Mixed Ehrhart polynomials

Christian Haase*
Fachbereich Mathematik und Informatik
Freie Universität Berlin, Germany
haase@math.fu-berlin.de

Martina Juhnke-Kubitzke†
Institut für Mathematik
Universität Osnabrück, Germany
juhnke-kubitzke@uni-osnabrueck.de

Raman Sanyal‡
FB 12 – Institut für Mathematik
Goethe-Universität Frankfurt, Germany
sanyal@math.uni-frankfurt.de

Thorsten Theobald§
FB 12 – Institut für Mathematik
Goethe-Universität Frankfurt, Germany
theobald@math.uni-frankfurt.de

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Abstract

For lattice polytopes $P_1, \ldots, P_k \subseteq \mathbb{R}^d$, Bihan (2016) introduced the discrete mixed volume $\mathrm{DMV}(P_1, \ldots, P_k)$ in analogy to the classical mixed volume. In this note we study the associated mixed Ehrhart polynomial $\mathrm{ME}_{P_1,\ldots,P_k}(n) = \mathrm{DMV}(nP_1, \ldots, nP_k)$. We provide a characterization of all mixed Ehrhart coefficients in terms of the classical multivariate Ehrhart polynomial. Bihan (2016) showed that the discrete mixed volume is always non-negative. Our investigations yield simpler proofs for certain special cases.

We also introduce and study the associated *mixed h^*-vector*. We show that for large enough dilates $rP_1, \ldots, rP_k$ the corresponding *mixed h^*-polynomial* has only real roots and as a consequence the mixed h^*-vector becomes non-negative.

Keywords: lattice polytope; (mixed) Ehrhart polynomial; discrete (mixed) volume; h^*-vector; real roots

1 Introduction

Given a lattice polytope $P \subseteq \mathbb{R}^d$, the number of lattice points $|P \cap \mathbb{Z}^d|$ is the **discrete volume** of P. It is well-known ([11]; see also [1, 2]) that the lattice point enumerator

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$E_P(n) = |nP \cap \mathbb{Z}^d|$ agrees with a polynomial, the so-called **Ehrhart polynomial** $E_P(n) \in \mathbb{Q}[n]$ of P, for all non-negative integers n. Recently, Bihan [5] introduced the notion of the **discrete mixed volume** of k lattice polytopes $P_1, \dots, P_k \subseteq \mathbb{R}^d$,

$$\mathrm{DMV}(P_1, \dots, P_k) \;:=\; \sum_{J \subseteq [k]} (-1)^{k-|J|} |P_J \cap \mathbb{Z}^d| \,, \tag{1}$$

where $P_J := \sum_{j \in J} P_j$ for $\varnothing \neq J \subseteq [k]$ and $P_\varnothing = \{0\}$.

In the present paper, we study the behavior of the discrete mixed volume under simultaneous dilation of the polytopes P_i. This furnishes the definition of a **mixed Ehrhart polynomial**

$$\mathrm{ME}_{P_1,\dots,P_k}(n) \;:=\; \mathrm{DMV}(nP_1, \dots, nP_k) \;=\; \sum_{J \subseteq [k]} (-1)^{k-|J|} E_{P_J}(n) \;\in\; \mathbb{Q}[n]. \tag{2}$$

Khovanskii [16] relates the evaluation $\mathrm{ME}_{P_1,\dots,P_k}(-1)$ to the arithmetic genus of a compactified complete intersection with Newton polytopes $P_1, \dots, P_k$. Danilov and Khovanskii [8] investigate the Hodge-Deligne polynomial $e(Z; u, v) \in \mathbb{Z}[u, v]$ of a complex algebraic variety Z which in case of a smooth projective variety agrees with the Hodge polynomial $\sum_{pq} (-1)^{p+q} h^{pq}(Z) u^p v^q$. In joint work with Sandra Di Rocco and Benjamin Nill the first author verified that $\mathrm{DMV}(P_1, \dots, P_k) = (-1)^{d-k} e(Z; 1, 0)$ where $Z \subset (\mathbb{C}^*)^d$ is the uncompactified complete intersection [9]. Using the mixed Ehrhart polynomial, this yields the reciprocity type result

$$\mathrm{ME}_{P_1,\dots,P_k}(-1) = (-1)^{d-k} e(\bar{Z}; 1, 0) \qquad \text{while} \qquad \mathrm{ME}_{P_1,\dots,P_k}(1) = (-1)^{d-k} e(Z; 1, 0) \,,$$

relating Z and its compactification $\bar{Z}$.

To the best of our knowledge, the origin of the mixed Ehrhart *polynomial* (2) goes back to Steffens and Theobald [22], see also [21]. In their work, a slight variant of (2), yet under the same name, has been employed as a very specific means to study higher-dimensional, mixed versions of Pick's formula in connection with the combinatorics of intersections of tropical hypersurfaces. The definition of the mixed Ehrhart polynomial therein differs from our definition by the exclusion of the empty set from the sum.

Here, we study mixed Ehrhart polynomials and their coefficients with respect to various bases of the vector space of polynomials of degree at most d. First, we show that in the usual monomial basis, the coefficients of mixed Ehrhart polynomials can be read off directly from the multivariate Ehrhart polynomial $E_{\mathbf{P}}(n_1, \dots, n_k)$, where $n_1, \dots, n_k \in \mathbb{Z}_{\geqslant 0}$ are non-negative integers (Theorem 2.4). This gives a meaning to the coefficients of $\mathrm{ME}_{P_1,\dots,P_k}(n)$. In particular this provides a simple proof that the coefficient of n^i vanishes for $i < k$ and also allows to give streamlined proofs for known characterizations of the two leading coefficients (Corollaries 2.5 and 2.6). We then deal with two prominent subclasses. For the case that all polytopes $P_1, \dots, P_k$ are equal, the mixed Ehrhart polynomial can be expressed in terms of the h^*-vector of P (see Proposition 2.7). And for the case that $P_1, \dots, P_k$ all contain the origin and satisfy $\dim(P_1 + \dots + P_k) = \dim P_1 + \dots + \dim P_k$, we can provide a combinatorial interpretation of $\mathrm{DMV}(P_1, \dots, P_k)$ (see Proposition 2.9).

