Upper Bounds for Stern's Diatomic Sequence and Related Sequences

Colin Defant

Department of Mathematics
University of Florida, U.S.A.

cdefant@ufl.edu

Abstract

Let $(s_2(n))_{n=0}^\infty$ denote Stern's diatomic sequence. For $n \geqslant 2$, we may view $s_2(n)$ as the number of partitions of $n - 1$ into powers of 2 with each part occurring at most twice. More generally, for integers $b, n \geqslant 2$, let $s_b(n)$ denote the number of partitions of $n - 1$ into powers of b with each part occurring at most b times. Using this combinatorial interpretation of the sequences $s_b(n)$, we use the transfer-matrix method to develop a means of calculating $s_b(n)$ for certain values of n. This then allows us to derive upper bounds for $s_b(n)$. In the special case $b = 2$, our bounds improve upon the current upper bounds for the Stern sequence. We then show that

$$\limsup_{n \to \infty} \frac{s_b(n)}{n^{\log_b \phi}} = \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}.$$

Keywords: Stern sequence; Fibonacci number; Lucas number; over-expansion; transfer-matrix method.

1   Introduction

Throughout this paper, F_m and L_m will denote the Fibonacci and the Lucas numbers. We have $F_{m+2} = F_{m+1} + F_m$ and $L_{m+2} = L_{m+1} + L_m$ for all integers m (including negative integers). We convene to use the initial values $F_1 = F_2 = 1$, $L_1 = 1$, and $L_2 = 3$. We also let $\phi = \dfrac{1 + \sqrt{5}}{2}$ and $\overline{\phi} = \dfrac{1 - \sqrt{5}}{2} = \dfrac{-1}{\phi}$. The symbol $\mathbb{N}$ will denote the set of positive integers.

Problem B1 of the 2014 William Lowell Putnam Competition defines a *base* 10 *over-expansion* of a positive integer N to be an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all i. We may generalize (and slightly modify) this notion to obtain the following definition.

Definition 1. Let $b \geqslant 2$ be an integer. A *base b over-expansion* of a positive integer N is a word $d_k d_{k-1} \cdots d_0$ over the alphabet $\{0, 1, \ldots, b\}$ such that $d_k \neq 0$ and $\sum_{i=0}^{k} d_i b^i = N$. We refer to the letter d_i as the ith *digit of the expansion*. It is well-known that each positive integer N has a unique base b over-expansion that does not contain the letter (or digit) b; we refer to this expansion as the *ordinary base b expansion* of N.

The *Stern-Brocot sequence*, also known as *Stern's diatomic sequence* or simply *Stern's sequence*, is defined by the simple recurrence relations

$$s(2n) = s(n) \quad \text{and} \quad s(2n+1) = s(n) + s(n+1)$$

for all nonnegative integers n, where $s(0) = 0$. This sequence has found numerous applications in number theory and combinatorics, and it has several interesting properties which relate it to the Fibonacci sequence. For $n \geqslant 2$, it is well-known that $s(n)$ is the number of base 2 over-expansions (also known as hyperbinary expansions) of $n - 1$ [3]. To generalize Stern's sequence, let $s_b(n)$ denote the number of base b over-expansions of $n - 1$. Equivalently, one may wish to think of $s_b(n)$ as the number of partitions of $n - 1$ into powers of b with each part occurring at most b times. We convene to let $s_b(0) = 0$ and $s_b(1) = 1$. The sequence $s_b(n)$ satisfies the recurrence relations $s_b(bn) = s_b(n)$, $s_b(bn + 1) = s_b(n) + s_b(n + 1)$, and $s_b(bn + i) = s_b(n + 1)$ for $i \in \{2, 3, \ldots, b - 1\}$. Equivalently,

$$s_b(n) = \begin{cases} s_b\left(\frac{n}{b}\right), & \text{if } n \equiv 0 \pmod{b}; \\ s_b\left(\frac{n-1}{b}\right) + s_b\left(\frac{n-1}{b} + 1\right), & \text{if } n \equiv 1 \pmod{b}; \\ s_b\left(\frac{n-i}{b} + 1\right), & \text{if } n \equiv i \pmod{b} \text{ and } 2 \leqslant i < b. \end{cases} \tag{1}$$

Using (1), one may easily prove the following lemma.

Lemma 2. *Let n be a positive integer. If $n \equiv 1 \pmod{b^2}$, then*

$$s_b(n) = s_b\left(\frac{n-1}{b^2}\right) + s_b\left(\frac{n+b-1}{b}\right).$$

If $n \equiv b + 1 \pmod{b^2}$, then

$$s_b(n) = s_b\left(\frac{n-1}{b}\right) + s_b\left(\frac{n+b^2-b-1}{b^2}\right).$$

Calkin and Wilf [2] determined that

$$0.958854\ldots = \frac{3^{\log_2 \phi}}{\sqrt{5}} \leqslant \limsup_{n \to \infty} \frac{s(n)}{n^{\log_2 \phi+}} \leqslant \frac{1 + \phi}{2} = 1.170820\ldots,$$

and they asked for the exact value of $\limsup \dfrac{s(n)}{n^{\log_2 \phi}}$. Here, we give upper bounds for the values of $s_b(n)$ for any integer $b \geqslant 2$, from which we will deduce that

$$\limsup_{n \to \infty} \frac{s_b(n)}{n^{\log_b \phi}} = \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}.$$

In particular, $\limsup\limits_{n \to \infty} \dfrac{s(n)}{n^{\log_2 \phi}} = \dfrac{3^{\log_2 \phi}}{\sqrt{5}}$. While preparing this manuscript, the author discovered that Coons and Tyler [4] had already determined this value of the supremum limit of $\dfrac{s(n)}{n^{\log_2 \phi}}$; in the same paper, they mention that this problem actually dates back to Berlekamp, Conway, and Guy in 1982 [1]. However, the bounds we derive apply to the more general family of sequences $s_b(n)$ and are stronger than those given in [4]. In addition, our methods of proof are quite different from those used in [4]. Coons and Tyler use clever analytic estimates to prove their results from the recursive definition of $s(n)$. By contrast, we will make heavy use of the interpretation of $s_b(n)$ as the number of base b over-expansions of $n-1$ in order to prove several of our most important results. In particular, we will combine this combinatorial interpretation of $s_b(n)$ with the transfer-matrix method in order to prove Theorem 12. In turn, Theorem 12 will allow us to determine the maximum values of $s_b(n)$ when n is restricted to certain intervals.

For each fixed $b \geqslant 2$, the sequence $s_b(n)$ has a somewhat periodic nature. Our upper bounds are local estimates within the periods; they refine the more global estimates given in [4] when $b = 2$.

2   Determining Maximum Values

Throughout this section, fix an integer $b \geqslant 2$. Our goal is to derive upper bounds for the numbers $s_b(n)$, particularly those values of n that are slightly larger than a power of b (we will make this precise soon). To do so, we will make use of the following sequence.

Definition 3. Let $h_1 = h_2 = 1$. For $m \geqslant 3$, let

$$h_m = 1 + \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} b^{m-2-2i}.$$

Alternatively, we may calculate h_m using the recurrence relation

$$h_m = \begin{cases} bh_{m-1} - b + 1, & \text{if } 2 \mid m; \\ bh_{m-1} + 1, & \text{if } 2 \nmid m \end{cases} \tag{2}$$

along with the initial value $h_1 = 1$. For example, $h_3 = b + 1$, $h_4 = b^2 + 1$, $h_5 = b^3 + b + 1$, and $h_6 = b^4 + b^2 + 1$. It is important to note that $h_m \equiv 1 \pmod{b}$ for all $m \in \mathbb{N}$. We state the following lemma for easy reference, although we omit the proof because it is fairly straightforward.

Lemma 4. *For any positive integer m,*

$$h_{m+1} - h_m = \frac{b^m + (-1)^m b}{b+1}.$$

The following lemma lists three simple but useful observations about the numbers $s_b(n)$. We omit the proof because it follows easily from (1).

Lemma 5. *Let n and k be nonnegative integers with $b^k \leqslant n \leqslant b^{k+1}$.*

   i. If $n = jb^k$ for some $j \in \{1, 2, \ldots, b-1\}$, then $s_b(n) = 1$.

   ii. If $k = 1$, then $s_b(n) \leqslant 2$, where equality holds if and only if $n \equiv 1 \pmod{b}$.

   iii. If $k \geqslant 1$ and $n \not\equiv 1 \pmod{b}$, then $s_b(n) = s_b(n')$ for some integer n' with $b^{k-1} \leqslant n' \leqslant b^k$.

Proposition 6. *Let k and n be nonnegative integers with $b^k \leqslant n < b^{k+1}$. We may write n uniquely in the form $n = jb^k + t$, where $j \in \{1, 2, \ldots, b-1\}$ and $t \in \{0, 1, \ldots, b^k - 1\}$. We have $s_b(n) \leqslant F_{k+2}$, where equality holds if and only if $t = h_k$ or $t = h_{k+1}$.*

Proof. That we may write n uniquely in the form $n = jb^k + t$ is trivial. The proof of the rest of the proposition is by induction on k. The case $k = 0$ is immediate from the first part of Lemma 5. The case $k = 1$ follows from the second part of the same lemma. Now, assume $k > 1$. We divide the proof into three cases.

Case 1: In this case, suppose $t = h_k$ or $t = h_{k+1}$. We will assume that $t = h_k$ and that k is even; a similar argument holds if k is odd or $t = h_{k+1}$. Thus, $n = jb^k + h_k$. Because k is even, we may use (2) to write $h_k = bh_{k-1} - b + 1$ and $h_{k-1} = bh_{k-2} + 1$. Furthermore, $h_k \equiv 1 \pmod{b^2}$ because k is even. Since $n \equiv h_k \equiv 1 \pmod{b^2}$, we have by Lemma 2 that

$$s_b(n) = s_b\left(\frac{n-1}{b^2}\right) + s_b\left(\frac{n+b-1}{b}\right) = s_b\left(jb^{k-2} + \frac{h_k - 1}{b^2}\right) + s_b\left(jb^{k-1} + \frac{h_k - 1}{b} + 1\right)$$

$$= s_b\left(jb^{k-2} + \frac{h_{k-1} - 1}{b}\right) + s_b\left(jb^{k-1} + h_{k-1}\right) = s_b\left(jb^{k-2} + h_{k-2}\right) + s_b\left(jb^{k-1} + h_{k-1}\right).$$

By induction on k, $s_b\left(jb^{k-2} + h_{k-2}\right) = F_k$ and $s_b\left(jb^{k-1} + h_{k-1}\right) = F_{k+1}$. Thus, $s_b(n) = F_k + F_{k+1} = F_{k+2}$, as desired.

Case 2: In this case, suppose $n \not\equiv 1 \pmod{b}$. By Lemma 5, there exists an integer n' with $b^{k-1} \leqslant n' \leqslant b^k$ such that $s_b(n) = s_b(n')$. If $n' = b^k$, then, using Lemma 5 again, $s_b(n) = 1 < F_{k+2}$. If $n' < b^k$, then it follows from induction on k that $s_b(n) = s_b(n') \leqslant F_{k+1} < F_{k+2}$.

Case 3: In this final case, suppose $n \equiv 1 \pmod{b}$ and that $t \neq h_k$ and $t \neq h_{k+1}$. Suppose, by way of contradiction, that $s_b(n) \geqslant F_{k+2}$. Since $t \equiv n \equiv 1 \pmod{b}$, we may write $t = bt' + 1$ for some integer $t' < b^{k-1}$. Using (1), we have

$$s_b(n) = s_b\left(\frac{n-1}{b}\right) + s_b\left(\frac{n-1}{b} + 1\right) = s_b(jb^{k-1} + t') + s_b(jb^{k-1} + t' + 1). \qquad (3)$$

If $t' \not\equiv 0, 1 \pmod{b}$, then we know from Lemma 5 and the fact that $b^{k-1} \leqslant jb^{k-1} + t' < jb^{k-1} + t' + 1 \leqslant b^k$ that $s_b(jb^{k-1} + t') = s_b(v_1)$ and $s_b(jb^{k-1} + t' + 1) = s_b(v_2)$ for some integers $v_1, v_2 \in [b^{k-2}, b^{k-1}]$. By (3) and induction on k, it follows that if $t' \not\equiv 0, 1 \pmod{b}$, then

$$s_b(n) = s_b(v_1) + s_b(v_2) = \leqslant F_k + F_k < F_{k+2}.$$

This is a contradiction, so $t' \equiv 0, 1 \pmod{b}$. We will assume that $t' \equiv 0 \pmod{b}$; a similar argument may be used to derive a contradiction in the case $t \equiv 1 \pmod{b}$. Write $t' = bt''$. Because $t' < b^{k-1}$ and $j \leqslant b - 1$, $jb^{k-1} + t' + 1 \leqslant b^k$. We know that $jb^{k-1} + t' + 1 \neq b^k$ because $t' \equiv 0 \pmod{b}$. Therefore, we have the inequalities $b^{k-2} \leqslant jb^{k-2} + t'' < b^{k-1} < jb^{k-1} + t' + 1 < b^k$. We see by (3) and induction that

$$s_b(n) = s_b(jb^{k-1}+t')+s_b(jb^{k-1}+t'+1) = s_b(jb^{k-2}+t'')+s_b(jb^{k-1}+t'+1) \leqslant F_k+F_{k+1}=F_{k+2},$$

where the equality $s_b(jb^{k-1}+t') = s_b(jb^{k-2}+t'')$ is immediate from (1). This last inequality must be an equality since we are assuming $s_b(n) \geqslant F_{k+2}$, so we must have $s_b(jb^{k-2}+t'') = F_k$ and $s_b(jb^{k-1}+t'+1) = F_{k+1}$. The induction hypothesis states that this is only possible if $t'' \in \{h_{k-2}, h_{k-1}\}$ and $t'+1 \in \{h_{k-1}, h_k\}$. Suppose first that $t'' = h_{k-2}$ and $t'+1 = h_{k-1}$. We have $h_{k-1} = bh_{k-2} + 1$, so it follows from (2) that k must be even. Using (2) again, we see that $h_k = bh_{k-1} - b + 1 = b(t'+1) - b + 1 = t$, which contradicts our assumption that $t \neq h_k$. Similarly, if $t'' = h_{k-1}$ and $t'+1 = h_k$, then we may derive the contradiction $t = h_{k+1}$. It is clearly impossible to have $t'' = h_{k-1}$ and $t'+1 = h_{k-1}$ since $t' = bt''$. Therefore, we are left to conclude that $t'' = h_{k-2}$ and $t'+1 = h_k$. If k is even, then we may use (2) to write

$$bt'' = t' = h_k - 1 = bh_{k-1} - b = b(bh_{k-2} + 1) - b = b^2 h_{k-2} = b^2 t'',$$

which is impossible. This means that k must be odd, so $h_k = bh_{k-1} + 1$ by (2). Since $h_k = t' + 1 = bt'' + 1 = bh_{k-2} + 1$, we conclude that $h_{k-1} = h_{k-2}$. It is easy to see from Definition 3 that this is only possible if $k = 3$. Hence, $t'' = h_1 = 1$, $t' = h_3 - 1 = b$, and $t = bt' + 1 = b^2 + 1$. However, this means that $t = h_4 = h_{k+1}$, which is our final contradiction because we assumed $t \notin \{h_k, h_{k+1}\}$. $\qquad\square$

Now that we know the maximum values of $s_b(n)$ for $b^k \leqslant n < b^{k+1}$, we may easily derive the following result.

Corollary 7. *We have*

$$\limsup_{n \to \infty} \frac{s_b(n)}{n^{\log_b \phi}} \geqslant \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}.$$

Proof. We will need Binet's formula for the Fibonacci numbers, which states that

$$F_m = \frac{\phi^m - (-\phi)^{-m}}{\sqrt{5}}$$

for all $m \in \mathbb{Z}$. For each positive integer k, let $u_k = b^k + h_k$. Let $i_k = (b^2 - 1)h_k - b^k$, and observe that

$$i_k = \begin{cases} -1, & \text{if } 2|k; \\ b^2 - b - 1, & \text{if } 2 \nmid k. \end{cases}$$

We have

$$\lim_{k\to\infty} \frac{\phi^{k+2} - (-\phi)^{-(k+2)}}{(b^{k+2} + i_k)^{\log_b \phi}} = \lim_{k\to\infty} \frac{1 - \phi^{-(k+2)}(-\phi)^{-(k+2)}}{\phi^{-(k+2)}(b^{k+2} + i_k)^{\log_b \phi}} = \lim_{k\to\infty} \frac{1 - \phi^{-(k+2)}(-\phi)^{-(k+2)}}{(1 + i_k/b^{k+2})^{\log_b \phi}} = 1.$$

By Proposition 6 and Binet's formula,

$$\frac{s_b(u_k)}{u_k^{\log_b \phi}} = \frac{F_{k+2}}{(b^k + h_k)^{\log_b \phi}} = \frac{\phi^{k+2} - (-\phi)^{-(k+2)}}{(b^k + h_k)^{\log_b \phi}\sqrt{5}} = \frac{\phi^{k+2} - (-\phi)^{-(k+2)}}{(b^k + \frac{b^k + i_k}{b^2 - 1})^{\log_b \phi}\sqrt{5}}$$

$$= \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}} \frac{\phi^{k+2} - (-\phi)^{-(k+2)}}{(b^{k+2} + i_k)^{\log_b \phi}},$$

so

$$\limsup_{n\to\infty} \frac{s_b(n)}{n^{\log_b \phi}} \geqslant \lim_{k\to\infty} \frac{s_b(u_k)}{u_k^{\log_b \phi}} = \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}. \qquad \square$$

We now wish to show that $\limsup\limits_{n\to\infty} \dfrac{s_b(n)}{n^{\log_b \phi}} \leqslant \dfrac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}$. If $b^k + h_k < n < b^{k+1}$ for some positive integer k, then we know from Proposition 6 that

$$\frac{s_b(n)}{n^{\log_b \phi}} \leqslant \frac{F_{k+2}}{n^{\log_b \phi}} = \frac{s_b(b^k + h_k)}{n^{\log_b \phi}} < \frac{s_b(b^k + h_k)}{(b^k + h_k)^{\log_b \phi}}.$$

We saw in the proof of Corollary 7 that

$$\lim_{k\to\infty} \frac{s_b(b^k + h_k)}{(b^k + h_k)^{\log_b \phi}} = \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}.$$

Hence, we need only find a sufficiently strong upper bound for $s_b(n)$ when $b^k \leqslant n \leqslant b^k + h_k$ for some positive integer k. For this purpose, we make the following definitions.

Definition 8. For nonnegative integers k, r, y, let

$$I(k, r, y) = \left\{ n \in \mathbb{N} \colon b^k < n \leqslant b^k + \sum_{i=0}^{y} b^{r-2i} \right\},$$

$$\mu(k, r, y) = \max\{s_b(n) \colon n \in I(k, r, y)\},$$

and

$$\nu(k, r, y) = \min\{n \in I(k, r, y) \colon s_b(n) = \mu(k, r, y)\}.$$

Our goal is to calculate $\mu(k, r, y)$ for any given nonnegative integers k, r, y that satisfy $2y < r < k - 1$. This will allow us to to derive tight upper bounds for all integers n that satisfy $b^k \leqslant n \leqslant b^k + h_k$ for some k by choosing appropriate values of r and y. One might think that a simple inductive argument based on the recurrence relation (1) should be able to derive our upper bounds quite effortlessly. However, the author has found that attempts to prove upper bounds for $s_b(n)$ using induction often fail or become

incredibly convoluted. Indeed, the reader may wish to look at [4] in order to appreciate the surprising amount of ingenuity that Coons and Tyler need for the derivation of their upper bounds (which are weaker than ours) in the specific case $b = 2$. Therefore, we shall prove a sequence of lemmas in order to develop more combinatorial means of calculating $\mu(k, r, y)$ and $\nu(k, r, y)$.

