subgraph of $G_{n,p}$ of order at most L . On the other hand, w.h.p. every subgraph of order L will have maximum density at most 2, which by Lemma 28 will be Waiter's game. \square

Lastly, we consider the Client-Waiter H -game, where H is a tree. Theorem 7 might lead us to think that the threshold for this game should be $\Theta(n^{-1})$, but in fact we will show that it is much lower. We start with a simple threshold for stars.

Claim 32. *Let S_k be the star with k edges. Then $n^{-2k/(2k-1)}$ is a threshold function for Client's win in $CW(G_{n,p}, S_k, 1)$.*

Proof. When $p \gg n^{-2k/(2k-1)}$, there will be w.h.p. a vertex v in $G_{n,p}$ with degree $2k - 1$. Client's strategy will be to take an edge at v every time such an edge is offered to him, thus getting S_k with v at the centre.

On the other hand, assume $p \ll n^{-2k/(2k-1)}$. Then w.h.p. every component of $G_{n,p}$ is a tree with at most $2k - 2$ edges. If Waiter plays every turn in a single component then there will not be a component in Client's graph with more than $k - 1$ edges, and in particular Client will not claim a copy of S_k . \square

The next claim shows that Waiter has a winning strategy in $CW(G_{n,p}, P_{k+1}, 1)$ provided $p \ll n^{-2^{k/2}/(2^{k/2}-1)}$. Thus trees of the same order might have different thresholds.

Claim 33. *Let P_{k+1} be the path with k edges. Then Waiter wins $CW(T_n, P_{k+1}, 1)$, where T_n is any tree of order $n < 2^{k/2}$.*

Proof. Observe that T_n contains at most $\binom{n}{2}$ copies of P_{k+1} . Indeed, each path in T_n is uniquely defined by its two end points. The claim now follows from the next Waiter's winning criterion by Bednarska-Bzdęga.

Lemma 34 (Corollary 1.4 in [7]). *For a set X and a family of subsets \mathcal{F} , if*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Waiter wins the $CW(X, \mathcal{F}, 1)$ game. \square

Though we cannot expect to find a single threshold probability for all trees of size k which depends only on k , we can still show that at any rate these probabilities must be much smaller than inverse linear.

Proof of Proposition 9. We may and will assume $k \geq 3$. Set $m = (k(q + 1))^2$ and let $T = T_{m,k}$ be the complete m -ary tree of height k . For an internal vertex $x \in V(T)$ let A_x be the set of edges from x towards the leaves of T , and let \mathcal{F} be the following family of edges,

$$\mathcal{F} = \{F \subseteq A_x : |F| = m - k + 1, \text{ for some internal vertex } x\}.$$

Since \mathcal{F} is $(m - k + 1)$ -uniform, and $|\mathcal{F}| = \sum_{i=0}^{k-1} m^i \binom{m}{k-1}$ we get that

$$\begin{aligned} \sum_{A \in \mathcal{F}} e^{-|A|/(q+1)} &\leq \binom{m}{k-1} \cdot \frac{m^k}{m-1} \cdot e^{-(m-k+1)/(q+1)} \\ &\leq \frac{m}{m-1} \cdot \left(\frac{em^2}{k-1}\right)^{k-1} e^{-(m-k+1)/(q+1)} \\ &\leq \left(\frac{em^2}{k-1}\right)^{k-1} e^{-(m-k)/(q+1)}. \end{aligned}$$

The last expression will be smaller than 1 when

$$(k-1)(q+1)(1 + 2 \ln m - \ln(k-1)) < m - k.$$

Using our choice for m and rearranging we get that the above condition will be true when $k \geq 3$ and $q \geq 1$, thus by Theorem 12 Client has a strategy such that by the end of the game he claimed at least k out-edges of any internal vertex in T , which means that he claimed a copy of $T_{k,k}$.