As a consequence of the main result from [5], it follows that $\mathrm{ME}_{P_1,\dots,P_k}(n) \geqslant 0$ for all $n \geqslant 0$ (see Theorem 3.1). This is accomplished by the use of irrational mixed decompositions and skilled estimates. Using our understanding of the coefficients of $\mathrm{ME}_{P_1,\dots,P_k}(n)$, we give direct proofs for this fact in the cases $k \in \{2, d-1, d\}$ and $P_1 = \cdots = P_k = P$ in Section 3. See also [14] for further developments in the context of valuations.

Expressing the (usual or mixed) Ehrhart polynomial in the basis $\{\binom{n+d-i}{d} : 0 \leqslant i \leqslant d\}$ gives rise to the definition of the (usual or mixed) h^*-vector and h^*-polynomial (see Section 4). By a famous result of Stanley [18], the usual h^*-vector is non-negative (componentwise). We illustrate that for the mixed h^*-vector this is not true in general (see Example 4.3). Yet, we show that this has to hold asymptotically for dilates $rP_1, rP_2, \dots, rP_k$ (Corollary 4.6). This follows from the stronger result that for $r \gg 0$ the mixed h^*-polynomial is real-rooted with roots converging to the roots of the d^{th} Eulerian polynomial (Theorem 4.5). This can be seen as the mixed analogue of Theorem 5.1 in [10] (see also [3, 7]). As a byproduct, we obtain that asymptotically the mixed h^*-vector becomes log-concave, unimodal and, as mentioned, in particular positive, except for its 0^{th} entry, which always equals 0 (see Corollary 4.6).

Our paper is structured as follows. In Section 2, we prove various structural properties of the mixed Ehrhart polynomial. In Section 3, we review Bihan's non-negativity result of the discrete mixed volume, particularly from the viewpoint of the mixed Ehrhart polynomial, and provide alternative proofs for some special cases. Finally, in Section 4 we study the h^*-vector and the h^*-polynomial of the mixed Ehrhart polynomial, and in particular show real-rootedness of the mixed h^*-polynomial and positivity of the h^*-vector for large dilates of lattice polytopes.

2 Structure of the mixed Ehrhart polynomial

In this section, we collect basic properties of the mixed Ehrhart polynomial. For some known results we provide new or simplified proofs. For the whole section, we fix a collection $\mathbf{P} = (P_1, \dots, P_k)$ of k lattice polytopes in $\mathbb{R}^d$ and we assume that $P_1 + \cdots + P_k$ is of full dimension d. The mixed Ehrhart polynomial, as introduced in (2), is by definition a univariate polynomial of degree $\leqslant d$, which can be written as

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; \mathrm{me}_d(\mathbf{P})n^d + \mathrm{me}_{d-1}(\mathbf{P})n^{d-1} + \cdots + \mathrm{me}_0(\mathbf{P}).$$

Example 2.1. *For the case of k copies of the d-dimensional unit cube, $P_1 = \cdots = P_k = [0,1]^d$, we have $E_{P_i}(n) = (n+1)^d$ and thus*

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (jn+1)^d \;=\; \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{i=0}^{d} \binom{d}{i} (jn)^i$$

by the binomial theorem. Hence,

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; \sum_{i=0}^{d} \binom{d}{i} n^i \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^i \,,$$

which gives $\mathrm{me}_i(\mathbf{P}) = \binom{d}{i}\Delta^k f(0)$ *with* $f(x) = x^i$, *where* $\Delta^k f(0)$ *denotes the* k^{th} *difference of the function* f *at 0 (see, e.g., [19, Sect. 1.4]). This can be expressed as* $\mathrm{me}_i(\mathbf{P}) = \binom{d}{i}k!S(i,k)$ *in the Stirling numbers* $S(i,k)$ *of the second kind (see [19, Prop. 1.4.2]).*

Remark 2.2. If $\mathbf{P} = (P_1, \dots, P_k)$ is a collection of lattice polytopes in $\mathbb{R}^d$ and $\dim P_i = 0$ for some i, then $\mathrm{ME}_{\mathbf{P}}(n) = 0$. To see this, assume that P_k is 0-dimensional. For any $J \subseteq [k-1]$, the polytopes P_J and $P_{J\cup\{k\}}$ are translates of each other and the corresponding terms in (2) occur with different signs.

As our first result, we give a description of the coefficients of the mixed Ehrhart polynomial in Theorem 2.4. For this we recall a result independently due to Bernstein and McMullen; see [13, Theorem 19.4].

Theorem 2.3 (Bernstein-McMullen)**.** *For lattice polytopes* $\mathbf{P} = (P_1, \dots, P_k)$ *in* $\mathbb{R}^d$, *the function*

$$E_{\mathbf{P}}(n_1, \dots, n_k) \;:=\; \left|(n_1P_1 + \cdots + n_kP_k) \cap \mathbb{Z}^d\right|$$

agrees with a multivariate polynomial for all $n_1, \dots, n_k \in \mathbb{Z}_{\geqslant 0}$. *The degree of* $E_{\mathbf{P}}$ *in* n_i *is* $\dim P_i$.