Lemma 9. *Let $a_t a_{t-1} \cdots a_0$ be the ordinary base b expansion of a positive integer n. Let $c_\ell c_{\ell-1} \cdots c_0$ be a base b over-expansion of n. Set $c_i = 0$ for all integers i with $\ell < i \leqslant t$. For any $m \in \{0, 1, \ldots, t\}$,*

$$c_m \in \begin{cases} \{0, b-1, b\}, & \text{if } a_m = 0; \\ \{0, 1, b\}, & \text{if } a_m = 1; \\ \{a_m - 1, a_m\}, & \text{otherwise.} \end{cases}$$

Proof. We know that $\displaystyle\sum_{i=0}^{t} a_i b^i = \sum_{i=0}^{t} c_i b^i$ because $a_t a_{t-1} \cdots a_0$ and $c_\ell c_{\ell-1} \cdots c_0$ are base b over-expansions of the same number. Therefore, for any $j \in \{0, 1, \ldots, t\}$,

$$\sum_{i=0}^{j} a_i b^i \equiv \sum_{i=0}^{j} c_i b^i \pmod{b^{j+1}}. \tag{4}$$

If $\displaystyle\sum_{i=0}^{j} a_i b^i > \sum_{i=0}^{j} c_i b^i$ for some $j \in \{0, 1, \ldots, t\}$, then it follows from (4) and the fact that $a_i \in \{0, 1, \ldots, b-1\}$ for all i that

$$\sum_{i=0}^{j} c_i b^i \leqslant -b^{j+1} + \sum_{i=0}^{j} a_i b^i \leqslant -b^{j+1} + \sum_{i=0}^{j} (b-1) b^i = -1,$$

which is impossible. Hence, $\displaystyle\sum_{i=0}^{j} a_i b^i \leqslant \sum_{i=0}^{j} c_i b^i$ for all $j \in \{0, 1, \ldots, t\}$. Choose some $m \in \{0, 1, \ldots, t\}$. Since $c_i \leqslant b$ for all i, we have

$$\sum_{i=0}^{m} c_i b^i \leqslant c_m b^m + \sum_{i=0}^{m-1} b \cdot b^i = (c_m + 1) b^m + \sum_{i=0}^{m-2} b^{i+1} < (c_m + 2) b^m. \tag{5}$$

It follows that

$$a_m b^m \leqslant \sum_{i=0}^{m} a_i b^i \leqslant \sum_{i=0}^{m} c_i b^i < (c_m + 2) b^m,$$

so $c_m \geqslant a_m - 1$. To complete the proof, we simply need to show that if $c_m > a_m$, then either $a_m = 0$ and $b_m \in \{b-1, b\}$ or $a_m = 1$ and $c_m = b$. Suppose $c_m > a_m$. We have

$$\sum_{i=0}^{m} a_i b^i = a_m b^m + \sum_{i=0}^{m-1} a_i b^i \leqslant a_m b^m + \sum_{i=0}^{m-1} c_i b^i < \sum_{i=0}^{m} c_i b^i$$

and

$$\sum_{i=0}^{m} a_i b^i \equiv \sum_{i=0}^{m} c_i b^i \pmod{b^{m+1}}$$

(by (4)), so we know from (5) that

$$\sum_{i=0}^{m} a_i b^i \leqslant -b^{m+1} + \sum_{i=0}^{m} c_i b^i < -b^{m+1} + (c_m + 2)b^m.$$

This implies that $a_m b^m < -b^{m+1} + c_m b^m + 2b^m$, so $c_m - a_m > b - 2$. This is impossible if $a_m \notin \{0, 1\}$ because $c_m \leqslant b$. If $a_m = 0$, we must have either $c_m = b - 1$ or $c_m = b$, and if $a_m = 1$, then we must have $c_m = b$. $\qquad\square$

Lemma 10. *Let k, r, y be nonnegative integers with $2y < r < k - 1$. If $a_t a_{t-1} \cdots a_0$ is the ordinary base b expansion of $\nu(k, r, y) - 1$, then $t = k$, $a_k = 1$, $a_j = a_0 = 0$ for all $j \in \{r + 1, \ldots, k - 1\}$, and $a_i \in \{0, 1\}$ for all $i \in \{0, 1, \ldots, k\}$.*

Proof. Let $\nu = \nu(k, r, y)$, and let $a_t a_{t-1} \cdots a_0$ be the ordinary base b expansion of $\nu - 1$. It follows from the fact that $\nu \in I(k, r, y)$ that $t = k$, $a_k = 1$, and $a_j = 0$ for all $j \in \{r + 1, \ldots, k - 1\}$. We still need to show that $a_0 = 0$ and $a_i \in \{0, 1\}$ for all $i \in \{0, 1, \ldots, k\}$. Suppose, for the sake of finding a contradiction, that $a_m \in \{2, 3, \ldots, b - 1\}$ for some $m \in \{0, 1, \ldots, k\}$. Because $a_k = 1$, we know that $m \in \{0, 1, \ldots, k - 1\}$. Let $\nu' = \nu - (a_m - 1)b^m$. The ordinary base b expansion of $\nu' - 1$ is simply the word obtained from $a_k a_{k-1} \cdots a_0$ by replacing the mth digit (which is a_m) with 1. Observe that $\nu' < \nu$ and $\nu' \in I(k, r, y)$ since

$$b^k < b^k + b^m = b^k + a_m b^m - (a_m - 1)b^m \leqslant \nu - (a_m - 1)b^m = \nu'.$$

Hence, by the definition of ν in Definition 8, $s_b(\nu') < s_b(\nu)$. Choose some base b over-expansion $c_\ell c_{\ell-1} \cdots c_0$ of $\nu - 1$, and let $f(c_\ell c_{\ell-1} \cdots c_0)$ be the word obtained from $c_\ell c_{\ell-1} \cdots c_0$ by replacing the mth digit (which is c_m) with $c_m - a_m + 1$. Lemma 9 implies that $c_m - a_m + 1 \in \{0, 1\}$ because $c_m \in \{a_m - 1, a_m\}$. Therefore, $f(c_\ell c_{\ell-1} \cdots c_0)$ is a base b over-expansion of $\nu' - 1$. We see that f is an injection from the set of base b over-expansions of $\nu - 1$ to the set of base b over-expansions of $\nu' - 1$. This contradicts the fact that $s_b(\nu') < s_b(\nu)$, so we conclude that $a_i \in \{0, 1\}$ for all $i \in \{0, 1, \ldots, k\}$.

We are left with the task of showing that $a_0 = 0$. Suppose $a_0 \neq 0$. By the preceding paragraph, we must have $a_0 = 1$. This means that $\nu \equiv 2 \pmod{b}$, so

$$s_b(\nu) = s_b\left(\frac{\nu - 2}{b} + 1\right) < s_b\left(\frac{\nu - 2}{b}\right) + s_b\left(\frac{\nu - 2}{b} + 1\right) = s_b(\nu - 1)$$

by (1). Since $\nu \equiv 2 \pmod{b}$ and $\nu > b^k$ by definition, we know that $\nu - 1 > b^k$. Therefore, $\nu - 1 \in I(k, r, y)$ and $s_b(\nu) < s_b(\nu - 1)$, which contradicts the definition of ν. It follows from this contradiction that $a_0 = 0$. $\qquad\square$

Lemma 10 hints that it is of interest to enumerate the base b over-expansions of positive integers whose ordinary base b expansions use only the digits 0 and 1. Let $e_0, e_1, \ldots, e_\ell$ be nonnegative integers with $e_0 < e_1 < \cdots < e_\ell$, and let $n = b^{e_0} + b^{e_1} + \cdots + b^{e_\ell}$. Let Q_i denote the operation that changes one base b over-expansion of n into another by increasing the ith digit of an expansion by b and decreasing the $(i+1)^{th}$ digit of the expansion by 1. The operation Q_i can only be used if the ith digit of the expansion under consideration is a 0. If we use the operation Q_i when the $(i+1)^{th}$ digit of the expansion is a 0, then this digit is immediately converted to a $b-1$ and the $(i+2)^{th}$ digit is reduced by 1. If the $(i+2)^{th}$ digit is also 0, then it is immediately converted to a $b-1$ and the $(i+3)^{th}$ digit is reduced by 1. This process continues until a nonzero digit is reduced by 1. After using the operation Q_i, any leading 0's are erased from the expansion. The transformation of 0's to $(b-1)$'s is analogous to the transformation of 0's to 9's that occurs when the number 1 is subtracted from the number 10000, resulting in 9999. As an example, only the operations Q_2, Q_4, Q_5, Q_6, and Q_8 may be used to transform the expansion 1010001011 into a new expansion. The operation Q_4 changes the expansion 1010001011 into the expansion $100(b-1)(b-1)b1011$. The operation Q_8 first changes the expansion 1010001011 into $0b10001011$; the leading 0 is then erased, yielding $b10001011$.

We will transform the ordinary base b expansion of n into a base b over-expansion $c_t c_{t-1} \cdots c_0$ of n by repeated use of the operations just described. At each step, we will choose the value of c_{e_i} for some $i \in \{0, 1, \ldots, \ell\}$ while simultaneously deciding the values of c_j for all $j \in \{e_{i-1}+1, e_{i-1}+2, \ldots, e_i-1\}$ (or all $j < e_i$ in the case $i = 0$). We proceed by permanently deciding the values of $c_0, c_1, \ldots, c_{e_0}$, then permanently deciding the values of $c_{e_0+1}, c_{e_0+2}, \ldots, c_{e_1}$, and so on until we decide the values of $c_{e_{\ell-1}+1}, c_{e_{\ell-1}+2}, \ldots, c_{e_\ell}$. We omit c_{e_ℓ} from the word $c_t c_{t-1} \cdots c_0$ if $c_{e_\ell} = 0$; in this case, $t = e_\ell - 1$. For the sake of providing a concrete example of this process, we will suppose that $\{e_0, e_1, \ldots, e_\ell\} = \{2, 4, 5, 9, 11\}$ and $b = 7$. Here, the ordinary base 7 expansion of n is 101000110100. Since $e_0 = 2$, we first choose the value of c_2 (while simultaneously deciding the values of c_0 and c_1). There is only one way to set $c_2 = 1$; namely, we keep the same expansion 101000110100. If we want to have $c_2 = 0$, then we could either perform the operation Q_0 to get the expansion 101000110067 or perform the operation Q_1 to get the expansion 101000110070. Similarly, if we want to have $c_2 = 7$, then we could either perform the operations Q_0 and then Q_2 to obtain 101000106767 or perform the operations Q_1 and then Q_2 to get 101000106770. That is, there are $e_0 = 2$ ways to set $c_2 = 0$ and $e_0 = 2$ ways to set $c_2 = 7$. By Lemma 9, $0, 1$, and 7 are the only possible values of c_2. For this example, we will assume we chose to use the operations Q_1 and Q_2 to get the expansion 101000106770. Next, we choose the value of c_4 because $e_1 = 4$. Because we used the operation Q_2, the 4th digit of the expansion was temporarily converted into a 0. This means that there is no way to set $c_4 = 1$ using the operations Q_i. There is one way to set $c_4 = 0$; just keep the expansion 101000106770. Similarly, there is one way to set $c_4 = 7$; just use the operation Q_4 to obtain 101000076770. We will assume that we make the former choice and keep the expansion 101000106770. Next, we choose the value of c_5 since $e_2 = 5$. The only way to set $c_5 = 1$ is to keep the expansion the same. Since we have already determined c_0, c_1, c_2, c_3, c_4, we cannot perform any of the operations Q_0, Q_1, Q_2, Q_3, Q_4 in order to
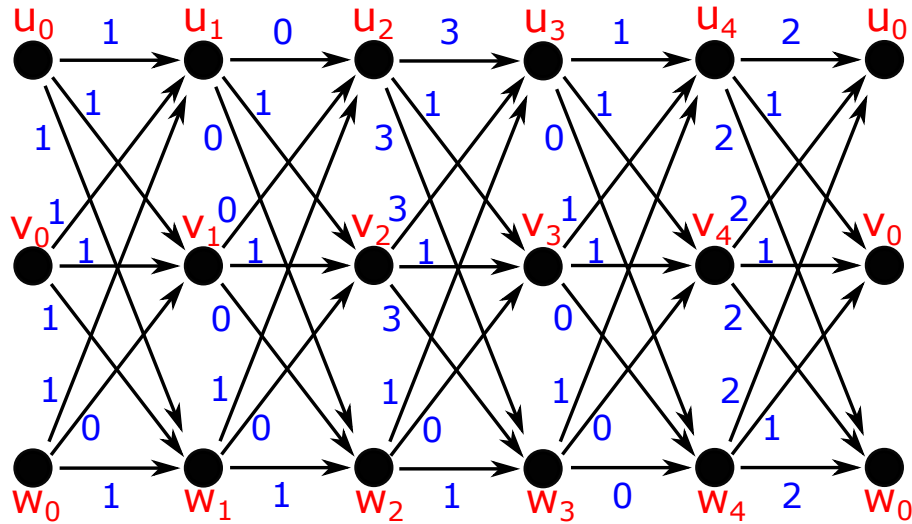
Figure 1: The edge-weighted digraph G for the given example. Note that each of the vertices u_0, v_0, w_0 is drawn twice.

temporarily reduce the 5th digit of the expansion by 1. In other words, c_5 is stuck with the value 1. We now have the expansion 101000106770, and we wish to choose the value of c_9 (since $e_3 = 9$) while simultaneously determining the values of c_6, c_7, c_8. Again, the only way to make $c_9 = 1$ is to keep the expansion 101000106770. To make $c_9 = 0$, we could use Q_6 to get 100667106770, use Q_7 to get 100670106770, or use Q_8 to get 100700106770. To set $c_9 = 7$, we could use Q_6 and then Q_9 to obtain 67667106770, use Q_7 and then Q_9 to get 67670106770, or use Q_8 and then Q_9 to get 67700106770. Hence, there are $e_3 - e_2 - 1 = 3$ ways to set $c_9 = 0$ and $e_3 - e_2 - 1 = 3$ ways to set $c_9 = 7$. We will assume that we choose to use Q_7 (and not Q_9) to obtain the expansion 100670106770. All that is left to do is decide the value of c_{11} (while simultaneously determining c_{10}). We cannot set $c_{11} = 7$ because there is no nonzero digit to the left of the 11th digit from which to "borrow." Alternatively, one could observe that if $c_{11} = 7$, then $c_{11}c_{10}\cdots c_0$ would be a base 7 over-expansion for a number strictly larger than n. We can, however, keep the expansion 100670106770 if we wish to set $c_{11} = 1$. There is one way (because $e_4 - e_3 - 1 = 1$) to set $c_{11} = 0$; simply use the operation Q_{10} to obtain the expansion 70670106770.

Figure 1 depicts an edge-weighted digraph G which encodes all the possible choices that we could have made in this example. The graph has only fifteen vertices, but we have drawn each of the vertices u_0, v_0, w_0 twice in order to improve the aesthetics of the image. The vertex u_i corresponds to choosing to set $c_{e_i} = 0$. The vertex v_i corresponds to setting $c_{e_i} = 1$. The vertex w_i corresponds to setting $c_{e_i} = 7$. The weights of the edges correspond to the number of choices possible. For example, if we have chosen to let $c_5 = 0$ (corresponding to the vertex u_2), then there are three ways to set $c_9 = 7$ (corresponding to the vertex w_3). Thus, there is an edge of weight 3 from u_2 to w_3. After setting $c_9 = 7$, it is impossible to set $c_{11} = 1$, so there is an edge of weight 0 from w_3 to v_4. The reader might ask why there are edges from u_4, v_4, w_4 to u_0, v_0, w_0. We include these edges because

we wish to interpret base 7 over-expansion of n in terms of closed walks in the graph G. There are edges of weight 2 from u_4, v_4, w_4 to u_0 because there are two ways to set $c_2 = 0$, regardless of the value of c_{11} (recall that we choose c_2 before choosing c_{11}). Similarly, there are edges of weight 1 from u_4, v_4, w_4 to v_0 and edges of weight 2 from u_4, v_4, w_4 to w_0.

Suppose we want to construct a base 7 over-expansion of n in which $c_2 = 0$, $c_4 = 1$, $c_5 = 1$, $c_9 = 7$, and $c_{11} = 0$. This choice of the values of c_{e_i} for $i \in \{0, 1, 2, 3, 4\}$ corresponds to the closed walk $(u_0, v_1, v_2, w_3, u_4, u_0)$ in G. The weight of this walk (which we calculate as the product of the weights of its edges) is 6, so there are 6 base 7 over-expansions of n with these specific values of $c_2, c_4, c_5, c_9, c_{11}$. As another example, there are no base 7 over-expansions of n in which $c_2 = 7$, $c_4 = 7$, $c_5 = 1$, $c_9 = 0$, and $c_{11} = 1$ because the weight of the closed walk $(w_0, w_1, v_2, u_3, v_4, w_0)$ is 0.

We are finally in a position to enumerate the base 7 over-expansions of n. To do so, we will need the following definition.

Definition 11. For any real t, let

$$M_t = \begin{pmatrix} t & 1 & t \\ t & 1 & t \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N_t = \begin{pmatrix} t & 1 & t \\ t & 1 & t \\ 0 & 0 & 0 \end{pmatrix}.$$

If we put the vertices of G in the order $u_0, v_0, w_0, u_1, v_1, w_1, \ldots, u_4, v_4, w_4$, then the adjacency matrix of G (written as a block matrix) is

$$A = \begin{pmatrix} O & M_{d_1} & O & O & O \\ O & O & M_{d_2} & O & O \\ O & O & O & M_{d_3} & O \\ O & O & O & O & B \\ C & O & O & O & O \end{pmatrix},$$

where $d_i = e_i - e_{i-1} - 1$,

$$B = \begin{pmatrix} e_\ell - e_{\ell-1} - 1 & 1 & 0 \\ e_\ell - e_{\ell-1} - 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} e_0 & 1 & e_0 \\ e_0 & 1 & e_0 \\ e_0 & 1 & e_0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix},$$

and O denotes the 3×3 zero matrix. The total number of base 7 over-expansions of n is equal to the sum of the weights of the closed walks of length 5 in G that start at u_0, v_0, or w_0. This, in turn, is equal to the sum of the first three diagonal entries of A^5. Using elementary linear algebra, we see that the first three rows and the first three columns of A^5 intersect in a 3×3 block given by $M_1 M_2 M_3 BC$. Therefore, the sum of the first three diagonal entries of A^5 is $\text{Tr}(M_1 M_2 M_3 BC)$, where Tr denotes the trace of a matrix. One easily calculates this value to be 158.

We now state this result in greater generality (and in a slightly different form). Sketching the proof of the following theorem, we trust the reader to see that the method used in the preceding example is representative of the method used in general.

Theorem 12. *Let $e_0, e_1, \ldots, e_\ell$ be nonnegative integers with $e_0 < e_1 < \cdots < e_\ell$ (where $\ell \geqslant 1$). Let $d_i = e_i - e_{i-1} - 1$ for all $i \in \{1, 2, \ldots, \ell\}$. The number of base b over-expansions of the number $b^{e_0} + b^{e_1} + \cdots + b^{e_\ell}$ is given by*

$$\mathrm{Tr}(N_{e_0} M_{d_1} M_{d_2} \cdots M_{d_\ell}).$$

Proof. Let $n = b^{e_0} + b^{e_1} + \cdots + b^{e_\ell}$. Let

$$A = \begin{pmatrix} O & M_{d_1} & O & \cdots & O & O & O \\ O & O & M_{d_2} & \cdots & O & O & O \\ O & O & O & \cdots & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O & M_{d_{\ell-1}} & O \\ O & O & O & \cdots & O & O & B \\ C & O & O & \cdots & O & O & O \end{pmatrix},$$

where

$$B = \begin{pmatrix} d_\ell & 1 & 0 \\ d_\ell & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} e_0 & 1 & e_0 \\ e_0 & 1 & e_0 \\ e_0 & 1 & e_0 \end{pmatrix}.$$

Let G be the edge-weighted digraph with vertex set $\{u_0, v_0, w_0, u_1, v_1, w_1, \ldots, u_\ell, v_\ell, w_\ell\}$ and adjacency matrix A. Suppose we wish to construct a base b over-expansion $c_t c_{t-1} \cdots c_0$ of n. As in the example above, specifying the values of $c_{e_0}, c_{e_1}, \ldots, c_{e_\ell}$ (each such value must be 0, 1, or b by Lemma 9) corresponds to choosing a closed walk of length $\ell + 1$ in G starting at u_0, v_0, or w_0. The weight of this walk is the number of base b over-expansions $c_{e_\ell} c_{e_\ell - 1} \cdots c_0$ of n that have the specified values of $c_{e_0}, c_{e_1}, \ldots, c_{e_\ell}$. Therefore, the total number of base b over-expansions of n is equal to the sum of the first three diagonal entries of $A^{\ell+1}$. The first three rows and the first three columns of $A^{\ell+1}$ intersect in the 3×3 block $M_{d_1} M_{d_2} \cdots M_{d_{\ell-1}} BC$, so the number of base b over-expansions of n is $\mathrm{Tr}(M_{d_1} M_{d_2} \cdots M_{d_{\ell-1}} BC)$. Now, one may easily calculate that $BC = M_{d_\ell} N_{e_0}$. Therefore, using the fact that $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ for any square matrices X, Y of the same size, we conclude that

$$\mathrm{Tr}(M_{d_1} M_{d_2} \cdots M_{d_{\ell-1}} BC) = \mathrm{Tr}(M_{d_1} M_{d_2} \cdots M_{d_{\ell-1}} M_{d_\ell} N_{e_0}) = \mathrm{Tr}(N_{e_0} M_{d_1} M_{d_2} \cdots M_{d_\ell}). \quad \square$$

Definition 13. We define $\Xi = \left\{ \begin{pmatrix} x_1 & z & x_1 \\ x_1 & z & x_1 \\ x_2 & x_3 & x_2 \end{pmatrix} : x_1, x_2, x_3, z \geqslant 0, z \neq 0 \right\}$ and $\Xi' = \Xi \cup \{I_3\}$, where I_3 is the 3×3 identity matrix.

