Packing polynomials on multidimensional integer sectors

Luis B. Morales
IIMAS, Universidad Nacional Autónoma de México,
Ciudad de México, 04510, México
lbm@unam.mx

Submitted: Jun 3, 2015; Accepted: Sep 28, 2016; Published: Oct 14, 2016
Mathematics Subject Classifications: 11C08, 05A10, 05A19, 06R05

Abstract

Denoting the real numbers and the nonnegative integers, respectively, by $\mathbf{R}$ and $\mathbf{N}$, let S be a subset of $\mathbf{N}^n$ for $n = 1, 2, \ldots$, and f be a mapping from $\mathbf{R}^n$ into $\mathbf{R}$. We call f a *packing function* on S if the restriction $f|_S$ is a bijection onto $\mathbf{N}$. For all positive integers $r_1, \ldots, r_{n-1}$, we consider the *integer sector* $I(r_1, \ldots, r_{n-1}) = \{(x_1, \ldots, x_n) \in N^n \mid x_{i+1} \leqslant r_i x_i \text{ for } i = 1, \ldots, n-1\}$. Recently, Melvyn B. Nathanson (2014) proved that for $n = 2$ there exist two quadratic packing polynomials on the sector $I(r)$. Here, for $n > 2$ we construct 2^{n-1} packing polynomials on multidimensional integer sectors. In particular, for each packing polynomial on $\mathbf{N}^n$ we construct a packing polynomial on the sector $I(1, \ldots, 1)$.

Keywords: Packing polynomials; s-diagonal polynomials; multidimensional lattice point enumeration

1 Introduction

In this paper, $\mathbf{N}$ and $\mathbf{R}$ denote, respectively, nonnegative integers and real numbers, $0 < n \in \mathbf{N}$, and let S be a subset of $\mathbf{N}^n$. A function f from $\mathbf{R}^n$ into $\mathbf{R}$ is a *packing function* on S if the restriction $f|_S$ is a bijection onto $\mathbf{N}$. Also $s(\mathbf{x}) = x_1 + \cdots + x_n$ when $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{N}^n$. Given w in $\mathbf{N}$, let $H(n, w) = \{\mathbf{x} \in \mathbf{N} \mid s(\mathbf{x}) \leqslant w\}$. A packing function f on $\mathbf{N}^n$ is called a *diagonal* mapping if f takes $H(n, w)$ bijectively onto $\{0, 1, \ldots, -1 + \binom{n+w}{n}\}$ for each $w \in \mathbf{N}$, or equivalently, if $f(\mathbf{x}) < f(\mathbf{y})$ whenever $\mathbf{x}, \mathbf{y} \in \mathbf{N}^n$ and $s(\mathbf{x}) < s(\mathbf{y})$ (see [4, 5]). Packing functions map arbitrarily large n-dimensional arrays into computer memory cells numbered $0, 1, \ldots$, and produce no conflicts in such a process (see [7]).

Let $n > 1$. For all positive real numbers $\alpha_1, \ldots, \alpha_{n-1}$, we define the *real sector*

$$S(\alpha_1, \ldots, \alpha_{n-1}) = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid 0 \leqslant x_{i+1} \leqslant \alpha_i x_i,\ i = 1, \ldots, n-1\}$$

and the *integer sector*

$$I(\alpha_1, \dots, \alpha_{n-1}) = \{(x_1, \dots, x_n) \in \mathbf{N}^n \mid 0 \leqslant x_{i+1} \leqslant \alpha_i x_i,\ i = 1, \dots, n-1\}.$$

We also define, respectively, the real and integer sectors

$$S(\infty, \dots, \infty) = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_i \geqslant 0,\ i = 1, \dots, n\}$$

and

$$I(\infty, \dots, \infty) = \{(x_1, \dots, x_n) \in \mathbf{N}^n \mid x_i \geqslant 0,\ i = 1, \dots, n\} = \mathbf{N}^n.$$

Given any permutation π on $\{1, \dots, n\}$ and any n-tuple $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$, define $\pi\mathbf{x} = (x_{\pi(1)}, \dots, x_{\pi(n)})$. Then, we say that two functions f and g on $\mathbf{R}^n$ are *equivalent* if there exists a permutation π such that for all $\mathbf{x}$, $f(\mathbf{x}) = g(\pi\mathbf{x})$. Hereafter, we will write permutations in cycle notation.

It is not hard to see that if $(x_1, \dots, x_n)$ is a variable vector and k is a positive integer, then the binomial coefficient $\binom{k+x_1+\cdots+x_n}{k}$ produces a degree k polynomial in the variables $x_1, \dots, x_n$.

Packing functions were introduced to literature by Cauchy [3]. Later, Cantor [1, 2] observed that the polynomial function

$$f(x_1, x_2) = \binom{x_1 + x_2 + 1}{2} + x_2 \tag{1}$$

is a bijection from $\mathbf{N}^2$ onto $\mathbf{N}$, hence a packing function on $I(\infty)$, in our terms. The polynomials f and $f\pi$, with $\pi = (1\ 2)$, are called the *Cantor polynomials.*

More generally, Skolem [**?**, **?**] constructed just one inequivalent n-dimensional packing polynomial on $I(\infty, \dots, \infty)$ for each $n > 0$. Morales and Lew [5] constructed 2^{n-2} inequivalent packing polynomials of dimension n for each $n > 1$. Morales [4] produced a family of packing polynomials, which includes the Morales-Lew polynomials. Finally, Sanchez [8] obtained a family of $(n-1)!$ inequivalent packing polynomials on $I(\infty, \dots, \infty)$, which includes the above family. All these polynomials are diagonal functions.

