An application of Hoffman graphs for spectral characterizations of graphs

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Abstract

In this paper, we present that the 2-clique extension of the $(t+1)\times(t+1)$-grid is determined by its spectrum if t is large enough. By applying results of Gavrilyuk and Koolen, this implies that the Grassmann graph $J_2(2D, D)$ is determined by its intersection array as a distance-regular graph if D is large enough. The main tool we are using is Hoffman graphs.

Keywords: Hoffman graph, graph eigenvalue, interlacing, walk-regular, spectral characterizations

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1 Introduction

In [8], Gavrilyuk and Koolen showed that for a distance-regular graph with the same intersection array as the Grassmann graph $J_q(2D, D)$, its subgraph $\triangle(x)$ induced on the neighbors of a fixed vertex x has the same spectrum as the q-clique extension of the $\frac{q^D-1}{q-1} \times \frac{q^D-1}{q-1}$-grid if $D \geqslant 3$. (For definitions of distance-regular graphs and related notions we refer to [3] and [7]. For other definitions we refer the reader to the next section.) Moreover, they showed that for a distance-regular graph, if its induced subgraph $\triangle(x)$ is isomorphic to the q-clique extension of the $\frac{q^D-1}{q-1} \times \frac{q^D-1}{q-1}$-grid for all vertices x, then the distance-regular graph is isomorphic to $J_q(2D, D)$.

In this paper, we will show that the 2-clique extension of the $(t+1) \times (t+1)$-grid is characterized by its spectrum if t is large enough. This implies that the Grassmann graph $J_2(2D, D)$ is characterized by its intersection array if D is large enough. Our main result is as follows:

Theorem 1. *Let G be a graph with spectrum*

$$\left\{(4t+1)^1,\ (2t-1)^{2t},\ (-1)^{(t+1)^2},\ (-3)^{t^2}\right\}.$$

Then there exists a positive constant C such that if $t \geqslant C$, then G is the 2-clique extension of the $(t+1) \times (t+1)$-grid.

Remark 2.

(*i*) The current estimates for C are unrealistic high, since the proof implicitly uses Ramsey theory.

(*ii*) In [1] it was shown that the 2-coclique extension of the square grid is usually not determined by its spectrum.

Another motivation to study the 2-clique extension of the $(t+1) \times (t+1)$-grid came from the study of connected regular graphs with four distinct eigenvalues. They have been previously studied in [4], and a key observation is that these graphs are walk-regular, which implies strong combinatorial information on the graph. The main work of this paper has a similar flavor as a result of Bang, Van Dam and Koolen [2]. They showed that the Hamming graph $H(3, q)$ is determined by its spectrum if $q \geqslant 36$. Note that the Hamming graph $H(3, q)$ is a connected regular graph with exactly four distinct eigenvalues.

The starting point for our work is a result by Koolen et al. [14]:

Theorem 3. *([14]) There exists a positive integer t such that any graph, that is cospectral with the 2-clique extension of $(t_1 \times t_2)$-grid is the slim graph of a 2-fat $\{\ ,\ ,\ \}$-line Hoffman graph for all $t_1 \geqslant t_2 \geqslant t$.*

As a direct consequence, we obtain the following corollary:

Corollary 4. *There exists a positive integer T such that any graph, that is cospectral with the 2-clique extension of $(t+1) \times (t+1)$-grid is the slim graph of a 2-fat $\{\ ,\ ,\ \}$-line Hoffman graph for all $t \geqslant T$.*

In view of Corollary 4, our Theorem 1 will follow from the following result:

Theorem 5. *Let G be a graph cospectral with the 2-clique extension of the $(t+1)\times(t+1)$-grid. If G is the slim graph of a 2-fat* $\{$, , $\}$*-line Hoffman graph, then G is the 2-clique extension of the $(t+1)\times(t+1)$-grid when $t > 4$.*

The main focus of this paper is to prove Theorem 5, and it is organized as follows. In Section 2, we review some preliminaries on graphs, interlacing and Hoffman graphs. Section 3 considers the graph cospectral with the 2-clique extension of the $(t+1)\times(t+1)$-grid, which is the slim graph of the Hoffman graph having possible indecomposable factors isomorphic to the Hoffman graphs in Figure 3. In Section 4, we forbid two of the mentioned Hoffman graphs to occur as indecomposable factors. In Section 5, the order of the quasi-cliques of the possible indecomposable factors is determined. Finally, in Section 6, we finish the proof of Theorem 5.

