Cluster automorphisms and the marked exchange graphs of skew-symmetrizable cluster algebras

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Abstract

Cluster automorphisms have been shown to have links to the mapping class groups of surfaces, maximal green sequences and to exchange graph automorphisms for skew-symmetric cluster algebras. In this paper we generalise these results to the skew-symmetrizable case by introducing a marking on the exchange graph. Many skew-symmetrizable matrices unfold to skew-symmetric matrices and we consider how cluster automorphisms behave under this unfolding with applications to coverings of orbifolds by surfaces.

Keywords: cluster algebra; quiver mutation; cluster automorphism; exchange graph; mapping class group.

1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in [14], and have since found applications across many types of mathematics. These are commutative subalgebras of $\mathbb{C}(x_1, \ldots, x_n)$ generated by rational functions constructed using a certain combinatorial procedure starting from an initial seed which produces that seed's mutation class.

In the same paper Fomin and Zelevinsky defined the exchange graph of a cluster algebra to better visualise the combinatorics of the mutation class. These graphs proved a useful tool in their classification of finite-type cluster algebras in [15] where these algebras were shown to correspond to Dynkin diagrams.

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Cluster algebras were shown to be closely related to triangulations of surfaces by Fomin, Shapiro and Thurston in [13], where a quiver is constructed from a given triangulation and quiver mutations correspond to flipping an edge in the triangulation. These quivers from surfaces play an important role in the classification of mutation-finite quivers, given by Felikson, Shapiro and Tumarkin in [12], as all such quivers are mutation-finite and there are only 11 other exceptional mutation classes.

In a similar fashion triangulations of orbifolds with orbifold points of order 2 were shown to correspond to mutation-finite diagrams by Felikson, Shapiro and Tumarkin in [10]. Unfoldings of diagrams, introduced by Felikson, Shapiro and Tumarkin in [11], then correspond to coverings of the orbifold by a triangulated surface, as shown in [10, Section 12].

Cluster automorphisms were introduced by Assem, Schiffler and Shramchenko in [1], for cluster algebras generated from quivers, as automorphisms of the cluster algebra taking clusters to clusters and acting as either the identity or the opposite function on quivers. These ideas were extended to cluster algebras generated from certain skew-symmetrizable matrices by Chang and Zhu in [7]. The group of cluster automorphisms of a cluster algebra arising from the triangulation of a surface was shown to be isomorphic to the mapping class group of this surface by Brüstle and Qiu in [4].

In their paper on labelled seeds and global mutations [18], King and Pressland showed that cluster automorphisms arise naturally when mutation classes are considered as orbits of labelled seeds under the action of a global mutation group M_n. The group of cluster automorphisms is a subgroup of the automorphisms of these mutation classes, Aut_{M_n}, which commute with this group action, and in fact for mutation-finite quivers these groups are isomorphic. We use the links between automorphisms of the exchange graph and the labelled exchange graph to prove that this group Aut_{M_n} is isomorphic to the group of exchange graph automorphisms:

Theorem 3.11. *For a labelled mutation class $\mathcal{S}^0$ with mutation class $\mathcal{S} = \mathcal{S}^0/\mathrm{Sym}(n)$ and exchange graph $\mathcal{E}(\mathcal{S})$*

$$\mathrm{Aut}_{M_n}(\mathcal{S}^0) \cong \mathrm{Aut}\,\mathcal{E}(\mathcal{S}).$$

Therefore for mutation-finite quivers, such as those from triangulations of a surface, exchange graph automorphisms are cluster automorphisms.

Corollary 4.7. *For a cluster algebra $\mathcal{A}$ constructed from a mutation-finite quiver with exchange graph $\mathcal{E}_{\mathcal{A}}$*

$$\mathrm{Aut}\,\mathcal{E}_{\mathcal{A}} \cong \mathrm{Aut}\,\mathcal{A}.$$

This result was proved in a different way by Chang and Zhu in [6] who also proved an extension of this to skew-symmetrizable matrices of type B_n and C_n for $n \geqslant 3$. However for other skew-symmetrizable matrices it is not true that exchange graph automorphisms are cluster automorphisms. It can be shown that the group of cluster automorphisms is isomorphic to a subgroup of the group of exchange graph automorphisms but in general there exist graph automorphisms which do not correspond to cluster automorphisms.

In order to generalise these results we introduce a marking on the exchange graph in such a way that any automorphism which fixes these markings does in fact correspond to a cluster automorphism.

Theorem 5.19. *Let $(\boldsymbol{x}, B)$ be a seed where B is a mutation-finite skew-symmetrizable matrix with cluster algebra $\mathcal{A}$ and marked exchange graph $\widehat{\mathcal{E}}_{\mathcal{A}}$ then*

$$\operatorname{Aut}\mathcal{A} = \operatorname{Aut}\widehat{\mathcal{E}}_{\mathcal{A}}.$$

Therefore the cluster automorphisms of any cluster algebra generated by mutation-finite skew-symmetrizable matrices can be studied using just the combinatorial properties of its marked exchange graph.

A skew-symmetrizable matrix associated to a good orbifold with order 2 orbifold points can be unfolded to a skew-symmetric matrix associated to a surface which covers the orbifold. In this case we show that automorphisms of the marked exchange graph induce automorphisms of the unfolded exchange graph.

Theorem 6.4. *Given a skew-symmetrizable matrix B which unfolds to a matrix Q, with corresponding marked exchange graphs $\widehat{\mathcal{E}}(B)$ and $\mathcal{E}(Q) = \widehat{\mathcal{E}}(Q)$,*

$$\operatorname{Aut}\widehat{\mathcal{E}}(B) \hookrightarrow \operatorname{Aut}\mathcal{E}(Q).$$

We finish the paper with a conjecture generalising a result of Brüstle and Qiu linking the tagged mapping class group of a surface with the cluster automorphisms of the corresponding surface cluster algebra.

Conjecture 7.6. *For a cluster algebra $\mathcal{A}$ arising from the triangulation of an orbifold $\mathcal{O}$*

$$\mathrm{MCG}_{\bowtie}(\mathcal{O}) \cong \operatorname{Aut}^{+}\mathcal{A}.$$

The structure of the paper is as follows: Section 2 gives basic definitions of cluster algebras and mutations while Section 3 looks at the exchange graph of a cluster algebra and includes proofs linking graph automorphisms and mutation class automorphisms. Section 4 recalls the definition of cluster automorphisms and various known results linking these to mutation class automorphisms and exchange graph automorphisms. The section ends by explaining how a maximal green sequence of an acyclic quiver can be used to construct a cluster automorphism.

In Section 5 we introduce the marked exchange graph which enables us to extend these results to cluster algebras from skew-symmetrizable matrices. We show that graph automorphisms fixing the marking are in one-to-one correspondence with cluster automorphisms.

In Section 6 we consider unfoldings of skew-symmetrizable matrices and show how the cluster automorphisms of a skew-symmetrizable cluster algebra induce cluster automorphisms of its unfolded cluster algebra. Section 7 looks at these ideas when the skew-symmetrizable cluster algebra is constructed from an orbifold and its unfolding gives a surface cluster algebra.

2 Mutations

A *skew-symmetric* matrix is a matrix A such that $A^T = -A$. A *skew-symmetrizable* matrix is a matrix B such that there exists some diagonal integer matrix D with positive diagonal entries for which BD is a skew-symmetric matrix. Such a matrix with the smallest entries is called the *symmetrizing matrix* of B.

A *quiver* is an oriented graph possibly with multiple arrows between two vertices and in this paper we always assume that it is restricted to having no loops or 2-cycles. If Q is a quiver, then its *opposite* Q^{op} is the quiver constructed by reversing the direction of all arrows in Q.

The restrictions on the definition of a quiver ensure that quivers are in one-to-one correspondence with skew-symmetric matrices. A given skew-symmetric matrix $B = (b_{i,j})_{i,j\in\{1,\ldots,n\}}$ defines a quiver with n vertices and $b_{i,j}$ arrows from the i-th vertex to the j-th vertex if $b_{i,j} > 0$.

A *diagram* is a weighted oriented graph which does not have multiple arrows between any two vertices, in addition to having no loops or 2-cycles, where the weights on the edges are positive integers. Similarly to quivers, if R is a diagram then its *opposite* R^{op} is constructed by reversing all arrows in R.

Unlike with quivers, there is no one-to-one correspondence between diagrams and skew-symmetrizable matrices. Given a skew-symmetrizable matrix $B = (b_{i,j})_{i,j\in\{1,\ldots,n\}}$ we can construct a diagram with n vertices and an arrow from the i-th vertex to the j-th vertex with weight $-b_{i,j}b_{j,i}$ if $b_{i,j} > 0$. Usually weights of 1 are omitted and just shown as an unweighted arrow. However a diagram only corresponds to a matrix if the product of weights along any chordless cycle is a perfect square and in this case may correspond to multiple matrices.

Throughout this paper we assume that all quivers and diagrams are connected. The results can be easily extended to disconnected diagrams, however care must be taken as different connected components could have their arrows reversed while other components do not, so the idea of an opposite diagram is less clear.

Let $\mathbb{K} = \mathbb{C}(x_1, \ldots, x_n)$. A *cluster* is a set of algebraically independent elements of $\mathbb{K}$, while a *labelled cluster* is a cluster with some ordering of its elements. The individual elements in a cluster are called *cluster variables*.

A *labelled seed* is a pair $(\mathbf{x}, B)$ where B is a skew-symmetrizable matrix and $\mathbf{x}$ is a labelled cluster. Each cluster variable in the cluster can be thought of as being attached to one of the matrix rows, or equivalently attached to one of the vertices of the corresponding quiver or diagram. A *seed* is a class of labelled seeds which differ only by permutations.

Throughout this paper we assume that the matrix in a seed is uniquely determined by its cluster. This has been proved for all cluster algebras of geometric type or generated from a non-degenerate matrix by Gekhtman, Shapiro and Vainshtein in [16]. In this case denote the matrix for a given cluster $\mathbf{x}$ by $B(\mathbf{x})$.

Definition 2.1. Given a labelled seed $u = (\mathbf{x}, B)$, where $\mathbf{x} = (\beta_1, \ldots, \beta_n)$ and $B = (b_{i,j})$, then the *mutation* μ_k acts on u to give $u \cdot \mu_k = (\mathbf{x}', B')$ where $\mathbf{x}' = (\beta_i', \ldots, \beta_n')$ and

$B' = \left(b'_{i,j}\right)$ given by

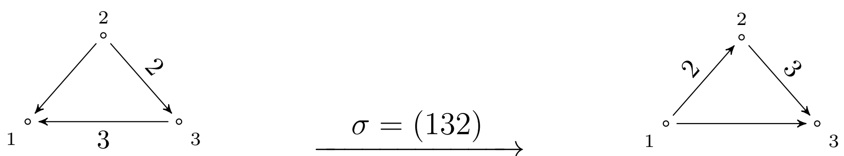
$$\beta'_i = \begin{cases} \beta_i & \text{if } i \neq k, \\ \frac{\prod_{b_{j,i}>0} \beta_j^{b_{j,i}} + \prod_{b_{j,i}<0} \beta_j^{-b_{j,i}}}{\beta_i} & \text{if } i = k, \end{cases}$$

$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k, \\ b_{i,j} + \frac{\left|b_{i,k}\right| b_{k,j} + b_{i,k} \left|b_{k,j}\right|}{2} & \text{otherwise.} \end{cases}$$

It is sometimes convenient to consider the local mutation $\mu_{\beta,\mathbf{x}}$ of a seed $(\mathbf{x}, B)$ corresponding to the mutation at the vertex associated to the cluster variable $\beta \in \mathbf{x}$. These local mutations act as functions on seeds, whereas global mutations act on labelled seeds.

Permutations act on a labelled seed $(\mathbf{x}, B)$, $\mathbf{x} = (\beta_1, \dots, \beta_n)$, $B = (b_{i,j})$ in the expected way taking the i-th vertex to the $\sigma(i)$-th vertex and the i-th cluster variable to the $\sigma(i)$-th cluster variable. Therefore $(\mathbf{x}, B) \cdot \sigma = (\mathbf{x}^\sigma, B^\sigma)$ where $\mathbf{x}^\sigma = \left(\beta_{\sigma^{-1}(1)}, \dots, \beta_{\sigma^{-1}(n)}\right)$ and $B^\sigma = \left(b^\sigma_{i,j}\right)$, $b^\sigma_{i,j} = b_{\sigma^{-1}(i),\sigma^{-1}(j)}$.