Recently Nathanson [6] proved that for $n = 2$ there exist two quadratic packing polynomials on the sector $I(r)$. Also he proved that packing polynomials on $I(\infty)$ are in bijection with packing polynomials on $I(1)$. For $n > 1$ we now construct 2^{n-1} packing polynomials—a generalization of Nathanson's—on $I(r_1 \dots, r_{n-1})$, such that $r_1, \dots, r_{n-1}$ are positive integers. Also, we prove that these 2^{n-1} packing polynomials have a kind of diagonal property. In particular, for each packing polynomial on $\mathbf{N}^n$ we construct a packing polynomial on $I(1, \dots, 1)$.

2 s-Diagonal functions on integer sectors

In this section we construct recursive subsets of multidimensional integer sectors. Using these sets we define s-diagonal packing functions on multidimensional integer sectors.

These functions are a generalization of the Nathanson's functions. Also we calculate the cardinalities of these sets, which are used to construct s-diagonal packing functions.

Given any nonnegative integer x, we define

$$E(1,x) = \{x\}.$$

For all positive integers r, we define

$$E_r(2,x) \;=\; \{(x,x_2) \in N^2 \mid x \in E(1,x_2),\ x_2 = 0,\dots,rx\}.$$

For $n > 2$ and all positive integers $r_1, r_2, \dots, r_{n-1}$,

$$\begin{aligned} E_{r_1,r_2,\dots,r_{n-1}}(n,x) \;=\; & \{(x,x_2,\dots,x_n) \in N^n \mid \\ & (x_2,\dots,x_n) \in E_{r_2,\dots,r_{n-1}}(n-1,x_2),\ x_2 = 0,\dots,r_1 x\}. \end{aligned} \tag{2}$$

It is not difficult to verify that if $x \neq x'$, then

$$E_{r_1,r_2,\dots,r_{n-1}}(n,x) \cap E_{r_1,r_2,\dots,r_{n-1}}(n,x') = \emptyset. \tag{3}$$

Note that our set $E_r(2,x)$ coincides with the set defined in [6, Theorem 7]. Moreover, if r_1, r_2 are positive integers, by direct calculation we get

$$E_{r_1,r_2}(3,x) = \{(x,0,0),(x,1,0),\dots,(x,1,1,r_2),\dots,(x,r_1x,0),\dots,(x,r_1x,r_1r_2x)\}.$$

Given any positive integers $r_1, r_2, \dots, r_{n-1}$, and any nonnegative integer x, we define

$$D_{r_1,r_2,\dots,r_{n-1}}(n,x) = \bigcup_{j=0}^{x} E_{r_1,r_2,\dots,r_{n-1}}(n,j). \tag{4}$$

It is easy to see that if $r_1, r_2, \dots, r_{n-1}$ are positive integers, then

$$I(r_1,\dots,r_{n-1}) = \bigcup_{x \in \mathbf{N}} E_{r_1,r_2,\dots,r_{n-1}}(n,x).$$

Moreover, from (2) and (4) we have

$$E_{r_1,r_2,\dots,r_{n-1}}(n,x) = \{(x,x_2,\dots,x_n) \in N^n \mid (x_2,\dots,x_n) \in D_{r_2,\dots,r_{n-1}}(n-1,r_1x)\} \tag{5}$$

for any $x \in \mathbf{N}$.

For use as the basis of an induction below, we define the 1-dimensional integer sector, I^1, as the set of nonnegative integers.

Let r be a positive integer. Then Nathanson's [6] packing polynomials on $I(r)$ have a special property: for each $x \in \mathbf{N}$, any Nathanson's polynomial maps $D_r(2,x)$ (resp. $E_r(2,x)$) bijectively onto $\{0,\dots,-1+|D_r(2,x)|\}$ (resp. $\{0,1,\dots,-1+|E_r(2,x)|\}$). In analogy with the definition of diagonal functions on $\mathbf{N}^2$, a packing function f on $I(r)$ is called an *s-diagonal* function if f has this property. We generalize this definition for packing functions on multidimensional integer sectors as follows.

Let $r_1, r_2, \ldots, r_{n-1}$ be positive integers. A packing function f on $I(r_1, \ldots, r_{n-1})$ is called a *s-diagonal* mapping if f maps $D_{r_1,r_2,\ldots,r_{n-1}}(n,x)$ bijectively onto $\{0, \ldots, -1 + |D_{r_1,r_2,\ldots,r_{n-1}}(n,x)|\}$. Also for each $n > 0$, we define $s\text{-}DB_{r_1,\ldots,r_{n-1}}(n)$ and $s\text{-}DP_{r_1,\ldots,r_{n-1}}(n)$ to be the sets of s-diagonal functions and s-diagonal polynomials defined on $\mathbf{R}^n$, respectively. ($s\text{-}DB^1(1)$ and $s\text{-}DP^1(1)$ denote the s-diagonal functions and s-diagonal polynomials on I^1.)

These definitions clearly imply the following lemma.

Lemma 1. *If $r_1, r_2, \ldots, r_{n-1}$ are positive integers, then*

(1) *If $f \in DB_{r_1,\ldots,r_{n-1}}(n)$, then $f(0, \ldots, 0) = 0$.*

(2) *$f \in DB_{r_1,\ldots,r_{n-1}}(n)$ if and only if $f(x_1, x_2, \ldots, x_n) < f(y_1, y_2, \ldots, y_n)$ whenever $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in I(r_1, \ldots, r_{n-1})$ and $x_1 < y_1$.*

(3) *If $f \in DB^1(1)$, then f is the identity map on I^1.*

Now we calculate the cardinalities of the sets $E_{r_1,r_2,\ldots,r_{n-1}}(n,x)$ and $D_{r_1,r_2,\ldots,r_{n-1}}(n,x)$.