2 Preliminaries

Throughout this paper we will consider only undirected graphs without loops or multiple edges. Suppose that Γ is a graph with vertex set $V(\Gamma)$ with $|V(\Gamma)| = n$ and edge set $E(\Gamma)$. Let A be the adjacency matrix of Γ, then the eigenvalues of Γ are the eigenvalues of A. Let $\lambda_0, \lambda_1, \ldots, \lambda_t$ be the distinct eigenvalues of Γ and m_i be the multiplicity of λ_i $(i = 0, 1, \ldots, t)$. Then the multiset $\{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_t^{m_t}\}$ is called the *spectrum* of Γ.

Two graphs are called *cospectral* if they have the same spectrum.

For a vertex x, let $\Gamma_i(x)$ be the set of vertices at distance i from x. When $i = 1$, we also denote it by $N_\Gamma(x)$. For two distinct vertices x and y, we denote the number of common neighbors between them by $\lambda_{x,y}$ if x and y are adjacent, and by $\mu_{x,y}$ if they are not.

Let t be a positive integer. Recall that a (t)-*clique* (or complete graph) is a graph (on t vertices) in which every pair of vertices is adjacent. We write K_t for the complete graph on t vertices. The $t\times t$-*grid* is the Cartesian product $K_t \square K_t$. Equivalently, it is the line graph of the complete bipartite graph $K_{t,t}$. The spectrum of the $t\times t$-grid is $\big\{(2t-2)^1, (t-2)^{2(t-1)}, (-2)^{(t-1)^2}\big\}$.

Recall that the *Kronecker product* $M_1 \otimes M_2$ of two matrices M_1 and M_2 is obtained by replacing the ij-entry of M_1 by $(M_1)_{i,j}M_2$ for all i and j. If τ and θ are eigenvalues of M_1 and M_2, then $\tau\theta$ is an eigenvalue of $M_1 \otimes M_2$ [9].

For an integer $q \geqslant 1$, the *q-clique extension of* Γ is the graph $\widetilde{\Gamma}$ obtained from Γ by replacing each vertex $x \in V(\Gamma)$ by a clique $\widetilde{X}$ with q vertices, such that $\tilde{x} \sim \tilde{y}$ (for $\tilde{x} \in \widetilde{X},\ \tilde{y} \in \widetilde{Y},\ \widetilde{X} \neq \widetilde{Y}$) in $\widetilde{\Gamma}$ if and only if $x \sim y$ in Γ. If $\widetilde{\Gamma}$ is the q-clique extension of Γ, then $\widetilde{\Gamma}$ has adjacency matrix $J_q \otimes (A + I_n) - I_{qn}$, where J_q is the all one matrix of size q and I_n is the identity matrix of size n.

In particular, if $q = 2$ and Γ has spectrum

$$\big\{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_t^{m_t}\big\}, \tag{1}$$

then it follows that the spectrum of $\widetilde{\Gamma}$ is

$$\left\{(2\lambda_0+1)^{m_0}, (2\lambda_1+1)^{m_1}, \ldots, (2\lambda_t+1)^{m_t}, (-1)^{(m_0+m_1+\cdots+m_t)}\right\}. \tag{2}$$

In case that Γ is a connected regular graph with valency k and with adjacency matrix A having exactly four distinct eigenvalues $\{\lambda_0 = k, \lambda_1, \lambda_2, \lambda_3\}$, then A satisfies the following (see for example [12]):

$$A^3 - \left(\sum_{i=1}^{3} \lambda_i\right) A^2 + \left(\sum_{1\leqslant i<j\leqslant 3} \lambda_i\lambda_j\right) A - \lambda_1\lambda_2\lambda_3 I = \frac{\prod_{i=1}^{3}(k-\lambda_i)}{n} J. \tag{3}$$

We also need to introduce an important spectral tool that will be used throughout this paper: eigenvalue interlacing.

Lemma 6. *([11, Interlacing]) Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. For some $m < n$, let S be a real $n \times m$ matrix with orthonormal columns, $S^T S = I$, and consider the matrix $B = S^T AS$, with eigenvalues $\mu_1 \geqslant \cdots \geqslant \mu_m$. Then,*

(*i*) *the eigenvalues of B interlace those of A, that is,*

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}, \qquad i = 1, \ldots, m. \tag{4}$$

(*ii*) *if the interlacing is tight, that is, there exists an integer $j \in \{1, 2, \ldots, m\}$ such that $\lambda_i = \mu_i$ for $1 \leqslant i \leqslant j$ and $\lambda_{n-m+i} = \mu_i$ for $j+1 \leqslant i \leqslant m$, then $SB = AS$.*

Two interesting particular cases of interlacing are obtained by choosing appropriately the matrix S. If $S = \begin{pmatrix} I & O \\ O & O \end{pmatrix}$, then B is just a principal submatrix of A. If $\pi = \{V_1, \ldots, V_m\}$ is a partition of the index set of the columns of A, we can take for $\widetilde{B}$ the so-called quotient matrix of A with respect to π. Let A be partitioned according to π as

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix},$$

where $A_{i,j}$ denotes the submatrix (block) of A formed by rows in V_i and columns in V_j. The *characteristic matrix* C is the $n \times m$ matrix whose j^{th} column is the characteristic vector of V_j $(j = 1, \ldots, m)$.

