Induced colorful trees and paths in large chromatic graphs

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Abstract

In a proper vertex coloring of a graph a subgraph is *colorful* if its vertices are colored with different colors. It is well-known (see for example in Gyárfás (1980)) that in every proper coloring of a k-chromatic graph there is a colorful path P_k on k vertices. The first author proved in 1987 that k-chromatic and *triangle-free* graphs have a path P_k which is an *induced* subgraph. N.R. Aravind conjectured that these results can be put together: in every proper coloring of a k-chromatic triangle-free graph, there is an induced colorful P_k. Here we prove the following weaker result providing some evidence towards this conjecture: For a suitable function $f(k)$, in any proper coloring of an $f(k)$-chromatic graph of girth at least five, there is an induced colorful path on k vertices.

Keywords: Induced subgraphs; Graph colorings

1 Introduction

A special case of a result of the first author in [7] says that every triangle-free k-chromatic graph G contains an induced path on k vertices. The following more general conjecture is attributed to N.R. Aravind in [2]. A path (or more generally a subgraph) in a proper coloring of G is called *colorful* if its vertices are colored with distinct colors.

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Conjecture 1. *In any proper coloring of any triangle free k-chromatic graph G there is an induced colorful path on k vertices.*

The main result of [2] is the proof of Conjecture 1 for the case when G has girth k. One can easily see that Conjecture 1 cannot be extended from paths to other trees. Indeed, the following example shows that there are graphs of arbitrary large chromatic number with proper colorings that contain no colorful $K_{1,3}$. For other similar problems on colorful paths see [3].

Example 1. *([5, 10]) Let SH_n be the graph whose vertex set is the set of $\binom{n}{3}$ triples of $[n]$ and where for $1 \leqslant i < j < k < \ell \leqslant n$, vertex (i, j, k) is adjacent to (j, k, ℓ). Coloring (i, j, k) with j, we have a proper coloring containing no colorful $K_{1,3}$ and the chromatic number of SH_n is unbounded.*

However, if we drop the colorful condition then (according to a well-known conjecture of the first author and Sumner [6, 12]) the existence of any induced subtree might be guaranteed in triangle-free graphs of sufficiently large chromatic number. If the triangle-free condition is strengthened, considering the family $\mathcal{G}_5$ of graphs with no cycles of length three and four, then the induced tree conjecture becomes easy, in fact large minimum degree can replace the chromatic bound.

Theorem 1. *(Gyárfás, Szemerédi, Tuza [8]). Let T_k be a tree on k vertices. Then every graph in $\mathcal{G}_5$ with minimum degree at least $k-1$ contains T_k as an induced subgraph.*

Assume we have a proper coloring on G. The color degree $cod_G(v)$ is the number of distinct colors appearing on the neighbors of v and $cod(G) = \min\{cod_G(v) : v \in V(G)\}$. Our first result is the following "colorful" variant of Theorem 1.

Theorem 2. *Let T_k be a tree on $k \geqslant 4$ vertices. Then every proper coloring of $G \in \mathcal{G}_5$ with $cod(G) \geqslant 2k-5$ contains T_k as an induced colorful subgraph.*

A related subject is to find induced subgraphs in *oriented* large chromatic triangle-free graphs, for old and new results see [1]. By a result of Chvátal [4], acyclic digraphs with no induced subgraph with edges $(1, 2), (2, 3), (4, 3)$ are perfect. On the other hand, triangle-free digraphs with no induced subgraph with edges $(1, 2), (3, 2), (3, 4)$ exist with an arbitrary large chromatic number (see [9]). In [9] it was asked what happens for the directed $P_4 = (1, 2), (2, 3), (3, 4)$? This was answered by Kierstead and Trotter [10] by constructing arbitrary large chromatic triangle-free oriented graphs without induced directed P_4. They also proved that if the clique size of a graph is fixed and its chromatic number is large then in every proper coloring and with orienting edges from smaller to larger color, there is *either* an induced colorful star S_k (a vertex with outdegree k) *or* an induced colorful directed path P_k. Here we present a result in a similar vein.