We omit the proofs of the following three lemmas because they are fairly straightforward.

Lemma 14. *The set Ξ is closed under matrix multiplication, and Ξ' is a monoid under matrix multiplication. For any nonnegative real number u, $M_u, N_u \in \Xi$.*

Lemma 15. *Let X be a 3×3 matrix with nonnegative real entries, and let $Y \in \Xi'$. If the second diagonal entry (the entry in the second row and the second column) of X is positive, then $\operatorname{Tr}(XY) > 0$.*

Lemma 16. *For any positive integer t,*

$$M_1^t = \begin{pmatrix} F_{2t} & F_{2t-1} & F_{2t} \\ F_{2t} & F_{2t-1} & F_{2t} \\ F_{2t-1} & F_{2t-2} & F_{2t-1} \end{pmatrix}.$$

The following lemma attempts to gain information about the ordinary base b expansion of $\nu(k, r, y) - 1$ when $2y < r < k - 1$. By Lemma 10, all of the digits in that expansion are 0's and 1's, so we may write $\nu(k, r, y) - 1 = b^{e_0} + b^{e_1} + \cdots + b^{e_\ell}$ for some nonnegative integers $e_0, e_1, \ldots, e_\ell$ that satisfy $e_0 < e_1 < \cdots < e_\ell = k$. Because $\nu(k, r, y) \in I(k, r, y)$, we have

$$b^k \leqslant \nu(k, r, y) - 1 < b^k + \sum_{i=0}^{y} b^{r-2i}.$$ It follows that $e_{\ell-1-j} \leqslant r - 2j$ for all $j \in \{0, 1, \ldots, y\}$, where equality cannot hold for all j. Hence, we may consider the smallest nonnegative integer λ such that $e_{\ell-1-\lambda} \leqslant r - 2\lambda - 1$.

Lemma 17. *Let k, r, y be nonnegative integers with $2y < r < k - 1$. Let $a_k a_{k-1} \cdots a_0$ be the ordinary base b expansion of $\nu(k, r, y) - 1$, and write $\nu(k, r, y) - 1 = b^{e_0} + b^{e_1} + \cdots + b^{e_\ell}$, where $e_0, e_1, \ldots, e_\ell$ are nonnegative integers that satisfy $e_0 < e_1 < \cdots < e_\ell = k$. Let λ be the smallest nonnegative integer such that $e_{\ell-1-\lambda} \leqslant r - 2\lambda - 1$. There does not exist $i \in \{1, 2, \ldots, e_{\ell-1-\lambda}\}$ such that $a_i = a_{i+1}$.*

Proof. Let $\nu = \nu(k, r, y)$. Note that it follows from the preceding paragraph that $\lambda \leqslant y$. Because $\nu - 1 = b^{e_0} + b^{e_1} + \cdots + b^{e_\ell}$, we have

$$a_i = \begin{cases} 1, & \text{if } i = e_j \text{ for some } j; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

for all $i \in \{0, 1, \ldots, k\}$. In particular, $e_0 > 0$ because $a_0 = 0$ by Lemma 10. We will let $d_j = e_j - e_{j-1} - 1$ for all $j \in \{1, 2, \ldots, \ell\}$. Suppose, by way of contradiction, that $a_i = a_{i+1}$ for some $i \in \{1, 2, \ldots, e_{\ell-1-\lambda}\}$, and let m be the smallest such index i. We have five cases to consider.

First, assume $a_1 = 1$ and $a_m = a_{m+1} = 0$. By the minimality of m, we see that $a_i = 0$ for all nonnegative even integers $i < m$ and $a_j = 1$ for all positive odd integers $j < m$. In addition, m must be even. Therefore, setting $t = \dfrac{m-2}{2}$, we have $e_j = 2j + 1$ for all $j \in \{0, 1, \ldots, t\}$. This means that $d_j = 1$ for all $j \in \{1, 2, \ldots, t\}$. It follows from (6) and the assumption that $a_m = a_{m+1} = 0$ that

$$e_t = m - 1 < m < m + 1 < e_{t+1} < e_{t+2} < \cdots < e_\ell = k. \tag{7}$$

Note that $t < \ell - 1 - \lambda$ because $e_t < m \in \{1, 2, \ldots, e_{\ell-1-\lambda}\}$. Let us write $\nu_1 = 1 + \sum_{i=0}^{t} b^{e_i+1} + \sum_{i=t+1}^{\ell} b^{e_i}$. We have

$$\nu_1 = 1 + \sum_{i=0}^{t} b^{e_i+1} + \sum_{i=t+1}^{\ell} b^{e_i} \leqslant 1 + \sum_{n=1}^{m} b^n + \sum_{i=t+1}^{\ell} b^{e_i} \leqslant 1 + \sum_{n=1}^{e_{t+1}-2} b^n + \sum_{i=t+1}^{\ell} b^{e_i}$$

$$< b^{e_{t+1}-1} + \sum_{i=t+1}^{\ell} b^{e_i} = b^{e_{t+1}-1} + \sum_{i=t+1}^{\ell-1-\lambda} b^{e_i} + \sum_{i=\ell-\lambda}^{\ell} b^{e_i}.$$

Recall from the paragraph immediately preceding this theorem that $e_{\ell-1-j} \leqslant r - 2j$ for all $j \in \{0, 1, \ldots, y\}$. Since λ is the smallest nonnegative integer such that $e_{\ell-1-\lambda} \leqslant r - 2\lambda - 1$, $e_{\ell-1-i} = r - 2i$ for all nonnegative integers $i < \lambda$. Furthermore, $e_\ell = k$, so we find that $\sum_{i=\ell-\lambda}^{\ell} b^{e_i} = b^k + \sum_{i=0}^{\lambda-1} b^{r-2i}$. Therefore,

$$\nu_1 < b^{e_{t+1}-1} + \sum_{i=t+1}^{\ell-1-\lambda} b^{e_i} + b^k + \sum_{i=0}^{\lambda-1} b^{r-2i} < b^{e_{\ell-1-\lambda}+1} + b^k + \sum_{i=0}^{\lambda-1} b^{r-2i}$$

$$\leqslant b^k + b^{r-2\lambda} + \sum_{i=0}^{\lambda-1} b^{r-2i} \leqslant b^k + \sum_{i=0}^{y} b^{r-2i},$$

where we have used the inequalities $e_{\ell-1-\lambda} \leqslant r - 2\lambda - 1$ and $\lambda \leqslant y$. This shows that $\nu_1 \in I(k, r, y)$, so $s_b(\nu_1) \leqslant s_b(\nu)$ by the definition of ν. Our goal is to derive a contradiction by showing that $s_b(\nu_1) > s_b(\nu)$. Let $B = M_{d_{t+2}} M_{d_{t+3}} \cdots M_{d_\ell}$. We saw that $t < \ell - 1 - \lambda$, so $t + 1 < \ell$. This shows that the matrix product defining B is nonempty. Lemma 14 implies that $B \in \Xi$. Now, $e_0 = 2(0) + 1 = 1$. By Theorem 12,

$$s_b(\nu) = \text{Tr}(N_1 M_{d_1} M_{d_2} \cdots M_{d_\ell}) = \text{Tr}(N_1 M_1^t M_{d_{t+1}} B).$$

We may write $\nu_1 - 1 = b^{e_0'} + b^{e_1'} + \ldots + b^{e_\ell'}$, where $e_j' = e_j + 1$ for all $j \in \{0, 1, \ldots, t\}$ and $e_j' = e_j$ for all $j \in \{t+1, \ldots, \ell\}$. Setting $d_j' = e_j' - e_{j-1}'$ for all $j \in \{1, 2, \ldots, \ell\}$, we have

$$s_b(\nu_1) = \text{Tr}(N_{e_0'} M_{d_1'} M_{d_2'} \cdots M_{d_\ell'}) = \text{Tr}(N_2 M_1^t M_{d_{t+1}-1} B)$$

by Theorem 12. Consequently,

$$s_b(\nu_1) - s_b(\nu) = \text{Tr}((N_2 M_1^t M_{d_{t+1}-1} - N_1 M_1^t M_{d_{t+1}}) B).$$

We remark that one must take care when considering the case $t = 0$. In this case, M_1^t is the 3×3 identity matrix. A straightforward calculation invoking Lemma 16 shows that

$$N_2 M_1^t M_{d_{t+1}-1} - N_1 M_1^t M_{d_{t+1}} = F_{2t+2} N_{d_{t+1}-2},$$

so
$$s_b(\nu_1) - s_b(\nu) = F_{2t+2} \operatorname{Tr}(N_{d_{t+1}-2}B).$$

Because $d_{t+1} = e_{t+1} - e_t - 1 \geqslant (m+2) - (m-1) - 1 = 2$ by (7), we see that $N_{d_{t+1}-2}$ is a matrix with nonnegative entries whose second diagonal entry is positive. By Lemma 15, $s_b(\nu_1) - s_b(\nu) > 0$, which is our desired contradiction.

Next, we assume $a_1 = 1$ and $a_m = a_{m+1} = 1$. The proof is similar to that given in the preceding paragraph. It follows from the minimality of m that $a_i = 0$ for all even nonnegative integers $i < m$ and $a_j = 1$ for all odd positive integers $j < m$. Also, m must be odd. Let $t = \dfrac{m-1}{2}$. We have $e_j = 2j+1$ for all $j \in \{0, 1, \ldots, t\}$ and

$$e_t = m < m+1 = e_{t+1} < e_{t+2} < \cdots < e_\ell = k.$$

We know from Lemma 10 that $a_j = 0$ for all $j \in \{r+1, \ldots, k-1\}$, so $m < r$. This implies that $e_{t+1} = m+1 \leqslant r < k = e_\ell$. Setting $\nu_2 = 1 + \sum\limits_{i=0}^{t-1} b^{e_i+1} + \sum\limits_{i=t+1}^{\ell} b^{e_i}$, we have

$$\nu_2 - 1 = \sum_{i=0}^{t-1} b^{e_i+1} + \sum_{i=t+1}^{\ell} b^{e_i} = \sum_{i=0}^{t-1} b^{e_{i+1}-1} + \sum_{i=t+1}^{\ell} b^{e_i}$$

$$= \sum_{i=1}^{t} b^{e_i-1} + \sum_{i=t+1}^{\ell} b^{e_i} < \nu - 1 < b^k + \sum_{i=0}^{y} b^{r-2i},$$

so $\nu_2 \in I(k, r, y)$ and $\nu_2 < \nu$. It follows from the definition of ν that $s_b(\nu_2) < s_b(\nu)$. As in the preceding case, we let $B = M_{d_{t+2}} M_{d_{t+3}} \cdots M_{d_\ell}$. The matrix product defining B is nonempty because $e_{t+1} < e_\ell$. In addition, $B \in \Xi$, so we may write

$$B = \begin{pmatrix} \xi_1 & \xi_2 & \xi_1 \\ \xi_1 & \xi_2 & \xi_1 \\ \xi_3 & \xi_4 & \xi_3 \end{pmatrix}$$

for some nonnegative real $\xi_1, \xi_2, \xi_3, \xi_4$ with $\xi_2 > 0$. Since $d_{t+1} = e_{t+1} - e_t - 1 = (m+1) - m - 1 = 0$, we have by Theorem 12 that

$$s_b(\nu) = \operatorname{Tr}(N_1 M_1^t M_0 B) \quad \text{and} \quad s_b(\nu_2) = \operatorname{Tr}(N_2 M_1^t B).$$

As in the previous paragraph, note that M_1^t is the 3×3 identity matrix if $t = 0$. By elementary calculations that make use of Lemma 16, we find that

$$N_2 M_1^t - N_1 M_1^t M_0 = F_{2t+1} \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,
$$s_b(\nu_2) - s_b(\nu) = \operatorname{Tr}((N_2 M_1^t - N_1 M_1^t M_0)B) = F_{2t+1}(\xi_3 + \xi_4) \geqslant 0,$$

which is a contradiction.

We now assume $a_1 = 0$, $a_2 = 1$, and $a_m = a_{m+1} = 0$. Invoking the minimality of m, we see that $a_i = 1$ for all positive even integers $i < m$ and $a_j = 1$ for all positive odd integers $j < m$. The index m must be odd. Let $t = \dfrac{m-3}{2}$. We find that $e_j = 2j + 2$ for all $j \in \{0, 1, \ldots, t\}$, so $d_j = 1$ for all $j \in \{1, 2, \ldots, t\}$. Coupling (6) with the assumption that $a_m = a_{m+1} = 0$ shows us that

$$e_t = m - 1 < m < m + 1 < e_{t+1} < e_{t+2} < \cdots < e_\ell = k. \tag{8}$$

Thus, $t < \ell - 1 - \lambda$ because $e_t < m \in \{1, 2, \ldots, e_{\ell-1-\lambda}\}$. Let $\nu_3 = 1 + b + \sum_{i=0}^{t} b^{e_i+1} + \sum_{i=t+1}^{\ell} b^{e_i}$. We find that

$$\nu_3 - 1 = b + \sum_{i=0}^{t} b^{e_i+1} + \sum_{i=t+1}^{\ell} b^{e_i} \leqslant \sum_{n=1}^{e_t+1} b^n + \sum_{i=t+1}^{\ell} b^{e_i}$$

$$\leqslant \sum_{n=1}^{e_{t+1}-2} b^n + \sum_{i=t+1}^{\ell} b^{e_i} < b^{e_{t+1}-1} + \sum_{i=t+1}^{\ell} b^{e_i} = b^{e_{t+1}-1} + \sum_{i=t+1}^{\ell-1-\lambda} b^{e_i} + \sum_{i=\ell-\lambda}^{\ell} b^{e_i}.$$

Because $e_\ell = k$ and $e_{\ell-1-i} = r - 2i$ for all nonnegative integers $i < \lambda$, we have

$$\sum_{i=\ell-\lambda}^{\ell} b^{e_i} = b^k + \sum_{i=0}^{\lambda-1} b^{r-2i}.$$

Consequently,

$$\nu_3 - 1 \leqslant b^{e_{t+1}-1} + \sum_{i=t+1}^{\ell-1-\lambda} b^{e_i} + b^k + \sum_{i=0}^{\lambda-1} b^{r-2i} < b^{e_{\ell-1-\lambda}+1} + b^k + \sum_{i=0}^{\lambda-1} b^{r-2i}$$

$$\leqslant b^k + b^{r-2\lambda} + \sum_{i=0}^{\lambda-1} b^{r-2i} \leqslant b^k + \sum_{i=0}^{y} b^{r-2i},$$

where we have used the inequalities $e_{\ell-1-\lambda} \leqslant r - 2\lambda - 1$ and $\lambda \leqslant y$. It follows that $\nu_3 \in I(k, r, y)$, so $s_b(\nu_3) \leqslant s_b(\nu)$. Let $B = M_{d_{t+2}} M_{d_{t+3}} \cdots M_{d_\ell}$. As in the previous two cases, the matrix product defining B is nonempty (because $t + 1 < \ell - \lambda \leqslant \ell$) and $B \in \Xi$. Note that $e_0 = 2(0) + 2 = 2$. We have $s_b(\nu) = \mathrm{Tr}(N_2 M_1^t M_{d_{t+1}} B)$ and $s_b(\nu_3) = \mathrm{Tr}(N_1 M_1^{t+1} M_{d_{t+1}-1} B)$, so

$$s_b(\nu_3) - s_b(\nu) = \mathrm{Tr}((N_1 M_1^{t+1} M_{d_{t+1}-1} - N_2 M_1^t M_{d_{t+1}}) B).$$

Using Lemma 16, one may easily show that

$$N_1 M_1^{t+1} M_{d_{t+1}-1} - N_2 M_1^t M_{d_{t+1}} = F_{2t+3} N_{d_{t+1}-2},$$

so
$$s_b(\nu_3) - s_b(\nu) = F_{2t+3} \operatorname{Tr}(N_{d_{t+1}-2}B).$$

Because $d_{t+1} = e_{t+1} - e_t - 1 \geqslant (m+2) - (m-1) - 1 = 2$ by (8), there follows that $N_{d_{t+1}-2}$ is a matrix with nonnegative entries whose second diagonal entry is positive. By Lemma 15, $s_b(\nu_3) - s_b(\nu) > 0$, a contradiction.

The fourth case we consider is that in which $a_1 = 0$, $a_2 = 1$, and $a_m = a_{m+1} = 1$. By now the reader is probably familiar with the general pattern of the proof. We use the minimality of m to say that $a_i = 1$ for all positive even integers $i < m$ and $a_j = 1$ for all positive odd integers $j < m$. The index m must be even. Let $t = \dfrac{m-2}{2}$, and observe that $e_j = 2j + 2$ for all $j \in \{0, 1, \ldots, t\}$ and that

$$e_t = m < m + 1 = e_{t+1} < e_{t+2} < \cdots < e_\ell = k.$$

We know that $m < r$ because Lemma 10 tells us that $a_j = 0$ for all $j \in \{r+1, \ldots, k-1\}$. Therefore, $e_{t+1} \leqslant r < k = e_\ell$. Let $\nu_4 = 1 + b + \sum_{i=0}^{t-1} b^{e_i+1} + \sum_{i=t+1}^{\ell} b^{e_i}$. We have

$$\nu_4 - 1 = b + \sum_{i=0}^{t-1} b^{e_i+1} + \sum_{i=t+1}^{\ell} b^{e_i} = b + \sum_{i=0}^{t-1} b^{e_{i+1}-1} + \sum_{i=t+1}^{\ell} b^{e_i}$$

$$= b + \sum_{i=1}^{t} b^{e_i-1} + \sum_{i=t+1}^{\ell} b^{e_i} = \sum_{i=0}^{t} b^{e_i-1} + \sum_{i=t+1}^{\ell} b^{e_i} < \nu - 1 < b^k + \sum_{i=0}^{y} b^{r-2i},$$

so $\nu_4 \in I(k, r, y)$ and $\nu_4 < \nu$. This implies that $s_b(\nu_4) < s_b(\nu)$. Let $B = M_{d_{t+2}} M_{d_{t+3}} \cdots M_{d_\ell}$. The matrix product defining B is nonempty because $e_{t+1} < e_\ell$, and $B \in \Xi$. Consequently,

$$B = \begin{pmatrix} \xi_1 & \xi_2 & \xi_1 \\ \xi_1 & \xi_2 & \xi_1 \\ \xi_3 & \xi_4 & \xi_3 \end{pmatrix}$$

for some nonnegative real $\xi_1, \xi_2, \xi_3, \xi_4$ with $\xi_2 > 0$. Because $d_{t+1} = e_{t+1} - e_t - 1 = (m+1) - m - 1 = 0$, we may use Theorem 12 to deduce that

$$s_b(\nu) = \operatorname{Tr}(N_2 M_1^t M_0 B) \quad \text{and} \quad s_b(\nu_4) = \operatorname{Tr}(N_1 M_1^{t+1} B).$$

One may show that

$$N_1 M_1^{t+1} - N_2 M_1^t M_0 = F_{2t+2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so
$$s_b(\nu_4) - s_b(\nu) = \operatorname{Tr}((N_1 M_1^{t+1} - N_2 M_1^t M_0)B) = F_{2t+2}(\xi_3 + \xi_4) \geqslant 0.$$

This is our desired contradiction.