Example 2.2. Given a 3 vertex seed $(\mathbf{x}, B)$ as in Figure 1 and permutation $\sigma = (132)$ then σ maps the first vertex and cluster variable to the third, second to first and third to second. Therefore $\beta^\sigma_1 = \beta_2 = \beta_{\sigma^{-1}(1)}$, $\beta^\sigma_2 = \beta_3$ and $\beta^\sigma_3 = \beta_1$. Similarly $B^\sigma_{1,2} = 2 = B_{2,3} = B_{\sigma^{-1}(1),\sigma^{-1}(2)}$ and $B^\sigma_{3,2} = -3 = B_{1,3}$.



$$B = \begin{pmatrix} 0 & -1 & -3 \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix} \xrightarrow{\sigma = (132)} B^\sigma = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 0 & 1 \\ -1 & -3 & 0 \end{pmatrix}$$

$$\mathbf{x} = (\beta_1, \beta_2, \beta_3) \qquad \mathbf{x}^\sigma = (\beta_2, \beta_3, \beta_1)$$

Figure 1: Example of a permutation $\sigma = (132)$ acting on a seed $(\mathbf{x}, B)$ to give $(\mathbf{x}, B) \cdot \sigma = (\mathbf{x}^\sigma, B^\sigma)$.

Definition 2.3 ([18, Section 1])**.** The *global mutation group* for seeds of rank n is given by

$$M_n = \left\langle \mu_1, \dots, \mu_n \mid \mu_i^2 = 1 \right\rangle \rtimes \mathrm{Sym}(n)$$

where the μ_i are mutations and $\mu_i \sigma = \sigma \mu_{\sigma(i)}$ for $\sigma \in \mathrm{Sym}(n)$.

The *labelled mutation class* $\mathcal{S}^0$ of a labelled seed $(\mathbf{x}, B)$ is the orbit of $(\mathbf{x}, B)$ under the action of M_n. The quotient by the symmetric group action gives the *mutation class*

$$\mathcal{S} = \mathcal{S}^0 \big/ \mathrm{Sym}(n).$$

Two seeds in the same mutation class are said to be *mutation-equivalent.*

Definition 2.4. The *cluster algebra* $\mathcal{A}(\mathcal{S})$ is the subalgebra of $\mathbb{K}$ generated by all cluster variables occurring in the seeds in $\mathcal{S}$.

A cluster algebra is said to be of *finite type* if there are a finite number of generating cluster variables in the mutation class, otherwise it is of *infinite type*. If there are a finite number of distinct matrices in the seeds of $\mathcal{S}$, then the cluster algebra and all the matrices are said to be *mutation-finite* or of *finite mutation type*, otherwise it is *mutation-infinite* or of *infinite mutation type*.

Definition 2.5 ([18, Section 2])**.** The mutation class automorphism group $\mathrm{Aut}_{M_n}(\mathcal{S}^0)$ is the group of bijections $\phi : \mathcal{S}^0 \to \mathcal{S}^0$ which commute with the action of M_n, so for all $s \in \mathcal{S}^0$, $g \in M_n$ and $\phi \in \mathrm{Aut}_{M_n}(\mathcal{S}^0)$

$$\phi(s \cdot g) = \phi(s) \cdot g.$$

3 Exchange graphs

Fomin and Zelevinsky in [14] developed the idea of the exchange graph of a cluster algebra to better visualise the relations in a mutation class. These were also an important tool in their classification of finite type cluster algebras in [15].

Definition 3.1. The *exchange graph* $\mathcal{E}(\mathcal{S})$ of a mutation class $\mathcal{S}$ is constructed with vertices for each seed in $\mathcal{S}$ and an edge between two seeds u and v if and only if there is a single local mutation μ such that $\mu(u) = v$.

The *labelled exchange graph* $\Delta(\mathcal{S}^0)$ of a labelled mutation class $\mathcal{S}^0$ is constructed with a vertex for each labelled seed in $\mathcal{S}^0$ and an edge labelled i between two labelled seeds u and v if and only if $u \cdot \mu_i = v$ (and conversely $v \cdot \mu_i = u$).

Example 3.2 ($\boldsymbol{A_2}$)**.** The exchange graph for the cluster algebra of type A_2 is the well known pentagon, as seen in Figure 3. The labelled exchange graph is a decagon shown in Figure 2, with the permutation acting by taking a seed to its antipodal seed.

Example 3.3 ($\boldsymbol{B_2}$)**.** The exchange graph for a cluster algebra of type B_2 is a hexagon, as shown in Figure 5. The labelled exchange graph however is the disjoint union of two hexagons as shown in Figure 4. The permutation interchanging the cluster variables in a labelled seed gives another labelled seed which cannot be obtained from the first though just mutations, so any labelled seed has a permuted counterpart in the other connected component.

Definition 3.4. The *exchange graph automorphism group* $\mathrm{Aut}\,\mathcal{E}(\mathcal{S})$ is the group of permutations σ of the vertex set of the exchange graph such that there is an edge between two vertices u and v if and only if there is an edge between $\sigma(u)$ and $\sigma(v)$.

The *labelled exchange graph automorphisms* in $\mathrm{Aut}\,\Delta(\mathcal{S}^0)$ must also preserve the labelling of the edges.

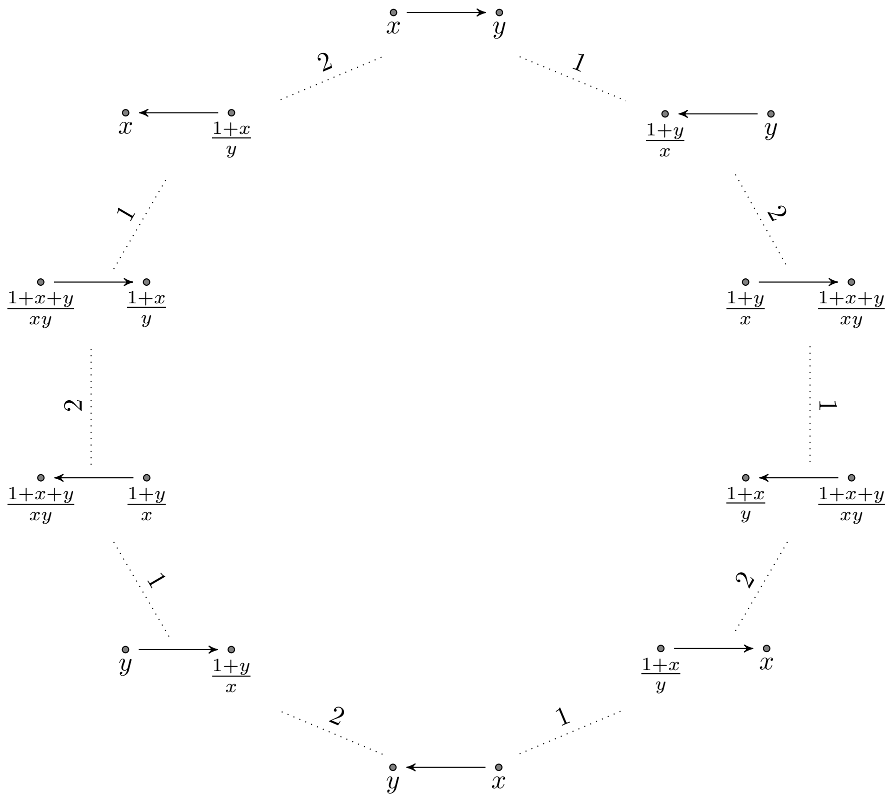


Figure 2: Labelled exchange graph for the mutation class of type A_2.

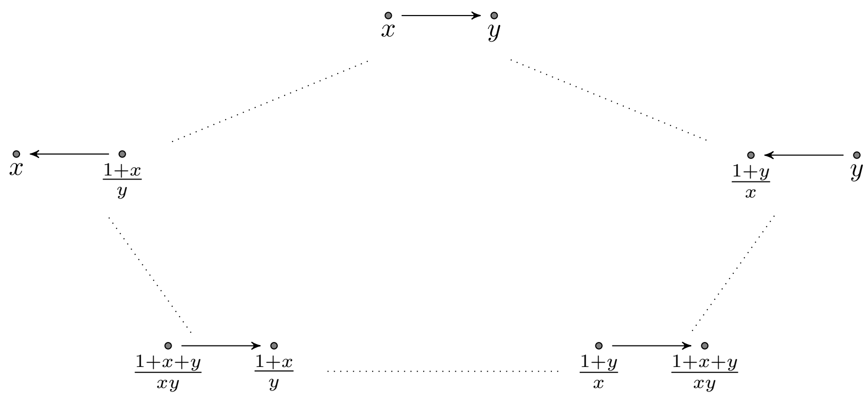


Figure 3: Exchange graph for the mutation class of type A_2.

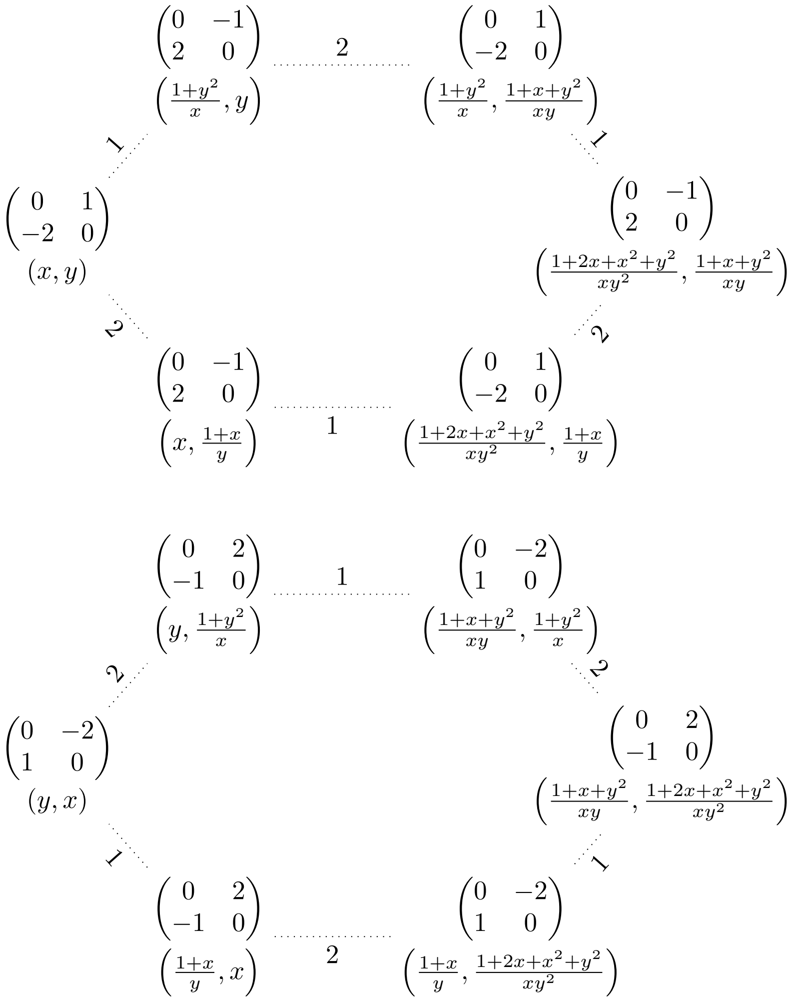


Figure 4: Labelled exchange graph for the mutation class of type B_2.

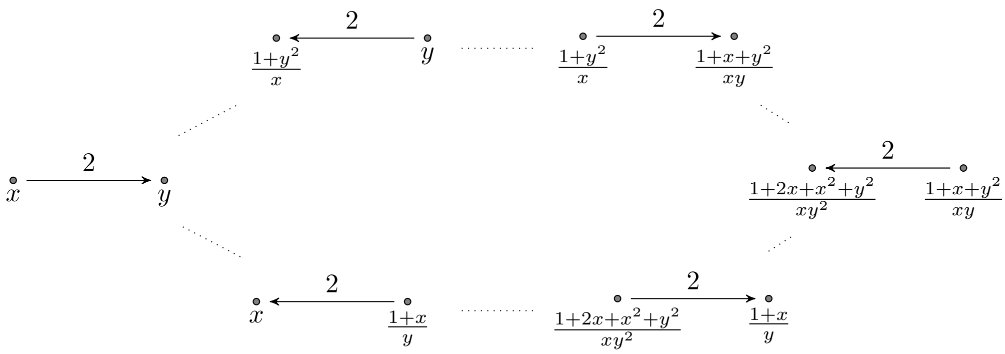


Figure 5: Exchange graph for the mutation class of type B_2.

Theorem 3.5. *For a labelled mutation class $\mathcal{S}^0$ with quotient $\mathcal{S}$ and corresponding exchange graphs $\Delta(\mathcal{S}^0)$ and $\mathcal{E}(\mathcal{S})$, then*

$$\operatorname{Aut}\mathcal{E}(\mathcal{S}) \hookrightarrow \operatorname{Aut}\Delta(\mathcal{S}^0).$$

Proof. To show this we construct a unique $\phi^\Delta \in \operatorname{Aut}\Delta(\mathcal{S}^0)$ for each $\phi \in \operatorname{Aut}\mathcal{E}(\mathcal{S})$. Let $\mathbf{x}(v)$ denote the cluster of a seed v.