Since $v_T = (m^{k+1} - 1)/(m - 1)$, for $p = n^{-(m^{k+1}+1)/m^{k+1}} \gg n^{-v_T/(v_T-1)}$ w.h.p. $G_{n,p}$ will contain T and thus $CW(G_{n,p}, T_{k,k}, q)$ will be Client's win. \square

5 Final words

We have investigated several Client-Waiter games played on the edges of the complete graph. We have found that the critical bias for the maximum-degree- k game is asymptotically between n/k and $2n/k$ (Proposition 1). A natural question is whether either of the bounds can be improved.

Question 35. Can we improve either bound of Proposition 1?

In the giant component game we discussed a phase transition taking place between $n/2$ and $1.6n$ (Theorem 2), where Client's achievement in this game drops from a linear-sized component to at most logarithmic. A more dramatic drop was observed in the path game: from linear length when $q < n/2$ to at most $\ln \ln n$ when $q > n$ (Theorem 3). For both games, but particularly in the giant component game, it is interesting to know if the phase transition can be more accurately located. In [8] Bednarska-Bzdęga et al. showed that in the Waiter-Client large component game the phase transition happens around $q = n$ (which is in accordance with the probabilistic intuition), so it is natural to expect that at least the $1.6n$ upper bound of the phase transition in the Client-Waiter large component game could be brought closer to n , especially since we feel that Client is somewhat weaker in most games in comparison to Waiter in the corresponding Waiter-Client game.

Question 36. Can we narrow the phase transitions described in Theorem 2(ii)+(iii) and Theorem 3(ii)+(iii)?

We have seen an improvement on the lower bound of the H -game played on the edge set of the complete graph (i.e. Proposition 6). Both the probability intuition and comparison to partial results for the Waiter-Client game ([9]) lead us to wonder whether the critical bias should be no higher than $O(n^{1/m(H)})$. An intermediate challenge could be the following.

Question 37. Is it true that for any integer $k \geq 3$ there is $c > 0$ such that Waiter wins $CW(K_n, K_k, q)$ when $q \geq cn^{2/(k-1)}$?

We have also studied the Client-Waiter H -game played on the edges of the random graph. We have seen that essentially there is little difference between this game and the corresponding Maker-Breaker game, and we extended the result of Nenadov, Steger and Stojaković ([28]) to include the biased version of the game, and the case of $H = K_3$ with bias at least 2. We also showed that when H is a tree the picture is more complex and highly depends on the exact structure of the tree, but nevertheless there is always some $\epsilon > 0$ for which Client wins the game w.h.p. when $p = n^{-1-\epsilon}$.

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Appendix A Two missing proofs from Section 4

The end of the proof of Proposition 22. Set $k = \lfloor m_2(H) \rfloor$ and $x = m_2(H) - k$. We consider two cases.

- a) $0 \leq x < 1/2$. Let v be a vertex with $d_H(v) = \delta(H)$. Since H is strictly 2-balanced we have

$$m_2(H \setminus \{v\}) = \frac{e_H - 1 - \delta(H)}{v_H - 3} < \frac{e_H - 1}{v_H - 2} = m_2(H),$$

which leads to $\delta(H) > m_2(H)$, and so $\delta(H) \geq k + 1$. Suppose G is a minimal graph which contradicts the proposition. If v is a vertex with $d_G(v) \leq 2(\delta(H) - 1)$, then Waiter can play his winning strategy on $G \setminus \{v\}$ (which exists by the minimality of G) and in the last $\delta(H) - 1$ turns offer the edges of v . We have found a winning strategy for Waiter on G , and that is a contradiction. Assume then that $\delta(G) \geq 2\delta(H) - 1 \geq 2k + 1$. But then $m(G) \geq k + 1/2 > m_2(H)$ and we have reached a contradiction again.

- b) $x \geq 1/2$. We consider further subcases.