Finally, we consider the case in which $a_1 = a_2 = 0$. In this case, $\nu \equiv 1 \pmod{b^3}$. Because $\nu - 1 < b^k + \sum_{i=0}^{y} b^{r-2i}$ and $\nu - 1 \equiv b^k + \sum_{i=0}^{y} b^{r-2i} \equiv 0 \pmod{b}$, it follows that $\nu - 1 \leqslant b^k - b + \sum_{i=0}^{y} b^{r-2i}$. This means that $\nu + b \in I(k, r, y)$, so $s_b(\nu + b) \leqslant s_b(\nu)$ by the definition of ν. As $\nu + b \equiv b + 1 \pmod{b^3}$, we find by (1) and Lemma 2 that

$$s_b(\nu + b) = s_b\left(\frac{(\nu + b) - 1}{b}\right) + s_b\left(\frac{(\nu + b) + b^2 - b - 1}{b^2}\right)$$

and

$$s_b\left(\frac{\nu - 1}{b^2} + 1\right) = s_b\left(\frac{\left(\frac{\nu-1}{b^2} + 1\right) - 1}{b}\right) + s_b\left(\frac{\left(\frac{\nu-1}{b^2} + 1\right) - 1}{b} + 1\right)$$

$$> s_b\left(\frac{\left(\frac{\nu-1}{b^2} + 1\right) - 1}{b}\right) = s_b\left(\frac{\nu - 1}{b^3}\right) = s_b\left(\frac{\nu - 1}{b}\right).$$

Therefore,

$$s_b(\nu + b) = s_b\left(\frac{(\nu + b) - 1}{b}\right) + s_b\left(\frac{(\nu + b) + b^2 - b - 1}{b^2}\right)$$

$$= s_b\left(\frac{\nu - 1}{b} + 1\right) + s_b\left(\frac{\nu - 1}{b^2} + 1\right) > s_b\left(\frac{\nu - 1}{b} + 1\right) + s_b\left(\frac{\nu - 1}{b}\right) = s_b(\nu).$$

With this contradiction, the proof is complete. $\qquad\square$

Lemma 18. *Preserving the notation from Lemma 17, we have* $e_{\ell-1-\lambda} = r - 2\lambda - 1$.

Proof. The proof is very similar to those of the first and third cases considered in the proof of Lemma 17. Suppose, by way of contradiction, that $e_{\ell-1-\lambda} \leqslant r - 2\lambda - 2$, and let $\nu' = 1 + \sum_{i=0}^{\ell-1-\lambda} b^{e_i+1} + \sum_{i=\ell-\lambda}^{\ell} b^{e_i}$ and $\nu'' = \nu' + b$. We have

$$\nu' - 1 < \nu'' - 1 = b + \sum_{i=0}^{\ell-1-\lambda} b^{e_i+1} + \sum_{i=\ell-\lambda}^{\ell} b^{e_i} \leqslant \sum_{n=1}^{e_{\ell-1-\lambda}+1} b^n + \sum_{i=\ell-\lambda}^{\ell} b^{e_i} \leqslant \sum_{n=1}^{r-2\lambda-1} b^n + \sum_{i=\ell-\lambda}^{\ell} b^{e_i}$$

$$< b^{r-2\lambda} + \sum_{i=\ell-\lambda}^{\ell} b^{e_i} = b^{r-2\lambda} + b^k + \sum_{i=0}^{\lambda-1} b^{r-2i} = b^k + \sum_{i=0}^{\lambda} b^{r-2i} \leqslant b^k + \sum_{i=0}^{y} b^{r-2i},$$

where we convene to let $\sum_{i=0}^{\lambda-1} b^{r-2i} = 0$ if $\lambda = 0$. Therefore, $\nu', \nu'' \in I(k, r, y)$. This means that $s_b(\nu'), s_b(\nu'') \leqslant s_b(\nu)$, so we will derive a contradiction by showing that $s_b(\nu') > s_b(\nu)$ or $s_b(\nu'') > s_b(\nu)$.

Let $B = M_{d_{\ell-\lambda+1}} M_{d_{\ell-\lambda+2}} \cdots M_{d_\ell}$, where we define B to be the 3×3 identity matrix if $\lambda = 0$. By Lemma 14, $B \in \Xi'$. To ease our notation, let $t = \ell - 1 - \lambda$ and $q = d_{\ell-\lambda}$. We have $s_b(\nu) = \operatorname{Tr}(N_{e_0} M_1^t M_q B)$, $s_b(\nu') = \operatorname{Tr}(N_{e_0+1} M_1^t M_{q-1} B)$, and $s_b(\nu'') = \operatorname{Tr}(N_1 M_{e_0-1} M_1^t M_{q-1} B)$. It is straightforward to show that $N_2 M_1^t M_{q-1} - N_1 M_1^t M_q = F_{2t+2} N_{q-2}$ and $N_1 M_1^{t+1} M_{q-1} - N_2 M_1^t M_q = F_{2t+3} N_{q-2}$. If $\lambda = 0$, then it follows from our assumption that $e_{\ell-1-\lambda} \leqslant r - 2\lambda - 2$ that

$$q = e_{\ell-\lambda} - e_{\ell-1-\lambda} - 1 = e_\ell - e_{\ell-1} - 1 = k - e_{\ell-1} - 1 \geqslant k - (r-2) - 1 \geqslant 3.$$

If $\lambda > 0$, then

$$q = e_{\ell-\lambda} - e_{\ell-1-\lambda} - 1 = e_{\ell-1-(\lambda-1)} - e_{\ell-1-\lambda} - 1 = (r - 2(\lambda-1)) - e_{\ell-1-\lambda} - 1$$
$$\geqslant (r - 2(\lambda-1)) - (r - 2\lambda - 2) - 1 = 3.$$

In either case, $q \geqslant 3$, so both $F_{2t+2} N_{q-2}$ and $F_{2t+3} N_{q-2}$ are 3×3 matrices with nonnegative real entries whose second diagonal entries are positive. It follows from Lemma 15 that if $e_0 = 1$, then

$$s_b(\nu') - s_b(\nu) = \operatorname{Tr}((N_2 M_1^t M_{q-1} - N_1 M_1^t M_q)B) = \operatorname{Tr}(F_{2t+2} N_{q-2} B) > 0.$$

Similarly, if $e_0 = 2$, then

$$s_b(\nu'') - s_b(\nu) = \operatorname{Tr}((N_1 M_1^{t+1} M_{q-1} - N_2 M_1^t M_q)B) = \operatorname{Tr}(F_{2t+3} N_{q-2} B) > 0.$$

This is a contradiction, so the proof is complete. $\qquad\square$

For nonnegative integers k, r, y with $2y < r < k - 1$, let $w = a_k a_{k-1} \cdots a_0$ be the ordinary base b expansion of $\nu(k, r, y) - 1$. We summarize here the information we have gained about this expansion. From Lemma 10, we know that $a_i \in \{0, 1\}$ for all i. We know from the same lemma that $a_k = 1$, $a_0 = 0$, and $a_j = 0$ for all $j \in \{r+1, \ldots, k-1\}$. Therefore, $w = 10^{k-r-1} v$, where v is a word of length $r+1$ over the binary alphabet $\{0, 1\}$ that ends in the letter (digit) 0. From Lemmas 17 and 18 and (6), we find that there exists some $\lambda \in \{0, 1, \ldots, y\}$ such that

A. $a_{r-2i} = 1$ and $a_{r-2i-1} = 0$ for all $i \in \{0, 1, \ldots, \lambda - 1\}$.

B. $a_{r-2\lambda} = 0$.

C. $a_{r-2\lambda-2i+1} = 1$ and $a_{r-2\lambda-2i} = 0$ for all $i \in \{1, 2, \ldots, \lfloor r/2 \rfloor - \lambda\}$.

That is,

$$v = \begin{cases} (10)^\lambda 0 (10)^{\lfloor r/2 \rfloor - \lambda} 0, & \text{if } 2 \nmid r; \\ (10)^\lambda 0 (10)^{\lfloor r/2 \rfloor - \lambda}, & \text{if } 2 \mid r. \end{cases}$$

This means that

$$w = \begin{cases} 10^{k-r-1} (10)^\lambda 0 (10)^{\lfloor r/2 \rfloor - \lambda} 0, & \text{if } 2 \nmid r; \\ 10^{k-r-1} (10)^\lambda 0 (10)^{\lfloor r/2 \rfloor - \lambda}, & \text{if } 2 \mid r. \end{cases} \tag{9}$$

This leads us to the following.

Definition 19. Let k, r, x be nonnegative integers such that $2x < r < k - 1$. Define the word $w(k, r, x)$ by

$$w(k,r,x) = \begin{cases} 10^{k-r-1}(10)^x 0(10)^{\lfloor r/2 \rfloor - x} 0, & \text{if } 2 \nmid r; \\ 10^{k-r-1}(10)^x 0(10)^{\lfloor r/2 \rfloor - x}, & \text{if } 2 \mid r, \end{cases}$$

and let $\gamma(k, r, x)$ be the positive integer whose ordinary base b expansion is $w(k, r, x)$.

Definition 20. For any integers k, r, x, define

$$V(k,r,x) = \frac{1}{5}((k-r)(2L_{r+2} - L_{r-4x+1}) + 2L_{r+1} + L_{r-4x+2}).$$

Lemma 21. *Let k, r, x, y be nonnegative integers such that $2x \leqslant 2y < r < k - 1$. We have*

$$\gamma(k,r,x) + 1 \in I(k,r,y) \quad and \quad s_b(\gamma(k,r,x) + 1) = V(k,r,x).$$

Proof. The ordinary base b expansion of $b^k + \sum_{i=0}^{y} b^{r-2i}$ is $10^{k-r-1}(10)^{y+1} 0^{r-2y-1}$, so one may see from Definition 19 and the hypothesis that $x \leqslant y$ that $\gamma(k,r,x) + 1 \leqslant b^k + \sum_{i=0}^{y} b^{r-2i}$. It should also be clear from Definition 19 that $b^k < \gamma(k,r,x) + 1$, so $\gamma(k,r,x) + 1 \in I(k,r,y)$. If we choose to write $\gamma(k,r,x) = \sum_{i=0}^{\ell} b^{e_i}$ for nonnegative integers $e_0, e_1, \ldots, e_\ell$ with $e_0 < e_1 \cdots < e_\ell$, then

$$e_0 = \begin{cases} 2, & \text{if } 2 \nmid r; \\ 1, & \text{if } 2 \mid r, \end{cases}$$

$$e_j = e_0 + 2j \text{ for all } j \in \{1, 2, \ldots, t\},$$

$$e_{t+j} = r - 2x + 2j = e_0 + 2t + 2j + 1 \text{ for all } j \in \{1, 2, \ldots, x\},$$

and

$$e_\ell = k,$$

where $t = \lfloor r/2 \rfloor - x - 1$. Observe that the letter (digit) 1 appears exactly $\lfloor r/2 \rfloor + 1$ times in the word $w(k, r, x)$, so it follows from (6) that $\ell = \lfloor r/2 \rfloor$ (implying that $t + 1 = \ell - x$). Let us first assume $x = 0$. Setting $d_j = e_j - e_{j-1} - 1$ for all $j \in \{1, 2, \ldots, \ell\}$, we have

$$d_j = \begin{cases} 1, & \text{if } j \in \{1, 2, \ldots, \ell - 1\}; \\ k - r, & \text{if } j = \ell. \end{cases}$$

We may invoke Theorem 12 to see that

$$s_b(\gamma(k,r,0) + 1) = \text{Tr}(N_{e_0} M_{d_1} M_{d_2} \cdots M_{d_\ell}) = \text{Tr}(N_{e_0} M_1^{\ell-1} M_{k-r}). \tag{10}$$

Let

$$T = \begin{pmatrix} F_r & F_{r-1} & F_r \\ F_r & F_{r-1} & F_r \\ 0 & 0 & 0 \end{pmatrix}.$$

If $2|r$, we may use Lemma 16 to find that

$$N_{e_0} M_1^{\ell-1} = N_1 M_1^{\frac{1}{2}(r-2)} = T.$$

Similarly, if $2 \nmid r$, then

$$N_{e_0} M_1^{\ell-1} = N_2 M_1^{\frac{1}{2}(r-3)} = T.$$

Therefore, regardless of the parity of r, we have $s_b(\gamma(k,r,0)+1) = \text{Tr}(TM_{k-r})$ by (10). Thus,

$$s_b(\gamma(k,r,0)+1) = \text{Tr}(TM_{k-r}) = (k-r)F_{r+1} + F_{r+2},$$

where the last equality is obtained by simply calculating the matrix TM_{k-r} by hand. One may easily show that $F_{r+1} = \dfrac{1}{5}(2L_{r+2} - L_{r+1})$ and $F_{r+2} = \dfrac{1}{5}(2L_{r+1} + L_{r+2})$, so

$$s_b(\gamma(k,r,0)+1) = \frac{1}{5}((k-r)(2L_{r+2} - L_{r+1}) + 2L_{r+1} + L_{r+2}) = V(k,r,0).$$

Let us now assume $x > 0$. Again, we set $d_j = e_j - e_{j-1} - 1$ for all $j \in \{1, 2, \ldots, \ell\}$, and we see that

$$d_j = \begin{cases} 1, & \text{if } j \in \{1, 2, \ldots, \ell - 1\} \setminus \{t+1\}; \\ 2, & \text{if } j = t+1; \\ k - r - 1, & \text{if } j = \ell. \end{cases}$$

By Theorem 12,

$$s_b(\gamma(k,r,x)+1) = \text{Tr}(N_{e_0} M_{d_1} M_{d_2} \cdots M_{d_\ell}) = \text{Tr}(N_{e_0} M_1^t M_2 M_1^{\ell-1-(t+1)} M_{k-r-1})$$

$$= \text{Tr}(N_{e_0} M_1^t M_2 M_1^{x-1} M_{k-r-1}).$$

If $2|r$, then Lemma 16 allows us to find that

$$N_{e_0} M_1^t = N_1 M_1^{r/2-x-1} = \begin{pmatrix} F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly, if $2 \nmid r$, then

$$N_{e_0} M_1^t = N_2 M_1^{(r-1)/2-x-1} = \begin{pmatrix} F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, no matter the parity of r, we have

$$N_{e_0} M_1^t M_2 M_1^{x-1} M_{k-r-1} = \begin{pmatrix} F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ 0 & 0 & 0 \end{pmatrix} M_2 M_1^{x-1} M_{k-r-1}.$$

Using Lemma 16 again, we find that

$$\begin{pmatrix} F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ F_{r-2x} & F_{r-2x-1} & F_{r-2x} \\ 0 & 0 & 0 \end{pmatrix} M_2 M_1^{x-1} M_{k-r-1} = \begin{pmatrix} \chi_1 & \chi_2 & \chi_1 \\ \chi_1 & \chi_2 & \chi_1 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\chi_1 = (k-r)(F_{r-2x+3}F_{2x} + F_{r-2x+1}F_{2x-1}) - F_{r-2x+3}F_{2x-2} - F_{r-2x+1}F_{2x-3}$$

and

$$\chi_2 = F_{r-2x+3}F_{2x} + F_{r-2x+1}F_{2x-1}.$$

Therefore,

$$s_b(\gamma(k,r,x) + 1) = \mathrm{Tr}(N_{e_0} M_1^t M_2 M_1^{x-1} M_{k-r-1}) = \chi_1 + \chi_2$$

$$= (k-r)(F_{r-2x+3}F_{2x} + F_{r-2x+1}F_{2x-1}) + F_{r-2x+3}F_{2x-1} + F_{r-2x+1}F_{2x-2}.$$

We now make use of the well-known identity

$$F_m F_n = \frac{1}{5}(L_{m+n} - (-1)^n L_{m-n})$$

to obtain

$$F_{r-2x+3}F_{2x} + F_{r-2x+1}F_{2x-1} = \frac{1}{5}(L_{r+3} - L_{r-4x+3} + L_r + L_{r-4x+2})$$

$$= \frac{1}{5}(L_{r+2} + L_{r+1} + L_r - L_{r-4x+1}) = \frac{1}{5}(2L_{r+2} - L_{r-4x+1})$$

and

$$F_{r-2x+3}F_{2x-1} + F_{r-2x+1}F_{2x-2} = \frac{1}{5}(L_{r+2} + L_{r-4x+4} + L_{r-1} - L_{r-4x+3})$$

$$= \frac{1}{5}(L_{r+1} + L_r + L_{r-1} + L_{r-4x+2}) = \frac{1}{5}(2L_{r+1} + L_{r-4x+2}).$$

Consequently,

$$s_b(\gamma(k,r,x) + 1) = (k-r)\left(\frac{1}{5}(2L_{r+2} - L_{r-4x+1})\right) + \frac{1}{5}(2L_{r+1} + L_{r-4x+2}) = V(k,r,x). \quad \square$$

Before proceeding to our main result, we need one final easy lemma.

Lemma 22. *For any integers* k, r, x,

$$V(k,r,x+1) - V(k,r,x) = \frac{1}{5}((k-r)(L_{r-4x+1} - L_{r-4x-3}) + L_{r-4x-2} - L_{r-4x+2}).$$

Proof. Referring to Definition 20, we have

$$V(k,r,x+1) = A + \frac{1}{5}((k-r)(-L_{r-4(x+1)+1}) + L_{r-4(x+1)+2})$$

and

$$V(k,r,x) = A + \frac{1}{5}((k-r)(-L_{r-4x+1}) + L_{r-4x+2}),$$

where $A = \frac{1}{5}((k-r)(2L_{r+2}) + 2L_{r+1})$. Thus,

$$V(k,r,x+1) - V(k,r,x) = \frac{1}{5}((k-r)(L_{r-4x+1} - L_{r-4x-3}) + L_{r-4x-2} - L_{r-4x+2}). \qquad \square$$

We are finally in a position to prove our main result.

Theorem 23. *Let k, r, y be nonnegatve integers such that $2y < r < k - 1$. If r is even, then*

$$\nu(k,r,y) = \gamma(k,r,y) + 1 \ \text{ and } \ \mu(k,r,y) = V(k,r,y).$$

If $k - 2 = r \equiv 3 \pmod 4$ and $y \geqslant \dfrac{r+1}{4}$, then

$$\nu(k,r,y) = \gamma(k,r,(r-3)/4) + 1 \ \text{ and } \ \mu(k,r,y) = V(k,r,(r-3)/4).$$

Otherwise,

$$\nu(k,r,y) = \gamma(k,r,\delta(r,y)) + 1 \ \text{ and } \ \mu(k,r,y) = V(k,r,\delta(r,y)),$$

where $\delta(r,y) = \min\left\{\left\lceil \frac{r-1}{4}\right\rceil, y\right\}$.