Choose a seed u in $\Delta(\mathcal{S}^0)$, then for each $i \in \{1,\dots,n\}$ there is a vertex $v^i = u \cdot \mu_i$ with a corresponding edge $u - v^i$ labelled i in the labelled exchange graph. The cluster $\mathbf{x}(u) = (\beta_1,\dots,\beta_i,\dots,\beta_n)$ then differs from the cluster $\mathbf{x}(v^i) = (\beta_1,\dots,\beta_i',\dots,\beta_n)$ in just the i-th cluster variable.

Under the quotient by the symmetric group action the labelled seed u gets mapped to a seed $[u]$ and in the exchange graph $\mathcal{E}(\mathcal{S})$ there are edges $[u]-[v^i]$ for each $i \in \{1,\dots,n\}$. Each unordered cluster $\mathbf{x}[v^i]$ differs from the unordered cluster $\mathbf{x}[u]$ in a single variable, just as the corresponding labelled clusters do.

The exchange graph automorphism ϕ maps $[u]$ to some seed $\phi[u]$ and preserves all edges in the graph, so $\phi[u]$ is connected to $\phi[v^i]$ for each i. Therefore each $\phi[v^i]$ is a single mutation from $\phi[u]$, and so the unordered cluster $\mathbf{x}(\phi[u]) = [\gamma_1,\dots,\gamma_{k_i},\dots,\gamma_n]$ differs from $\mathbf{x}(\phi[v^i]) = [\gamma_1,\dots,\gamma_{k_i}',\dots,\gamma_n]$ in a single cluster variable.

Set the image $\phi^\Delta(u)$ to be the seed defined by the labelled cluster $\mathbf{x}\big(\phi^\Delta(u)\big) = (\gamma_{k_1},\gamma_{k_2},\dots,\gamma_{k_n})$, obtained by choosing an order of the cluster $\mathbf{x}(\phi[u])$ such that the i-th variable of $\mathbf{x}\big(\phi^\Delta(v^i)\big)$ is the corresponding γ_{k_i}', while all other variables are the same as for $\phi^\Delta(u)$. This ensures that the edge between $\phi^\Delta(u)$ and $\phi^\Delta(v^i)$ is labelled i. Repeat this procedure with initial seed v^i to get the ordering of the seeds connected to $\phi^\Delta(v^i)$.

Continuing this construction for all seeds in $\Delta(\mathcal{S}^0)$ constructs images under ϕ^Δ for all seeds in the labelled exchange graph. For any two seeds s,t connected by an edge labelled k in $\Delta(\mathcal{S}^0)$ this construction ensures that the images $\phi^\Delta(s)$ and $\phi^\Delta(t)$ are also connected by an edge labelled k, and so ϕ^Δ is indeed an automorphism of the labelled exchange graph. □

Example 3.6. Consider the automorphism ϕ of the B_2 exchange graph $\mathcal{E}$ shown in Figure 5 given by a clockwise rotation by angle $\frac{\pi}{3}$. This automorphism pulls back to an automorphism ϕ^Δ of the labelled exchange graph Δ shown in Figure 4.

To determine the automorphism ϕ^Δ, choose an initial labelled seed $u = (\mathbf{x}, B)$ where $\mathbf{x} = (x,y)$. The automorphism ϕ maps the corresponding cluster $[x,y]$ to $[\frac{1+y^2}{x}, y]$ and the mutation μ_1 takes (x,y) to $(x,y)\cdot\mu_1 = \Big(\frac{1+y^2}{x}, y\Big)$, whose corresponding cluster $[\frac{1+y^2}{x}, y]$ is mapped to $[\frac{1+y^2}{x}, \frac{1+x+y^2}{xy}]$ by ϕ, as shown in Figure 6.

Denote by $\phi^\Delta \in \operatorname{Aut}\Delta$ the automorphism which corresponds to $\phi \in \operatorname{Aut}\mathcal{E}$ and denote the quotient by the symmetric group action as $\pi : \mathcal{S}^0 \to \mathcal{S}$. Then $\phi^\Delta(u)$ is a labelled seed in $\mathcal{S}^0$ such that $\pi\big(\phi^\Delta(u)\cdot\mu_1\big) = \phi(\pi(u\cdot\mu_1))$. Hence the cluster variable which differs between $[\frac{1+y^2}{x}, y]$ and $[\frac{1+y^2}{x}, \frac{1+x+y^2}{xy}]$ needs to appear in the first position of the labelled

$$\begin{tikzcd}
\left(x, \frac{1+x}{y}\right) \arrow[r, dotted, dash, "\mu_2"] \arrow[d, "\pi"] & (x,y) \arrow[r, dotted, dash, "\mu_1"] \arrow[d, "\pi"] \arrow[dddd, dashed, bend right, "\phi^\Delta"'] & \left(\frac{1+y^2}{x}, y\right) \arrow[d, "\pi"] \\
[x, \frac{1+x}{y}] \arrow[r, dotted, dash] \arrow[d, "\phi"] & [x,y] \arrow[r, dotted, dash] \arrow[d, "\phi"] & [\frac{1+y^2}{x}, y] \arrow[d, "\phi"] \\
[x,y] \arrow[r, dotted, dash] & [\frac{1+y^2}{x}, y] \arrow[r, dotted, dash] \arrow[dd, "\text{choose order}"] & [\frac{1+y^2}{x}, \frac{1+x+y^2}{xy}] \\
 & & \\
(y,x) \arrow[r, dotted, dash, "\mu_2"] \arrow[uu, "\pi"] & \left(y, \frac{1+y^2}{x}\right) \arrow[r, dotted, dash, "\mu_1"] & \left(\frac{1+x+y^2}{xy}, \frac{1+y^2}{x}\right) \arrow[uu, "\pi"]
\end{tikzcd}$$

Figure 6: Commutative diagram of maps involved in Example 3.6.

cluster of $\phi^\Delta(u)$ and so

$$\phi^\Delta(\mathbf{x}) = \left(y, \frac{1+y^2}{x}\right).$$

This shows that the rotation of $\mathcal{E}$ actually corresponds to an automorphism of Δ which interchanges the two components of the graph (see Figure 4) as well as rotating each component.

Note that this automorphism takes the diagram $D = \circ \xrightarrow{2} \circ$ to its opposite D^{op}, however the matrix $B = \left(\begin{smallmatrix} 0 & 1 \\ -2 & 0 \end{smallmatrix}\right)$ is not taken to $-B$, but rather to $-B^T$.

Example 3.7. Consider the exchange graph $\mathcal{E}$ of the mutation class of type A_2 shown in Figure 3, with the labelled exchange graph Δ in Figure 2. An order 5 clockwise $\frac{2\pi}{5}$ rotation ϕ of $\mathcal{E}$ is an exchange graph automorphism and so induces an automorphism ϕ^Δ of Δ.

The cluster $\mathbf{x} = [x, y]$ maps to $\phi(\mathbf{x}) = [\frac{1+y}{x}, y]$, so the labelled cluster $\hat{\mathbf{x}} = (x, y)$ would be mapped to either $\left(\frac{1+y}{x}, y\right)$ or $\left(y, \frac{1+y}{x}\right)$. To determine which, consider the labelled clusters adjacent to (x, y):

$$(x,y) \cdot \mu_1 = \left(\frac{1+y}{x}, y\right);$$

$$(x,y) \cdot \mu_2 = \left(x, \frac{1+x}{y}\right).$$

The cluster $[\frac{1+y}{x}, y]$ is mapped to $[\frac{1+y}{x}, \frac{1+x+y}{xy}]$ so we need to choose an ordering for $\phi^\Delta(\hat{\mathbf{x}})$ such that $\phi^\Delta(\hat{\mathbf{x}}) \cdot \mu_1$ corresponds to the same ordering of $[\frac{1+y}{x}, \frac{1+x+y}{xy}]$. These two clusters $\phi(\mathbf{x}) = [\frac{1+y}{x}, y]$ and $[\frac{1+y}{x}, \frac{1+x+y}{xy}]$ differ by replacing y with $\frac{1+x+y}{xy}$, while μ_1 changes the cluster variable in the first position, therefore the required ordering is

$$\phi^\Delta(\hat{\mathbf{x}}) = \left(y, \frac{1+y}{x}\right) \quad \text{and} \quad \phi^\Delta(\hat{\mathbf{x}}) \cdot \mu_1 = \left(\frac{1+x+y}{xy}, \frac{1+y}{x}\right).$$

This shows that ϕ induces the automorphism of Δ given by clockwise $\frac{6\pi}{5}$ rotation, which again has order 5.

Remark 3.8. It is not true in general that $\operatorname{Aut}\Delta(\mathcal{S}^0) \cong \operatorname{Aut}\mathcal{E}(\mathcal{S})$, as $\Delta(\mathcal{S}^0)$ can have a number of connected components which are identified under the quotient by the symmetric group action. Any automorphism which changes a single connected component while fixing all others would therefore not project down to an automorphism of $\mathcal{E}(\mathcal{S})$. For example, in the case of the cluster algebra of type B_2, the labelled exchange graph automorphism given by rotating the top hexagon in Figure 4 while fixing the bottom hexagon would not give any valid exchange graph automorphism.

Given $\phi \in \operatorname{Aut}\mathcal{E}(\mathcal{S})$ then $\phi^\Delta \in \operatorname{Aut}\Delta(\mathcal{S}^0)$ is constructed in such a way that for $\pi : \mathcal{S}^0 \to \mathcal{S}$ the quotient by the symmetric group action, $u \in \mathcal{S}^0$ a labelled seed and μ_k a single global mutation,

$$\begin{aligned} \phi(\pi(u)) &= \pi\big(\phi^\Delta(u)\big), \\ \phi(\pi(u \cdot \mu_k)) &= \pi\big(\phi^\Delta(u) \cdot \mu_k\big). \end{aligned}$$

Proposition 3.9. *The inclusion* $\operatorname{Aut}\mathcal{E}(\mathcal{S}) \hookrightarrow \operatorname{Aut}\Delta(\mathcal{S}^0)$ *is a homomorphism, that is* $(\psi\phi)^\Delta = \psi^\Delta\phi^\Delta$ *for any exchange graph automorphisms* $\psi, \phi \in \operatorname{Aut}\mathcal{E}(\mathcal{S})$.

Proof. Choose a labelled seed $u \in \mathcal{S}^0$ then

$$\pi\Big((\psi\phi)^\Delta(u)\Big) = (\psi\phi)(\pi(u)) = \psi(\phi(\pi(u))) = \psi\big(\pi\big(\phi^\Delta(u)\big)\big) = \pi\big(\psi^\Delta\big(\phi^\Delta(u)\big)\big).$$

This shows that the labelled seeds $(\psi\phi)^\Delta(u)$ and $\psi^\Delta\phi^\Delta(u)$ are the same up to permutation, however for any $k \in \{1, \dots, n\}$

$$\begin{aligned} \pi\Big((\psi\phi)^\Delta(u) \cdot \mu_k\Big) = (\psi\phi)(\pi(u \cdot \mu_k)) = \psi(\phi(\pi(u \cdot \mu_k))) &= \psi\big(\pi\big(\phi^\Delta(u) \cdot \mu_k\big)\big) \\ &= \pi\big(\psi^\Delta\big(\phi^\Delta(u) \cdot \mu_k\big)\big) \end{aligned}$$

so after mutation in the k-th vertex $(\psi\phi)^\Delta(u)$ and $\psi^\Delta\phi^\Delta(u)$ are the still same up to permutation. The only way that the k-th mutation affects two labelled seeds in the same way is if the labelled seeds are in fact equal and not permutations of one another, so

$$(\psi\phi)^\Delta(u) = \psi^\Delta\phi^\Delta(u) \quad \text{for any} \quad u \in \mathcal{S}^0$$

and therefore $(\psi\phi)^\Delta = \psi^\Delta\phi^\Delta$. □

Proposition 3.10. *Let* $\phi \in \operatorname{Aut}\mathcal{E}(\mathcal{S})$ *with pullback* $\phi^\Delta \in \operatorname{Aut}\Delta(\mathcal{S}^0)$, *then for any labelled seed* u *and any permutation* σ

$$\phi^\Delta(u \cdot \sigma) = \phi^\Delta(u) \cdot \sigma.$$

Therefore although it looks like the construction of ϕ^Δ from ϕ depends on the initial choice of ordering of u, any other ordering just gives a permutation of ϕ^Δ.

Proof. In $\Delta(\mathcal{S}^0)$ there are edges $u - v^i$ for each mutation μ_i, applying σ gives edges $u\cdot\sigma - v^i\cdot\sigma$ for each $\mu_{\sigma(i)}$. When projected $\mathbf{x}(\phi[u]) = \mathbf{x}(\phi[u\cdot\sigma])$ and $\mathbf{x}(\phi[v^i]) = \mathbf{x}(\phi[v^i\cdot\sigma])$ for each i.