Denote the cardinalities of the sets $E_{r_1,r_2,\ldots,r_{n-1}}(n,x)$ and $D_{r_1,r_2,\ldots,r_{n-1}}(n,x)$ by the functions $T_{r_1,r_2,\ldots,r_{n-1}}(x)$ and $Q_{r_1,r_2,\ldots,r_{n-1}}(x)$, respectively. Also we define $T_{r_1,r_2,\ldots,r_{n-1}}(-1) = Q_{r_1,r_2,\ldots,r_{n-1}}(-1) = 0$. The next lemma calculates the cardinality of these sets.

Lemma 2. *If $r_1, r_2, \ldots, r_{n-1}$ are positive integers and x is any nonnegative integer, then*

$$T_{r_1,r_2,\ldots,r_{n-1}}(x) = \sum_{i_2=0}^{r_1 x} \cdots \sum_{i_n=0}^{r_{n-1} i_{n-1}} 1, \tag{6}$$

$$Q_{r_1,r_2,\ldots,r_{n-1}}(x) = \sum_{i_1=0}^{x} \sum_{i_2=0}^{r_1 i_1} \cdots \sum_{i_n=0}^{r_{n-1} i_{n-1}} 1, \tag{7}$$

$$Q_{r_1,r_2,\ldots,r_{n-1}}(x) = Q_{r_1,r_2,\ldots,r_{n-1}}(x-1) + Q_{r_2,\ldots,r_{n-1}}(r_1 x). \tag{8}$$

Proof. Clearly, $|E(1,x)| = 1$. For $n > 1$, relation (3) and the induction hypothesis imply that

$$T_{r_1,r_2,\ldots,r_{n-1}}(x) = \sum_{j=0}^{r_1 x} T_{r_2,\ldots,r_{n-1}}(j) = \sum_{j=0}^{r_1 x} \sum_{i_3=0}^{r_2 j} \cdots \sum_{i_n=0}^{r_{n-1} i_{n-1}} 1.$$

Thus (6) holds. From (3), (4) and (6) we obtain

$$\begin{aligned} Q_{r_1,r_2,\ldots,r_{n-1}}(x) &= |D_{r_1,r_2,\ldots,r_{n-1}}(n,x)| = \sum_{i_1=0}^{x} |E_{r_1,r_2,\ldots,r_{n-1}}(n,i_1)| = \sum_{i_1=0}^{x} T_{r_1,r_2,\ldots,r_{n-1}}(i_1) \\ &= \sum_{i_1=0}^{x} \sum_{i_2=0}^{r_1 i_1} \cdots \sum_{i_n=0}^{r_{n-1} i_{n-1}} 1. \end{aligned}$$

Thus (7) holds.

By (7) we have that

$$\begin{aligned}Q_{r_1,r_2,\ldots,r_{n-1}}(x) &= \sum_{i_1=0}^{x}\sum_{i_2=0}^{r_1 i_1}\cdots\sum_{i_{n-1}=0}^{r_{n-1}i_{n-1}} 1 = \sum_{i_1=0}^{x-1}\sum_{i_2=0}^{r_1 i_1}\cdots\sum_{i_{n-1}=0}^{r_{n-2}i_{n-2}} 1 + \sum_{i_2=0}^{r_1 x}\sum_{i_3=0}^{r_2 i_2}\cdots\sum_{i_n=0}^{r_{n-1}i_{n-1}} 1\\ &= Q_{r_1,r_2,\ldots,r_{n-1}}(x-1) + Q_{r_2,\ldots,r_{n-1}}(r_1 x).\end{aligned}$$

This completes the proof. □

Using binomial coefficient identities, we can check that

$$\begin{aligned}T_{1,\ldots,1}(x) &= \binom{n-1+x}{n-1},\\ Q_{1,\ldots,1}(x) &= \binom{n+x}{n}.\end{aligned}$$

3 Polynomial formula

In this section we express the functions $T_{r_1,r_2,\ldots,r_{n-1}}(x)$ and $Q_{r_1,r_2,\ldots,r_{n-1}}(x)$ as polynomials in the variable x. In order to prove this, for any positive integer r we first construct a family of polynomials in the variable x,

$$\left\{P_{r,k}(x) = \sum_{s=1}^{k} c_{k,s}^{r}\binom{x}{s} \mid k = 1,\ldots,n\right\},$$

whose coefficients $c_{k,j}^{r}$ satisfy the recurrence relations

$$c_{k,j}^{r} = \sum_{s=1}^{k-j+1}\binom{r}{s}c_{k-s,j-1}^{r} \text{ for } k > j \geqslant 2 \text{ with } c_{k,1}^{r} = \binom{r+1}{k} \;\; k \geqslant 2 \text{ and } c_{1,1}^{r} = r \tag{9}$$

(when no confusion arises we omit the superscript r). Since $c_{1,1} = r$ and $c_{k,k} = rc_{k-1,k-1}$ for $k \geqslant 2$, it follows that $c_{k,k} = r^k$ for $k \geqslant 1$.