Proof. We saw in the discussion immediately preceding Definition 19 that $\nu(k,r,y) = \gamma(k,r,\lambda) + 1$ for some $\lambda \in \{0, 1, \ldots, y\}$; we simply need to determine the value of λ. By Definition 8, λ is the element of $\{0, 1, \ldots, y\}$ such that $s_b(\gamma(k,r,\lambda) + 1)$ is maximized and $\gamma(k,r,\lambda) + 1 \in I(k,r,y)$. Lemma 21 tells us that $\gamma(k,r,x) + 1 \in I(k,r,y)$ and $s_b(\gamma(k,r,x) + 1) = V(k,r,x)$ for all $x \in \{0, 1, \ldots, y\}$, so

$$\mu(k,r,y) = \max\{V(k,r,x) \colon x \in \{0, 1, \ldots, y\}\}.$$

That is, λ must be the element of the set $\{0, 1, \ldots, y\}$ that maximizes $V(k,r,\lambda)$ (if there are multiple such elements, we choose the smallest in accordance with the definition of $\nu(k,r,y)$ in Definition 8). We first assume r is even. We wish to show that $\lambda = y$. This is immediate if $y = 0$, so we will assume $y > 0$. Choose some $x \in \{0, 1, \ldots, y - 1\}$. We make use of an easily-proven fact about Lucas numbers that states that if m, n are odd integers (not necessarily positive) with $m < n$, then $L_m < L_n$. In this case, we use the assumption that r is even to see that $r - 4x - 3$ and $r - 4x + 1$ are odd integers with $r - 4x - 3 < r - 4x + 1$. Therefore, $L_{r-4x+1} - L_{r-4x-3} > 0$. Because $k - r \geqslant 2$ by hypothesis, we may use Lemma 22 to find that

$$V(k,r,x+1) - V(k,r,x) \geqslant \frac{1}{5}(2(L_{r-4x+1} - L_{r-4x-3}) + L_{r-4x-2} - L_{r-4x+2})$$

$$= \frac{1}{5}(L_{r-4x-1} - L_{r-4x-5}).$$

As $r - 4x - 5$ and $r - 4x - 1$ are odd integers with $r - 4x - 5 < r - 4x - 1$, we see that

$$V(k, r, x+1) - V(k, r, x) > 0.$$

Because x was arbitrary, this shows that

$$V(k, r, 0) < V(k, r, 1) < \cdots < V(k, r, y),$$

so $\lambda = y$.

We now assume r is odd. Let x_1, x_2 be integers such that $x_1 \leqslant \dfrac{r-5}{4} < \dfrac{r-1}{4} \leqslant x_2$. We will make use of the fact that $L_m = L_{-m}$ for any even integer m. Because $x_1 \leqslant \dfrac{r-5}{4}$, we obtain the inequalities $r - 4x_1 + 1 > r - 4x_1 - 3 > 0$. Therefore, $L_{r-4x_1+1} - L_{r-4x_1-3} > 0$. Because $k - r \geqslant 2$, we see from Lemma 22 that

$$V(k, r, x_1 + 1) - V(k, r, x_1) \geqslant \frac{1}{5}(2(L_{r-4x_1+1} - L_{r-4x_1-3}) + L_{r-4x_1-2} - L_{r-4x_1+2})$$

$$= \frac{1}{5}(L_{r-4x_1-1} - L_{r-4x_1-5}).$$

We know that $L_{r-4x_1-1} - L_{r-4x_1-5} > 0$ because $r - 4x_1 - 5 \geqslant 0$. Thus,

$$V(k, r, x_1 + 1) > V(k, r, x_1). \tag{11}$$

If $y < \lceil \frac{r-1}{4} \rceil$, then every element of the set $\{0, 1, \ldots, y - 1\}$ is less than $\dfrac{r-5}{4}$, so (11) shows that

$$V(k, r, 0) < V(k, r, 1) < \cdots < V(k, r, y).$$

This shows that if $y < \lceil \frac{r-1}{4} \rceil$, then

$$\lambda = y = \delta(r, y),$$

which agrees with the statement of the theorem. Therefore, we will henceforth assume that $y \geqslant \lceil \frac{r-1}{4} \rceil$ so that $\delta(r, y) = \lceil \frac{r-1}{4} \rceil$. It follows from the inequality $x_2 \geqslant \dfrac{r-1}{4}$ that $4x_2 - r + 3 > 4x_2 - r - 1 \geqslant -2$. If $4x_2 - r - 1 \geqslant 0$, then

$$L_{4x_2-r-1} - L_{4x_2-r+3} < 0.$$

If $4x_2 - r - 1 = -2$, then

$$L_{4x_2-r-1} - L_{4x_2-r+3} = L_{-2} - L_2 = 0.$$

Hence,

$$L_{4x_2-r-1} - L_{4x_2-r+3} \leqslant 0.$$

Because $r - 4x_2 + 1$ and $r - 4x_2 - 3$ are even,

$$L_{r-4x_2+1} - L_{r-4x_2-3} = L_{4x_2-r-1} - L_{4x_2-r+3} \leqslant 0.$$

Using Lemma 22 and the fact that $k - r \geqslant 2$, we have

$$V(k, r, x_2 + 1) - V(k, r, x_2) \leqslant \frac{1}{5}(2(L_{r-4x_2+1} - L_{r-4x_2-3}) + L_{r-4x_2-2} - L_{r-4x_2+2})$$

$$= \frac{1}{5}(L_{r-4x_2-1} - L_{r-4x_2-5}).$$

Because $4x_2 - r + 5 > 4x_2 - r + 1 \geqslant 0$ and $r - 4x_2 - 1, r - 4x_2 - 5$ are even, we have

$$L_{r-4x_2-1} - L_{r-4x_2-5} = L_{4x_2-r+1} - L_{4x_2-r+5} < 0.$$

This shows that

$$V(k, r, x_2 + 1) < V(k, r, x_2). \tag{12}$$

We now have three short cases to consider. In the first case, assume $r \equiv 1 \pmod 4$. We have shown through (11) and (12) that

$$V(k, r, m) > V(k, r, m - 1) > V(k, r, m - 2) > \cdots$$

and

$$V(k, r, m) > V(k, r, m + 1) > V(k, r, m + 2) > \cdots,$$

where $m = \dfrac{r - 1}{4}$. This implies that $\dfrac{r - 1}{4}$ is the unique value of x that maximizes $V(k, r, x)$. Recall that we have assumed that $y \geqslant \lceil \frac{r-1}{4} \rceil$, so $\dfrac{r - 1}{4} \in \{0, 1, \ldots, y\}$. Consequently,

$$\lambda = \frac{r - 1}{4} = \delta(r, y).$$

For the second case, we suppose that $r \equiv 3 \pmod 4$ and $r < k - 2$. We need to show that $\lambda = \delta(r, y)$. We may use (11) and (12) to see that

$$V(k, r, n) > V(k, r, n - 1) > V(k, r, n - 2) > \cdots$$

and

$$V(k, r, n + 1) > V(k, r, n + 2) > V(k, r, n + 3) > \cdots,$$

where $n = \dfrac{r - 3}{4}$. By Lemma 22,

$$V(k, r, n + 1) - V(k, r, n) = \frac{1}{5}((k - r)(L_4 - L_0) + L_1 - L_5)$$

$$= \frac{1}{5}((k - r)(7 - 2) + 1 - 11) = k - r - 2 > 0,$$

so $V(k, r, n) < V(k, r, n + 1)$. This shows that $n + 1 = \frac{r+1}{4}$ is the unique value of x that maximizes $V(k, r, x)$. Because $y \geqslant \left\lceil \frac{r-1}{4} \right\rceil = \frac{r+1}{4}$, it follows that $\frac{r+1}{4} \in \{0, 1, \ldots, y\}$. Therefore,

$$\lambda = \tfrac{r+1}{4} = \left\lceil \tfrac{r-1}{4} \right\rceil = \delta(r, y).$$

For the third and final case, we assume $k - 2 = r \equiv 3 \pmod 4$. We wish to show that $\lambda = \frac{r-3}{4}$. Using (11) and (12), we obtain

$$V(k, r, n) > V(k, r, n - 1) > V(k, r, n - 2) > \cdots$$

and

$$V(k, r, n + 1) > V(k, r, n + 2) > V(k, r, n + 3) > \cdots,$$

where $n = \dfrac{r - 3}{4}$. Lemma 22 tells us that

$$V(k, r, n + 1) - V(k, r, n) = \frac{1}{5}((k - r)(L_4 - L_0) + L_1 - L_5)$$

$$= \frac{1}{5}((k - r)(7 - 2) + 1 - 11) = k - r - 2 = 0,$$

so $V(k, r, n) = V(k, r, n + 1)$. Therefore, $V(k, r, x)$ obtains its maximum when $x = n$ and when $x = n+1$. This means that $\mu(k, r, y) = V(k, r, n)$. Either $\nu(k, r, y) = \gamma(k, r, n)+1$ or $\gamma(k, r, n+1)+1$. Note that $\gamma(k, r, n)+1 < \gamma(k, r, n+1)+1$ by Definition 19. Since $\nu(k, r, y)$ is defined to be the *smallest* element of $I(k, r, y)$ such that $s_b(\nu(k, r, y) + 1) = \mu(k, r, y)$, it follows that $\nu(k, r, y) = \gamma(k, r, n) + 1$. That is, $\lambda = n = \dfrac{r - 3}{4}$. $\qquad\square$

We end this section with a definition and a theorem that will prove useful in the next section.

Definition 24. For any integers k and m with $2 \leqslant m \leqslant k$, let $G_k(m) = b^k + h_m$ and $H_k(m) = F_{m+2} + (k - m)F_m$.

Although it is possible to give an easy inductive proof of the following theorem using the simple Fibonacci-like recurrence

$$H_k(m) = H_{k-1}(m - 1) + H_{k-2}(m - 2), \tag{13}$$

we prefer a proof based on Theorem 12.

Theorem 25. *For any integers k and m with $2 \leqslant m \leqslant k$, we have $s_b(G_k(m)) = H_k(m)$.*

Proof. Referring to Definition 3, we see that we may write

$$G_k(m) - 1 = b^k + \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} b^{m-2-2i} = \sum_{i=0}^{\ell} b^{e_i},$$

where

$$e_0 = m - 2 - 2 \left\lfloor \frac{m-3}{2} \right\rfloor = \begin{cases} 2, & \text{if } 2 \mid m; \\ 1, & \text{if } 2 \nmid m, \end{cases}$$

$\ell = \left\lfloor \frac{m-1}{2} \right\rfloor$, $e_\ell = k$, and $e_j = e_0 + 2j$ for all $j \in \{0, 1, \ldots, \ell - 1\}$. For example,

$$G_{12}(7) - 1 = b^{12} + \sum_{i=0}^{\left\lfloor \frac{7-3}{2} \right\rfloor} b^{7-2-2i} = b^1 + b^3 + b^5 + b^{12}.$$

Defining d_i as in Theorem 12, we have $d_j = 1$ for all $j \in \{1, 2, \ldots, \ell - 1\}$ and

$$d_\ell = e_\ell - e_{\ell-1} - 1 = k - (e_0 + 2(\ell-1)) - 1 = k - \left(\left(m - 2 - 2 \left\lfloor \frac{m-3}{2} \right\rfloor \right) + 2 \left(\left\lfloor \frac{m-1}{2} \right\rfloor - 1 \right) \right) - 1$$

$$= k - m + 1.$$

By Theorem 12, $s_b(G_k(m)) = \text{Tr}(N_{e_0} M_1^{\ell-1} M_{k-m+1})$. Using Lemma 16, one may show that

$$N_{e_0} M_1^{\ell-1} M_{k-m+1} = \begin{pmatrix} \rho_1 & \rho_2 & \rho_1 \\ \rho_1 & \rho_2 & \rho_1 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\rho_1 = (k - m - 1)(e_0 F_{2\ell} + F_{2\ell-1}) + e_0 F_{2\ell+2} + F_{2\ell+1} \quad \text{and} \quad \rho_2 = e_0 F_{2\ell} + F_{2\ell-1}.$$

Therefore,

$$s_b(G_k(m)) = \rho_1 + \rho_2 = (k - m)(e_0 F_{2\ell} + F_{2\ell-1}) + e_0 F_{2\ell+2} + F_{2\ell+1}.$$

If m is even, then $e_0 = 2$ and $\ell = \dfrac{m-2}{2}$, so

$$s_b(G_k(m)) = (k - m)(2F_{m-2} + F_{m-3}) + 2F_m + F_{m-1} = (k - m)F_m + F_{m+2} = H_k(m).$$

If m is odd, then $e_0 = 1$ and $\ell = \dfrac{m-1}{2}$, so

$$s_b(G_k(m)) = (k - m)(F_{m-1} + F_{m-2}) + F_{m+1} + F_m = (k - m)F_m + F_{m+2} = H_k(m). \quad \square$$

3 More Manageable Bounds

In the previous section, we derived fairly strong upper bounds (relative to those previously known) for the values of $s_b(n)$ for those integers n satisfying $b^k \leqslant n \leqslant b^k + h_k$ for some integer $k \geqslant 3$. Unfortunately, these bounds are somewhat cumbersome. In this section, we will weaken them in order to make them cleaner and more easily applicable.

Suppose k, m, and n are integers such that $2 \leqslant m < k$ and $G_k(m) < n < G_k(m+1)$. We wish to use Theorem 23 to show that the point $(n, s_b(n))$ lies below the line segment connecting the points $(G_k(m), H_k(m))$ and $(G_k(m+1), H_k(m+1))$. This leads us to the following definition.

Definition 26. For any integers k and m with $2 \leqslant m < k$, define a function $J_{k,m} \colon \mathbb{R} \to \mathbb{R}$ by

$$J_{k,m}(x) = \frac{H_k(m+1) - H_k(m)}{G_k(m+1) - G_k(m)}(x - G_k(m)) + H_k(m).$$

The graph of $J_{k,m}(x)$ is the line passing through $(G_k(m), H_k(m))$ and $(G_k(m+1), H_k(m+1))$. Our aim is to show that $s_b(n) < J_{k,m}(n)$ for all $n \in \{G_k(m) + 1, \ldots, G_k(m+1) - 1\}$. Observe that we may rewrite $J_{k,m}(x)$ as

$$J_{k,m}(x) = \frac{F_{m-1}(k-m)}{h_{m+1} - h_m}(x - G_k(m)) + H_k(m)$$

because $H_k(m+1) - H_k(m) = (k-m-1)F_{m+1} + F_{m+3} - (k-m)F_m - F_{m+2} = F_{m-1}(k-m)$ and $G_k(m+1) - G_k(m) = h_{m+1} - h_m$.

Lemma 27. *If m and x are positive integers with $m \geqslant 4$ and $x < G_{m-1}(m-3)$, then $s_b(x) \leqslant F_m$.*

Proof. Let m and x be as in the statement of the lemma. If $x < b^{m-1}$, then it follows from Proposition 6 that $s_b(x) \leqslant F_m$. If $x = b^{m-1}$, then $s_b(x) = 1 < F_m$ by Lemma 5. Therefore, we will assume $b^{m-1} < x$. Note that the inequalities $b^{m-1} < x < G_{m-1}(m-3)$ force $m \geqslant 6$. Since

$$b^{m-1} < x \leqslant G_{m-1}(m-3) - 1 = b^{m-1} + \sum_{i=0}^{\lfloor \frac{m-6}{2} \rfloor} b^{m-5-2i},$$

it follows from Definition 8 that $x \in I\left(m-1, m-5, \lfloor \frac{m-6}{2} \rfloor\right)$. Referring to Definition 8 again, we find that $s_b(x) \leqslant \mu\left(m-1, m-5, \lfloor \frac{m-6}{2} \rfloor\right)$. Therefore, we simply need to show that

$$\mu\left(m-1, m-5, \lfloor \tfrac{m-6}{2} \rfloor\right) \leqslant F_m. \tag{14}$$

Suppose m is odd. It follows from Theorem 23 and Definition 20 that

$$\mu\left(m-1, m-5, \lfloor \tfrac{m-6}{2} \rfloor\right) = \mu\left(m-1, m-5, \tfrac{m-7}{2}\right) = V\left(m-1, m-5, \tfrac{m-7}{2}\right)$$

$$= \frac{1}{5}(4(2L_{m-3} - L_{10-m}) + 2L_{m-4} + L_{11-m}).$$

Because m is odd, $L_{10-m} = -L_{m-10}$ and $L_{11-m} = L_{m-11}$. Therefore,

$$\mu\left(m-1, m-5, \lfloor \tfrac{m-6}{2} \rfloor\right) = \frac{1}{5}(4(2L_{m-3} + L_{m-10}) + 2L_{m-4} + L_{m-11}).$$

For any integer t, let

$$\eta_t = \frac{1}{5}(4(2L_{t-3} + L_{t-10}) + 2L_{t-4} + L_{t-11}) \quad \text{and} \quad \kappa_t = 11F_{t-5}.$$

It is easy to see that the recurrence relations $\eta_{t+4} = 3\eta_{t+2} - \eta_t$ and $\kappa_{t+4} = 3\kappa_{t+2} - \kappa_t$ hold for all integers t. In addition, one may verify that $\kappa_5 = \eta_5 = 0$ and $\kappa_7 = \eta_7 = 11$. This implies that $\kappa_t = \eta_t$ for all odd integers t. In particular,

$$\mu\left(m-1, m-5, \left\lfloor \tfrac{m-6}{2} \right\rfloor\right) = \eta_m = \kappa_m = 11F_{m-5} < F_m,$$

where the inequality $11F_{m-5} < F_m$ is easily-proven for all odd $m > 6$ (for example, by showing that $11F_{m-5} = 22F_{m-7} + 11F_{m-8} < 21F_{m-7} + 13F_{m-8} = F_m$). This proves (14) when m is odd.

Next, assume m is even. By Theorem 23,

$$\mu\left(m-1, m-5, \left\lfloor \tfrac{m-6}{2} \right\rfloor\right) = V\left(m-1, m-5, \delta\left(m-5, \left\lfloor \tfrac{m-6}{2} \right\rfloor\right)\right),$$

where

$$\delta\left(m-5, \left\lfloor \tfrac{m-6}{2} \right\rfloor\right) = \min\left\{ \left\lceil \tfrac{(m-5)-1}{4} \right\rceil, \left\lfloor \tfrac{m-6}{2} \right\rfloor \right\} = \left\lceil \tfrac{m-6}{4} \right\rceil.$$

If $m \equiv 0 \pmod 4$, then $\left\lceil \tfrac{m-6}{4} \right\rceil = \tfrac{m-4}{4}$, so

$$\mu\left(m-1, m-5, \left\lfloor \tfrac{m-6}{2} \right\rfloor\right) = V\left(m-1, m-5, \tfrac{m-4}{4}\right) = \frac{1}{5}(4(2L_{m-3} - L_0) + 2L_{m-4} + L_1)$$

$$= \frac{1}{5}(8L_{m-3} + 2L_{m-4} - 7) < \frac{1}{5}(8L_{m-3} + 2L_{m-4}).$$

Similarly, if $m \equiv 2 \pmod 4$, then $\left\lceil \tfrac{m-6}{4} \right\rceil = \tfrac{m-6}{4}$, so

$$\mu\left(m-1, m-5, \left\lfloor \tfrac{m-6}{2} \right\rfloor\right) = V\left(m-1, m-5, \tfrac{m-6}{4}\right) = \frac{1}{5}(4(2L_{m-3} - L_2) + 2L_{m-4} + L_3)$$

$$= \frac{1}{5}(8L_{m-3} + 2L_{m-4} - 8) < \frac{1}{5}(8L_{m-3} + 2L_{m-4}).$$

In either case, we may use the identity $F_u = \frac{1}{5}(L_{u+1} + L_{u-1})$ to see that

$$\mu\left(m-1, m-5, \left\lfloor \tfrac{m-6}{2} \right\rfloor\right) < \frac{1}{5}(8L_{m-3} + 2L_{m-4}) = \frac{1}{5}(6L_{m-3} + 2L_{m-2}) = \frac{1}{5}(4L_{m-3} + 2L_{m-1})$$

$$= 2 \cdot \frac{1}{5}(L_{m-3} + L_{m-1}) + \frac{1}{5}(L_{m-3} + L_{m-4} + L_{m-5}) = 2F_{m-2} + \frac{1}{5}(L_{m-2} + L_{m-5})$$

$$< 2F_{m-2} + \frac{1}{5}(L_{m-2} + L_{m-4}) = 2F_{m-2} + F_{m-3} = F_m.$$

This proves (14) in the case when m is even, so the proof is complete. $\qquad\square$

Throughout the proofs of the following three lemmas, we assume $b = 2$. For example, since $G_k(4) = b^k + h_4 = b^k + b^2 + 1$, it will be understood that $G_k(4) = 2^k + 5$.