The clusters $\mathbf{x}(\phi[u]) = [a,\ldots,k_i,\ldots]$ and $\mathbf{x}(\phi[v^i]) = [a,\ldots,k_i',\ldots]$ differ in a single cluster variable k_i to k_i'. In the construction of $\phi^\Delta(u)$ we specified an ordering ρ_u on $\mathbf{x}(\phi[u])$ such that the i-th variable $\rho_u(\mathbf{x}(\phi[u]))_i = k_i$ for each i. To construct $\phi^\Delta(u\cdot\sigma)$ we need an ordering $\rho_{u\cdot\sigma}$ such that the $\sigma(i)$-th variable $\rho_{u\cdot\sigma}(\mathbf{x}(\phi[u]))_{\sigma(i)} = k_i$ so that the position of the variable which changes matches the label on the edge in Δ. Therefore $\mathbf{x}(\phi^\Delta(u\cdot\sigma)) = \rho_{u\cdot\sigma}(\mathbf{x}(\phi[u])) = \sigma(\rho_u(\mathbf{x}(\phi[u]))) = \sigma(\mathbf{x}(\phi^\Delta(u))) = \mathbf{x}(\phi^\Delta(u)\cdot\sigma)$. □

So far in this section we have proved properties of automorphisms of the exchange graph of a cluster algebra. In the remainder of this paper we use these results to compare these exchange graph automorphisms to other automorphisms related to the cluster algebra.

Theorem 3.11. *For a labelled mutation class $\mathcal{S}^0$ with mutation class $\mathcal{S} = \mathcal{S}^0/\mathrm{Sym}(n)$ and exchange graph $\mathcal{E}(\mathcal{S})$*

$$\mathrm{Aut}_{M_n}(\mathcal{S}^0) \cong \mathrm{Aut}\,\mathcal{E}(\mathcal{S}).$$

Proof. Let $\phi \in \mathrm{Aut}_{M_n}(\mathcal{S}^0)$ and let ψ be the transformation of $\mathcal{E}(\mathcal{S})$ given by $\psi([u]) = [\phi(u)]$. The automorphism ϕ commutes with permutations so the choice of order of u does not matter, because for any other choice of order u' there is some permutation σ such that $u' = u\cdot\sigma$ and then $[\phi(u')] = [\phi(u\cdot\sigma)] = [\phi(u)\cdot\sigma] = [\phi(u)]$.

For any two seeds u and $v = u\cdot\mu$ related by a single mutation μ there is an edge $[u] - [v]$ in $\mathcal{E}(\mathcal{S})$. Then $\psi([v]) = \psi([u\cdot\mu]) = [\phi(u\cdot\mu)] = [\phi(u)\cdot\mu] = \tilde{\mu}[\phi(u)] = \tilde{\mu}\psi([u])$ where $\tilde{\mu}$ is the single local mutation on $[u]$ corresponding to the global mutation μ on u. Hence there is an edge $\psi[u] - \psi[v]$ in $\mathcal{E}(\mathcal{S})$, so $\psi \in \mathrm{Aut}\,\mathcal{E}(\mathcal{S})$ and $\mathrm{Aut}_{M_n}(\mathcal{S}^0) \subset \mathrm{Aut}\,\mathcal{E}(\mathcal{S})$.

To show the converse, let $\psi \in \mathrm{Aut}\,\mathcal{E}(\mathcal{S})$ which pulls back to $\psi^\Delta \in \mathrm{Aut}\,\Delta(\mathcal{S}^0)$ by Theorem 3.5. Let $\phi : \mathcal{S}^0 \to \mathcal{S}^0$ be the map given by $u \mapsto \psi^\Delta(u)$. Any element of M_n can be written as a product of mutations and permutations, so to prove $\phi \in \mathrm{Aut}_{M_n}(\mathcal{S}^0)$ it suffices to show that ϕ commutes with any permutation and any mutation.

Let σ be a permutation, then by Proposition 3.10

$$\phi(\sigma u) = \psi^\Delta(\sigma u) = \sigma\psi^\Delta(u) = \sigma\phi(u).$$

Let u and $v = u\cdot\mu$ be two labelled seeds related by a single mutation, then ψ^Δ is an automorphism of $\Delta(\mathcal{S}^0)$, so

$$\phi(u\cdot\mu) = \psi^\Delta(u\cdot\mu) = \psi^\Delta(u)\cdot\mu = \phi(u)\cdot\mu.$$ □

4 Cluster automorphisms

Cluster automorphisms were introduced by Assem, Schiffler and Shramchenko in [1]. In their paper the authors computed some particular examples of automorphism groups and drew links between automorphisms of the cluster algebra of a surface and the mapping class group of that surface. This correspondence was later proved by Brüstle and Qiu in [4] for all surfaces except a select few, as discussed in Section 7.

Definition 4.1 ([1])**.** A $\mathbb{K}$-automorphism f is a *cluster automorphism* of $\mathcal{A}(\mathcal{S})$ if there exists a seed $(\mathbf{x}, B)$ in $\mathcal{S}$ such that

1. $f(\mathbf{x})$ is a cluster.
2. for every $x \in \mathbf{x}$ we have $f(\mu_{x,\mathbf{x}}(\mathbf{x})) = \mu_{f(x),f(\mathbf{x})}(f(\mathbf{x}))$.

Cluster automorphisms were originally only defined for skew-symmetric matrices and hence quivers, but the same definitions and some results apply to skew-symmetrizable matrices as well. The cluster automorphism groups in this setting were first studied by Chang and Zhu in [6] and [7]. Recall that throughout this paper we assume that the cluster $\mathbf{x}$ of a seed uniquely determines the seed's matrix, and in this case the matrix is denoted $B(\mathbf{x})$.

Lemma 4.2 ([1, Lemma 2.3],[6, Lemma 2.9])**.** *If f is a $\mathbb{K}$-automorphism, then f is a cluster automorphism if and only if there exists a seed $(\boldsymbol{x}, B)$ such that $f(\boldsymbol{x})$ is a cluster and $B(f(\boldsymbol{x})) = B$ or $-B$.*

The definition of a cluster automorphism only requires that there exists a single seed such that the image is a seed and the automorphism is compatible with mutations of that seed, however the compatibility with mutations allows these properties to be extended to all seeds in the cluster algebra.

Proposition 4.3 ([1, Prop 2.4])**.** *Let f be a cluster automorphism of a cluster algebra $\mathcal{A}$, then f satisfies the conditions in Definition 4.1 and Lemma 4.2 for every seed in $\mathcal{A}$.*

This therefore gives two ways of thinking of cluster automorphisms as either automorphisms taking clusters to clusters which are compatible with mutations or as automorphisms which fix exchange matrices (up to multiplication by -1).

Definition 4.4. A cluster automorphism which fixes exchange matrices is called a *direct* cluster automorphism, whereas those which send an exchange matrix B to $-B$ are called *inverse* cluster automorphisms.

Cluster automorphisms form a group, so let $\operatorname{Aut}\mathcal{A}$ denote the group of all cluster automorphisms of $\mathcal{A}$, and $\operatorname{Aut}^+\mathcal{A}$ be the subgroup of direct cluster automorphisms.

Proposition 4.5 ([1, Lemma 2.9, Theorem 2.11])**.** *Let $\mathcal{A}$ be a cluster algebra generated by an exchange matrix B. If B is mutation-equivalent to $-B$ then $\operatorname{Aut}^+\mathcal{A}$ is a normal subgroup of $\operatorname{Aut}\mathcal{A}$ with index 2, otherwise $\operatorname{Aut}^+\mathcal{A} = \operatorname{Aut}\mathcal{A}$.*

Cluster automorphisms arise naturally in the labelled seed and global mutation setting introduced by King and Pressland, with the following correspondence:

Theorem 4.6 ([18, Corollary 6.3])**.** *If $\mathcal{S}$ is the mutation class of a seed $(\boldsymbol{x}, Q)$ where Q is a skew-symmetric mutation-finite matrix then*

$$\operatorname{Aut}_{M_n}(\mathcal{S}^0) \cong \operatorname{Aut}\mathcal{A}(\mathcal{S}).$$

Combining Theorem 4.6 with Theorem 3.11 gives the following:

Corollary 4.7. *For a cluster algebra $\mathcal{A}$ constructed from a mutation-finite quiver with exchange graph $\mathcal{E}_{\mathcal{A}}$*

$$\operatorname{Aut}\mathcal{E}_{\mathcal{A}} \cong \operatorname{Aut}\mathcal{A}.$$

Chang and Zhu provide an alternative proof of this in [6] and extend the result to certain finite type skew-symmetrizable matrices:

Theorem 4.8 ([6, Theorem 3.7])**.** *If $\mathcal{S}$ is the mutation class of a seed $(\boldsymbol{x}, B)$ where B is a skew-symmetrizable matrix of Dynkin type B_n or C_n for $n \geqslant 3$ then*

$$\operatorname{Aut}\mathcal{A}(\mathcal{S}) = \operatorname{Aut}\mathcal{E}_{\mathcal{A}}(\mathcal{S}).$$

4.1 Examples: Maximal green sequences

Maximal green sequences are certain sequences of mutations of a given quiver. First studied by Keller in [17] in relation to quantum dilogarithms they have subsequently been used to study BPS states in theoretical physics (see for example [5]).

In their paper on maximal green sequences, Brüstle, Dupont and Pérotin proved that any maximal green sequence for some quiver Q takes it to a quiver which is isomorphic to Q [3, Proposition 2.10]. Hence this sequence of mutations will give an element of the mutation group $\mu_{i_1} \cdot \ldots \cdot \mu_{i_k} \in M_n$ which takes a cluster to a cluster and a quiver Q to an isomorphic quiver Q', with some permutation σ which acts on the vertices of Q' to give Q. Then $\mu_{i_1} \cdot \ldots \cdot \mu_{i_k} \cdot \sigma \in M_n$ fixes the quiver and therefore induces a cluster automorphism.

Definition 4.9 ([3, Definition 2.4])**.** Given a quiver Q, its *framed quiver* $\hat{Q}$ (respectively *coframed* quiver $\check{Q}$) is the quiver constructed from Q by adding an additional vertex $\hat{i}$ and an additional arrow $i \to \hat{i}$ (resp. $\hat{i} \to i$) for each vertex i of Q.

These additional vertices are considered frozen vertices of the (co)framed quiver. For a quiver Q call this set of frozen vertices of the quiver Q_0^F.

Definition 4.10 ([3, Definition 2.5])**.** Given a quiver Q with framed quiver $\hat{Q}$, a non-frozen vertex i of a quiver R in the mutation class of $\hat{Q}$ is called *green* (resp. *red*) if for each $j \in R_0^F$ there is no arrow $j \to i$ (resp. no arrow $i \to j$) in the quiver R.

Every (non-frozen) vertex in a quiver of the mutation class of $\hat{Q}$ is either green or red [3, Theorem 2.6]. A maximal green sequence is then a sequence of mutations at green vertices which continues until every non-frozen vertex is red.

Example 4.11. The quiver of type A_2 has framed and coframed quivers as shown in Figure 7. This quiver has two maximal green sequences given by $\mu_1 \cdot \mu_2$ and $\mu_2 \cdot \mu_1 \cdot \mu_2$ which are illustrated in Figure 8.

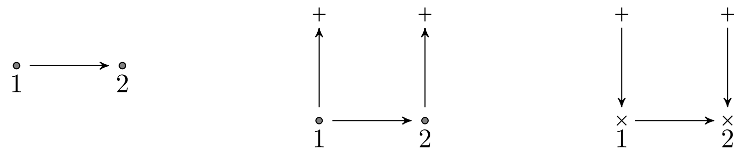


Figure 7: The quiver of type A_2 (left), with its framed quiver (center) and coframed quiver (right). Green vertices are shown as circles, red vertices as crosses and frozen vertices as plusses.

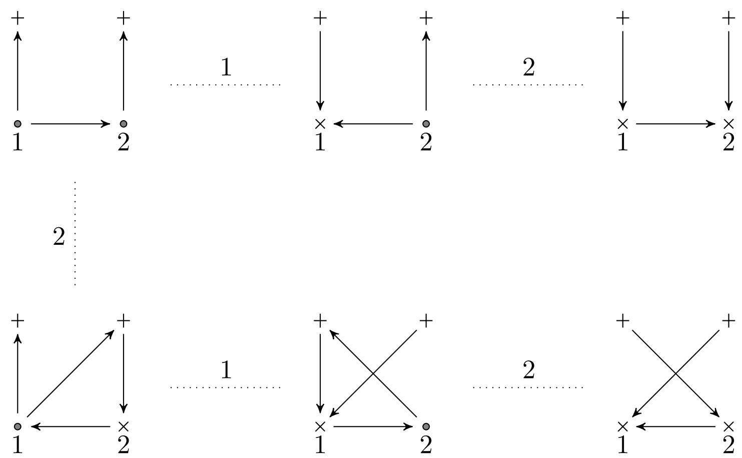


Figure 8: The two maximal green sequences of the quiver of type A_2 starting with its framed quiver. The top green sequence is $\mu_1 \cdot \mu_2$ and the bottom is $\mu_2 \cdot \mu_1 \cdot \mu_1$. The two resulting quivers are both isomorphic to the coframed quiver of the quiver of type A_2. Green vertices are shown as circles, red vertices as crosses and frozen vertices as plusses.