- i. If $k \geq 3$ then

$$\begin{aligned} \left\lceil \frac{m(G)}{2} \right\rceil &\leq \left\lceil \frac{m_2(H)}{2} \right\rceil \leq \left\lceil \frac{k+1}{2} \right\rceil \leq k-1 \leq m_2(H) - 3/2 \\ &= \frac{e_H - 1}{v_H - 2} - 3/2 = \frac{v_H}{v_H - 2} (m(H) - 1/v_H) - 3/2 \\ &= m(H) + \frac{2}{v_H - 2} m(H) - \frac{1}{v_H - 2} - 3/2 \\ &= m(H) + \frac{2e_H - v_H - \frac{3}{2}(v_H^2 - 2v_H)}{v_H(v_H - 2)} \\ &= m(H) + \frac{2e_H - \frac{3}{2}v_H^2 + 2v_H}{v_H(v_H - 2)} < m(H), \end{aligned}$$

where in the last inequality we used that $e_H \leq \binom{v_H}{2} < \frac{3}{4}v_H^2 - v_H$ (since $v_H \geq 4$). By Lemma 24 Waiter has a winning strategy.

ii. If $e_H < v_H^2/4$, then $\frac{e_H}{v_H} + 1/2 > \frac{e_H-1}{v_H-2}$, and we get

$$\left\lceil \frac{m(G)}{2} \right\rceil \leq \left\lceil \frac{m_2(H)}{2} \right\rceil \leq \left\lceil \frac{k+1}{2} \right\rceil \leq k \leq m_2(H) - 1/2 < m(H),$$

and again by Lemma 24 Waiter has a winning strategy.

iii. Suppose that $e_H \geq \lceil v_H^2/4 \rceil$ and $k < 3$ and $v_H \geq 5$. Then

$$m_2(H) = \frac{e_H - 1}{v_H - 2} \geq \frac{\lceil v_H^2/4 \rceil - 1}{v_H - 2} \geq 2.$$

But since we assume $x \geq 1/2$ then actually $m_2(H) \geq 2.5$. For any subgraph $G' \subseteq G$, we have $e_{G'}/(v_{G'} - 1) = e_{G'}/v_{G'} + e_{G'}/(v_{G'}(v_{G'} - 1)) \leq e_{G'}/v_{G'} + 1/2$, which together with $m_2(H) < 3$ leads to

$$ar(G) \leq m(G) + 1/2 \leq m_2(H) + 1/2 < 4.$$

On the other hand, using $v_H \geq 5$, we derive

$$ar(H) \geq \frac{e_H}{v_H - 1} = \frac{m_2(H)(v_H - 2) + 1}{v_H - 1} \geq 2.$$

We got that $\lceil ar(G)/2 \rceil < ar(H)$ and by Lemma 23 Waiter wins.

iv. The remaining case is $v_H = 4$. In this case $H = C_4$ or $H = K_4$, as those are the only strictly 2-balanced graphs on 4 vertices. The latter can be proved by an adaptation of Lemma 2.1 in [26], while in the former we have $ar(C_4) = 4/3$, and $ar(G) \leq m(G) + 1/2 \leq m_2(C_4) + 1/2 \leq 2$, and again Waiter wins by Lemma 23.

□

Proof of Lemma 31. We call a triangle T *unproblematic* if at least two of its edges are open or one of its edges is open and it shares a half-open edge with triangle T_1 which has at least one open edge. Otherwise we call T *problematic*. Fix some K_3 -core of $G = G_{n,p}$ and let G' be a bi-connected component of this K_3 -core. We describe a process to construct G' from the empty graph by repeatedly attaching triangles.