Lemma 28. *For any positive integers t and x with $t \geqslant \lfloor \log_2 x \rfloor + 2$, we have*

$$s_2(2^t + x) \leqslant s_2(x)(t + 1 - \lfloor \log_2 x \rfloor).$$

Proof. The proof is by induction on t. The inequality $t \geqslant \lfloor \log_2 x \rfloor + 2$ forces $t \geqslant 2$. If $t = 2$, then the inequality $t \geqslant \lfloor \log_2 x \rfloor + 2$ forces $x = 1$ so that

$$s_2(2^t + x) = s_2(5) = 3 = s_2(1)(2 + 1 - \lfloor \log_2 1 \rfloor) = s_2(x)(t + 1 - \lfloor \log_2 x \rfloor).$$

Now, suppose $t \geqslant 3$. If x is even, then it follows from (1) and induction on t that

$$s_2(2^t + x) = s_2(2^{t-1} + x/2) \leqslant s_2(x/2)((t-1) + 1 - \lfloor \log_2(x/2) \rfloor) = s_2(x)(t + 1 - \lfloor \log_2 x \rfloor).$$

Therefore, we may assume x is odd. By induction on t, we have

$$s_2(2^{t-1} + (x-1)/2) \leqslant s_2((x-1)/2)((t-1) + 1 - \lfloor \log_2((x-1)/2) \rfloor)$$

$$= s_2((x-1)/2)(t + 1 - \lfloor \log_2 x \rfloor)$$

and

$$s_2(2^{t-1} + (x+1)/2) \leqslant s_2((x+1)/2)((t-1) + 1 - \lfloor \log_2((x+1)/2) \rfloor)$$

$$\leqslant s_2((x+1)/2)(t + 1 - \lfloor \log_2 x \rfloor).$$

Therefore, it follows from (1) that

$$s_2(2^t + x) = s_2\left(2^{t-1} + (x-1)/2\right) + s_2\left(2^{t-1} + (x+1)/2\right)$$

$$\leqslant (s_2((x-1)/2) + s_2((x+1)/2))(t + 1 - \lfloor \log_2 x \rfloor) = s_2(x)(t + 1 - \lfloor \log_2 x \rfloor). \qquad \square$$

Lemma 29. *If $k, m,$ and n are integers such that $4 \leqslant m < k$ and*

$$2^k + 2^{m-1} \leqslant n < 2^k + G_{m-1}(m-3),$$

then $s_2(n) < H_k(m)$.

Proof. Let $k, m,$ and n be as stated in the lemma. Let $x = n - 2^k$, and note that $\lfloor \log_2 x \rfloor = m - 1$. We may combine Lemmas 27 and 28 to get

$$s_2(n) = s_2(2^k + x) \leqslant s_2(x)(k + 1 - \lfloor \log_2 x \rfloor) = s_2(x)(k - m + 2)$$

$$\leqslant F_m(k - m + 2) < F_m(k - m) + F_{m+2} = H_k(m). \qquad \square$$

Lemma 30. *If $k, m,$ and n are integers such that $m \in \{5, 7\}$, $m < k$, and $n = 2^k + G_{m-1}(m-3)$, then $s_2(n) < J_{k,m}(n)$.*

Proof. The proof is very straightforward. We make use of the fact that $s_2(17) = 5$ and $s_2(69) = 14$. Suppose $m = 5$. Then $k \geqslant 6$ and $n = 2^k + G_4(2) = 2^k + 2^4 + h_2 = 2^k + 17$ by hypothesis. Using Lemma 28 and the remark following Definition 26, we have

$$s_2(n) = s_2(2^k + 17) \leqslant s_2(17)(k + 1 - \lfloor \log_2 17 \rfloor) = 5(k - 3) < \frac{34}{5}(k - 5) + 13$$

$$= \frac{3(k-5)}{21 - 11}((2^k + 17) - (2^k + 11)) + 13 + 5(k - 5)$$

$$= \frac{F_4(k-5)}{h_6 - h_5}(n - G_k(5)) + F_7 + F_5(k-5) = J_{k,5}(n).$$

Next, suppose $m = 7$. Then $k \geqslant 8$ and $n = 2^k + G_6(4) = 2^k + 2^6 + 2^2 + 1 = 2^k + 69$. Using Lemma 28 and the paragraph following Definition 26 once again yields

$$s_2(n) = s_2(2^k + 69) \leqslant s_2(69)(k + 1 - \lfloor \log_2 69 \rfloor) = 14(k-5) < \frac{104}{21}(k-7) + 34 + 13(k-7)$$

$$= \frac{8(k-7)}{85 - 43}((2^k + 69) - (2^k + 43)) + 34 + 13(k-7)$$

$$= \frac{F_6(k-7)}{h_8 - h_7}(n - G_k(7)) + F_9 + F_7(k-7) = J_{k,7}(n). \qquad \square$$

We now return to our base assumption that b may represent any integer larger than 1.

Lemma 31. *Let m and y be integers such that $m \geqslant 4$ and $1 \leqslant y \leqslant \lfloor \frac{m-2}{2} \rfloor$. We have $L_{m-4y-1} \leqslant L_{m-5}$ and $L_{m-4y} \geqslant -L_{m-6}$.*

Proof. First, suppose m is even. We use the fact that $L_u < L_v$ for any odd integers u, v with $u < v$ to see that the maximum possible value of L_{m-4y-1} is obtained when $y = 1$. That is, $L_{m-4y-1} \leqslant L_{m-5}$. It is easy to see that $L_u \geqslant 2$ for all even integers u, so $L_{m-4y} \geqslant 2 > -2 \geqslant -L_{m-6}$.

Next, suppose m is odd. We use the fact that $L_{-u} = L_u < L_v = L_{-v}$ for any nonnegative even integers u, v with $u < v$ to find that the bilateral sequence $\ldots, L_{-4}, L_{-2}, L_0, L_2, L_4, \ldots$ is strictly convex. Therefore, the maximum possible value of L_{m-4y-1} occurs when $y = 1$ or when $y = \lfloor \frac{m-2}{2} \rfloor = \frac{m-3}{2}$. Because $L_{m-4(1)-1} = L_{m-5} = L_{5-m} = L_{m-4((m-3)/2)-1}$, the maximum possible value of L_{m-4y-1} is L_{m-5}. As $L_u < L_v$ for any odd integers u, v with $u < v$, the smallest possible value of L_{m-4y} occurs when $y = \frac{m-3}{2}$. That is, $L_{m-4y} \geqslant L_{m-4((m-3)/2)} = L_{6-m}$. Now, $L_{-u} = -L_u$ for all odd integers u, so $L_{m-4y} \geqslant L_{6-m} = -L_{m-6}$. $\qquad \square$

Lemma 32. *If m and y are integers such that $m \geqslant 4$ and $1 \leqslant y \leqslant \lfloor \frac{m-2}{2} \rfloor$, then*

$$\frac{L_{m-2} + L_{m-4y}}{5F_{m-1}} > \sqrt{5} - 2.$$

Furthermore, if $m \geqslant 4$ and $m \notin \{5, 7\}$, then

$$\frac{L_{m-2} + L_{m-4}}{5F_{m-1}} \geqslant \frac{8}{21}.$$

Proof. We will use the closed-form expressions $F_n = \dfrac{\phi^n - \overline{\phi}^n}{\sqrt{5}}$ and $L_n = \phi^n + \overline{\phi}^n$ for the Fibonacci and Lucas numbers. We also make use of the fact that $L_u \geqslant 2$ for any even integer u. For any integer j, we have

$$\frac{L_{m-j}}{5F_{m-1}} - \frac{1}{\phi^{j-1}\sqrt{5}} = \frac{\phi^{m-j} + \overline{\phi}^{m-j}}{\left(\phi^{m-1} - \overline{\phi}^{m-1}\right)\sqrt{5}} - \frac{1}{\phi^{j-1}\sqrt{5}} = \frac{1}{\sqrt{5}} \frac{\overline{\phi}^{m-j}\left(\phi^{j-1} + \overline{\phi}^{j-1}\right)}{\phi^{j-1}\left(\phi^{m-1} - \overline{\phi}^{m-1}\right)}$$

$$= \frac{1}{5\phi^{j-1}} \frac{\overline{\phi}^{m-j} L_{j-1}\sqrt{5}}{\phi^{m-1} - \overline{\phi}^{m-1}} = \frac{L_{j-1}}{5\phi^{j-1}F_{m-1}} \overline{\phi}^{m-j} = (-1)^{m-j} \frac{L_{j-1}}{5\phi^{m-1}F_{m-1}},$$

where the last equality follows from the identity $\overline{\phi} = \frac{-1}{\phi}$. Therefore,

$$\frac{L_{m-j}}{5F_{m-1}} = \frac{1}{\phi^{j-1}\sqrt{5}} + (-1)^{m-j}\frac{L_{j-1}}{5\phi^{m-1}F_{m-1}}. \tag{15}$$

If m is even, then we may set $j = 2$ in (15) and invoke the inequality $L_{m-4y} \geqslant 2$ (since $m - 4y$ is even) to find that

$$\frac{L_{m-2} + L_{m-4y}}{5F_{m-1}} \geqslant \frac{L_{m-2} + 2}{5F_{m-1}} > \frac{L_{m-2}}{5F_{m-1}} = \frac{1}{\phi\sqrt{5}} + (-1)^{m-2}\frac{L_1}{5\phi^{m-1}F_{m-1}}$$

$$= \frac{1}{\phi\sqrt{5}} + \frac{1}{5\phi^{m-1}F_{m-1}} > \frac{1}{\phi\sqrt{5}} > \sqrt{5} - 2.$$

Let us now suppose m is odd. By Lemma 31,

$$\frac{L_{m-2} + L_{m-4y}}{5F_{m-1}} \geqslant \frac{L_{m-2} - L_{m-6}}{5F_{m-1}} = \frac{L_{m-3} + L_{m-5}}{5F_{m-1}}.$$

Using (15), once with $j = 3$ and once with $j = 5$, yields

$$\frac{L_{m-3} + L_{m-5}}{5F_{m-1}} = \frac{1}{\phi^2\sqrt{5}} + (-1)^{m-3}\frac{L_2}{5\phi^{m-1}F_{m-1}} + \frac{1}{\phi^4\sqrt{5}} + (-1)^{m-5}\frac{L_4}{5\phi^{m-1}F_{m-1}}$$

$$> \frac{1}{\phi^2\sqrt{5}} + \frac{1}{\phi^4\sqrt{5}} = \sqrt{5} - 2$$

because $(-1)^{m-3}\dfrac{L_2}{5\phi^{m-1}F_{m-1}}$ and $(-1)^{m-5}\dfrac{L_4}{5\phi^{m-1}F_{m-1}}$ are positive. This completes the proof of the first inequality in the statement of the lemma.

Assume, now, that $m \geqslant 4$ and $m \notin \{5, 7\}$. We may use (15), once with $j = 2$ and again with $j = 4$, to see that

$$\frac{L_{m-2} + L_{m-4}}{5F_{m-1}} = \frac{1}{\phi\sqrt{5}} + (-1)^{m-2}\frac{L_1}{5\phi^{m-1}F_{m-1}} + \frac{1}{\phi^3\sqrt{5}} + (-1)^{m-4}\frac{L_3}{5\phi^{m-1}F_{m-1}}$$

$$= \frac{1}{\phi\sqrt{5}} + \frac{1}{\phi^3\sqrt{5}} + \frac{(-1)^m(L_1 + L_3)}{5\phi^{m-1}F_{m-1}} = \frac{1}{\phi\sqrt{5}} + \frac{1}{\phi^3\sqrt{5}} + \frac{(-1)^m}{\phi^{m-1}F_{m-1}}.$$

For $m \geqslant 4$ and $m \notin \{5, 7\}$, it is easy to see that $\dfrac{(-1)^m}{\phi^{m-1}F_{m-1}}$ attains its minimum when $m = 9$. Thus,

$$\frac{L_{m-2} + L_{m-4}}{5F_{m-1}} \geqslant \frac{1}{\phi\sqrt{5}} + \frac{1}{\phi^3\sqrt{5}} + \frac{(-1)^9}{\phi^8 F_8} = \frac{8}{21},$$

where this last equality is easily verified. Alternatively, we could have proven the second inequality stated in the lemma using the fact that $\dfrac{L_{m-2} + L_{m-4}}{5F_{m-1}} = \dfrac{F_{m-3}}{F_{m-1}}$. $\qquad\square$

Lemma 33. *Let k and m be integers such that $k \geqslant 5$ and $2 \leqslant m \leqslant k - 1$. Define a function $\Theta_{k,m} \colon \mathbb{R} \to \mathbb{R}$ by*

$$\Theta_{k,m}(x) = \frac{J_{k,m}(b^k + x)}{(b^k + x)^{\log_b \phi}}.$$

For any $x \geqslant 0$,

$$\frac{d}{dx}\Theta_{k,m}(b^k + x) \geqslant 0.$$

Proof. We begin with the easily-proven fact that $\dfrac{F_{m-2}}{F_{m-1}} \leqslant \dfrac{2}{3}$ for all integers $m \geqslant 4$. From this, we have

$$\frac{H_k(m)}{(k-m)F_{m-1}} = \frac{(k-m)F_m + F_{m+2}}{(k-m)F_{m-1}} \leqslant \frac{(k-m)F_m + (k-m)F_{m+2}}{(k-m)F_{m-1}}$$

$$= \frac{F_m + F_{m+2}}{F_{m-1}} = \frac{4F_{m-1} + 3F_{m-2}}{F_{m-1}} = 4 + 3\frac{F_{m-2}}{F_{m-1}} \leqslant 6.$$

One may easily show that

$$b^2 + b^{-1}\log_b \phi \geqslant 6\log_b \phi$$

for any choice of an integer $b \geqslant 2$ (if $b \geqslant 3$, this follows from the observation that $b^2 > 6\log_b \phi$). In addition, using Definition 3 and Lemma 4, we have $h_m \geqslant b^{m-2}$ and

$$h_{m+1} - h_m \leqslant \frac{b^m + b}{b+1} \leqslant \frac{b^m + b^{m-1}}{b+1} = b^{m-1}.$$

Therefore,

$$b^k + h_m \log_b \phi \geqslant b^k + b^{m-2}\log_b \phi = b^{m-1}(b^{k-m+1} + b^{-1}\log_b \phi) \geqslant b^{m-1}(b^2 + b^{-1}\log_b \phi)$$

$$\geqslant 6b^{m-1}\log_b \phi \geqslant 6(h_{m+1} - h_m)\log_b \phi \geqslant \frac{H_k(m)}{(k-m)F_{m-1}}(h_{m+1} - h_m)\log_b \phi.$$

Let $T_k(m) = \dfrac{(k-m)F_{m-1}}{h_{m+1} - h_m}$ so that we obtain $b^k + h_m \log_b \phi \geqslant \dfrac{H_k(m)}{T_k(m)}\log_b \phi$. Choose some $x \geqslant 0$. We have

$$xT_k(m)(1 - \log_b \phi) \geqslant 0 \geqslant T_k(m)\left[\frac{H_k(m)}{T_k(m)}\log_b \phi - h_m \log_b \phi - b^k\right]$$

$$= [H_k(m) - T_k(m)h_m]\log_b \phi - b^k T_k(m).$$

Adding $xT_k(m)\log_b \phi + b^k T_k(m)$ to each side of this last inequality yields

$$(b^k + x)T_k(m) \geqslant \log_b \phi \left(xT_k(m) + [H_k(m) - T_k(m)h_m]\right),$$

which we may rewrite as

$$(b^k + x)T_k(m) - \log_b \phi \left[H_k(m) + T_k(m)(x - h_m)\right] \geqslant 0. \tag{16}$$

Now, referring to the paragraph immediately following Definition 26, we see that $J_{k,m}(b^k+x) = T_k(m)(x-h_m)+H_k(m)$. Therefore, (16) becomes $(b^k+x)T_k(m) - J_{k,m}(b^k+x)\log_b \phi \geqslant 0$. In addition, $\dfrac{d}{dx}J_{k,m}(b^k+x) = T_k(m)$. Consequently,

$$\frac{d}{dx}\frac{J_{k,m}(b^k+x)}{(b^k+x)^{\log_b \phi}} = \frac{(b^k+x)^{\log_b \phi}T_k(m) - J_{k,m}(b^k+x)\log_b \phi(b^k+x)^{\log_b \phi-1}}{(b^k+x)^{2\log_b \phi}}$$

$$= \frac{(b^k+x)T_k(m) - J_{k,m}(b^k+x)\log_b \phi}{(b^k+x)^{\log_b \phi+1}} \geqslant 0. \qquad \square$$

We are finally ready to prove the main results of this section.

Theorem 34. *If k,m, and n are integers such that $2 \leqslant m < k$ and $G_k(m) < n < G_k(m+1)$, then*

$$s_b(n) < J_{k,m}(n).$$

Proof. By Definition 3 and Definition 24,

$$n \leqslant G_k(m+1) - 1 = b^k + \sum_{i=0}^{\left\lfloor \frac{m-2}{2} \right\rfloor} b^{m-1-2i}.$$

Therefore, we may let y be the smallest element of the set $\left\{0,1,\ldots,\left\lfloor \frac{m-2}{2} \right\rfloor\right\}$ such that $n \leqslant b^k + \sum_{i=0}^{y} b^{m-1-2i}$. Referring to Definition 8, we find that $s_b(n) \leqslant \mu(k,m-1,y)$ because $n \in I(k,m-1,y)$. Hence, it suffices to prove that

$$\mu(k,m-1,y) < J_{k,m}(n). \tag{17}$$

It follows from Theorem 23, Definition 20, Definition 24, and the identity $F_u = \dfrac{1}{5}(L_{u+1} + L_{u-1})$ that

$$\mu(k,m-1,0) = V(k,m-1,0) = \frac{1}{5}((k-m+1)(2L_{m+1} - L_m) + 2L_m + L_{m+1})$$

$$= (k-m)\left(\frac{1}{5}(2L_{m+1} - L_m)\right) + \frac{1}{5}(2L_{m+2} + L_{m-1})$$

$$= (k-m)\left(\frac{1}{5}(L_{m+1} + L_{m-1})\right) + \frac{1}{5}(L_{m+3} + L_{m+1}) = (k-m)F_m + F_{m+2} = H_k(m).$$

This shows that if $y = 0$, then

$$s_b(n) \leqslant \mu(k,m-1,0) = H_k(m) = J_{k,m}(G_k(m)) < J_{k,m}(n),$$

where we have used the trivial fact that $J_{k,m}$ is an increasing function. Therefore, we may assume $y > 0$. Because of the way we chose y, this means that $n > b^k + \sum\limits_{i=0}^{y-1} b^{m-1-2i} \geqslant b^k + b^{m-1}$. Note that this also forces $m \geqslant 4$ because $y \leqslant \left\lfloor \frac{m-2}{2} \right\rfloor$. By a similar token, it follows from Lemma 29 that

$$s_b(n) = s_2(n) \leqslant H_k(m) = J_{k,m}(G_k(m)) < J_{k,m}(n)$$

if $b = 2$ and $n < 2^k + G_{m-1}(m - 3)$. Therefore, we may assume $n \geqslant 2^k + G_{m-1}(m - 3)$ if $b = 2$. If $b = 2$, $m \in \{5, 7\}$, and $n = 2^k + G_{m-1}(m - 3)$, then the desired result is simply Lemma 30. This means that if $b = 2$ and $m \in \{5, 7\}$, then we may assume $n \geqslant 2^k + G_{m-1}(m - 3) + 1$. We have two cases to consider.

Case 1: In this case, assume that $\mu(k, m - 1, y) = V(k, m - 1, y)$. Recall Definition 20 to see that

$$V(k, m - 1, y) = \frac{1}{5}((k - m + 1)(2L_{m+1} - L_{m-4y}) + 2L_m + L_{m-4y+1})$$

$$= \frac{1}{5}((k - m)(2L_{m+1} - L_{m-4y}) + 2L_{m+2} + L_{m-4y-1}).$$

We now use the fact that $L_{m-4y-1} < L_{m-1}$ (which follows immediately from Lemma 31) as well as the identity $\frac{1}{5}(L_{u+1} + L_{u-1}) = F_u$ to see that

$$V(k, m - 1, y) < \frac{1}{5}((k - m)(2L_{m+1} - L_{m-4y}) + 2L_{m+2} + L_{m-1})$$

$$= \frac{1}{5}(k - m)(2L_{m+1} - L_{m-4y}) + \frac{1}{5}(L_{m+3} + L_{m+1}) = \frac{1}{5}(k - m)(2L_{m+1} - L_{m-4y}) + F_{m+2}.$$

Because

$$J_{k,m}(n) = \frac{F_{m-1}(k - m)}{h_{m+1} - h_m}(n - G_k(m)) + H_k(m)$$

$$= \frac{F_{m-1}(k - m)}{h_{m+1} - h_m}(n - b^k - h_m) + F_m(k - m) + F_{m+2},$$

we simply need to show that

$$\frac{1}{5}(k - m)(2L_{m+1} - L_{m-4y}) \leqslant \frac{F_{m-1}(k - m)}{h_{m+1} - h_m}(n - b^k - h_m) + F_m(k - m)$$

in order to obtain (17). After dividing each side by $k - m$ and using the identity

$$\frac{1}{5}(2L_{m+1} - L_{m-4y}) = F_{m+1} - \frac{1}{5}(L_{m-2} + L_{m-4y}),$$

we find that this last inequality becomes

$$F_{m+1} - \frac{1}{5}(L_{m-2} + L_{m-4y}) \leqslant F_{m-1}\left(1 - \frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m}\right) + F_m,$$

which we may rewrite as

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} \leqslant \frac{L_{m-2} + L_{m-4y}}{5F_{m-1}} \tag{18}$$

after subtracting F_{m+1} from each side and then rearranging terms. We will prove (18) in each of the following two subcases.