Example. For $n = 3$, we have

$$P_{r,1}(i) = ri, \quad P_{r,2}(i) = r^2\binom{i}{2} + \binom{r+1}{2}i, \quad P_{r,3}(i) = r^3\binom{i}{3} + r^3\binom{i}{2} + \binom{r+1}{3}i.$$

Let r and i be two positive integers and t be any integer such that $0 \leqslant t < r$. It is not hard to check that for every positive integer k,

$$\sum_{j=0}^{r}\binom{j}{k} = \binom{r+1}{k+1}, \tag{10}$$

$$\sum_{j=0}^{ri+t}\binom{j}{k} = \sum_{j=0}^{ri}\binom{j}{k} + \sum_{j=1}^{t}\binom{ir+j}{k}, \tag{11}$$

$$\sum_{j=0}^{ri}\binom{j}{k} = \sum_{j=0}^{i-1}\sum_{\ell=1}^{r}\binom{rj+\ell}{k} = \sum_{j=0}^{i-1}\sum_{\ell=1}^{r}\sum_{s=0}^{rj+\ell-1}\binom{s}{k-1}. \tag{12}$$

These binomial coefficient identities will be used in the proof of the following lemma.

Lemma 3. *Let r be a positive integer. Then, for any positive integer i and any integer $0 \leqslant t < r$,*

$$\sum_{j=0}^{ri+t}\binom{j}{k} = P_{r,k+1}(i)+\sum_{h=1}^{k}\binom{t}{h}P_{r,k+1-h}(i)+\binom{t+1}{k+1}.$$

Proof. We give an inductive proof on k. Assume that $k=1$. Then by (10)-(12) we obtain

$$\begin{aligned}\sum_{j=0}^{ri+t}\binom{j}{1} &= \sum_{j=0}^{ri}\binom{j}{1}+\sum_{j=1}^{t}\binom{ri+j}{1}=\sum_{j=0}^{i-1}\sum_{s=1}^{r}[rj+s]+\sum_{j=1}^{t}[ri+j]\\ &= \sum_{j=0}^{i-1}\left[r^2j+\binom{r+1}{2}\right]+tri+\binom{t+1}{2}\\ &= \binom{i}{2}r^2+\binom{r+1}{2}i+tri+\binom{t+1}{2}.\end{aligned}$$

Since $\binom{i}{2}r^2+\binom{r+1}{2}i=P_{r,2}(i)$ and $ir=P_{r,1}(i)$, the formula holds for $k=1$.

Now suppose that the induction hypothesis is true for $k-1$. Note that (11) allows us to divide the proof in two parts:

Part A. Here we prove that $\sum_{j=0}^{ri}\binom{j}{k}=P_{r,k+1}(i)$. From (10) and (12) we obtain

$$\sum_{j=0}^{ri}\binom{j}{k} = \sum_{j=0}^{i-1}\sum_{\ell=1}^{r}\binom{rj+\ell}{k}=\sum_{j=0}^{i-1}\sum_{\ell=1}^{r}\sum_{m=0}^{rj+\ell-1}\binom{m}{k-1}.$$

Then, by induction hypothesis

$$\begin{aligned}\sum_{j=0}^{ri}\binom{j}{k} &= \sum_{j=0}^{i-1}\sum_{\ell=1}^{r}\left[P_{r,k}(j)+\sum_{h=1}^{k-1}\binom{\ell-1}{h}P_{r,k-h}(j)+\binom{\ell}{k}\right]\\ &= \sum_{j=0}^{i-1}\left[rP_{r,k}(j)+\sum_{h=1}^{k-1}\binom{r}{h+1}P_{r,k-h}(j)+\binom{r+1}{k+1}\right]\\ &= \sum_{j=0}^{i-1}\left[\sum_{s=1}^{k}\binom{r}{1}c_{k,s}\binom{j}{s}+\sum_{h=1}^{k-1}\sum_{s=1}^{k-h}\binom{r}{h+1}c_{k-h,s}\binom{j}{s}+\binom{r+1}{k+1}\right]\\ &= \sum_{j=0}^{i-1}\left[\sum_{h=1}^{k}\binom{r}{h}c_{k+1-h,1}\binom{j}{1}+\sum_{h=1}^{k-1}\binom{r}{h}c_{k+1-h,2}\binom{j}{2}+\cdots+\binom{r}{1}c_{k,k}\binom{j}{k}\right.\\ &\qquad\left.+\binom{r+1}{k+1}\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{i-1}\left[\sum_{s=1}^{k}\sum_{h=1}^{k+1-s}\binom{r}{h}c_{k+1-h,s}\binom{j}{s}+\binom{r+1}{k+1}\right]\\
&= \sum_{s=1}^{k}\sum_{h=1}^{k+1-s}\binom{r}{h}c_{k+1-h,s}\sum_{j=0}^{i-1}\binom{j}{s}+\sum_{j=0}^{i-1}\binom{r+1}{k+1}\\
&= \sum_{s=1}^{k}\sum_{h=1}^{k+1-s}\binom{r}{h}c_{k+1-h,s}\binom{i}{s+1}+\binom{r+1}{k+1}i\\
&= \sum_{s=2}^{k+1}\sum_{h=1}^{k+1-s+1}\binom{r}{h}c_{k+1-h,s-1}\binom{i}{s}+\binom{r+1}{k+1}i.
\end{aligned}
$$

However, from (9) we get $c_{k+1,s} = \sum_{h=1}^{k+1-s+1}\binom{r}{h}c_{k+1-h,s-1}$ for $k+1 > s \geqslant 2$. Therefore

$$
\sum_{j=0}^{ri}\binom{j}{k} = \sum_{s=2}^{k+1}c_{k+1,s}\binom{i}{s}+\binom{r+1}{k+1}i = P_{r,k+1}(i).
$$

Thus we have proved part A.