In particular, $E_{\mathbf{P}}(n_1, \dots, n_k)$ has total degree $d = \dim(P_1 + \cdots + P_k)$. We write this polynomial as

$$E_{\mathbf{P}}(n_1, \dots, n_k) \;=\; \sum_{\alpha\in\mathbb{Z}^k_{\geqslant 0}} e_\alpha(\mathbf{P})\underline{n}^\alpha \tag{3}$$

where $\underline{n}^\alpha = n_1^{\alpha_1}\cdots n_k^{\alpha_k}$ for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k_{\geqslant 0}$.

Theorem 2.4. *For* $\mathbf{P} = (P_1, \dots, P_k)$ *a collection of lattice polytopes in* $\mathbb{R}^d$

$$\mathrm{me}_i(\mathbf{P}) \;=\; \sum_\alpha e_\alpha(\mathbf{P}),$$

where the sum runs over all $\alpha \in \mathbb{Z}^k_{\geqslant 1}$ *such that* $|\alpha| := \alpha_1 + \cdots + \alpha_k = i$. *In particular,* $\mathrm{me}_i(P) = 0$ *for all* $0 \leqslant i < k$.

Proof. For a subset $J \subseteq [k]$ we write $\mathbb{1}_J \in \{0,1\}^k$ for its characteristic vector. Observe that

$$\mathrm{DMV}(P_1, \dots, P_k) \;=\; \sum_{J\subseteq[k]} (-1)^{k-|J|}E_{\mathbf{P}}(\mathbb{1}_J).$$

Setting

$$A(\alpha) \;=\; \sum_{J\subseteq[k]} (-1)^{k-|J|}\mathbb{1}_J^\alpha \quad \text{where} \quad \mathbb{1}_J^\alpha = \prod_{i=1}^k (\mathbb{1}_J)_i^{\alpha_i},$$

presentation (3) then implies

$$\mathrm{DMV}(P_1, \dots, P_k) \;=\; \sum_{\alpha\in\mathbb{Z}^k_{\geqslant 0}} e_\alpha(\mathbf{P})A(\alpha).$$

Using that $0^0 = 1$, it is easy to verify that $A(\alpha) = 1$ if $\alpha_i > 0$ for all $i = 1, \ldots, k$ and $A(\alpha) = 0$ otherwise. Hence,

$$\mathrm{ME}_{P_1,\ldots,P_k}(n) = \mathrm{DMV}(nP_1, \ldots, nP_k) \;=\; \sum_{\alpha \in \mathbb{Z}^k_{\geqslant 0}} e_\alpha(n\mathbf{P})A(\alpha) \;=\; \sum_{i=0}^{d} \Big(\sum_{\substack{\alpha \in \mathbb{Z}^k_{\geqslant 1} \\ |\alpha| = i}} e_\alpha(\mathbf{P}) \Big) n^i \,,$$

where the last step used that e_α is homogeneous of degree $(\alpha_1, \ldots, \alpha_d)$. □

The proof also recovers Lemma 4.10 from [22] with different techniques. In order to give an interpretation to some of the coefficients of $\mathrm{ME}_{\mathbf{P}}(n)$, let us write

$$E_{\mathbf{P}}(n_1, \ldots, n_k) \;=\; \sum_{i=0}^{d} E^i_{\mathbf{P}}(n_1, \ldots, n_k)$$

where $E^i_{\mathbf{P}}(n_1, \ldots, n_k)$ is homogeneous in $n_1, \ldots, n_k$ of degree i. By the same reasoning as in the case of the usual Ehrhart polynomial (see, e.g., [2, Sect. 3.6]), we note

$$\begin{aligned} E^d_{\mathbf{P}}(n_1, \ldots, n_k) \;&=\; \mathrm{vol}_d(n_1P_1 + \cdots + n_kP_k) \\ &=\; \sum_{\substack{\alpha \in \mathbb{Z}^k_{\geqslant 0} \\ |\alpha| = d}} \tbinom{d}{\alpha_1, \ldots, \alpha_k} \mathrm{MV}_d(P_1[\alpha_1], \ldots, P_k[\alpha_k]) \underline{n}^\alpha. \end{aligned} \tag{4}$$

In the above expression, the (normalized) coefficient MV is called the **mixed volume**; see, e.g., [13, Chapter 6]. Moreover, the notation $\mathrm{MV}_d(P_1[\alpha_1], \ldots, P_k[\alpha_k])$ means that the polytope P_i is taken α_i times, and (4) comes from a well-known property of mixed volumes (see [17, Section 5.1]). Based on (4) together with Theorem 2.4, we obtain streamlined proofs for the following known characterizations of the two highest mixed Ehrhart coefficients (see [22] for the case $k = d - 1$ and [21, Lemma 3.7 and 3.8]).

Corollary 2.5 (Steffens, Theobald). *Let $\mathbf{P} = (P_1, \ldots, P_k)$ be lattice polytopes such that $d = \dim(P_1 + \cdots + P_k)$. Then*

$$\mathrm{me}_d(\mathbf{P}) \;=\; \sum_{\alpha} \tbinom{d}{\alpha_1, \ldots, \alpha_k} \mathrm{MV}_d(P_1[\alpha_1], \ldots, P_k[\alpha_k]),$$

where the sum runs over all $\alpha \in \mathbb{Z}^k_{\geqslant 1}$ with $|\alpha| = d$. In particular, $\mathrm{ME}_{\mathbf{P}}(n)$ is a polynomial of degree exactly d.