If the initial labelled seed is $(Q, \mathbf{x})$ with cluster $\mathbf{x} = (x, y)$, then the resulting cluster after these green sequences induces a cluster automorphism as shown below. The sequence $\mu_2 \cdot \mu_1 \cdot \mu_2$ does not give the same quiver, but after the permutation (12) it does:

$$\big(Q, (x, y)\big) \cdot \mu_1 \cdot \mu_2 = \left(Q, \left(\frac{1+y}{x}, \frac{1+x+y}{xy}\right)\right) = \big(Q, (x, y)\big) \cdot \mu_2 \cdot \mu_1 \cdot \mu_2 \cdot (12).$$

These both give the same cluster automorphism $x \mapsto \frac{1+y}{x}$ and $y \mapsto \frac{1+x+y}{xy}$.

5 Generalising automorphisms to skew-symmetrizable case

Theorems 4.6 and 4.8 show that cluster automorphisms are linked to the automorphisms of the exchange graph for mutation-finite skew-symmetric matrices as well as a specific family of skew-symmetrizable matrices. However, in general the exchange graph automorphism group for any mutation-finite skew-symmetrizable matrix is larger than the cluster automorphism group.

An example of this would be the exchange graph automorphism of the mutation class of B_2 considered in Example 3.6. This graph automorphism does not correspond to a cluster automorphism as the initial matrix B is sent to $-B^T \neq \pm B$.

In this section we aim to generalise the results of the previous section to the skew-symmetrizable case. To do this we introduce additional structure on the exchange graph, which defines a marked exchange graph. This extra structure ensures that any graph automorphism fixing this structure corresponds to a cluster automorphism. In this way the study of cluster automorphisms can be reduced to the combinatorial study of graph automorphisms.

5.1 Marked exchange graph

Let B be a skew-symmetrizable matrix, with symmetrizing matrix D. If μ_i is any mutation, then D is also the symmetrizing matrix for $B \cdot \mu_i$. Similarly for any permutation σ the permuted matrix $D \cdot \sigma = \operatorname{diag}(d_i^\sigma) = \operatorname{diag}\big(d_{\sigma^{-1}(i)}\big)$ is the symmetrizing matrix for $B \cdot \sigma = B^\sigma$.

Definition 5.1. The *marked labelled exchange graph* of a mutation class generated by $u = (\mathbf{x}, B)$ where B is a skew-symmetrizable matrix with symmetrizing matrix $D = \operatorname{diag}(d_i)$ is the labelled exchange graph with an additional marking on each edge. Each edge corresponds to a global mutation μ_i for some i, so mark that edge with the symmetrizing entry d_i.

If a permutation σ acts on u to give a labelled seed in a different component of $\Delta(\mathcal{S}^0)$, then mark the i-th edges with d_i^σ, where $D \cdot \sigma = \operatorname{diag}(d_i^\sigma)$.

In the exchange graph $\mathcal{E}$ each edge no longer corresponds to a global mutation μ_i, but rather to a local mutation $\mu_{\beta,\mathbf{x}}$ at a specific cluster variable β in a cluster $\mathbf{x}$.

For a permutation σ and permuted seed $(\mathbf{x}, B) \cdot \sigma = (\mathbf{x}^\sigma, B^\sigma)$, then the edge $\mu_{\sigma(i)}$ adjacent to this seed corresponds to the local mutation $\mu_{\beta^\sigma_{\sigma(i)},[\mathbf{x}^\sigma]} = \mu_{\beta_i,[\mathbf{x}]}$ as $\beta^\sigma_{\sigma(i)} =$

$\beta_{\sigma^{-1}(\sigma(i))} = \beta_i$ and $[\mathbf{x}^\sigma] = [\mathbf{x}]$. This edge $\mu_{\sigma(i)}$ is marked with $d^\sigma_{\sigma(i)} = d_{\sigma^{-1}(\sigma(i))} = d_i$ and hence in the quotient the edge $\mu_{\beta_i,\mathbf{x}}$ has a consistent marking, so the following is well-defined.

Definition 5.2. Let $\widehat{\mathcal{E}}(\mathcal{S})$ be the *marked exchange graph* of a mutation class $\mathcal{S}$ given by taking the quotient of the marked labelled exchange graph with respect to the symmetric group action.

Alternatively let B be a skew-symmetrizable matrix, with symmetrizing matrix D and let R be the diagram corresponding to B so each row in B represents a vertex in R. Each diagonal entry in D can be thought of as being attached to that row's vertex of R, and the edge in $\widehat{\mathcal{E}}$ representing mutation in that vertex should be marked with this diagonal entry.

Example 5.3 ($\boldsymbol{B_3}$). The marked exchange graph of the cluster algebra of type B_3 is shown in Figure 9. The cluster variables are not written out in full, rather only the denominators are shown with a bar above except for the initial cluster variables x_1, x_2 and x_3 which are shown with a bar underneath. Each vertex is adjacent to two dotted edges and one dashed edge.

Choosing a matrix in the mutation class, the symmetrizing matrix is $\operatorname{diag}(2,1,1)$:

$$\begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The dotted edges correspond to mutations in the vertices with symmetrizing entry 1, while the dashed edge corresponds to the mutation in the vertex with symmetrizing entry 2.

In this case, any automorphism of the unmarked exchange graph sends dashed edges to dashed edges, so automatically preserves the markings and hence $\operatorname{Aut}\mathcal{E} = \operatorname{Aut}\widehat{\mathcal{E}}$.

Example 5.4 ($\boldsymbol{B_2}$). The marked exchange graph of the cluster algebra of type B_2 is shown in Figure 10, where dotted edges correspond to mutations in vertices with symmetrizer 1 and dashed edges correspond to symmetrizer 2. The initial matrix for the cluster $[x,y]$ was chosen to be $\left(\begin{smallmatrix} 0 & 1 \\ -2 & 0 \end{smallmatrix}\right)$ with symmetrizing matrix $\operatorname{diag}(1,2)$.

The automorphism considered in Example 3.6, given by a rotation of angle $\frac{\pi}{3}$, does not fix the markings in the graph, so is not an automorphism of the marked graph.

Remark 5.5. For any skew-symmetric matrix the symmetrizing matrix is the identity, so all markings would be the same and $\mathcal{E}_\mathcal{A} = \widehat{\mathcal{E}}_\mathcal{A}$.

Remark 5.6. For a cluster algebra of Dynkin type B_n or C_n, for $n \geqslant 3$, the marking on the exchange graph does not limit the number of automorphisms, so $\operatorname{Aut}\mathcal{E}_\mathcal{A} = \operatorname{Aut}\widehat{\mathcal{E}}_\mathcal{A}$. This follows from Theorem 3.7 in Chang and Zhu's paper [6] linking exchange graph automorphisms and cluster automorphisms.

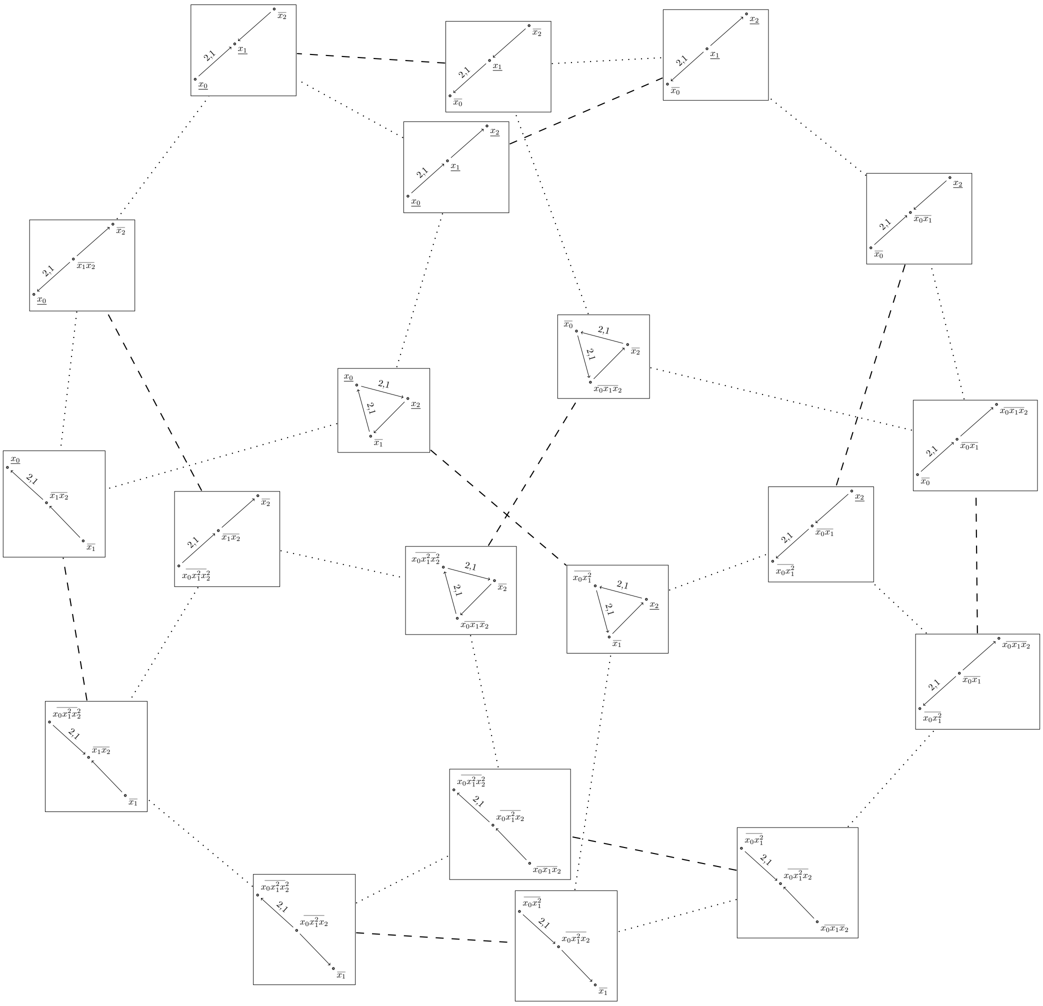


Figure 9: Marked exchange graph of type B_3. Dotted edges correspond to a symmetrizing entry of 1, while dashed edges correspond to 2. Only denominators are shown in the cluster variables with a bar above each, unless the cluster variable is one of x_1, x_2 or x_3 where the variable is shown with a bar underneath.

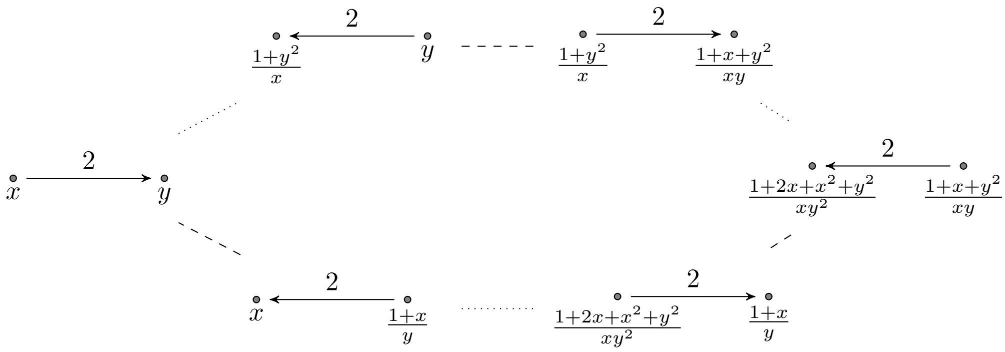


Figure 10: Marked exchange graph for cluster algebra of type B_2. Dotted edges correspond to mutations in a vertex with symmetrizer 1 while dashed edges correspond to symmetrizer 2.

5.2 Geodesic loops

Definition 5.7 ([8, Def. 2.25])**.** Let $\mathcal{E}$ be an exchange graph of a seed $u = (x, B)$ with vertices labelled $(v_i)_{i\in\{1,\dots,n\}}$. For a subset of vertices $\{v_k\}$ the *frozenisation* of u with respect to $\{v_k\}$ is the mutation class constructed by freezing all vertices in $\{v_k\}$.

It is often more convenient to consider the *cofrozenisation* of u with respect to $\{v_k\}$, denoted $u\backslash\{v_k\}$, which is constructed by freezing all vertices in u except those in $\{v_k\}$. This is then a frozenisation of u with respect to $\{v_i\} - \{v_k\}$.

Definition 5.8 ([15, Section 2])**.** A *geodesic loop* $\mathcal{L} = \mathcal{L}_u^{a,b}$ is the exchange graph of a cofrozenisation $u\backslash\{a, b\}$ which leaves only two vertices a and b unfrozen. A loop is then either a polygon with $4, 5, 6$ or 8 sides or an infinite line, which embeds into the exchange graph of the mutation class of u.