Subcase 1: In this subcase, assume $b = 2$ and $y = 1$. Recall that we mentioned at the beginning of the proof that we may assume $n \geqslant 2^k + G_{m-1}(m-3)$ when $b = 2$. Furthermore, we may assume $n \geqslant 2^k + G_{m-1}(m-3) + 1$ if $m \in \{5, 7\}$. If $m = 5$, then

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} \leqslant \frac{h_{m+1} - (G_{m-1}(m-3) + 1)}{h_{m+1} - h_m} = \frac{h_6 - (G_4(2) + 1)}{h_6 - h_5}$$

$$= \frac{(2^4 + 2^2 + 1) - (2^4 + 1 + 1)}{(2^4 + 2^2 + 1) - (2^3 + 2 + 1)} = \frac{3}{10} < \frac{1}{3} = \frac{L_3 + L_1}{5F_4} = \frac{L_{m-2} + L_{m-4y}}{5F_{m-1}}.$$

If $m = 7$, then

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} \leqslant \frac{h_{m+1} - (G_{m-1}(m-3) + 1)}{h_{m+1} - h_m} = \frac{h_8 - (G_6(4) + 1)}{h_8 - h_7}$$

$$= \frac{(2^6 + 2^4 + 2^2 + 1) - (2^6 + 2^2 + 1 + 1)}{(2^6 + 2^4 + 2^2 + 1) - (2^5 + 2^3 + 2 + 1)} = \frac{5}{14} < \frac{3}{8} = \frac{L_5 + L_3}{5F_6} = \frac{L_{m-2} + L_{m-4y}}{5F_{m-1}}.$$

This proves (18) if $m \in \{5, 7\}$, so we will assume $m \notin \{5, 7\}$. By Definition 3,

$$h_{m+1} = 1 + \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} 2^{m-1-2i} = 1 + 2^{m-1} + 2^{m-3} + \sum_{i=2}^{\lfloor \frac{m-2}{2} \rfloor} 2^{m-1-2i}$$

$$= 1 + 2^{m-1} + 2^{m-3} + \sum_{i=0}^{\lfloor \frac{m-6}{2} \rfloor} 2^{m-5-2i} = 2^{m-3} + 2^{m-1} + h_{m-3} = 2^{m-3} + G_{m-1}(m-3).$$

Therefore, using Lemma 4 to write $h_{m+1} - h_m = \dfrac{2^m + (-1)^m \cdot 2}{3}$, we have

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} \leqslant \frac{h_{m+1} - G_{m-1}(m-3)}{h_{m+1} - h_m} = \frac{2^{m-3}}{h_{m+1} - h_m} = 3\frac{2^{m-3}}{2^m + (-1)^m \cdot 2}.$$

If m is even, then

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} \leqslant 3\frac{2^{m-3}}{2^m + 2} < 3\frac{2^{m-3}}{2^m} = \frac{3}{8} < \frac{8}{21}.$$

If m is odd, then $m \geqslant 9$, so

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} \leqslant 3\frac{2^{m-3}}{2^m - 2} = \frac{3}{8 - 2^{4-m}} \leqslant \frac{3}{8 - 2^{4-9}} = \frac{32}{85} < \frac{8}{21}.$$

Either way, $\dfrac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \dfrac{8}{21}$. Lemma 32 tells us that $\dfrac{8}{21} \leqslant \dfrac{L_{m-2} + L_{m-4}}{5F_{m-1}}$, so we obtain (18).

Subcase 2: In this subcase, suppose either $y > 1$ or $b \neq 2$. It follows from our choice of y that $n > b^k + \displaystyle\sum_{i=0}^{y-1} b^{m-1-2i}$. Using Definition 3 and Lemma 4, we see that

$$h_{m+1} - (n - b^k) = 1 + \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} b^{m-1-2i} - (n - b^k) < 1 + \sum_{i=0}^{\infty} b^{m-1-2i} - (n - b^k)$$

$$< 1 + \sum_{i=0}^{\infty} b^{m-1-2i} - \sum_{i=0}^{y-1} b^{m-1-2i} = 1 + \sum_{i=y}^{\infty} b^{m-1-2i} = 1 + \frac{b^{m+1-2y}}{b^2 - 1}$$

and

$$h_{m+1} - h_m \geqslant \frac{b^m - b}{b + 1}.$$

Therefore,

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \frac{1 + b^{m+1-2y}/(b^2 - 1)}{(b^m - b)/(b+1)} = \frac{b^2 + b^{m+1-2y} - 1}{(b^m - b)(b - 1)}. \tag{19}$$

If we treat m as a continuous real variable, then

$$\frac{\partial}{\partial m} \left(\frac{b^2 + b^{m+1-2y} - 1}{(b^m - b)(b - 1)} \right) = \frac{1}{b - 1} \frac{\partial}{\partial m} \left(\frac{b^2 + b^{m+1-2y} - 1}{b^m - b} \right)$$

$$= \frac{1}{b - 1} \frac{(b^m - b)b^{m+1-2y} \log b - (b^2 + b^{m+1-2y} - 1)b^m \log b}{(b^m - b)^2}$$

$$= \frac{\log b}{(b - 1)(b^m - b)^2} (-b^{m+2-2y} - b^{m+2} + b^m) < 0.$$

This shows that $\dfrac{b^2 + b^{m+1-2y} - 1}{(b^m - b)(b - 1)}$ is decreasing in m.

Suppose $y \geqslant 2$. This forces $m \geqslant 6$ because $y \leqslant \lfloor \frac{m-2}{2} \rfloor$. Because $\dfrac{b^2 + b^{m+1-2y} - 1}{(b^m - b)(b - 1)}$ is decreasing in m, we see from (19) that

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \frac{b^2 + b^{m+1-2y} - 1}{(b^m - b)(b - 1)} \leqslant \frac{b^2 + b^{7-2y} - 1}{(b^6 - b)(b - 1)} \leqslant \frac{b^2 + b^3 - 1}{(b^6 - b)(b - 1)}.$$

If $b = 2$, then

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \frac{b^2 + b^3 - 1}{(b^6 - b)(b - 1)} = \frac{11}{62} < \sqrt{5} - 2.$$

If $b \geqslant 3$, then

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \frac{b^2 + b^3 - 1}{(b^6 - b)(b - 1)} < \frac{b^3 + b^2 + b + 1 + b^{-1}}{(b^6 - b)(b - 1)}$$

$$= \frac{b^4 + b^3 + b^2 + b + 1}{b^2(b^5 - 1)(b - 1)} = \frac{1}{b^2(b - 1)^2} \leqslant \frac{1}{3^2(3 - 1)^2} = \frac{1}{36} < \sqrt{5} - 2.$$

No matter the value of b, we find from Lemma 32 that

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \sqrt{5} - 2 < \frac{L_{m-2} + L_{m-4}y}{5F_{m-1}},$$

which is (18).

We have proven (18) when $y \geqslant 2$, so assume $y = 1$. We assumed either $y > 1$ or $b \neq 2$, so we must have $b \geqslant 3$. Since $\dfrac{b^2 + b^{m+1-2y} - 1}{(b^m - b)(b - 1)}$ is decreasing in m and $m \geqslant 4$, it follows from (19) that

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \frac{b^2 + b^{m+1-2y} - 1}{(b^m - b)(b - 1)} \leqslant \frac{b^2 + b^{5-2y} - 1}{(b^4 - b)(b - 1)} = \frac{b^2 + b^3 - 1}{(b^4 - b)(b - 1)}.$$

If $b = 3$, then

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \frac{b^2 + b^3 - 1}{(b^4 - b)(b - 1)} = \frac{35}{156} < \sqrt{5} - 2,$$

which proves (18) with the help of Lemma 32. If $b \geqslant 4$, then

$$\frac{h_{m+1} - (n - b^k)}{h_{m+1} - h_m} < \frac{b^2 + b^3 - 1}{(b^4 - b)(b - 1)} < \frac{b^3 + b^2 + b}{(b^4 - b)(b - 1)} = \frac{b^2 + b + 1}{(b^3 - 1)(b - 1)}$$

$$= \frac{1}{(b - 1)^2} \leqslant \frac{1}{(4 - 1)^2} = \frac{1}{9} < \sqrt{5} - 2,$$

which proves (18) once again.

Case 2: Here, assume $\mu(k, m - 1, y) \neq V(k, m - 1, y)$. Referring to Theorem 23, we see that $\nu(k, m-1, y) = \gamma(k, m-1, x) + 1$ and $\mu(k, m-1, y) = V(k, m-1, x)$ for some $x < y$. One finds from Definition 19 that $\gamma(k, m - 1, x) + 1 > b^k + \sum_{i=0}^{x-1} b^{m-1-2i}$. By Lemma 21, $\gamma(k, m - 1, x) + 1 \in I(k, m - 1, x)$. In other words, if we let y' be the smallest element of the set $\left\{ 0, 1, \ldots, \left\lfloor \frac{m-2}{2} \right\rfloor \right\}$ such that $\gamma(k, m - 1, x) + 1 \leqslant b^k + \sum_{i=0}^{y'} b^{m-1-2i}$, then $y' = x$. By definition, $\mu(k, m - 1, x)$ is the maximum value of $s_b(j)$ as j ranges over all elements of $I(k, m - 1, x)$. Because $\gamma(k, m - 1, x) + 1 \in I(k, m - 1, x)$, this means that

$$\mu(k, m - 1, x) \geqslant s_b(\gamma(k, m - 1, x) + 1) = s_b(\nu(k, m - 1, y)) = \mu(k, m - 1, y),$$

where we have used Definition 8 to deduce the last equality. Similarly, $\mu(k, m-1, y)$ is the maximum value of $s_b(j)$ as j ranges over all elements of $I(k, m-1, y)$. Since $x < y$, $I(k, m-1, x) \subseteq I(k, m-1, y)$. Therefore, $\mu(k, m-1, x) \leqslant \mu(k, m-1, y)$. This shows that

$$\mu(k, m-1, x) = \mu(k, m-1, y) = V(k, m-1, x).$$

Now, choose some integer $n' \in \left(b^k + \sum_{i=0}^{x-1} b^{m-1-2i}, b^k + \sum_{i=0}^{x} b^{m-1-2i} \right]$ (note that $\gamma(k, m-1, x) + 1$ is in this interval). Because x is the smallest element of the set $\left\{ 0, 1, \ldots, \left\lfloor \frac{m-2}{2} \right\rfloor \right\}$ satisfying $n' \leqslant b^k + \sum_{i=0}^{x} b^{m-1-2i}$ and $\mu(k, m-1, x) = V(k, m-1, x)$, it follows from Case 1 that $s_b(n') < J_{k,m}(n')$. We may set $n' = \gamma(k, m-1, x) + 1$, so $s_b(\gamma(k, m-1, x) + 1) < J_{k,m}(\gamma(k, m-1, x) + 1)$. Using the fact that $\gamma(k, m-1, x) + 1 \leqslant b^k + \sum_{i=0}^{x} b^{m-1-2i} \leqslant b^k + \sum_{i=0}^{y-1} b^{m-1-2i} < n$, we have

$$\mu(k, m-1, y) = s_b(\gamma(k, m-1, x) + 1) < J_{k,m}(\gamma(k, m-1, x) + 1) < J_{k,m}(n).$$

This proves (17) and completes the proof of the theorem. $\qquad \square$

Corollary 35. *We have*

$$\limsup_{n \to \infty} \frac{s_b(n)}{n^{\log_b \phi}} = \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}.$$

Proof. Corollary 7 states that

$$\limsup_{n \to \infty} \frac{s_b(n)}{n^{\log_b \phi}} \geqslant \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}},$$

so we will now prove the reverse inequality. For each integer $k \geqslant 3$, let $u_k = G_k(k) = b^k + h_k$. Let $\theta(x) = \lceil \log_b(x) \rceil - 1$ for each $x > 0$. Recall that we showed in the proof of Corollary 7 that

$$\lim_{k \to \infty} \frac{s_b(u_k)}{u_k^{\log_b \phi}} = \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}}. \tag{20}$$

We will show that

$$\frac{s_b(n)}{n^{\log_b \phi}} \leqslant \frac{s_b(u_{\theta(n)})}{(u_{\theta(n)})^{\log_b \phi}} \quad \text{for all } n > b^5. \tag{21}$$

It will then follow from (20) and (21) that

$$\limsup_{n \to \infty} \frac{s_b(n)}{n^{\log_b \phi}} \leqslant \limsup_{n \to \infty} \frac{s_b(u_{\theta(n)})}{(u_{\theta(n)})^{\log_b \phi}} = \limsup_{k \to \infty} \frac{s_b(u_k)}{u_k^{\log_b \phi}} = \frac{(b^2 - 1)^{\log_b \phi}}{\sqrt{5}},$$

which will complete the proof. In order to derive (21), let us choose some integer $n > b^5$. Let $\theta = \theta(n)$, and note that $b^\theta < n \leqslant b^{\theta+1}$. It follows from Proposition 6 that

$$s_b(n) \leqslant F_{\theta+2} = s_b(b^\theta + h_\theta) = s_b(u_\theta)$$

(if $n = b^{\theta+1}$, then Proposition 6 does not apply, but the inequality still holds because $s_b(n) = 1 < F_{\theta+2}$). If $n \geqslant u_\theta$, then this shows that

$$\frac{s_b(n)}{n^{\log_b \phi}} \leqslant \frac{s_b(u_\theta)}{n^{\log_b \phi}} \leqslant \frac{s_b(u_\theta)}{u_\theta^{\log_b \phi}},$$

which is the inequality we seek to prove. Therefore, we will assume $n < u_\theta$.

For any $t \in \{2, 3, \ldots, \theta - 1\}$, we may use Lemma 33 to see that

$$\Theta_{\theta,t}(G_\theta(t)) = \Theta_{\theta,t}(b^\theta + h_t) \leqslant \Theta_{\theta,t}(b^\theta + h_{t+1}) = \Theta_{\theta,t}(G_\theta(t+1)),$$

where we have preserved the notation from that lemma. Furthermore, with the help of Definition 26, we find that

$$\Theta_{\theta,t}(G_\theta(t+1)) = \frac{J_{\theta,t}(G_\theta(t+1))}{G_\theta(t+1)^{\log_b \phi}} = \frac{H_\theta(t+1)}{G_\theta(t+1)^{\log_b \phi}} = \frac{J_{\theta,t+1}(G_\theta(t+1))}{G_\theta(t+1)^{\log_b \phi}} = \Theta_{\theta,t+1}(G_\theta(t+1)) \tag{22}$$

for all $t \in \{2, 3, \ldots, \theta - 1\}$. Therefore,

$$\Theta_{\theta,t}(G_\theta(t)) \leqslant \Theta_{\theta,t+1}(G_\theta(t+1))$$

for all $t \in \{2, 3, \ldots, \theta - 1\}$. That is,

$$\Theta_{\theta,2}(G_\theta(2)) \leqslant \Theta_{\theta,3}(G_\theta(3)) \leqslant \cdots \leqslant \Theta_{\theta,\theta}(G_\theta(\theta)). \tag{23}$$

Because $b^\theta < n < u_\theta = G_\theta(\theta)$, there exists some $m \in \{2, 3, \ldots, \theta - 1\}$ such that $G_\theta(m) \leqslant n < G_\theta(m+1)$. In other words, we may write $n = b^\theta + x$ for some $x \in \{h_m, h_m + 1, \ldots, h_{m+1} - 1\}$. By Lemma 33 and (22),

$$\Theta_{\theta,m}(n) = \Theta_{\theta,m}(b^\theta + x) \leqslant \Theta_{\theta,m}(b^\theta + h_{m+1}) = \Theta_{\theta,m}(G_\theta(m+1)) = \Theta_{\theta,m+1}(G_\theta(m+1)).$$

It follows from (23) that $\Theta_{\theta,m+1}(G_\theta(m+1)) \leqslant \Theta_{\theta,\theta}(G_\theta(\theta))$, so

$$\Theta_{\theta,m}(n) \leqslant \Theta_{\theta,\theta}(G_\theta(\theta)). \tag{24}$$

If $n = G_\theta(m)$, then it follows easily from Theorem 25 and Definition 26, that

$$s_b(n) = s_b(G_\theta(m)) = H_\theta(m) = J_{\theta,t}(G_\theta(m)) = J_{\theta,m}(n).$$

If $n \neq G_\theta(m)$, then $G_\theta(m) < n < G_\theta(m+1)$, so Theorem 34 shows that $s_b(n) < J_{\theta,m}(n)$. Either way, $s_b(n) \leqslant J_{\theta,m}(n)$. Hence, using (24), we obtain

$$\frac{s_b(n)}{n^{\log_b \phi}} \leqslant \frac{J_{\theta,m}(n)}{n^{\log_b \phi}} = \Theta_{\theta,m}(n) \leqslant \Theta_{\theta,\theta}(G_\theta(\theta)) = \frac{J_{\theta,\theta}(G_\theta(\theta))}{G_\theta(\theta)^{\log_b \phi}} = \frac{s_b(G_\theta(\theta))}{G_\theta(\theta)^{\log_b \phi}} = \frac{s_b(u_\theta)}{u_\theta^{\log_b \phi}},$$

which proves (21). $\qquad \square$

Theorem 36. *For any real $x \geqslant 0$ and any integer $k \geqslant 3$, let*

$$\beta(x) = \log_b((b^2 - 1)x + 1)$$

and

$$f_k(x) = \frac{1}{\sqrt{5}}\left[(k - \beta(x))(\phi^{\beta(x)} + \phi^{-\beta(x)}) + \phi^{\beta(x)+2} + \phi^{-\beta(x)-2}\right].$$

If k and n are positive integers such that $k \geqslant 3$ and $b^k < n \leqslant b^k + h_k$, then $s_b(n) < f_k(n - b^k)$.