Part B. Here we prove that $\sum_{j=1}^{t}\binom{ir+j}{k} = \sum_{h=1}^{k}\binom{t}{h}P_{r,k+1-h}(i)+\binom{t+1}{k+1}$. By (10) and (12) we get

$$
\sum_{j=1}^{t}\binom{ir+j}{k} = \sum_{j=1}^{t}\sum_{s=1}^{ir+j-1}\binom{s}{k-1}.
$$

Then, by induction hypothesis

$$
\begin{aligned}
\sum_{j=1}^{t}\binom{ir+j}{k} &= \sum_{j=1}^{t}\left[P_{r,k}(i)+\sum_{h=1}^{k-1}\binom{j-1}{h}P_{r,k-h}(i)+\binom{j}{k}\right]\\
&= tP_{r,k}(i)+\sum_{h=1}^{k-1}\binom{t}{h+1}P_{r,k-h}(i)+\binom{t+1}{k+1}\\
&= \sum_{h=1}^{k}\binom{t}{h}P_{r,k+1-h}(i)+\binom{t+1}{k+1}.
\end{aligned}
$$

Thus we have proved part B.

Finally, parts A, B and (11) imply the lemma. □

In the following theorem we show that the functions $T_{r_1,r_2,\ldots,r_{n-1}}(x)$ and $Q_{r_1,r_2,\ldots,r_{n-1}}(x)$ can be extended to polynomials in the variable x. For use as the basis of an induction below, since $E(1,x) = \{x\}$ and $D(1,x) = \{0,\ldots,x\}$, we can define $T^1(x) = 1$ and $Q^1(x) = x+1$.

Theorem 4. *Let $r_1, r_2, \ldots, r_{n-1}$ be positive integers. Then*

$$\begin{aligned} T_{r_1,r_2,\ldots,r_{n-1}}(x) &= c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-1}=1}^{i_{n-2}+1} c_{i_{n-2}+1,i_{n-1}}^{r_1} \binom{x}{i_{n-1}} \qquad (13) \\ &\quad + T_{r_1,\ldots,r_{n-2}}(x), \\ Q_{r_1,r_2,\ldots,r_{n-1}}(x) &= c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-1}=1}^{i_{n-2}+1} c_{i_{n-2}+1,i_{n-1}}^{r_1} \binom{x+1}{i_{n-1}+1} \qquad (14) \\ &\quad + Q_{r_1,\ldots,r_{n-2}}(x). \end{aligned}$$

Moreover, both functions $T_{r_1,r_2,\ldots,r_{n-1}}(x)$ and $Q_{r_1,r_2,\ldots,r_{n-1}}(x)$ are polynomials in x of degree $n-1$ and n, respectively.

Proof. From Lemma 2 we get

$$T_{r_1}(x) = \sum_{i_1=0}^{r_1 x} 1 = r_1 x + 1 = c_{1,1}^{r_1}\binom{x}{1} + T^1(x), \quad Q_{r_1}(x) = c_{1,1}^{r_1}\binom{x+1}{2} + x + 1.$$

Thus the lemma holds for $n = 2$. Now suppose that $n > 2$. It follows from Lemma 2 and the induction hypothesis that

$$\begin{aligned} T_{r_1,r_2,\ldots,r_{n-1}}(x) &= \sum_{j_2=0}^{r_1 x} \sum_{j_3=0}^{r_2 j_2} \cdots \sum_{j_n=0}^{r_{n-1} j_{n-1}} 1 = \sum_{j_2=0}^{r_1 x} T_{r_2,\ldots,r_{n-1}}(j_2) \\ &= \sum_{j_2=0}^{r_1 x} c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-2}=1}^{i_{n-3}+1} c_{i_{n-3}+1,i_{n-2}}^{r_2} \binom{j_2}{i_{n-2}} \\ &\quad + \sum_{j_2=0}^{r_1 x} T_{r_2,\ldots,r_{n-2}}(j_2) \\ &= c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-2}=1}^{i_{n-3}+1} c_{i_{n-3}+1,i_{n-2}}^{r_2} \sum_{j_2=0}^{r_1 x} \binom{j_2}{i_{n-2}} \\ &\quad + \sum_{j_2=0}^{r_1 x} T_{r_2,\ldots,r_{n-2}}(j_2). \qquad (15) \end{aligned}$$

However, from Lemma 3 we get

$$\sum_{j_2=0}^{r_1 x} \binom{j_2}{i_{n-2}} = P_{r_1,i_{n-2}+1}(x) = \sum_{i_{n-1}=1}^{i_{n-2}+1} c_{i_{n-2}+1,i_{n-1}}^{r_1} \binom{x}{i_{n-1}}.$$

Using the above identity in (15) we obtain

$$\begin{aligned} & c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-2}=1}^{i_{n-3}+1} c_{i_{n-3}+1,i_{n-2}}^{r_2} \sum_{j_2=0}^{r_1 x} \binom{j_2}{i_{n-2}} \\ = \; & c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-2}=1}^{i_{n-3}+1} c_{i_{n-3}+1,i_{n-2}}^{r_2} \sum_{i_{n-1}=1}^{i_{n-2}+1} c_{i_{n-2}+1,i_{n-1}}^{r_1} \binom{x}{i_{n-1}}. \end{aligned} \tag{16}$$

On the other hand, from (6) we have

$$\sum_{j_2=0}^{r_1 x} T_{r_2,\ldots,r_{n-2}}(j_2). = \sum_{j_2=0}^{r_1 x} \sum_{i_3=0}^{r_2 j_2} \cdots \sum_{i_{n-1}=0}^{r_{n-2} i_{n-2}} 1 = T_{r_1,r_2,\ldots,r_{n-2}}(x). \tag{17}$$

Then (15), (16) and (17) imply relation (13). Clearly, Equation (16) gives a polynomial in x. Moreover, by induction hypothesis $T_{r_1,r_2,\ldots,r_{n-2}}(x)$ is also a polynomial in x. Therefore $T_{r_1,r_2,\ldots,r_{n-2}}(x)$ is a polynomial in x. From (13) it is clear that $i_{n-1} \leqslant n-1$; hence the degree of this polynomial is $n-1$.