Proof. By Theorem 2.4, $\mathrm{me}_d(\mathbf{P})$ is the sum of the coefficients of $E_{\mathbf{P}}(n_1, \ldots, n_d)$ of total degree d. These are coefficients of $E^d_{\mathbf{P}}(n_1, \ldots, n_k) = \mathrm{vol}(n_1P_1 + \cdots + n_kP_k)$ that are given by the mixed volumes. For the second claim, it is sufficient to note that since the sum $P_1 + \cdots + P_k$ is full-dimensional, $E^d_{\mathbf{P}}$ is not identically zero and all coefficients are non-negative. □

For $k = d$, the previous corollary recovers a result of Bernstein [4]

$$\mathrm{ME}_{P_1,\ldots,P_d}(n) = d!\,\mathrm{MV}_d(P_1,\ldots,P_d)\,n^d. \tag{5}$$

A similar reasoning also allows us to give an expression for the second highest coefficient $\mathrm{me}_{d-1}(P)$. For an integral linear functional $a \in (\mathbb{Z}^d)^\vee$, the rational subspace $a^\perp \subseteq \mathbb{R}^d$ is equipped with a lattice $a^\perp \cap \mathbb{Z}^d$ and a volume form vol_a so that a fundamental parallelpiped has unit volume. We denote the face of a polytope $P \subseteq \mathbb{R}^d$ where a is maximized by P^a and consider it (after translation) as a polytope inside $a^\perp$. Similarly, we write MV_a (DMV_a) for the (discrete) mixed volume of a collection of polytopes in $a^\perp$ computed using vol_a.

Corollary 2.6 (Steffens, Theobald)**.** *Let $\mathbf{P} = (P_1,\ldots,P_k)$ be a collection of d-dimensional lattice polytopes in $\mathbb{R}^d$. Then*

$$\mathrm{me}_{d-1}(\mathbf{P}) \;=\; \frac{1}{2}\sum_a \sum_{\substack{\alpha\in\mathbb{Z}^k_{\geqslant 1}\\ |\alpha|=d-1}} \binom{d-1}{\alpha_1,\ldots,\alpha_k}\mathrm{MV}_a(P_1^a[\alpha_1],\ldots,P_k^a[\alpha_k]),$$

where a ranges over the primitive facet normals of $P_1 + \cdots + P_k$.

Proof. For a full-dimensional lattice polytope $P \subseteq \mathbb{R}^d$, the second highest Ehrhart coefficient can be expressed as

$$e_{d-1}(P) = \sum_a \frac{1}{2}\mathrm{vol}_a(P^a),$$

where a ranges over all primitive vectors of facets of P (see, e.g., [2, Thm. 5.6]). This, of course, is a finite sum as P has only finitely many facets. It follows that

$$\begin{aligned}\mathrm{me}_{d-1}(\mathbf{P}) \;&=\; \sum_{J\subseteq[k]}(-1)^{k-|J|}\sum_a \frac{1}{2}\mathrm{vol}_a(P_J^a)\\ &=\; \frac{1}{2}\sum_a\sum_{J\subseteq[k]}(-1)^{k-|J|}\mathrm{vol}_a(P_J^a).\end{aligned}$$

Since $e_d(P) = \mathrm{vol}_d(P)$ for a d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$, (2) implies that the inner sum in the previous expression equals $\mathrm{me}_{d-1}(P_1^a,\ldots,P_k^a)$. The result now follows from Corollary 2.5 applied to $\mathbf{P}^a = (P_1^a,\ldots,P_k^a)$. Observe that if a is a facet normal for P_J, then a is a facet normal for P_I for all $I \supseteq J$. Hence, the above sum is over all primitive facet normals of $P_1 + \cdots + P_k$. $\square$

Next, we will show that for the special case $P_1 = P_2 = \cdots = P_k = P$, the mixed Ehrhart polynomial can be expressed in terms of the h^*-vector of P; see Section 4. For now, let us mention that Stanley [18] showed that there exist non-negative integers $h_i^*(P)$ such that

$$E_P(n) \;=\; h_0^*(P)\binom{n+d}{d} + h_1^*(P)\binom{n+d-1}{d} + \cdots + h_d^*(P)\binom{n}{d}. \tag{6}$$

In terms of this presentation we get:

Proposition 2.7. *Let $P \subseteq \mathbb{R}^d$ be a d-dimensional lattice polytope and let $\mathbf{P} = (P, \ldots, P)$ be a collection of k copies of P. Then*

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; \sum_{j=0}^{d} \left(\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{in+d-j}{d} \right) h_j^*(P).$$

In particular,

$$\mathrm{DMV}(\mathbf{P}) \;=\; \sum_{j=0}^{d} \binom{d-j}{d-k} h_j^*(P).$$

Proof. From the definition of the mixed Ehrhart polynomial, we infer

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E_{iP}(n).$$

Using that $E_{iP}(n) = E_P(in)$ and the expression of $E_P(n)$ as in (6), this yields

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; \sum_{j=0}^{d} h_j^*(P) \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{in+d-j}{d}.$$

This shows the first claim. For the second claim it suffices to observe that $\mathrm{DMV}(\mathbf{P}) = \mathrm{ME}_{\mathbf{P}}(1)$ and to check that, in this case, the inner sum in the above expression equals $\binom{d-j}{d-k}$ by the binomial identity in [12, (5.24)]. $\square$

This is reminiscent to the relation between f-vectors and h-vectors in the enumerative theory of simplicial polytopes. Investigating this analogy further yields a theory of discrete mixed valuations; see [14].