Proof. Let $k \geqslant 3$ be an integer. Let $\alpha = \dfrac{b^k + b^2 - b - 1}{b^2 - 1}$, and note that $h_k \leqslant \alpha$. We will show that f_k is increasing and concave down on the open interval $(1, \alpha)$. The desired inequality will then follow quite easily. Observe that we may write $f_k(x) = g_k(\beta(x))$, where

$$g_k(x) = \frac{1}{\sqrt{5}}\left[(k - x)(\phi^x + \phi^{-x}) + \phi^{x+2} + \phi^{-x-2}\right].$$

If $x \leqslant \log_b(b^k + b^2 - b)$, then

$$k - x \geqslant k - \log_b(b^k + b^2 - b) = -\log_b\left(1 + \frac{b - 1}{b^{k-1}}\right) \geqslant -\frac{b - 1}{b^{k-1}\log b} \geqslant -\frac{b - 1}{b^2 \log b} \geqslant -\frac{1}{4\log 2},$$

where we have used the inequality $\log_b(1 + u) \leqslant \dfrac{u}{\log b}$ that holds for all $u > -1$ (as well as the inequalities $k \geqslant 3$ and $b \geqslant 2$). Therefore, if $2 \leqslant x \leqslant \log_b(b^k + b^2 - b)$, then

$$g_k'(x) = \frac{1}{\sqrt{5}}\left[(k - x)(\phi^x - \phi^{-x})\log\phi - (\phi^x + \phi^{-x}) + (\phi^{x+2} - \phi^{-x-2})\log\phi\right]$$

$$\geqslant \frac{1}{\sqrt{5}}\left[-\frac{1}{4\log 2}(\phi^x - \phi^{-x})\log\phi - (\phi^x + \phi^{-x}) + (\phi^{x+2} - \phi^{-x-2})\log\phi\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^x\left(\phi^2\log\phi - \frac{\log\phi}{4\log 2} - 1\right) + \phi^{-x}\left(-\phi^{-2}\log\phi + \frac{\log\phi}{4\log 2} - 1\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[C_1\phi^x + C_2\phi^{-x}\right],$$

where

$$C_1 = \phi^2\log\phi - \frac{\log\phi}{4\log 2} - 1 \approx 0.086\ldots$$

and

$$C_2 = -\phi^{-2}\log\phi + \frac{\log\phi}{4\log 2} - 1 \approx -1.010\ldots.$$

If $x \geqslant 2$, then $C_1\phi^x \geqslant C_1\phi^2$ and $C_2\phi^{-x} \geqslant C_2\phi^{-2}$. This means that

$$g_k'(x) \geqslant \frac{1}{\sqrt{5}}\left[C_1\phi^x + C_2\phi^{-x}\right] \geqslant C_1\phi^2 + C_2\phi^{-2} > 0$$

whenever $2 \leqslant x \leqslant \log_b(b^k + b^2 - b)$. Now, if $1 \leqslant x \leqslant \alpha$, then $2 \leqslant \beta(x) \leqslant \log_b(b^k + b^2 - b)$ and $\beta'(x) > 0$. Consequently,

$$f_k'(x) = \frac{d}{dx} g_k(\beta(x)) = g_k'(\beta(x)) \beta'(x) > 0$$

for all $x \in [1, \alpha]$. This shows that f_k is increasing on the open interval $(1, \alpha)$.

We now wish to show that f_k is concave down on the interval $(1, \alpha)$. We first calculate

$$\beta'(x) = \frac{1}{\log b} \frac{b^2 - 1}{(b^2 - 1)x + 1}$$

and

$$\beta''(x) = -\frac{1}{\log b} \left(\frac{b^2 - 1}{(b^2 - 1)x + 1} \right)^2 = -(\beta'(x))^2 \log b$$

to obtain

$$f_k''(x) = \frac{d^2}{d^2 x} g_k(\beta(x)) = \frac{d}{dx} (g_k'(\beta(x)) \beta'(x)) = g_k'(\beta(x)) \beta''(x) + (\beta'(x))^2 g_k''(\beta(x))$$

$$= (\beta'(x))^2 (g_k''(\beta(x)) - g_k'(\beta(x)) \log b).$$

Therefore, to show that f_k is concave down on the interval $(1, \alpha)$, we just need to show that $g_k'(\beta(x)) \log b > g_k''(\beta(x))$ for all $x \in (1, \alpha)$. To do so, it suffices to show that

$$\frac{\sqrt{5} \log b}{\log \phi} g_k'(x) > \frac{\sqrt{5}}{\log \phi} g_k''(x) \tag{25}$$

for all $x \in (2, \log_b(b^k + b^2 - b))$.

Suppose $x \in (2, \log_b(b^k + b^2 - b))$. We have

$$-(\phi^x + \phi^{-x}) + (\phi^{x+2} - \phi^{-x-2}) \log \phi = \phi^x (\phi^2 \log \phi - 1) - \phi^{-x} (\phi^{-2} \log \phi + 1)$$

$$\geqslant \phi^2 (\phi^2 \log \phi - 1) - \phi^{-x} (\phi^{-2} \log \phi + 1) \geqslant \phi^2 (\phi^2 \log \phi - 1) - \phi^{-2} (\phi^{-2} \log \phi + 1) > 0,$$

so

$$g_k'(x) = \frac{1}{\sqrt{5}} \left[(k - x)(\phi^x - \phi^{-x}) \log \phi - (\phi^x + \phi^{-x}) + (\phi^{x+2} - \phi^{-x-2}) \log \phi \right]$$

$$> \frac{1}{\sqrt{5}} \left[(k - x)(\phi^x - \phi^{-x}) \log \phi \right].$$

In other words,

$$\frac{\sqrt{5} \log b}{\log \phi} g_k'(x) > (k - x)(\phi^x - \phi^{-x}) \log b. \tag{26}$$

Using the inequality $\log_b(1 + u) \leqslant \dfrac{u}{\log b}$, which holds for all $u > -1$, we find that

$$k - x > k - \log_b(b^k + b^2 - b) = -\log_b \left(1 + \frac{b - 1}{b^{k-1}} \right) \geqslant -\frac{b - 1}{b^{k-1} \log b},$$

so it follows from the assumption that $k \geqslant 3$ that

$$k - x > -\frac{b-1}{b^2 \log b}. \tag{27}$$

In addition, since $x > 2$,

$$\phi^x(\log b - \log \phi) - \phi^{-x}(\log b + \log \phi) > \phi^2(\log b - \log \phi) - \phi^{-2}(\log b + \log \phi) > 0. \tag{28}$$

Combining (26), (27), and (28) with the fact that

$$\frac{\sqrt{5}}{\log \phi} g_k''(x) = \log \phi \left[(k-x)(\phi^x + \phi^{-x}) - 2\frac{\phi^x - \phi^{-x}}{\log \phi} + (\phi^{x+2} + \phi^{-x-2}) \right]$$

yields

$$\frac{\sqrt{5} \log b}{\log \phi} g_k'(x) - \frac{\sqrt{5}}{\log \phi} g_k''(x) >$$

$$(k-x)(\phi^x - \phi^{-x}) \log b - (k-x)(\phi^x + \phi^{-x}) \log \phi + 2(\phi^x - \phi^{-x}) - (\phi^{x+2} + \phi^{-x-2}) \log \phi$$

$$= (k-x)(\phi^x(\log b - \log \phi) - \phi^{-x}(\log b + \log \phi)) + \phi^x(2 - \phi^2 \log \phi) - \phi^{-x}(2 + \phi^{-2} \log \phi)$$

$$> -\frac{b-1}{b^2 \log b}(\phi^x(\log b - \log \phi) - \phi^{-x}(\log b + \log \phi)) + \phi^x(2 - \phi^2 \log \phi) - \phi^{-x}(2 + \phi^{-2} \log \phi)$$

$$= A_1 \phi^x - A_2 \phi^{-x},$$

where

$$A_1 = 2 - \phi^2 \log \phi - \frac{b-1}{b^2 \log b}(\log b - \log \phi)$$

and

$$A_2 = 2 + \phi^{-2} \log \phi - \frac{b-1}{b^2 \log b}(\log b + \log \phi).$$

Because $b \geqslant 2$, it is easy to see that

$$A_1 > 2 - \phi^2 \log \phi - \frac{b-1}{b^2} \geqslant 2 - \phi^2 \log \phi - \frac{1}{4} > 0.49 \quad \text{and} \quad A_2 < 2 + \phi^{-2} \log \phi < 2.19.$$

Since $x > 2$,

$$\frac{\sqrt{5} \log b}{\log \phi} g_k'(x) - \frac{\sqrt{5}}{\log \phi} g_k''(x) > A_1 \phi^x - A_2 \phi^{-x} > 0.49 \phi^x - 2.19 \phi^{-x} > 0.49 \phi^2 - 2.19 \phi^{-2} > 0,$$

which proves (25).

We have shown that f_k is increasing and concave down on the interval $(1, \alpha)$. Now, suppose m is odd and $3 \leqslant m \leqslant k$. Because m is odd, one may easily show that $h_m = 1 + \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} b^{m-2-2i} = \frac{b^m + b^2 - b - 1}{b^2 - 1}$. Since

$$G_k(m) - b^k = h_m = \frac{b^m + b^2 - b - 1}{b^2 - 1} > \frac{b^m - 1}{b^2 - 1},$$

we may use the fact that f_k is increasing on $(1, \alpha)$ to see that

$$f_k(G_k(m) - b^k) > f_k\left(\frac{b^m - 1}{b^2 - 1}\right) = g_k\left(\beta\left(\frac{b^m - 1}{b^2 - 1}\right)\right) = g_k(m)$$

$$= \frac{1}{\sqrt{5}}\left[(k - m)(\phi^m + \phi^{-m}) + \phi^{m+2} + \phi^{-m-2}\right]$$

$$= (k - m)\frac{\phi^m - (-1/\phi)^m}{\sqrt{5}} + \frac{\phi^{m+2} - (-1/\phi)^{m+2}}{\sqrt{5}} = (k - m)F_m + F_{m+2} = H_k(m).$$

Similarly, if m is even and $2 \leqslant m \leqslant k$, then

$$f_k(G_k(m) - b^k) = f_k(h_m) = f_k\left(\frac{b^m - 1}{b^2 - 1}\right) = g_k\left(\beta\left(\frac{b^m - 1}{b^2 - 1}\right)\right) = g_k(m)$$

$$= \frac{1}{\sqrt{5}}\left[(k - m)(\phi^m + \phi^{-m}) + \phi^{m+2} + \phi^{-m-2}\right]$$

$$> \frac{1}{\sqrt{5}}\left[(k - m)(\phi^m - \phi^{-m}) + \phi^{m+2} - \phi^{-m-2}\right]$$

$$= (k - m)\frac{\phi^m - (-1/\phi)^m}{\sqrt{5}} + \frac{\phi^{m+2} - (-1/\phi)^{m+2}}{\sqrt{5}} = (k - m)F_m + F_{m+2} = H_k(m).$$

Hence, $f_k(G_k(m) - b^k) > H_k(m)$ for all $m \in \{2, 3, \ldots, k\}$. We may now prove that $s_b(n) < f_k(n - b^k)$ for all $n \in \{b^k + 1, b^k + 2, \ldots, b^k + h_k\}$. Choose such an integer n. If $n = G_k(m)$ for some $m \in \{2, 3, \ldots, k\}$, then it follows from Theorem 25 and the preceding discussion that

$$s_b(n) = H_k(m) < f_k(n - b^k).$$

Therefore, we will assume $n \neq G_k(m)$ for all $m \in \{2, 3, \ldots, k\}$. Note that there exists some $m \in \{2, 3, \ldots, k - 1\}$ such that $G_k(m) < n < G_k(m + 1)$. Let $\mathcal{C}$ be the curve $\{(x, f_k(x - b^k)) : G_k(m) \leqslant x \leqslant G_k(m + 1)\}$, and let $\mathcal{L} = \{J_{k,m}(x) : G_k(m) \leqslant x \leqslant G_k(m + 1)\}$ be the line segment connecting the points $(G_k(m), H_k(m))$ and $(G_k(m+1), H_k(m+1))$. Because $f_k(x)$ is concave down on the interval $(1, \alpha)$, the curve $\mathcal{C}$ is concave down. Since $f_k(G_k(m) - b^k) > H_k(m)$ and $f_k(G_k(m + 1) - b^k) > H_k(m + 1)$, the curve $\mathcal{C}$ must lie above the line segment $\mathcal{L}$. In particular, $J_{k,m}(n) < f_k(n - b^k)$. By Theorem 34, $s_b(n) < J_{k,m}(n) < f_k(n - b^k)$, as desired. $\qquad \square$

4 Concluding Remarks

We wish to acknowledge some of the potential uses and extensions of results derived in this paper. First, we note that Theorem 12 allows us to derive explicit formulas for the values of $s_b(n)$ for integers n whose ordinary base b expansions have certain forms. For example, we were able to invoke Theorem 12 in the proof of Theorem 25 in order to show
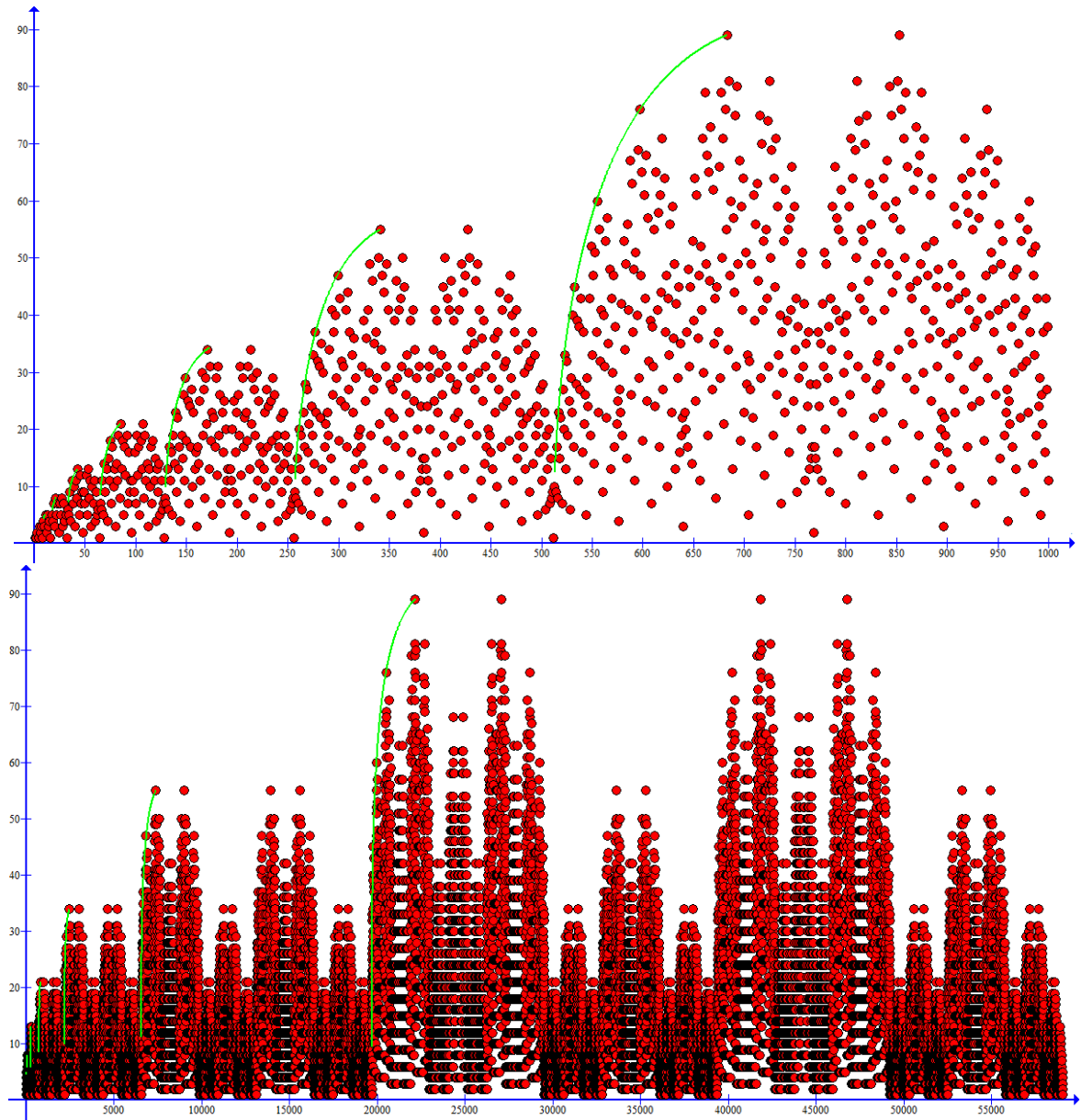
Figure 2: Two plots of $s_b(n)$ for $1 \leqslant n \leqslant b^{10}$. The top plot uses the value $b = 2$, while the bottom uses $b = 3$. For each $k \in \{3, 4, \ldots, 9\}$, the graph of $f_k(x - b^k)$ for $b^k \leqslant x \leqslant b^k + h_k$ is shown in green.
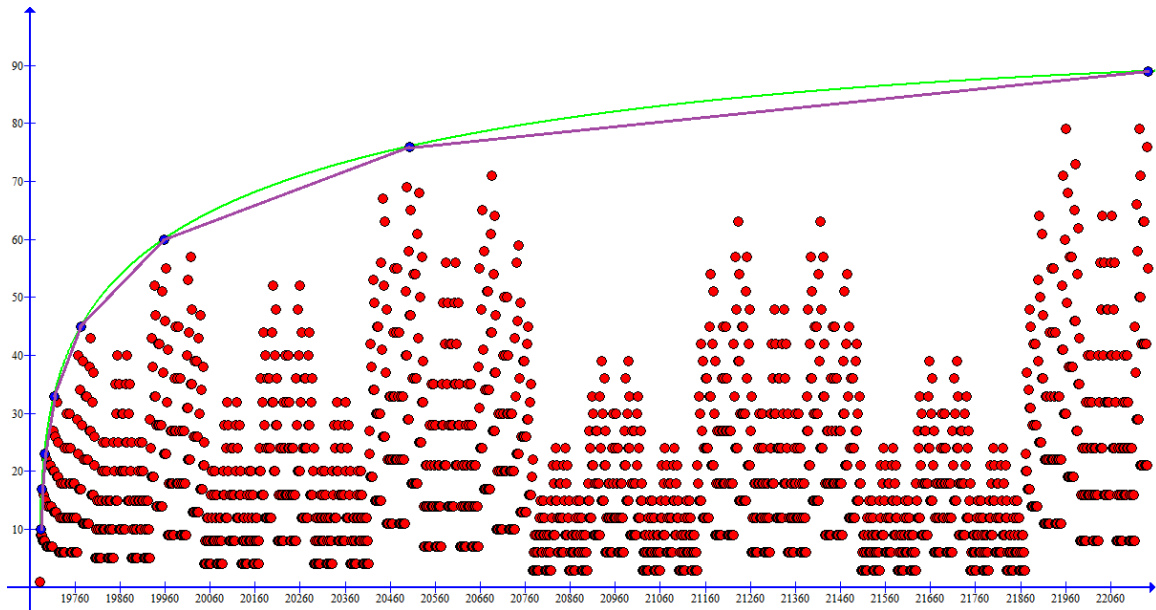
Figure 3: A plot of $s_3(n)$ for $3^9 \leqslant n \leqslant 3^9 + h_9$ (where h_9 is defined with $b = 3$). The graph of $f_k(x - 3^9)$ is colored green. The upper bound from Theorem 34 is the polygonal path colored purple. The points $(G_9(m), H_9(m))$ for $m \in \{2, 3, \ldots, 9\}$ are colored blue.

that $s_b(G_k(m)) = H_k(m)$. As another example, it is possible to use Theorem 12 to show that

$$s_b(1 + b^{x_1} + b^{x_1+x_2} + b^{x_1+x_2+x_3}) = x_1 x_2 x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 + x_2 - 1 \qquad (29)$$

for any positive integers x_1, x_2, x_3. The equation (29) appears with several similar identities (many of which can be deduced from Theorem 12) in [3].

Second, arguments based on symmetry and periodicity may be used to extend the upper bounds given by Theorems 23, 34, and 36. For example, referring to the top image in Figure 2, one will see that the plot of $s_2(n)$ forms several "mound" shapes. However, only the left sides of the mounds are bounded above by the green curves. It is known [5, page 2] that if $k \in \mathbb{N}$, then

$$s_2(2^k + x) = s_2(2^{k+1} - x) \text{ for all } x \in \{1, 2, \ldots, 2^k\}. \qquad (30)$$

This allows us to obtain upper bounds over the right sides of the mounds for free. More precisely, since we know from Theorem 36 that $s_2(2^k + x) < f_k(x)$ for all $x \in \{1, 2, \ldots, h_k\}$, it follows from (30) that $s_2(2^{k+1} - x) < f_k(x)$ for all such x. Using identities similar to (30) for arbitrary values of b, one may extend our upper bounds for $s_b(n)$ to a larger range of values of n.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. Winning ways for your mathematical plays, Vol. 1: Games in general, Academic Press, London, 1982.

[2] N. J. Calkin and H. S. Wilf. Binary partitions of integers and Stern-Brocot-like trees, unpublished (1998), updated version August 5, 2009, 19 pages.

[3] L. Carlitz. A problem in partitions related to the Stirling numbers. *Bull. Amer. Math. Soc.* 70 (1964), 275–278.

[4] M. Coons and J. Tyler. The maximal order of Stern's diatomic sequence. *Mosc. J. Comb. Number Theory.* 4 (2014), 3–13.

[5] S. Northshield. Stern's diatomic sequence $0, 1, 1, 2, 1, 3, 2, 3, 1, 4, \ldots$, *Amer. Math. Monthly.* 117 (2010), 581–598.