Theorem 3. *Let k be a positive integer and T_k be a tree on k vertices. There exists a function $f(k)$ such that the following holds. If $G \in \mathcal{G}_5$ with $\chi(G) \geqslant f(k)$ then in any proper coloring of G and in any acyclic orientation of G there is either an induced colorful T_k or an induced directed path P_k.*

Note that in Theorem 3 the orientation of T_k is not prescribed (but P_k is the directed path). Also, P_k is induced but not necessarily colorful. However, if G is oriented so that for $c(v) < c(w)$ we have $(v, w) \in E(G)$, P_k must be colorful as well. Selecting this acyclic orientation and $T_k = P_k$, we get from Theorem 3 the following weakened form of Conjecture 1.

Corollary 1. *In any proper coloring of an $f(k)$-chromatic graph $G \in \mathcal{G}_5$, G contains an induced colorful path on k vertices.*

To get closer to Conjecture 1 it would be very desirable to forbid only triangles (and allow four-cycles) in Corollary 1. It is worth considering the following problem.

Problem 1. *Let k be a positive integer and T_k be a tree on k vertices. Is there a function $f(k)$ such that the following holds? If G is a triangle-free graph with $\chi(G) \geqslant f(k)$ then in any proper coloring of G with $\chi(G)$ colors, there is an induced colorful T_k.*

Problem 1 seems certainly difficult since it contains the Gyárfás - Sumner conjecture. The case when T_k is a path should be easier, it is weaker than Conjecture 1. Note that the condition that the proper coloring of G must use $\chi(G)$ colors, eliminates Example 1. In fact, Problem 1 has an affirmative answer for any k-vertex star with $f(k) = k$, since a k-chromatic graph must contain a vertex adjacent to all other color classes in any k-coloring and these neighbors form an independent set since G is triangle-free.

2 Proofs

Proof of Theorem 2. We construct an induced colorful T_k by induction. For $k = 4$ we have two trees to construct from the condition that $cod(G) \geqslant 3$. For each of the two trees the proof is obvious and we omit the details.

For the inductive step, assume v is a leaf of a tree T_k with neighbor w in T_k. Let T^* be the tree $T_k - v$. By induction we find T^* as an induced colorful subgraph of G so $w \in V(T^*) \subseteq V(G)$. By the condition on the color degree, w is adjacent to a set $S \subset V(G) \setminus V(T^*)$ such that $|S| \geqslant k-3$, S is colorful and $\{c(v) : v \in S\} \cap \{c(v) : v \in T^*\} = \emptyset$. No edge of G goes from S to any vertex of $T^* - \{w\}$ that is at distance one or two from w in T^* since $G \in \mathcal{G}_5$. There are at most $k-4$ vertices of T^* that are at distance at least three from w in T^* and each of them sends at most one edge to S since G is C_4-free. Thus at least one vertex in S is nonadjacent to any vertex of $T^* - \{w\}$ and it extends T^* to a tree isomorphic to T_k and it is an induced colorful subgraph of G. □

Proof of Theorem 3. Let c be a proper coloring of G, G^* is an orientation of G. Assume that we have an ordering $<$ on $V(G)$, then the forward color degree $fcod_G(v)$ is the number of distinct colors appearing on the neighbors of v that are larger than v, i.e.

$$fcod_G(v) = |\{c(w) : vw \in E(G), v < w\}|.$$

We shall prove that the following function $f(k)$ is suitable.

$$f(k) = \begin{cases} k & \text{if } 1 \leqslant k \leqslant 4 \\ \\ g^2(k) & \text{if } k \geqslant 5, \end{cases}$$

where

$$g(k) = \begin{cases} k & \text{if } 1 \leqslant k \leqslant 4 \\ \\ (2k-6)g(k-1)+1 & \text{if } k \geqslant 5. \end{cases}$$

For $k \leqslant 3$ the theorem holds with the first alternative: a 1-chromatic graph has a vertex, a 2-chromatic graph has an edge, 3-chromatic graphs without triangles have odd induced cycles of length at least 5 which must contain colorful induced P_3. Thus we assume $k \geqslant 4$.

Assume first that G has a subgraph G' such that $cod_{G'}(v) \geqslant 2k-5$ for every $v \in G'$. By Theorem 2 we find a colorful induced $T_k = P_k$ in G'.

Now we may assume that G has no subgraph G' such that $cod_{G'}(v) \geqslant 2k-5 = d$ for every $v \in G'$. Thus we can define an ordering π on $V(G)$ by repeatedly selecting vertices $v_1, \dots, v_i, \dots$ such that $fcod_{G_i}(v_i) < d$ for all $v_i \in V(G)$, where G_i is the subgraph of G induced by the vertices $(V(G) \setminus \{v_1, \dots, v_{i-1}\})$.