The *distance* between a geodesic loop $\mathcal{L}$ and any vertex v in $\mathcal{E}$ is the (possibly zero) minimum number of edges in $\mathcal{E}$ between v and any vertex in $\mathcal{L}$.

The *length* of a geodesic loop $\mathrm{Len}(\mathcal{L}) \in \{4, 5, 6, 8, \infty\}$ is the number of edges in the loop.

Geodesic loops as subgraphs of a larger exchange graph give rise to the following sets, which encode the information about a given seed represented by a vertex of the exchange graph. The following construction is a slight notational variation of the one given by Chang and Zhu in Definition 3.1 of [6].

Definition 5.9. Let u be a seed of rank n in an exchange graph, then define $N^0(u)$ to be the set of $\binom{n}{2}$ numbers given by the length of all geodesic loops distance 0 from u. Similarly define $N^1(u)$ to be the set of $n\binom{n-1}{2}$ numbers given by the lengths of all geodesic loops distance 1 from u.

Remark 5.10. An exchange graph automorphism $\phi \in \mathrm{Aut}\,\mathcal{E}$ induces an automorphism $\phi^\Delta \in \mathrm{Aut}\,\Delta$ and in this way ϕ induces a map ϕ_v which takes cluster variables in a seed u to variables in $\phi(u)$.

A geodesic loop $\mathcal{L}_u^{a,b}$ in an exchange graph $\mathcal{E}$ must get mapped to another geodesic loop of the same length by any exchange graph automorphism, however it is not clear that the image of $\mathcal{L}_u^{a,b}$ will be generated by the cofrozenisation $\phi(u)\backslash\{\phi_v(a), \phi_v(b)\}$ rather than another cofrozenisation with two different unfrozen vertices in u. The following Lemma explains that this must always be the case.

Lemma 5.11. *Let u be a seed in a cluster algebra $\mathcal{A}$ and $\phi \in \operatorname{Aut}\mathcal{E}_{\mathcal{A}}$. For any two vertices a and b the geodesic loop $\mathcal{L}_u^{a,b}$ is isomorphic to its image $\mathcal{L}_{\phi(u)}^{\phi_v(a),\phi_v(b)}$.*

Proof. Choose some ordering on u so that the vertices $a = v_i$ and $b = v_j$ are indexed by i and j respectively, then the length of the geodesic loop specifies a relation $u = u \cdot \mu_i \mu_j \mu_i \cdots$.

For example if the loop has length 6, then $u = u \cdot (\mu_i \mu_j)^3$, whereas if the length is 5 then $u = u \cdot \mu_i \mu_j \mu_i \mu_j \mu_i$.

The exchange graph automorphism ϕ corresponds to some $\phi_{M_n} \in \operatorname{Aut}_{M_n}$ which commutes with the action of M_n. Hence

$$\phi_{M_n}(u) = \phi_{M_n}(u \cdot \mu_i \mu_j \cdots) = \phi_{M_n}(u) \cdot \mu_i \mu_j \cdots$$

so the geodesic loop $\mathcal{L}_{\phi(u)}^{\phi_v(a),\phi_v(b)}$ has the same length as the geodesic loop $\mathcal{L}_u^{a,b}$, and hence the two loops are isomorphic. □

Exchange graph automorphisms preserve the combinatorial structure around a seed. As these automorphisms are compatible with mutations the above result could be extended to the exchange graphs of cofrozenisations with any number of unfrozen vertices.

Lemma 5.12. *If $\phi \in \operatorname{Aut}\mathcal{E}$ is an exchange graph automorphism, with u a seed and $v = \phi(u)$ its image, then $N^0(u) = N^0(v)$ and $N^1(u) = N^1(v)$.*

Lemma 5.13. *Given a mutation-finite diagram with at least 3 vertices, the exchange graph of a frozenisation leaving just two vertices unfrozen determines the weight on the arrow between the two unfrozen vertices.*

Proof. The exchange graph of the frozenisation leaving just two vertices a and b unfrozen is a geodesic loop $\mathcal{L}^{a,b}$ with length $\operatorname{Len}(\mathcal{L}) \in \{4, 5, 6, 8, \infty\}$.

If $\operatorname{Len}(\mathcal{L}) = 4$ then the vertices have no arrow between them, while if $\operatorname{Len}(\mathcal{L}) = 5$ there is a single unweighted arrow. If $\operatorname{Len}(\mathcal{L}) = 6$ then there is an arrow weighted 2 and $\operatorname{Len}(\mathcal{L}) = 8$ shows there is an arrow weighted 3.

The highest edge weight in a mutation-finite diagram (with more than 2 vertices) is 4, so $\operatorname{Len}(\mathcal{L}) = \infty$ implies that there is an arrow weighted 4. □

Remark 5.14. For any 2-vertex diagram B, an edge weight of 4 or more will always give $\operatorname{Len}(\mathcal{L}) = \infty$, so the exchange graph cannot determine this weight. However the only diagrams mutation-equivalent to B are B and B^{op}, so all diagrams in the same mutation class have the same edge weight.

5.3 Exchange graph automorphism effects on diagrams and matrices

Lemma 5.15. *An exchange graph automorphism $\phi \in \operatorname{Aut}\mathcal{E}$ takes a seed $u = (x, B)$ to another seed $v = \phi(u) = (x', B')$ where the unoriented diagram of B' is the same as the unoriented diagram of B.*

Proof. Fix any two vertices u_0 and u_1 in u. Under ϕ_v these vertices are mapped to corresponding vertices $\phi_v(u_0) = v_0$ and $\phi_v(u_1) = v_1$ in v.

The weight on (or absence of) the arrow between u_0 and u_1 determines the exchange graph $\mathcal{E}_u$ of the cofrozenisation $u\backslash\{u_0, u_1\}$. By Lemma 5.11, $\mathcal{E}_u$ is isomorphic to the exchange graph $\mathcal{E}_v$ of the cofrozenisation $v\backslash\{v_0, v_1\}$. Hence this exchange graph determines the arrow between v_0 and v_1 by Lemmas 5.12 and 5.13, which necessarily must be the same as that between u_0 and u_1. $\square$

This shows that the unoriented diagrams of two seeds related by an exchange graph automorphism must be the same. To see how exchange graphs automorphisms affect the orientations of the arrows we need to consider frozenisations with three unfrozen vertices.

Lemma 5.16. *For any seed $u = (x, B)$ with 3 vertices in an exchange graph of a mutation-finite skew-symmetrizable diagram, the diagram of B is determined by the sets $N^0(u)$ and $N^1(u)$, up to reversing all arrows.*

Proof. The unoriented diagram of B is determined by $N^0(u) = \{n_i\}$, where each $n_i \in \{4, 5, 6, 8, \infty\}$ determines a weighted arrow, or absence of arrow, between two vertices.

The orientation of B (up to reversing all arrows) is given by $N^1(u)$ as shown in Tables 1, 2 and 3, where all mutation-finite 3-vertex diagrams are illustrated along with their defining sets N^0 and N^1. Hence the pair (N^0, N^1) defines a unique diagram, up to reversing all arrows. $\square$

In the case $N^0(u) = \{4, 4, \infty\}$ the diagram is of the form:

$$\begin{tikzcd} & \circ & \\ \circ \arrow[rr, "k"'] & & \circ \end{tikzcd}$$

where the weight satisfies $k \geqslant 4$ and so the diagram is not uniquely determined. However if $k > 4$ then the resulting diagram will never appear as a subdiagram of any larger mutation-finite diagram. This is precisely the setup used in the proofs below and so $N^0(u) = \{4, 4, \infty\}$ is always assumed to correspond to a diagram of the form:

$$\begin{tikzcd} & \circ & \\ \circ \arrow[rr, "4"'] & & \circ \end{tikzcd}$$

Proposition 5.17. *Let $\phi \in \operatorname{Aut}\mathcal{E}$ be an exchange graph automorphism and $u = (\boldsymbol{x}, B)$ a seed where B is a mutation-finite skew-symmetrizable matrix with corresponding connected diagram R. In the image $\phi(u) = (\boldsymbol{x}', B')$, the diagram R' corresponding to the matrix B' is either R or R^{op}.*

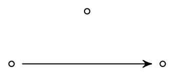
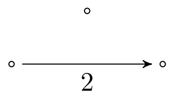
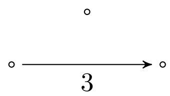
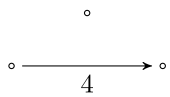
Diagram	N^0	N^1
	$\{4, 4, 5\}$	$\{4, 4, 5\}$
	$\{4, 4, 6\}$	$\{4, 4, 6\}$
	$\{4, 4, 8\}$	$\{4, 4, 8\}$
	$\{4, 4, \infty\}$	$\{4, 4, \infty\}$
	$\{4, 4, 4\}$	$\{4, 4, 4\}$

Table 1: Disconnected 3-vertex diagrams determined by values of N^0.

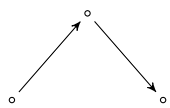
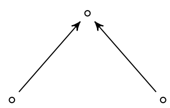
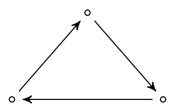
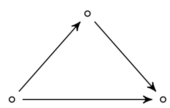
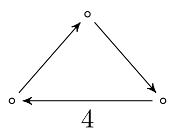
Diagram	N^0	N^1
	$\{4, 5, 5\}$	$\{5, 5, 5\}$
	$\{4, 5, 5\}$	$\{4, 5, 5\}$
	$\{5, 5, 5\}$	$\{4, 4, 4\}$
	$\{5, 5, 5\}$	$\{5, 5, \infty\}$
	$\{5, 5, \infty\}$	$\{5, 5, 5\}$

Table 2: Connected skew-symmetric 3-vertex diagrams determined by values of N^0 and N^1.

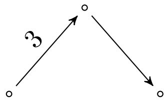
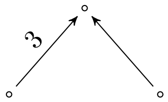
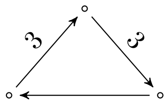
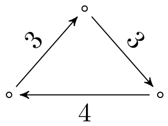
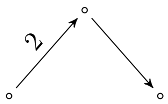
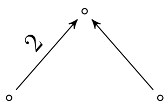
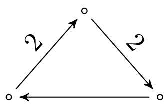
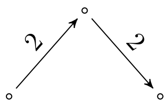
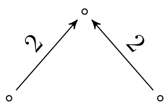
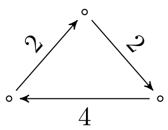
Diagram	N^0	N^1
	$\{4,5,8\}$	$\{5,8,8\}$
	$\{4,5,8\}$	$\{4,5,8\}$
	$\{5,8,8\}$	$\{4,4,\infty\}$
	$\{8,8,\infty\}$	$\{5,8,8\}$
	$\{4,5,6\}$	$\{5,6,6\}$
	$\{4,5,6\}$	$\{4,5,6\}$
	$\{5,6,6\}$	$\{4,4,5\}$
	$\{4,6,6\}$	$\{4,6,\infty\}$
	$\{4,6,6\}$	$\{4,6,6\}$
	$\{6,6,\infty\}$	$\{4,6,6\}$

Table 3: Connected skew-symmetrizable 3-vertex diagrams determined by values of N^0 and N^1.

Proof. Choose any 3 vertices a, b, c in u, then by Lemma 5.11 there is an isomorphism $\mathcal{E}(u\backslash\{a,b,c\}) \cong \mathcal{E}(\phi(u)\backslash\{\phi(a),\phi(b),\phi(c)\})$ and $N^0(u) = N^0(\phi(u))$, $N^1(u) = N^1(\phi(u))$. Therefore by Lemma 5.16 the subdiagram S of R consisting just of the arrows between a, b and c is the same as the subdiagram S' of R' consisting of the arrows between $\phi(a), \phi(b)$ and $\phi(c)$, up to reversing all arrows.

Choose a fourth vertex d and consider the 3-vertex subdiagram S_a on the vertex set $\{b, c, d\}$. By the same reasoning as above the image $S'_a = \phi(S_a)$ must be the same, but possibly with all arrows reversed. However both S' and S'_a share the edge between vertices $\phi(b)$ and $\phi(c)$, so if $S' = S^{\mathrm{op}}$ then $S'_a = S_a^{\mathrm{op}}$ whereas if $S' = S$ then $S'_a = S_a$.

As R is connected, by successively choosing different vertices, the whole diagram R' must either be the same as R or R^{op}. □

This shows that any exchange graph automorphism takes clusters to clusters and a diagram to itself or its opposite. However this is not enough to show that these automorphisms are cluster automorphisms, as this requires the matrix B of the diagram to be sent to $\pm B$. For this we require the markings on the exchange graph.

Proposition 5.18. *Given a marked exchange graph automorphism $\phi \in \operatorname{Aut}\widehat{\mathcal{E}}$ and a seed $u = (x, B)$ with image $\phi(u) = (x', B')$, then the matrix $B' = B$ or $-B$.*

Proof. Let R be the diagram associated to B, and let R' be the diagram associated to B'. Let D_B be the symmetrizing matrix for B, each vertex v_k in u has a symmetrizing multiplier, which marked exchange graph automorphisms preserve, so each vertex $\phi_v(v_k)$ in $\phi(u)$ has the same symmetrizing multiplier as v_k and $D_B = D_{B'}$.