Now we show (14). From (6) and (7) we deduce that $Q_{r_1,\ldots,r_{m-1}}(x) = \sum_{y=0}^{x} T_{r_1,\ldots,r_{n-1}}(y)$. Then Using (13), we can see that

$$\begin{aligned} Q_{r_1,\ldots,r_{n-1}}(x) &= \sum_{y=1}^{x} \left[c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-1}=1}^{i_{n-2}+1} c_{i_{n-2}+1,i_{n-1}}^{r_1} \binom{y}{i_{n-1}} + T_{r_1,\ldots,r_{n-2}}(y) \right] \\ &= c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-1}=1}^{i_{n-2}+1} c_{i_{n-2}+1,i_{n-1}}^{r_1} \sum_{y=1}^{x} \left[\binom{y}{i_{n-1}} + T_{r_1,\ldots,r_{n-2}}(y) \right] \\ &= c_{1,1}^{r_{n-1}} \sum_{i_2=1}^{2} c_{2,i_2}^{r_{n-2}} \sum_{i_3=1}^{i_2+1} c_{i_2+1,i_3}^{r_{n-3}} \cdots \sum_{i_{n-1}=1}^{i_{n-2}+1} c_{i_{n-2}+1,i_{n-1}}^{r_1} \binom{x+1}{i_{n-1}+1} + Q_{r_1,\ldots,r_{n-2}}(x). \end{aligned}$$

In a similar way as in the proof of (13), we can show that $Q_{r_1,\ldots,r_{n-1}}(x)$ is a polynomial in x of degree n. □

4 s-Diagonal polynomials on integer sectors

Here, for $n > 1$ we construct recursive 2^{n-1} s-diagonal packing polynomials on multidimensional integer sectors, using the polynomial formula of the function $Q_{r_1,r_2,\ldots,r_{n-1}}$. The recursive construction begins with the s-diagonal identity polynomial on the integer sector I^1.

Let $r_1, r_2, \ldots, r_{n-1}$ be any positive integers and f be a real-valued function on $\mathbf{N}^{n-1}$. We define two operators F and G that transform the function f into two functions on $\mathbf{N}^n$, as follows

$$Ff(x_1, \ldots, x_n) = Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1) + f(x_2, \ldots, x_n), \tag{18}$$

$$Gf(x_1, \ldots, x_n) = Q_{r_1,r_2,\ldots,r_{n-1}}(x_1) - 1 - f(x_2, \ldots, x_n). \tag{19}$$

Let r be a positive integer. By Lemma 1(3), if $\mathcal{I} \in DB^1(1)$, then $\mathcal{I}$ is the identity map on I^1 (=$\mathbf{N}$). Thus by direct calculation, $F\mathcal{I}$ and $G\mathcal{I}$ are the same Nathanson polynomials defined in [6, Theorem 7]. Also [6, Theorem 7] yields that $F\mathcal{I},\ G\mathcal{I} \in DB_r(2)$.

Theorem 5. *Let $r_1, r_2, \ldots, r_{n-1}$ be positive integers. If $n > 1$ and $f \in DB_{r_2,\ldots,r_{n-1}}(n-1)$, then $Ff,\ Gf \in DB_{r_1,r_2,\ldots,r_{n-1}}(n)$.*

Proof. To prove the theorem, we use a double induction on n and x_1. For $n = 2$ and x_1 arbitrary the statement follows from the preceding remark. By definition, $E_{r_1,\ldots,r_{n-1}}(n, 0) = \{(0, \ldots, 0)\}$, so (4) implies that $D_{r_1,\ldots,r_{n-1}}(n, 0) = \{(0, \ldots, 0)\}$. Then the result is true for all $n > 2$ and $x_1 = 0$.

Now assume that $n > 2$ and $x_1 > 0$. Then by induction hypothesis, Ff and Gf are bijections from $D_{r_1,\ldots,r_{n-1}}(n, x_1 - 1)$ onto $\{0, 1, \ldots, -1 + Q_{r_1,\ldots,r_{n-1}}(x_1 - 1)\}$. However, by (3) and (4) we have that $D_{r_1,\ldots,r_{n-1}}(n, x_1)$ is the disjoint union of $D_{r_1,\ldots,r_{n-1}}(n, x_1 - 1)$ and $E_{r_1,r_2,\ldots,r_{n-1}}(n, x_1)$. Hence to complete the proof of the theorem, we need only show that Ff and Gf bijectively map $E_{r_1,r_2,\ldots,r_{n-1}}(n, x_1)$ onto $\{Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1), \ldots, -1 + Q_{r_1,r_2,\ldots,r_{n-1}}(x_1)\}$.