Example 2.8. *Let $P = [0,1]^3 \subseteq \mathbb{R}^3$ be the 3-dimensional unit cube. Then the usual h^*-vector of P equals $(1,4,1,0)$. By Proposition 2.7, the discrete mixed volume of the collection (P,P) is given by*

$$\mathrm{DMV}(P,P) = \binom{3-0}{3-2} \cdot 1 + \binom{3-1}{3-2} \cdot 4 + \binom{3-2}{3-2} \cdot 1 = 12.$$

The mixed Ehrhart polynomial of (P,P) equals $\mathrm{ME}_{P,P}(t) = 6t^3 + 6t^2$, which is consistent with Example 2.1.

Proposition 2.9. *Let $\mathbf{P} = (P_1, \ldots, P_k)$ be a collection of lattice polytopes, each containing 0, such that $\dim(P_1 + \cdots + P_k) \;=\; \dim P_1 + \cdots + \dim P_k$. Then $\mathrm{DMV}(P_1, \ldots, P_k)$ counts the number of lattice points $z \in (P_1 + \cdots + P_k) \cap \mathbb{Z}^d$ that are not contained in a subsum P_J for $J \neq [k]$.*

Proof. By the hypothesis, $P = P_1 + \cdots + P_k$ is affinely isomorphic to the Cartesian product $P_1 \times \cdots \times P_k$. For any $J \subseteq [k]$, this implies that P_J is the intersection of P with the linear span of P_J. And for $I, J \subseteq [k]$, we see that the intersection $P_I \cap P_J$ is exactly $P_{I \cap J}$. Hence

$$\mathrm{DMV}(P_1, \ldots, P_k) \;=\; |P \cap \mathbb{Z}^d| \;-\; \sum_{J \neq [k]} (-1)^{k-|J|} |P_J \cap \mathbb{Z}^d| \;=\; \Big|\Big(P \setminus \bigcup_{J \neq [k]} P_J\Big) \cap \mathbb{Z}^d\Big| .$$

□

For the case that all polytopes are segments, the previous proposition can be used to recover (5).

3 Non-negativity of the discrete mixed volume and the mixed Ehrhart polynomial

Bihan [5] showed the following fundamental non-negativity result.

Theorem 3.1 (Bihan)**.** *Let $P_1, \ldots, P_k \subseteq \mathbb{R}^d$ be lattice polytopes. Then*

$$\mathrm{DMV}(P_1, \ldots, P_k) \;\geqslant\; 0\,.$$

Thus the mixed Ehrhart polynomial evaluates to non-negative integers for all positive integers n.

For a proof of Theorem 3.1, Bihan develops a theory of irrational mixed decompositions which yields a technical induction on dimension. The discrete mixed volume is a particular combinatorial mixed valuation in the sense of [14]. It is shown in [14] that the discrete mixed volume is monotone with respect to inclusion, which implies Theorem 3.1. The proofs in [14] are less technical but set in the context of the polytope algebra. In this section we give simple and geometrically sound proofs for the special cases $k \in \{2, d-1, d\}$ and $P_1 = \cdots = P_k = P$. For the case $k = d$ this is a consequence of (5). For $P_1 = \cdots = P_k = P$ this follows from Proposition 2.7 together with Stanley's result on the non-negativity of the h^*-vector; see [18, 15].

Direct proof of Theorem 3.1 for $k = 2$. Let P_1 and P_2 be lattice polytopes in $\mathbb{R}^d$. Since the mixed Ehrhart polynomial is invariant under translation of the polytopes, we may assume that 0 is a common vertex of P_1 and P_2, $P_1 \cap P_2 = \{0\}$ and that there is a hyperplane H weakly separating P_1 from P_2. Hence $P_1 \cup P_2 \subseteq P_1 + P_2$. It follows that

$$E_{P_1+P_2}(n) \;\geqslant\; E_{P_1}(n) + E_{P_2}(n) - 1,$$

for all $n \in \mathbb{Z}_{\geqslant 0}$, i.e., $\mathrm{ME}_{P_1,P_2}(n) \geqslant 0$ for all $n \in \mathbb{Z}_{\geqslant 0}$. □

For $k = d - 1$, a direct proof idea has already been used in the framework of tropical geometry and d-dimensional Pick-type formulas in [22]. The following is a proof along similar lines.

Direct proof of Theorem 3.1 for $k = d-1$. By Theorem 2.4, the mixed Ehrhart polynomial of a collection of lattice polytopes $\mathbf{P} = (P_1, \ldots, P_{d-1}) \subseteq \mathbb{R}^d$ is of the form

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; \mathrm{me}_d(\mathbf{P})n^d + \mathrm{me}_{d-1}(\mathbf{P})n^{d-1}.$$

It suffices to show that $\mathrm{me}_d(\mathbf{P})$ and $\mathrm{me}_{d-1}(\mathbf{P})$ are non-negative. Since the mixed volume is multilinear and symmetric in its entries, Corollary 2.5 yields

$$\mathrm{me}_d(\mathbf{P}) \;=\; \tfrac{d!}{2}\mathrm{MV}_d\left(P_1, P_2, \ldots, P_{d-1}, \sum_{i=1}^{d-1} P_i\right).$$

For the remaining coefficient, Corollary 2.6 together with the fact that $k = d-1$ yields

$$\mathrm{me}_{d-1}(\mathbf{P}) \;=\; \frac{(d-1)!}{2}\sum_a \mathrm{MV}_a(P_1^a, \ldots, P_{d-1}^a),$$

where the sum is over all primitive facet normals of $P_1 + \cdots + P_{d-1}$. $\square$