- 1: Let T_0 be a triangle in G'
- 2: $k \leftarrow 0$; $\hat{G} \leftarrow T_0$
- 3: **while** $\hat{G} \neq G'$ **do**
- 4: $k \leftarrow k + 1$
- 5: **if** \hat{G} contains a triangle which is unproblematic in \hat{G} **then**
- 6: let $\ell < k$ be the smallest index such that T_ℓ is an unproblematic triangle in \hat{G}
- 7: **if** there is a triangle $T \subset G'$ such that T contains one of T_ℓ 's open edges **then**
- 8: $T_k = T$

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9:         else
10:            let  $e \in T_\ell$  be a half-open edge of  $T_\ell$ 
11:            if there is a triangle  $T \subset G'$ ,  $T \notin \hat{G}$  and  $T$  contains  $e$  then
12:                 $T_k = T$ 
13:            else
14:                let  $T' \subset \hat{G}$  be the other triangle which contains  $e$ 
15:                let  $T_k \subset G'$  such that  $T_k$  contains one of  $T'$ 's open edges
16:            end if
17:        end if
18:    else
19:        let  $T_k$  be a triangle in  $G'$  that is not contained in  $\hat{G}$  and intersects  $\hat{G}$  in at least
        one edge
20:    end if
21:     $\hat{G} \leftarrow \hat{G} \cup T_k$ 
22: end while

```

We need to show that w.h.p. the highest value of k is bounded by some constant. For $i \geq 1$, let T_i be the triangle added to \hat{G} at the i -th step, and let \hat{G}_i be the graph \hat{G} just after adding T_i . If T_i intersected \hat{G}_{i-1} in exactly one edge we call T_i *regular*, while if it intersected \hat{G}_{i-1} in three vertices we call it *degenerate*. Denote by $\text{reg}(i)$ and $\text{deg}(i)$ the number of regular, resp. degenerate, triangles in \hat{G}_i . Furthermore, for $1 \leq i \leq \ell$ we say that T_i is *fully-open* at time ℓ if T_i has a vertex which is not touched by any other triangle of \hat{G}_ℓ (notice that T_i is necessarily regular in this case). Denote by $f(\ell)$ the number of fully-open triangles at time ℓ .

Claim 38. *For every $\ell \geq 1$, assuming the process does not stop before the ℓ -th step, we have*

$$f(\ell) \geq \frac{1}{2} \cdot \text{reg}(\ell) - 3 \cdot \text{deg}(\ell).$$

Proof. Denote the right hand side of the above by $\varphi(\ell) := \text{reg}(\ell)/2 - 3\text{deg}(\ell)$. We will use induction to show that the following stronger statement holds for any $\ell \geq 1$

$$f(\ell) \geq \begin{cases} \varphi(\ell), & \text{if } T_\ell \text{ is regular} \\ \varphi(\ell) + 1, & \text{if } T_\ell \text{ is degenerate.} \end{cases}$$

This is true for $\ell = 1$ since T_1 must be regular and $f(1) = 1 > 1/2$. At $\ell = 2$ triangle T_0 still has two open edges and T_2 must share one of them. We have two options: if T_2 is regular then it is fully-open and $f(2) = 2 > \varphi(2) = 1$, and if T_2 is degenerate then $f(2) = 0 > \varphi(2) + 1 = -1.5$. Suppose now that we are at the ℓ -th step, $\ell \geq 3$. If T_ℓ is degenerate, then since T_ℓ shares edges with at most two fully-open triangles we have $f(\ell) \geq f(\ell-1) - 2 \geq \text{reg}(\ell-1)/2 - 3\text{deg}(\ell-1) - 2 = \text{reg}(\ell)/2 - 3(\text{deg}(\ell) - 1) - 2 = \varphi(\ell) + 1$. Otherwise, assume that T_ℓ is regular. Consider two cases

- If T_ℓ does not connect to a fully-open triangle then since T_ℓ is regular and fully-open, $f(\ell) = f(\ell-1) + 1 \geq \varphi(\ell-1) + 1 = \varphi(\ell) + 1/2$.