Then, the *quotient matrix* of A with respect to π is the $m \times m$ matrix $\widetilde{B}$ whose entries are the average row sums of the blocks $A_{i,j}$ of A, more precisely:

$$(\widetilde{B})_{i,j} = \frac{1}{|V_i|}(C^T AC)_{i,j}.$$

The partition π is called *equitable* (or *regular*) if each block $A_{i,j}$ of A has constant row (and column) sum, that is, $C\widetilde{B} = AC$.

Lemma 7. *Let A be a real symmetric matrix with π as a partition of the index set of its columns. Suppose $\widetilde{B}$ is the quotient matrix of A with respect to π, then the following holds:*

(i) *The eigenvalues of $\widetilde{B}$ interlace the eigenvalues of A.*

(ii) *If the interlacing is tight, then the partition π is equitable.*

By an equitable partition of a graph, we always mean an equitable partition of its adjacency matrix.

Lemma 8. *[9, Theorem 9.3.3] If π is an equitable partition of a graph Γ with quotient matrix $\widetilde{B}$, then the eigenvalues of $\widetilde{B}$ interlace the eigenvalues of A tightly.*

2.1 Hoffman graphs

We will need the following properties and definitions related to Hoffman graphs.

Definition 9. A *Hoffman graph* $\mathfrak{h}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labeling map $\mu : V \to \{f, s\}$, satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

(ii) vertices with label f are pairwise non-adjacent.

We call a vertex with label s a *slim vertex*, and a vertex with label f a *fat vertex*. We denote by $V_s = V_s(\mathfrak{h})$ (resp. $V_f(\mathfrak{h})$) the set of slim (resp. fat) vertices of $\mathfrak{h}$.

For a vertex x of $\mathfrak{h}$, we define $N^s_{\mathfrak{h}}(x)$ (resp. $N^f_{\mathfrak{h}}(x)$) the set of slim (resp. fat) neighbors of x in $\mathfrak{h}$. If every slim vertex of a Hoffman graph $\mathfrak{h}$ has a fat neighbor, then we call $\mathfrak{h}$ *fat*. And if every slim vertex has at least t fat neighbors, we call $\mathfrak{h}$ *t-fat*. In a similar fashion, we define $N^f(x_1, x_2) = N^f_{\mathfrak{h}}(x_1, x_2)$ to be the set of common fat neighbors of two slim vertices x_1 and x_2 in $\mathfrak{h}$ and $N^s(F_1, F_2) = N^s_{\mathfrak{h}}(F_1, F_2)$ to be the set of common slim neighbors of two fat vertices F_1 and F_2 in $\mathfrak{h}$.

The *slim graph* of a Hoffman graph $\mathfrak{h}$ is the subgraph of H induced by $V_s(\mathfrak{h})$.

A Hoffman graph $\mathfrak{h}_1 = (H_1, \mu_1)$ is called an *induced Hoffman subgraph* of $\mathfrak{h} = (H, \mu)$, if H_1 is an induced subgraph of H and $\mu_1(x) = \mu(x)$ for all vertices x of H_1.

Let W be a subset of $V_s(\mathfrak{h})$. An *induced Hoffman subgraph of* $\mathfrak{h}$ *generated by* W, denoted by $\langle W \rangle_{\mathfrak{h}}$, is the Hoffman subgraph of $\mathfrak{h}$ induced by $W \bigcup \{f \in V_f(\mathfrak{h}) \mid f \sim w \text{ for some } w \in W\}$.

A *quasi-clique* is a subgraph of the slim graph of $\mathfrak{h}$ induced by the neighborhood of a fat vertex of $\mathfrak{h}$. If a quasi-clique is induced by the neighborhood of fat vertex F, we say it is the quasi-clique corresponding to F and denote it by $Q_{\mathfrak{h}}(F)$.

Definition 10. For a Hoffman graph $\mathfrak{h} = (H, \mu)$, let A be the adjacency matrix of H

$$A = \begin{pmatrix} A_s & C \\ C^T & O \end{pmatrix}$$

in a labeling in which the fat vertices come last. The *special matrix* $\mathcal{S}(\mathfrak{h})$ of $\mathfrak{h}$ is the real symmetric matrix $\mathcal{S}(\mathfrak{h}) := A_s - CC^T$. The eigenvalues of $\mathfrak{h}$ are the eigenvalues of $\mathcal{S}(\mathfrak{h})$.