4 Mixed h^*-polynomials

For a d-dimensional lattice polytope P, let

$$\mathrm{Ehr}_P(z) \;=\; \sum_{n \geqslant 0} E_P(n) z^n$$

be the **Ehrhart series** of P. As $E_P(n)$ is a polynomial in n of degree d, there exist integers $h_0^*(P), \ldots, h_d^*(P)$ such that

$$\mathrm{Ehr}_P(z) \;=\; \frac{\sum_{j=0}^d h_j^*(P) z^j}{(1-z)^{d+1}};$$

see, e.g., [1, 13]. The polynomial in the numerator is called the ***h**-polynomial** of P (also known as δ-polynomial [20]) and denoted by $h_P^*(z)$. Similarly, the coefficient vector $h^*(P) := (h_0^*(P), \ldots, h_d^*(P))$ is called the ***h**-vector** of P. Stated differently, the Ehrhart polynomial of a d-dimensional lattice polytope P can be expressed as in (6). By Stanley's non-negativity theorem [18], the coefficients $h_0^*(P), h_1^*(P), \ldots, h_d^*(P)$ are non-negative integers. Moreover, it is known that

$$h_0^*(P) = 1\,, \quad h_1^*(P) = |P \cap \mathbb{Z}^d| - (d+1)\,, \quad h_d^*(P) = |\,\mathrm{int}(P) \cap \mathbb{Z}^d|\,, \tag{7}$$

where $\mathrm{int}(P)$ denotes the interior of the polytope P.

The definition of mixed Ehrhart polynomials prompts the notion of a **mixed *h**-vector** $h^*(\mathbf{P}) = (h_0^*(\mathbf{P}), \ldots, h_d^*(\mathbf{P}))$ of a collection $\mathbf{P} = (P_1, \ldots, P_k)$ of lattice polytopes, which is given by

$$\mathrm{ME}_{\mathbf{P}}(n) \;=\; h_0^*(\mathbf{P})\binom{n+d}{d} + \cdots + h_d^*(\mathbf{P})\binom{n}{d}, \tag{8}$$

where $d = \dim(P_1+\cdots+P_k)$. In this section, we study properties of $h^*(\mathbf{P})$ for a collection of *full-dimensional* lattice polytopes, i.e., $\dim P_i = d$ for all i. Though Stanley's h^*-non-negativity does not extend to the mixed h^*-vector as we will see below, we show that for large enough dilates $r\mathbf{P} := (rP_1, \ldots, rP_k)$, $r \gg 0$, the corresponding **mixed h^*-polynomial**

$$h^*_{r\mathbf{P}}(z) \;=\; h^*_0(r\mathbf{P}) + \cdots + h^*_d(r\mathbf{P})z^d$$

has only real roots and hence the mixed h^*-vector is log-concave. This is in line with results of Diaconis and Fulman [10] for the usual h^*-vector.

Remark 4.1. Note that the contribution of the index set $J = \varnothing$ in the discrete mixed volume (1) is $(-1)^k$. For the **mixed Ehrhart series** $\sum_{n\geqslant 0} \mathrm{ME}_{P_1,\ldots,P_k}(n)z^n$ this induces for the index set $J = \varnothing$ the contribution

$$\sum_{n\geqslant 0}(-1)^k z^n \;=\; (-1)^k\frac{1}{1-z} \;=\; (-1)^k\frac{(1-z)^d}{(1-z)^{d+1}} = (-1)^k\frac{\sum_{i=0}^d \binom{d}{i}(-1)^i z^i}{(1-z)^{d+1}}. \tag{9}$$

Since we assume that all polytopes P_i have the same dimension, linearity and Remark 4.1 allow to write the mixed h^*-vector as

$$h^*_i(\mathbf{P}) \;=\; \sum_{\varnothing\neq J\subseteq[k]}(-1)^{k-|J|}h^*_i(P_J) + (-1)^{k+i}\binom{d}{i} \tag{10}$$

for $0 \leqslant i \leqslant d$. The next lemma collects some elementary properties of mixed h^*-vectors.

Lemma 4.2. *Let $\mathbf{P} = (P_1, P_2, \ldots, P_k) \subseteq \mathbb{R}^d$ be a collection of d-dimensional lattice polytopes. Then:*

(i) *$h^*_0(\mathbf{P}) = 0$.*

(ii) *If $k = 2$, then $h^*_1(P_1, P_2) = \mathrm{DMV}(P_1, P_2)$. In particular, $h^*_1(P_1, P_2) \geqslant 0$.*

Proof. (i) Since, for any lattice polytope $P \subseteq \mathbb{R}^d$ we have $h^*_0(P) = 1$ (see (7)), it follows from (10) that

$$h^*_0(\mathbf{P}) = \sum_{\varnothing\neq J\subseteq[k]}(-1)^{k-|J|} + (-1)^k = 0.$$

(ii) We know from (7) that for a d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$

$$h^*_1(P) = |P\cap\mathbb{Z}^d| - (d+1) = E_P(1) - (d+1).$$

This fact combined with (10) yields

$$\begin{aligned} h^*_1(P_1,P_2) &= (E_{P_1+P_2}(1) - (d+1)) - (E_{P_1}(1) - (d+1) + E_{P_2}(1) - (d+1)) - d \\ &= E_{P_1+P_2}(1) - E_{P_1}(1) - E_{P_2}(1) + 1 \\ &= \mathrm{DMV}(P_1,P_2), \end{aligned}$$

where the last equality follows from (2). For the second claim, it suffices to note that $\mathrm{DMV}(P_1, P_2)$ is non-negative (see Section 3). ☐

A natural question when dealing with integer vectors is whether all entries are non-negative. The previous lemma provides some positive results in this direction. However, as opposed to the h^*-vector of a single lattice polytope, the next example shows that in general it is not true that the coefficients of the mixed h^*-vector are non-negative.