By Proposition 5.17, R' is the same as R or R^{op} with symmetrizing matrix $D_{B'} = D_B$ which defines the skew-symmetrizable matrix $B' = B$ or $-B$. □

These results ensure that a marked exchange graph automorphism fixes matrices in seeds and so correspond to cluster automorphisms. In this way we generalise Corollary 4.7 to all mutation-finite skew-symmetrizable matrices.

Theorem 5.19. *For a seed $(\boldsymbol{x}, B)$ where B is a mutation-finite skew-symmetrizable matrix with mutation class $\mathcal{S}$, cluster algebra $\mathcal{A} = \mathcal{A}(\mathcal{S})$ and marked exchange graph $\widehat{\mathcal{E}}_{\mathcal{A}}$ then*

$$\operatorname{Aut}\mathcal{A} = \operatorname{Aut}\widehat{\mathcal{E}}_{\mathcal{A}}.$$

Proof. A cluster automorphism $f \in \operatorname{Aut}\mathcal{A}$ satisfies the following properties:

- $f(\mathbf{x})$ is a cluster
- f is compatible with mutations
- $B(f(\mathbf{x})) \cong B$ or $-B$

for all seeds (x, B) in the mutation class $\mathcal{S}$. Such an automorphism induces an automorphism of the exchange graph, and as f sends a matrix B to $\pm B$ it also fixes the symmetrizing matrix so fixes the marking on the exchange graph. Therefore $f \in \operatorname{Aut}\widehat{\mathcal{E}}_{\mathcal{A}}$ and $\operatorname{Aut}\mathcal{A} \subset \operatorname{Aut}\widehat{\mathcal{E}}_{\mathcal{A}}$.

To show that $\operatorname{Aut}\widehat{\mathcal{E}}_{\mathcal{A}} \subset \operatorname{Aut}\mathcal{A}$ let $(\mathbf{x}, B)$ be a labelled seed, with mutation class $\mathcal{S}^0$ and quotient $\mathcal{S}$. If $\phi \in \operatorname{Aut}\widehat{\mathcal{E}}(\mathcal{S}) \subset \operatorname{Aut}\mathcal{E}(\mathcal{S})$, then by Theorem 3.5 this pulls back to an automorphism $\phi^{\Delta} \in \operatorname{Aut}\Delta(\mathcal{S}^0)$. Then the image $\phi^{\Delta}(\mathbf{x}) = (y_1, \dots, y_n)$ where $\mathbf{x} = (x_1, \dots, x_n)$ gives an automorphism $f : \mathbb{C}(x_1, \dots, x_n) \to \mathbb{C}(x_1, \dots, x_n)$ defined by $f(x_i) = y_i$.

This f then corresponds to ϕ, so $f(\mathbf{x})$ is a cluster and it remains to show that $B(f(\mathbf{x})) = \pm B = \pm B(\mathbf{x})$, however this follows from Proposition 5.18 so $f \in \operatorname{Aut}\mathcal{A}$. $\square$

6 Unfoldings

Many skew-symmetrizable matrices B have unfoldings to skew-symmetric matrices C, which extend to seeds, where a given seed in $\mathcal{S}(B)$ unfolds to a seed in $\mathcal{S}(C)$. The corresponding exchange graphs are related, with the marked exchange graph $\widehat{\mathcal{E}}(B)$ embedding into the exchange graph $\mathcal{E}(C)$ provided edges marked in certain ways split into multiple edges.

Definition 6.1 ([11, Section 4])**.** Given a skew-symmetrizable $n \times n$ matrix $B = (b_{i,j})$ with symmetrizing matrix $D = \operatorname{diag}(d_i)$, let $m = \sum_{j=1}^{n} d_j$ and partition the set $\{1, \dots, m\}$ into n disjoint consecutive index sets E_i such that $|E_j| = d_j$ for all j.

Construct a skew-symmetric $m \times m$ matrix C where:

1. The sum of entries in each column of each $E_i \times E_j$ block equals $b_{i,j}$.
2. If $b_{i,j} > 0$ then all entries in the $E_i \times E_j$ block are non-negative.
3. All entries in each $E_i \times E_i$ block are zero.

Given $i \in \{1, \dots, n\}$ and any $j, k \in E_i$ the corresponding mutations μ_j and μ_k commute. The i-th *composite mutation* $\widetilde{\mu}_i$ of C is given by

$$\widetilde{\mu}_i = \prod_{j \in E_i} \mu_j.$$

The matrix C is the *unfolding* of B if the matrix $C' = C \cdot (\widetilde{\mu}_{k_1}\widetilde{\mu}_{k_2} \cdots \widetilde{\mu}_{k_r})$ satisfies the conditions 1 and 2 above with respect to the matrix $B' = B \cdot (\mu_{k_1}\mu_{k_2} \cdots \mu_{k_r})$ for any sequence of mutations μ_{k_i} with corresponding composite mutations $\widetilde{\mu}_{k_i}$.

A labelled seed $([\beta_i], B)$, with skew-symmetrizable matrix B, unfolds in the same way to $([\gamma_i], C)$ where C is the unfolding of B. The j-th row in B corresponds to the cluster variable β_j and this row unfolds to d_j rows in C, hence β_j unfolds to d_j cluster variables $\{\gamma_{j_1}, \dots, \gamma_{j_{d_j}}\}$.

Remark 6.2. A diagram has a finite number of distinct matrix representations, each of which may give different unfoldings, or may not admit any unfolding. Almost all mutation-finite matrices have an unfolding.

Definition 6.3. Given a permutation $\sigma \in \mathrm{Sym}(n)$ of the initial seed, construct the *composite permutation* $\widetilde{\sigma} \in \mathrm{Sym}(m)$ to be the permutation given by:

$$\begin{tikzcd}
\{1,\dots,m\} \arrow[r, equal] & E_1, E_2, \dots, E_n \arrow[d, mapsto] \\
\{\widetilde{\sigma}^{-1}(1),\dots,\widetilde{\sigma}^{-1}(m)\} \arrow[u, mapsto, "\widetilde{\sigma}"] \arrow[r, equal] & E_{\sigma^{-1}(1)}, E_{\sigma^{-1}(2)}, \dots, E_{\sigma^{-1}(n)}
\end{tikzcd}$$

Theorem 6.4. *Given a skew-symmetrizable matrix B which unfolds to a matrix C, with corresponding marked exchange graphs $\widehat{\mathcal{E}}(B)$ and $\mathcal{E}(C) = \widehat{\mathcal{E}}(C)$, then*

$$\mathrm{Aut}\, \widehat{\mathcal{E}}(B) \hookrightarrow \mathrm{Aut}\, \mathcal{E}(C).$$

Proof. Choose an initial $n \times n$ labelled seed $u = ([\beta_i], B)$ which unfolds to the $m \times m$ labelled seed $([\gamma_i], C)$ with index sets E_k for $k = 1, \dots, n$ and $\beta_i \rightsquigarrow \{\gamma_j\}_{j \in E_i}$.

Let $\phi \in \mathrm{Aut}\, \widehat{\mathcal{E}}(B)$ be an exchange graph automorphism, then ϕ corresponds to both a cluster automorphism $f \in \mathrm{Aut}\, \mathcal{A}_B$ of the cluster algebra $\mathcal{A}_B$, constructed from the initial seed u, and to a mutation class automorphism $\phi_M \in \mathrm{Aut}_{M_n} \mathcal{S}^0(B)$. This mutation class automorphism in turn corresponds to an element of M_n, so there is a sequence of r mutations μ_{k_i} and a permutation σ such that

$$\phi_M(u) = u \cdot (\mu_{k_1} \mu_{k_2} \cdots \mu_{k_r} \sigma).$$

All such automorphisms are constructed to have the same action on the initial seed u, so

$$\phi(u) = \big([\bar{\beta}_i], \pm B\big) = ([f(\beta_i)], \pm B) = \phi_M(u) = u \cdot (\mu_{k_1} \mu_{k_2} \cdots \mu_{k_r} \sigma).$$

In the unfolding, each mutation μ_{k_i} corresponds to the composite mutation $\widetilde{\mu}_{k_i}$ and the permutation σ corresponds to the composite permutation $\widetilde{\sigma}$, so the following commutes:

$$\begin{tikzcd}
([\beta_i], B) \arrow[r, "\phi"] \arrow[d, "\text{unfold}"'] & ([\beta_i], B) \cdot (\mu_{k_1} \cdots \mu_{k_r} \sigma) \arrow[d, "\text{unfold}"] & = ([\bar{\beta}_i], \pm B) \\
([\gamma_i], C) \arrow[r, dash] & ([\gamma_i], C) \cdot (\widetilde{\mu}_{k_1} \cdots \widetilde{\mu}_{k_r} \widetilde{\sigma}) & = ([\bar{\gamma}_i], \pm C)
\end{tikzcd}$$

The automorphism ϕ corresponds to a cluster automorphism, so the matrix of the image of u is $\pm B$. The seed $\phi(u) = \big([\bar{\beta}_i], \pm B\big)$ unfolds to $([\bar{\gamma}_i], \pm C)$ and hence $(\widetilde{\mu}_{k_1} \cdots \widetilde{\mu}_{k_r} \widetilde{\sigma}) \in M_m$ acts on $([\gamma_i], C)$ to give a seed with the same matrix up to sign, so corresponds to a cluster automorphism of the cluster algebra constructed with $([\gamma_i], C)$ as the initial seed, and hence to an automorphism of the exchange graph $\mathcal{E}(C)$. $\square$

Corollary 6.5. *By Theorem 5.19 the marked exchange graph automorphisms correspond to cluster automorphisms, so for a skew-symmetrizable matrix B which unfolds to C and with corresponding cluster algebras $\mathcal{A}_B$ and $\mathcal{A}_C$, Theorem 6.4 implies*

$$\mathrm{Aut}\, \mathcal{A}_B \hookrightarrow \mathrm{Aut}\, \mathcal{A}_C.$$

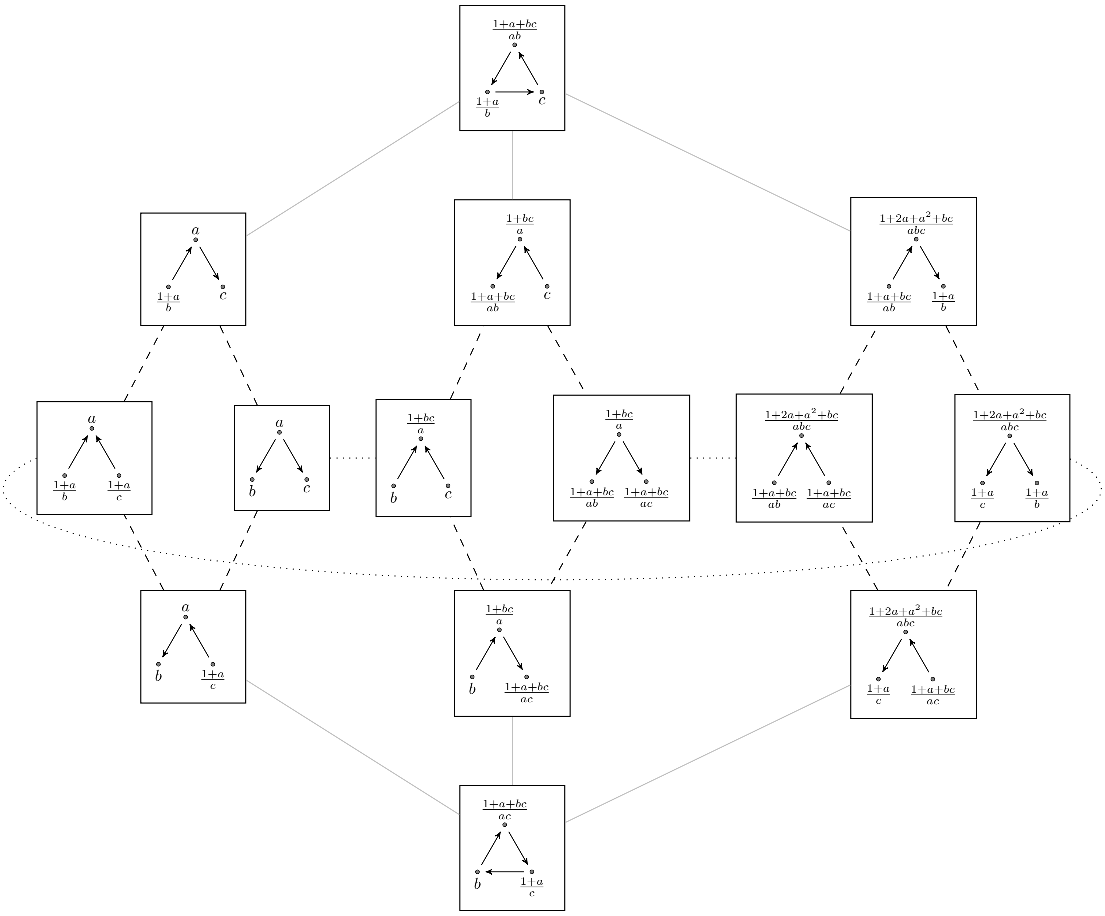


Figure 11: Exchange graph of the mutation class of type A_3. The dotted and dashed edges show how the marked exchange graph of type B_2 shown in Figure 10 unfolds. A dashed edge in Figure 10 corresponds to the composite mutation denoted by a consecutive pair of dashed edges in this figure.