On the other, by the hypothesis, f bijectively maps

$$D_{r_2,\ldots,r_{n-1}}(n - 1, r_1x_1) \text{ onto } \{0, 1, \ldots, -1 + Q_{r_2,\ldots,r_{n-1}}(r_1x_1)\}.$$

Then it is easy to see that $Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1) + f$ and $Q_{r_1,r_2,\ldots,r_{n-1}}(x_1) - 1 - f$ bijectively map $D_{r_2,\ldots,r_{n-1}}(n - 1, r_1x_1)$ onto

$$\{Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1), \ldots, -1 + Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1) + Q_{r_2,\ldots,r_{n-1}}(r_1x_1)\} \text{ and } \tag{20}$$
$$\{-Q_{r_2,\ldots,r_{n-1}}(r_1x_1) + Q_{r_1,r_2,\ldots,r_{n-1}}(x_1), \ldots, -1 + Q_{r_1,r_2,\ldots,r_{n-1}}(x_1)\}, \tag{21}$$

respectively. By (8) we get

$$Q_{r_1,r_2,\ldots,r_{n-1}}(x_1) = Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1) + Q_{r_2,\ldots,r_{n-1}}(r_1x_1), \tag{22}$$
$$Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1) = -Q_{r_2,\ldots,r_{n-1}}(r_1x_1) + Q_{r_1,r_2,\ldots,r_{n-1}}(x_1). \tag{23}$$

Then (20), (21), (22) and (23) imply that $Q_{r_1,r_2,\ldots,r_{n-1}}(x_1-1)+f$ and $Q_{r_1,r_2,\ldots,r_{n-1}}(x_1)-1-f$ are bijections from $D_{r_2,\ldots,r_{n-1}}(n - 1, r_1x_1)$ onto

$$\{Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1), \ldots, -1 + Q_{r_1,r_2,\ldots,r_{n-1}}(x_1)\}. \tag{24}$$

However, by (5), (18), (19) and (24) we deduce that both Af and Bf bijectively map $E_{r_1,r_2,\ldots,r_{n-1}}(n, x_1)$ onto $\{Q_{r_1,r_2,\ldots,r_{n-1}}(x_1 - 1), \ldots, -1 + Q_{r_1,r_2,\ldots,r_{n-1}}(x_1)\}$. This completes the proof. □

Theorem 6. *Let $r_1, r_2, \ldots, r_{n-1}$ be positive integers. If $n > 1$ and $f \in DP_{r_2,\ldots,r_{n-1}}(n - 1)$ then*

$$Ff(1, 0, \ldots, 0) = 1, \qquad Gf(1, 0, \ldots, 0) = Q_{r_1,\ldots,r_{n-1}}(2) - 1. \tag{25}$$

Moreover, if $n > 1$ then Ff and Gf are distinct functions.

Proof. From Theorems 4 and 5 we have that $Ff,\ Gf \in DP_{r_1,r_2,\ldots,r_{n-1}}(n)$. Lemma 1, (18) and (19) yield relations (25). By definition, if $n > 1$, then $Q_{r_1,r_2,\ldots,r_{n-1}}(2) = |D_{r_1,r_2,\ldots,r_{n-1}}(n,1)| \geqslant 3$. This result and (25) imply the last statement. □

Given any positive integers $r_1, r_2, \ldots, r_{n-1}$ with $n > 0$, then by Theorems 4 and 5 we can define a family of s-diagonal degree n polynomials, $QDr_1, r_2, \ldots, r_{n-1}(n)$, on $I(r_1, \ldots, r_{n-1})$ such that $QD^1(1) = DP^1(1)$, and for $n > 1$,

$$QD_{r_1,r_2,\ldots,r_{n-1}}(n) = \{Ff,\ Gf \mid f \in QD_{r_2,\ldots,r_{n-1}}(n-1)\ \}.$$

Note that if f and g are two different functions in $DP_{r_2,\ldots,r_{n-1}}(n-1)$, then by (25), $Ff \neq Gf$. Therefore, it follows from Theorem 6 that $|QD_{r_1,r_2,\ldots,r_{n-1}}(n)| = 2^{n-1}$.

Example. Let r_1 and r_2 be any nonnegative integers and $\mathcal{I} \in DB^1(1)$. By direct calculation, the four s-diagonal polynomials of the set $QD_{r_1,r_2}(3)$ are

$$\begin{aligned}
FF\mathcal{I}(x_1,x_2,x_3) &= r_2r_1^2\binom{x_1}{3} + r_2\binom{r_1+1}{2}\binom{x_1}{2} + r_1\binom{x_1}{2} + x_1 + r_2\binom{x_2}{2} + x_2 + x_3,\\
FG\mathcal{I}(x_1,x_2,x_3) &= r_2r_1^2\binom{x_1}{3} + r_2\binom{r_1+1}{2}\binom{x_1}{2} + r_1\binom{x_1}{2} + x_1 + r_2\binom{x_2+1}{2} + x_2\\
&\quad +1-1-x_3,\\
GF\mathcal{I}(x_1,x_2,x_3) &= r_2r_1^2\binom{x_1+1}{3} + r_2\binom{r_1+1}{2}\binom{x_1+1}{2} + r_1\binom{x_1+1}{2} + x_1 + 1 - 1\\
&\quad -\left[r_2\binom{x_2}{2} + x_2 + x_3\right],\\
GG\mathcal{I}(x_1,x_2,x_3) &= r_2r_1^2\binom{x_1+1}{3} + r_2\binom{r_1+1}{2}\binom{x_1+1}{2} + r_1\binom{x_1+1}{2} + x_1 + 1 - 1\\
&\quad -\left[r_2\binom{x_2+1}{2} + x_2 + 1 - 1 - x_3\right].
\end{aligned}$$

5 s-Diagonal polynomials on $I(1,\ldots,1)$

In this section we study the relation between packing polynomials on $I(1,\ldots,1)$ and on $I(\infty,\ldots,\infty)$. Here $\mathcal{I}$ denotes the identity map on $\mathbf{N}$.