Example 4.3. *Consider the collection* $\mathbf{P} = (\Delta_d, \ldots, \Delta_d)$ *consisting of* k *copies of the standard* d*-simplex. For* $(d+1-i)k < d+1$ *none of the dilates* $(d+1-i)P_J = (d+1-i)|J|\Delta_d$ *has interior lattice points. Hence,* $h_i^*(P_J) = 0$ *so that* $h_i^*(\mathbf{P}) = (-1)^{k+i}\binom{d}{i}$ *can be negative (see (10)). A specific case is* $h^*(\Delta_3, \Delta_3) = (0, 3, 4, -1)$.

Though we have just seen that there do exist collections of polytopes with negative mixed h^*-vector entries, observe that for $m \geqslant 2$ all entries of $h^*(m\Delta_3, m\Delta_3) = (0, m^3 + 2m, 4m^3, m^3 - 2m^2)$ are indeed non-negative, and the leading coefficients in m are the Eulerian numbers $(0, 1, 4, 1)$ (see below).

We therefore propose to study the following question.

Question 4.4. *Let* $\mathbf{P} = (P_1, \ldots, P_k) \subseteq \mathbb{R}^d$ *be a collection of* d*-dimensional lattice polytopes. Under which conditions are all (or certain) entries of* $h^*(\mathbf{P})$ *non-negative? Is it true that* $h_i^*(\mathbf{P}) \geqslant (-1)^{k+i}\binom{d}{i}$*? What can be said if the polytopes are allowed to be of arbitrary dimension?*

In Corollary 4.6 we will show that asymptotically, i.e., if one considers high enough dilations of d-dimensional polytopes, the mixed h^*-vector $h^*(\mathbf{P})$ always becomes non-negative. This suggests that it might be enough to require the lattice polytopes to contain "sufficiently many" interior points.

Before we provide our main result of this section, we recall the definition of the d^{th} **Eulerian polynomial** $A_d(z) = \sum_{k=1}^{d} A(d,k) z^k$. One, out of several, combinatorial approaches to define $A_d(z)$ is the following:

$$\sum_{n \geqslant 0} n^d z^n \;=\; \frac{A_d(z)}{(1-z)^{d+1}}.$$

It is known that the Eulerian polynomials $A_d(z)$ have only simple and real roots and all roots are negative. The following result, which is a generalization of Theorem 5.1 in [10], provides a relation between Eulerian polynomials and mixed h^*-polynomials of lattice polytopes.

Theorem 4.5. *Let* $\mathbf{P} = (P_1, \ldots, P_k)$ *be a collection of* d*-dimensional lattice polytopes. Then, as* $r \to \infty$,

$$\frac{h^*_{r\mathbf{P}}(z)}{r^d} \;\longrightarrow\; \sum_{J \subseteq [k]} (-1)^{k-|J|} \mathrm{vol}(P_J)\, A_d(z).$$

Note that for $k = d$, (5) implies the result for $r = 1$.

Proof. For a d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$, the Ehrhart polynomial in the usual basis takes the form

$$E_P(n) \;=\; e_d(P)n^d + e_{d-1}(P)n^{d-1} + \cdots + e_0(P)$$

where $e_d(P) = \mathrm{vol}(P)$. For the Ehrhart series we compute

$$\sum_{n\geqslant 0} E_P(n)z^n \;=\; \sum_{i=0}^{d} e_i(P)\sum_{n\geqslant 0} n^i z^n \;=\; \frac{\sum_{i=0}^{d} e_i(P)(1-z)^{d-i}A_i(z)}{(1-z)^{d+1}}.$$

By comparing $E_{rP}(n)$ with $E_P(rn)$, we see that $e_i(rP) = r^i e_i(P)$ and hence

$$h^*_{rP}(z) \;=\; r^d \mathrm{vol}(P)A_d(z) + \sum_{i=0}^{d-1} r^i e_i(P)(1-z)^{d-i}A_i(z).$$

Now from (10) together with the fact that $rP_\varnothing = \{0\}$ for all $r > 0$, we infer

$$\frac{h^*_{r\mathbf{P}}(z)}{r^d} \;=\; \sum_{J\subseteq[k]} (-1)^{k-|J|}\frac{h^*_{rP_J}(z)}{r^d} \;\xrightarrow{r\to\infty}\; \sum_{J\subseteq[k]} (-1)^{k-|J|}\mathrm{vol}(P_J)\, A_d(z),$$

which shows the claim. □

For $k = 1$ this also yields the result of Diaconis and Fulman [10, Theorem 5.1] on the asymptotic behavior of the usual h^*-polynomial of a lattice polytope. Similar results have also been achieved by Brenti and Welker [7] and Beck and Stapledon [3].

Before we provide some almost immediate consequences of the previous result, we recall that a sequence $(a_0, a_1, \ldots, a_d)$ of real numbers is called **log-concave** if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for all $1 \leqslant i \leqslant d-1$. The sequence $(a_0, a_1, \ldots, a_d)$ is called **unimodal** if there exists a $0 \leqslant \ell \leqslant d$ such that $a_0 \leqslant \cdots \leqslant a_\ell \geqslant \cdots \geqslant a_d$.