- If T_ℓ does connect to a fully-open triangle then $f(\ell) = f(\ell-1)$. If $T_{\ell-1}$ was degenerate then $f(\ell-1) \geq \varphi(\ell-1) + 1 = \varphi(\ell) + 1/2$. Assume then that $T_{\ell-1}$ is regular and connected to T' . If T' was not fully open then $f(\ell) = f(\ell-2) + 1 \geq \varphi(\ell-2) + 1 \geq (\text{reg}(\ell) - 2)/2 - 3\text{deg}(\ell) + 1 = \varphi(\ell)$. Assume then that T' was fully open. If $T_{\ell-2}$ was degenerate then $f(\ell) = f(\ell-2) \geq \varphi(\ell-2) + 1 = \varphi(\ell)$. Assume that $T_{\ell-2}$ was regular and connected to T'' . Again if T'' was not fully open we are done, otherwise T'' is fully open and $T_{\ell-2}$ is connected to one of its open edges, e . At time $\ell-1$, T'' is still unproblematic since it has an open edge and e is half-open and $T_{\ell-2}$ is fully open. Then by our algorithm, and since T' is fully open, it must be that $T' = T_{\ell-2}$. But then at time ℓ , T'' is still unproblematic and therefore T_ℓ must connect to $T_{\ell-2}$. But $T_{\ell-2}$ is no longer fully open, which is a contradiction. \square

Returning to the proof of the lemma, suppose we are in the i -th step of the process described above. We first bound the probabilities of finding certain triangles. If there is an unproblematic triangle then there are at most four edges to which a new regular triangle can connect, and at most n possibilities for the extra vertex. Thus the probability of finding a new regular triangle given that we have an unproblematic triangle is at most $\Pr_{\text{reg,unprob.}} \leq 4np^2 \leq 4c^2 < 1/2$, if we choose $c < 1/\sqrt{8}$. Whenever a degenerate triangle is added we add at least one new edge to the graph, which at the i -th step has at most $3i$ vertices. Thus the probability of finding a degenerate triangle at the i -th step is at most $P_{\text{deg}}(i) \leq \binom{3i}{3}p \leq (3i)^3cn^{-1/2}$. Set $L = 42$ and $\ell_0 = 4\log_2 n$. Let X be the random variable of the number of different runnings of this process on $G_{n,p}$, that contain at least 7 degenerate triangles in the first ℓ_0 steps. After the L -th step, and as long as we have fewer than 7 degenerate triangles, we must have unproblematic triangles, because for any $\ell > L$, by Claim 38 $f(\ell) \geq (\ell-6)/2 - 18 > 0$. So the probability for a regular triangle after the L -th step and before the 7-th degenerate triangle appears is at most $1/2$. Denote by ℓ' the moment in which the 7-th degenerate triangle appears, then we have

$$\mathbb{E}[X] \leq \binom{n}{3} \sum_{8 \leq \ell' \leq \ell_0} \binom{\ell' - 1}{6} (27\ell_0^3 cn^{-1/2})^7 \cdot 2^{-(\ell' - L - 6)} = o(1).$$

Now denote by Y the random variable of the number of different runnings of this process on $G_{n,p}$ that last more than ℓ_0 steps and contain fewer than 7 degenerate triangles in the first ℓ_0 steps. We get

$$\mathbb{E}[Y] \leq \binom{n}{3} \sum_{k=0}^6 \binom{\ell_0}{k} (27\ell_0^3 cn^{-1/2})^k \cdot 2^{-(\ell_0 - L - k)} = o(1),$$

by our choice of ℓ_0 . So w.h.p. $X = Y = 0$, which means that all the processes last for less than ℓ_0 steps and contain at most 6 degenerate triangles. Denote by ℓ_e the length of such a process. Then it must be that $f(\ell_e) = 0$, or we would still have fully-open triangles at step ℓ_e . Thus by Claim 38

$$0 = f(\ell_e) \geq \text{reg}(\ell_e)/2 - 3\text{deg}(\ell_e) \geq (\ell_e - 6)/2 - 18 \implies \ell_e \leq 42. \quad \square$$