Let f be a real-valued function on $\mathbf{N}^{n-1}$. Morales and Lew [5] defined the operators A and B that transform the function f into two functions on $\mathbf{N}^n$, as follows

$$Af(x_1,\ldots,x_n) = \binom{n-1+x_1+\cdots+x_n}{n} + f(x_2,\ldots,x_n), \quad (26)$$

$$Bf(x_1,\ldots,x_n) = \binom{n+x_1+\cdots+x_n}{n} - 1 - f(x_{n-1},\ldots,x_1). \quad (27)$$

They proved that if f is a diagonal polynomial on $\mathbf{N}^{n-1}$, then both Af and Bf are diagonal polynomials on $\mathbf{N}^n$. In particular, they proved that both $A\mathcal{I}$ and $B\mathcal{I}$ are the same

Cantor polynomial (1). Thus they constructed 2^{n-2} inequivalent diagonal polynomials on $I(\infty,\ldots,\infty)$.

For any positive $n>1$, we define a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ whose matrix with respect to the standard base is

$$\Lambda_n=\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1\\ 0 & 1 & 1 & \ldots & 1 & 1\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1 & 1\\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

It is easy to see that

$$\Lambda_n^{-1}=\begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0\\ 0 & 1 & -1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1 & -1\\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Theorem 7. *If f is a diagonal polynomial on $I(\infty,\ldots,\infty)$ then $f\circ\Lambda_n^{-1}$ is an s-diagonal polynomial on $I(1,\ldots,1)$. Moreover, if f is an s-diagonal polynomial on $I(1,\ldots,1)$, then $f\circ\Lambda_n$ is a diagonal polynomial on $I(\infty,\ldots,\infty)$.*

Proof. Clearly Λ_n^{-1} is a bijection from $I(1,\ldots,1)$ onto $I(\infty,\ldots,\infty)$. Let $\mathbf{x}=(x_1,\ldots,x_n)$, $\mathbf{y}=(y_1,\ldots,y_n)\in I(\infty,\ldots,\infty)$. Then $x_1<y_1$ if and only if $s(\Lambda_n^{-1}(\mathbf{x}))<s(\Lambda_n^{-1}(\mathbf{x}))$. Therefore, if f is a diagonal polynomial on $I(\infty,\ldots,\infty)$, then $f\circ\Lambda_n^{-1}$ is an s-diagonal polynomial on $I(1,\ldots,1)$. The last statement is proved in a similar way. $\square$

Example. The four s-diagonal polynomials on $I(1,1)$ satisfy

$$\begin{aligned}&FF\mathcal{I}(x_1,x_2,x_3)=AA\mathcal{I}\Lambda^{-1}(x_1,x_2,x_3),\ FG\mathcal{I}(x_1,x_2,x_3)=AA\mathcal{I}(1)(23)\Lambda^{-1}(x_1,x_2,x_3)\\ &GF\mathcal{I}(x_1,x_2,x_3)=BA\mathcal{I}(13)(2)\Lambda^{-1}(x_1,x_2,x_3),\\ &GG\mathcal{I}(x_1,x_2,x_3)=BA\mathcal{I}(132)\Lambda^{-1}(x_1,x_2,x_3).\end{aligned}$$

These identities are not hard to prove. For example if $(x_1,x_2,x_3)\in I(1,1)$, then

$$\begin{aligned}BA\mathcal{I}(13)(2)\Lambda^{-1}(x_1,x_2,x_3) &= BA\mathcal{I}(13)(2)(x_1-x_2,x_2-x_3,x_3)\\ &= BA\mathcal{I}(x_3,x_2-x_3,x_1-x_2)\\ &= \binom{3+x_1}{3}-1-A\mathcal{I}(x_2-x_3,x_3)\\ &= \binom{3+x_1}{3}-1-\binom{1+x_2}{2}-x_3=GF\mathcal{I}(x_1,x_2,x_3).\end{aligned}$$

Acknowledgements

The author would like to thank anonymous referees for their suggestions to improve the paper.

References

[1] G. Cantor. Ein Beitrag zur mannigfaltigkeitslehre. *J. Reine Angew. Math. (Crelle's Journal)*, 84:242–258, 1878.

[2] G. Cantor. Beitrage zur begrundung der transfiniter mengenlehre. Math. Ann. 46:481–512, 1885; translated as *Contributions to the Founding of the Theory of Transfinite Numbers*, Dover, New York, 1952.

[3] A.L. Cauchy. Cours d'analyse de l'École Royale Polytechnique, 1re partie: Analyse algébrique, 1'Imprimerie Royale, Paris 1821, reprinted by WissenschFftliche Buchgesellschfft, Darmstadt, pages 140–158, 1968

[4] L.B. Morales. Diagonal polynomials and diagonal orders on multidimensional lattices. *Theory of Comput. Systems*, 30(4):367–382, 1997.

[5] L.B. Morales, and J.S. Lew. An enlarged of family packing polynomials on multidimensional lattices. *Math. Systems Theory*, 29(3):293–303, 1996.

[6] M.B. Nathanson. Cantor polynomials for semigroup sectors. *J. Algebra Appl.* 13(5) 1350165 (14 pages), 2014.

[7] A. L. Rosenberg. Storage mappings for extendible arrays. *Current Trends in Programming Methodology*, Vol. IV, (R. T. Yeh ed.), Prentice–Hall, Englewood Cliffs, NJ, 1978, pp. 263–311.

[8] A. Sanchez-Flores. A family of $(n-1)!$ diagonal polynomials orders on N^n. *Order*, 12(2):173–187, 1997.