The oriented graph G^* can be written as $G_1 \cup G_2$ where both graphs have vertex set $V(G)$ and G_1 contains the edges of G^* oriented forward with respect to π and G_2 contains the edges of G^* oriented backward with respect to π. By the Gallai - Vitaver - Roy theorem ([11], Ex. 9.9) we can find a directed path P_t in one of the two G_i-s with $t = g(k)$. Indeed, for $k = 4$, $g(4) = f(4) = 4$ thus G is 4-chromatic, it contains P_4 which is induced since $G \in \mathcal{G}_5$. For $k > 4$, we use that $f(k) \leqslant \chi(G^*) = \chi(G_1 \cup G_2) \leqslant \chi(G_1)\chi(G_2)$, thus $\chi(G_i) \geqslant t = \sqrt{f(k)} = g(k)$ for some $i \in \{1, 2\}$. Assume that P_t is oriented forward in the ordering π.

Lemma 1. *$P = P_t$ contains an induced P_k starting from the first vertex of P.*

Proof. For $k = 4$ the lemma is obvious since in a graph $G \in \mathcal{G}_5$ any P_4 is induced. Assuming that it is true for some $k-1 \geqslant 4$, consider the at most $d-1 = 2k-6$ forward edges from the starting point $v \in P$ to other points $w_1, \dots, w_s$ of P where $s \leqslant 2k-6$. This partitions $P-v$ into at most $2k-6$ disjoint paths, $Q_1 = w_1 \dots, Q_2 = w_2 \dots, Q_s = w_s \dots,$ one of them, Q_j, must contain at least $\frac{g(k)-1}{2k-6} = g(k-1)$ vertices. By induction, Q_j contains an induced P_{k-1} from its first vertex w_j. No edge of G is oriented from Q_j to v since the orientation is acyclic. Also, apart from vw_j, no edge of G is oriented from v to Q_j by the definition of Q_j. Thus vP_{k-1} is the required induced P_k. This proves the lemma. □

The proof of Theorem 3 is now finished, observing that if P_t is oriented backward in the ordering π, Lemma 1 should be used in "backward" version, stating that $P = P_t$ contains an induced P_k ending in the first vertex of P. □

References

[1] P. Aboulker, J. Bang-Jensen, N. Bousquet, P. Charbit, F. Havet, F. Maffray, J. Zemora, χ-bounded families of oriented graphs, arXiv:1605.07411v1.

[2] J. Babu, M. Basavaraju, L. S. Chandran, M. C. Francis, On induced colorful paths in triangle-free graphs, arXiv:1604.06070v1.

[3] S. Bessy, N. Bousquet, Colorful paths for 3-chromatic graphs, arXiv:1503.00965.

[4] V. Chvátal, Perfectly ordered graphs, *in: Topics on Perfect Graphs, Ann. Discrete Math.* **21** (1984) 63-65.

[5] P. Erdős, Z. Füredi, A. Hajnal, P. Komjáth, V. Rödl, Á. Seress, Coloring graphs with locally few colors, *Discrete Math.* **39** (1986) 21-34.

[6] A. Gyárfás, On Ramsey covering numbers, *Infinite and Finite Sets, Coll. Math. Soc. János Bolyai* **10** (1973) 801-816.

[7] A. Gyárfás, Problems from the world surounding perfect graphs, Zastowania Matematyki Applicationes Mathematicae **XIX, 3-4** (1987) 413-444.

[8] A. Gyárfás, E. Szemerédi, Zs. Tuza, Induced subtrees in graphs of large chromatic number, *Discrete Math.* **30** (1980) 235-244.

[9] A. Gyárfás, Problem 115, *Discrete Math.* **79** (1989-90) 109-110.

[10] H.A. Kierstead, W.T. Trotter, Colorful induced subgraphs, *Discrete Math.* **101** (1992) 165-169.

[11] L. Lovász, Combinatorial Problems and Exercises, *North Holland*, 1993.

[12] D. P. Sumner, Subtrees of a graph and chromatic number, *in: G. Chartrand, ed., The Theory and Applications of Graphs* (Wiley, New York, 1981) 557-576.