Example 6.6. The matrix B representing the Dynkin diagram of type B_2

$$B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \quad \text{unfolds to} \quad C = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

the matrix representing a quiver of Dynkin type A_3. The symmetrizing matrix of B is given by $D = \operatorname{diag}(1,2)$ so the B_2 marked exchange graph shown in Figure 10 embeds into the exchange graph of type A_3 shown in Figure 11. The dashed edges in Figure 10 correspond to the pairs of dashed edges representing composite mutations in Figure 11. Dotted edges in Figure 10 correspond to single dotted edges in Figure 11.

The seed $([x,y],B)$ unfolds to the seed $([a,b,c],C)$ and the cluster variables of these two seeds are related with

$$x \rightsquigarrow a, \quad y \rightsquigarrow \{b,c\}$$

as the symmetrizing matrix $\operatorname{diag}(1,2)$ ensures that y unfolds to two cluster variables.

The automorphism $\phi \in \operatorname{Aut}\widehat{\mathcal{E}}(B_2)$ given by rotation by $\frac{2\pi}{3}$ takes the seed $[x,y]$ to $\left[\frac{1+y^2}{x}, \frac{1+x+y^2}{xy}\right]$ and corresponds to the cluster automorphism $f \in \operatorname{Aut}\mathcal{A}(B_2)$ given by

$$f(x) = \frac{1+y^2}{x}, \quad f(y) = \frac{1+x+y^2}{xy}.$$

This automorphism induces an automorphism of the exchange graph of A_3 given by a rotation along the embedded $\widehat{\mathcal{E}}(B_2)$ fixing the seeds with cyclic quivers and takes $[a,b,c]$ to $\left[\frac{1+bc}{a}, \frac{1+a+bc}{ab}, \frac{1+a+bc}{ac}\right]$ which corresponds to the cluster automorphism $g \in \operatorname{Aut}\mathcal{A}(A_3)$ given by

$$g(a) = \frac{1+bc}{a}, \quad g(b) = \frac{1+a+bc}{ab}, \quad g(c) = \frac{1+a+bc}{ac}.$$

However the automorphism could also correspond to the cluster automorphism $\tilde{g} \in \operatorname{Aut}\mathcal{A}(A_3)$ where

$$\tilde{g}(a) = g(a), \quad \tilde{g}(b) = \frac{1+a+bc}{ac}, \quad \tilde{g}(c) = \frac{1+a+bc}{ab}.$$

There is a single non-identity $\mathcal{E}(A_3)$ exchange graph automorphism which fixes the embedded $\widehat{\mathcal{E}}(B_2)$, given by a reflection in the circle of the embedded subgraph and interchanging the two seeds with cyclic quivers. This then corresponds to the cluster automorphism $h \in \operatorname{Aut}\mathcal{A}(A_3)$ given by

$$h(a) = a, \quad h(b) = c, \quad h(c) = b$$

such that $\tilde{g} = g \circ h = h \circ g$.

Theorem 6.4 shows that cluster automorphisms of $\mathcal{A}_B$ commute with unfolding the seeds, so a direct cluster automorphism $\phi \in \operatorname{Aut}\mathcal{A}_B$ preserves the exchange matrix B, which when unfolded to $\psi \in \operatorname{Aut}\mathcal{A}_C$ must also preserve the exchange matrix C and so is also a direct cluster automorphism.

Corollary 6.7. $\operatorname{Aut}^+\mathcal{A}_B \hookrightarrow \operatorname{Aut}^+\mathcal{A}_C$.

7 Mapping class groups

In their paper introducing cluster automorphisms [1] Assem, Schiffler and Shramchenko introduced the tagged mapping class group for surfaces with punctures. This group has been shown to coincide with the group of direct cluster automorphisms of the surface's corresponding cluster algebra.

Definition 7.1. Given a surface with marked points (S, M) the *mapping class group* of the surface is given by

$$\mathrm{MCG}(S,M) = {\mathrm{Homeo}^{+}(S,M)}\Big/{\mathrm{Homeo}^{0}(S,M)}.$$

Here $\mathrm{Homeo}^{+}(S,M)$ is the group of orientation-preserving homeomorphisms from S to itself which sends the set M to itself, but does not necessarily fix M nor the boundary of S pointwise, and $\mathrm{Homeo}^{0}(S,M)$ is the subgroup of homeomorphisms which are isotopic to the identity such that the isotopy fixes M pointwise.

The cluster structure given by triangulations of a surface with marked points was first studied by Fomin, Shapiro and Thurston in [13], where they show that flips of arcs in a triangulation coincide with mutations. However such a triangulation could contain self-folded triangles, and therefore arcs that cannot be flipped; to get around this problem, the authors introduced taggings on the arcs. A *tagged arc* is an arc which does not cut out a once-punctured monogon, where the enpoints are tagged either *plain* or *notched*, such that any endpoints on ∂S are tagged plain and if the endpoints of an arc coincide then they must be tagged the same.

Two tagged arcs are *compatible* if either their underlying arcs are the same and then at least one endpoint must be tagged in the same way, or the underlying arcs are not equal but are compatible. In this case, if they share an endpoint, the arcs must be tagged in the same way at that endpoint. A tagged triangulation is a maximal collection of compatible tagged arcs and a tagged flip is then defined in the same way as for triangulations, where a tagged arc is replaced with the unique other compatible tagged arc and these flips again correspond to mutations. See [13, Section 7] or [1, Section 4] for more details.

Definition 7.2. The *tagged mapping class group* of a surface (S, M) with p punctures is the semidirect product of the standard mapping class group of the surface with $\mathbb{Z}_2^p$,

$$\mathrm{MCG}_{\bowtie}(S,M) = \mathbb{Z}_2^p \rtimes \mathrm{MCG}(S,M),$$

where the elements of $\mathrm{MCG}(S,M)$ act as diffeomorphisms on the surface and elements of $\mathbb{Z}_2^p$ switch or preserve the tags on the tagged triangulation at each puncture.

Theorem 7.3 ([1, Theorem 4.11])**.** *Let (S, M) be a surface with p punctures, with corresponding cluster algebra $\mathcal{A}$, then*

1. $\mathrm{MCG}(S)$ *is isomorphic to a subgroup of* $\mathrm{Aut}^{+}\mathcal{A}$*.*

2. *If $p \geqslant 2$ or $\partial S \neq \emptyset$ then $\mathrm{MCG}_{\bowtie}(S)$ is isomorphic to a subgroup of $\mathrm{Aut}^+ \mathcal{A}$.*

They showed that for discs and annuli without punctures as well as for certain discs with 1 or 2 punctures then the tagged mapping class group is isomorphic to the group of direct cluster automorphisms of the corresponding cluster algebra. The authors conjectured that this would be the case for almost all surfaces with marked points. Brüstle and Qiu proved that this conjecture is true in [4]:

Theorem 7.4 ([4, Theorem 4.7])**.** *Let (S, M) be a surface with marked points which is not*

1. *a once-punctured disc with 2 or 4 marked points on the boundary*
2. *a twice-punctured disc with 2 marked points on the boundary*

then

$$\mathrm{MCG}_{\bowtie}(S, M) = \mathrm{Aut}^+ \mathcal{A}.$$

Theorem 7.3 shows that $\mathrm{MCG}_{\bowtie}(S, M) \hookrightarrow \mathrm{Aut}^+ \mathcal{A}$, so the proof of Theorem 7.4 needs to show that this injection is surjective. This follows from the result below proved by Bridgeland and Smith:

Proposition 7.5 ([2, Prop. 8.5])**.** *Suppose (S, M) is a surface which is not one of:*

1. *a sphere with $\leqslant 5$ marked points;*
2. *an unpunctured disc with $\leqslant 3$ marked points on the boundary;*
3. *a disc with a single puncture and one marked point on the boundary;*
4. *a once-punctured disc with 2 or 4 marked points on the boundary;*
5. *a twice-punctured disc with 2 marked points on the boundary,*

then two tagged triangulations of (S, M) differ by an element of $\mathrm{MCG}_{\bowtie}(S, M)$ if and only if the associated quivers are isomorphic.

7.1 Unfoldings and covering maps

Diagrams correspond to triangulations of orbifolds in the same way that quivers correspond to triangulations of surfaces. A covering of the orbifold by a surface corresponds to an unfolding of the diagram to a quiver, in such a way that composite mutations of the quiver correspond to triangle flips in the triangulation of the surface, as discussed in [10].

In their paper on the growth rate of cluster algebras, Felikson, Shapiro, Thomas and Tumarkin [9] defined the mapping class group of a cluster algebra $\mathrm{MCG}(\mathcal{A})$ to be the elements of M_n which fix the initial exchange matrix up to a quotient by those elements of M_n which fix the initial seed. Elements of this group would then fix the initial exchange

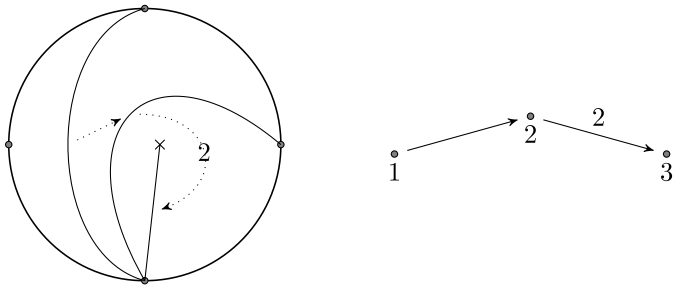


Figure 12: Triangulation of an orbifold (left) with associated diagram (right). The interior orbifold point is shown as a cross. The associated diagram is also shown with dotted arrows inside the triangulation.

matrix and map the initial cluster to some other cluster in the mutation class, and hence would induce a direct cluster automorphism.

Fix a marked orbifold $\mathcal{O}$ with m punctures. In [9, Remark 4.15] the cluster mapping class group is argued to either contain the orbifold's mapping class group as a proper normal subgroup with quotient $\mathrm{MCG}(\mathcal{A})/\mathrm{MCG}(\mathcal{O}) \cong \mathbb{Z}_2^m$ (when $m > 1$, or when $m = 1$ and the boundary non-empty) or be isomorphic to the orbifold mapping class group (when $m = 0$, or when $m = 1$ and the boundary is empty).

The additional $\mathbb{Z}_2$ for each interior marked point corresponds to the additional taggings in the definition of $\mathrm{MCG}_{\bowtie}(\mathcal{O})$ and so suggests that the following would be true:

Conjecture 7.6. *For a cluster algebra $\mathcal{A}$ arising from the triangulation of an orbifold $\mathcal{O}$*

$$\mathrm{MCG}_{\bowtie}(\mathcal{O}) \cong \mathrm{Aut}^+ \mathcal{A}.$$

Example 7.7. Consider the orbifold $\mathcal{O}$ constructed from the disc with four marked points on the boundary and a single orbifold point in the interior, as shown in Figure 12.

This orbifold has no punctures, so the tagged mapping class group is equal to the mapping class group. Any element of the mapping class group must fix the orbifold point and permute the four boundary marked points. The only such permutations are rotations around the boundary, as any reflection would not preserve the orientation, hence the mapping class group is isomorphic to $\mathbb{Z}_4$ generated by a rotation by angle $\frac{\pi}{2}$.

This orbifold corresponds to the cluster algebra of Dynkin type B_3, which can be generated by the diagram in Figure 12. The cluster automorphism group of $\mathcal{A}_{B_3}$ is the dihedral group with 8 elements:

$$\mathrm{Aut}\,\mathcal{A}_{B_3} \cong D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2,$$

where $\mathbb{Z}_4$ is generated by the automorphism given by the action of $\mu_1\mu_2\mu_3$ on the initial cluster and $\mathbb{Z}_2$ by $\mu_1\mu_3$. This can be seen as the automorphisms of the marked exchange graph shown in Figure 9 where the 4 squares are permuted while fixing the markings.

The direct cluster automorphisms are those in the subgroup $\mathbb{Z}_4$ of the cluster automorphism group, and so

$$\mathrm{Aut}^+ \mathcal{A}_{B_3} \cong \mathbb{Z}_4 \cong \mathrm{MCG}(\mathcal{O}) = \mathrm{MCG}_{\bowtie}(\mathcal{O}).$$

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