Corollary 4.6. *Let $\mathbf{P} = (P_1, P_2, \ldots, P_k)$ be a collection of d-dimensional lattice polytopes in $\mathbb{R}^d$. Then there exists a positive integer R (depending on $\mathbf{P}$) such that for $r \geqslant R$:*

(i) *the mixed h^*-polynomial $h^*_{r\mathbf{P}}(z)$ has only real roots $\beta^{(1)}(r) < \beta^{(2)}(r) < \cdots < \beta^{(d-1)}(r) < \beta^{(d)}(r) < 0$ with $\lim_{r\to\infty} \beta^{(i)}(r) = \rho^{(i)}$ for $1 \leqslant i \leqslant d$. Here, $\rho^{(1)} < \rho^{(2)} < \cdots < \rho^{(d)} = 0$ denote the roots of $A_d(z)$.*

(ii) *$h^*_i(r\mathbf{P}) > 0$ for $1 \leqslant i \leqslant d$.*

(iii) *$h^*(r\mathbf{P})$ is log-concave and unimodal.*

Proof. (i) First observe that, by Theorem 4.5, the roots of $h^*_{r\mathbf{P}}(z)$ converge to the roots of $A_d(z)$. Moreover, since the roots of $A_d(z)$ are known to be all distinct and negative, there exists a positive integer R such that for $r \geqslant R$, $h^*_{r\mathbf{P}}(z)$ has only simple and real (negative) roots. Otherwise, since complex roots come up in pairs (if τ is a complex root, also its complex conjugate $\bar{\tau}$ is a root), the Eulerian polynomial $A_d(z)$ would be forced to have a root of multiplicity 2, which yields a contradiction.

(ii) Using (2) and the fact that $e_d(P) = \mathrm{vol}(P)$ for a d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$, we recognize the right-hand side in Theorem 4.5 as $\mathrm{me}_d(\mathbf{P})A_d(z)$. By Corollary 2.5, $\mathrm{me}_d(\mathbf{P})$ is a non-negative linear combination of mixed volumes of $P_1, \ldots, P_k$ and thus positive. Since the coefficients of the Eulerian polynomials (besides the constant term which is 0) are all positive as well, Theorem 4.5 then implies that the sequence $\left(\frac{h^*_{r\mathbf{P}}(z)}{r^d}\right)_{r\geqslant 1}$ converges to a polynomial, whose coefficients, except for the constant term (which equals 0), are positive. Hence, there has to exist a positive integer R (depending on $\mathbf{P}$) such that for $r \geqslant R$ all but the constant coefficient of $h^*_{r\mathbf{P}}(z)$ are positive. This shows (ii).

(iii) Using the first two parts, we know that there exists a positive integer R such that for $r \geqslant R$ the polynomial $h^*_{r\mathbf{P}}(z)$ has, except for the constant term, positive coefficients and is real-rooted. Theorem 1.2.1 of [6] implies that $h^*(r\mathbf{P})$ is log-concave. Since by the choice of r, this sequence does not have internal zeros, it is unimodal by [6, Section 2.5]. $\square$

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References

[1] A. Barvinok. *Integer Points in Polyhedra*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[2] M. Beck and S. Robins. *Computing the Continuous Discretely*. Springer, New York, 2007.

[3] M. Beck and A. Stapledon. On the log-concavity of Hilbert series of Veronese subrings and Ehrhart series. *Math. Z.*, 264(1):195–207, 2010.

[4] D. N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*, 9(3):1–4, 1975.

[5] F. Bihan. Irrational mixed decomposition and sharp fewnomial bounds for tropical polynomial systems. *Discrete Comput. Geom.*, 55(4):907–933, 2016.

[6] F. Brenti. Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.*, 81(413), 1989.

[7] F. Brenti and V. Welker. The Veronese construction for formal power series and graded algebras. *Adv. in Appl. Math.*, 42(4):545–556, 2009.

[8] V. I. Danilov and A. G. Khovanskii. Newton polyhedra and an algorithm for computing Hodge–Deligne numbers. *Math. USSR Izvestiya*, 29(2):279–298, 1987.

[9] S. Di Rocco, C. Haase, and B. Nill. Discrete mixed volume and Hodge-Deligne numbers. Preprint, September 2016, 7 pages, arXiv:1609.09115.

[10] P. Diaconis and J. Fulman. Carries, shuffling, and symmetric functions. *Adv. in Appl. Math.*, 43(2):176–196, 2009.

[11] E. Ehrhart. Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux. *J. Reine Angew. Math.*, 226:1–29, 1967.

[12] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, Reading, MA, 1990.

[13] P. M. Gruber. *Convex and Discrete Geometry*. Springer, Berlin, 2007.

[14] K. Jochemko and R. Sanyal. Combinatorial mixed valuations. Preprint, May 2016, 16 pages, arXiv:1605.07431.

[15] K. Jochemko and R. Sanyal. Combinatorial positivity of translation-invariant valuations and a discrete Hadwiger theorem. To appear in *J. Eur. Math. Soc.*, 2015.

[16] A. G. Khovanskii. Newton polyhedra, and the genus of complete intersections. *Funktsional. Anal. i Prilozhen*, 12(1):51–61, 1978.

[17] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2nd edition, 2014.

[18] R. P. Stanley. Decompositions of rational convex polytopes. *Ann. Discrete Math.*, 6:333–342, 1980.

[19] R. P. Stanley. *Enumerative Combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[20] A. Stapledon. Inequalities and Ehrhart δ-vectors. *Trans. Amer. Math. Soc.*, 361(10):5615–5626, 2009.

[21] R. Steffens. *Mixed Volumes, Mixed Ehrhart Theory and Applications to Tropical Geometry and Linkage Configurations*. PhD thesis, Goethe-Universität Frankfurt, 2009.

[22] R. Steffens and T. Theobald. Combinatorics and genus of tropical intersections and Ehrhart theory. *SIAM J. Discrete Math.*, 24(1):17–32, 2010.