Note that $\mathfrak{h}$ is not determined by $\mathcal{S}$, since different $\mathfrak{h}$ may have the same special matrix $\mathcal{S}$. Observe also that if there are no fat vertices, then $\mathcal{S}(\mathfrak{h}) = A_s$ is just the standard adjacency matrix.

Now we introduce two key concepts in this work: the direct sum of Hoffman graphs and line Hoffman graphs.

Definition 11. (Direct sum of Hoffman graphs) Let $\mathfrak{h}$ be a Hoffman graph and $\mathfrak{h}^1$ and $\mathfrak{h}^2$ be two induced Hoffman subgraphs of $\mathfrak{h}$. The Hoffman graph $\mathfrak{h}$ is the *direct sum* of $\mathfrak{h}^1$ and $\mathfrak{h}^2$, that is $\mathfrak{h} = \mathfrak{h}^1 \bigoplus \mathfrak{h}^2$, if and only if $\mathfrak{h}^1, \mathfrak{h}^2$ and $\mathfrak{h}$ satisfy the following conditions:

(*i*) $V(\mathfrak{h}) = V(\mathfrak{h}^1) \bigcup V(\mathfrak{h}^2)$;

(*ii*) $\{V_s(\mathfrak{h}^1), V_s(\mathfrak{h}^2)\}$ is a partition of $V_s(\mathfrak{h})$;

(*iii*) if $x \in V_s(\mathfrak{h}^i)$, $f \in V_f(\mathfrak{h})$ and $x \sim f$, then $f \in V_f(\mathfrak{h}^i)$;

(*iv*) if $x \in V_s(\mathfrak{h}^1)$ and $y \in V_s(\mathfrak{h}^2)$, then x and y have at most one common fat neighbor, and they have exactly one common fat neighbor if and only if they are adjacent.

Let us show an example of how to construct a direct sum of two Hoffman graphs.

Example 12. Let $\mathfrak{h}_1, \mathfrak{h}_2$ and $\mathfrak{h}_3$ be the Hoffman graphs represented in Figure 1.

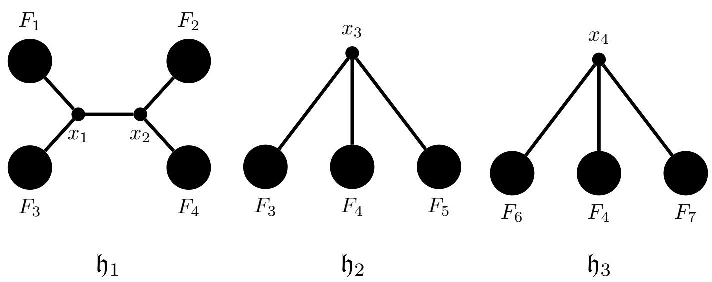


Figure 1

Then $\mathfrak{h}_1 \bigoplus \mathfrak{h}_2$, $\mathfrak{h}_1 \bigoplus \mathfrak{h}_3$ and $\mathfrak{h}_2 \bigoplus \mathfrak{h}_3$ are shown in Figure 2.

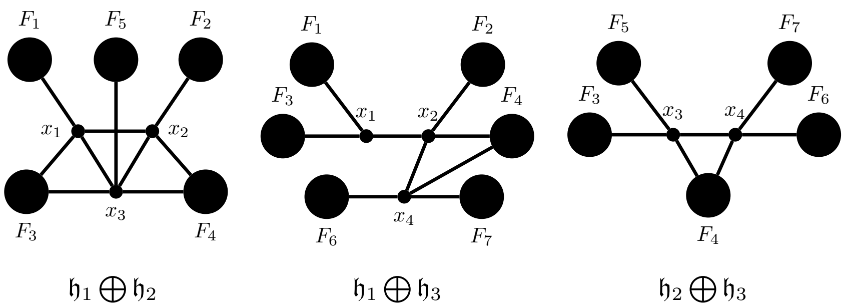


Figure 2

Definition 13. If a Hoffman graph $\mathfrak{h}$ is the direct sum of Hoffman graphs $\mathfrak{h}_1$ and $\mathfrak{h}'$, then we call the Hoffman graph $\mathfrak{h}$ *decomposable* and the Hoffman graph $\mathfrak{h}_1$ a *factor* of $\mathfrak{h}$.

Definition 14. Let $\mathfrak{G}$ be a family of Hoffman graphs. A Hoffman graph $\mathfrak{h}$ is called a *$\mathfrak{G}$-line Hoffman graph* if $\mathfrak{h}$ is an induced Hoffman subgraph of Hoffman graph $\mathfrak{h}' = \oplus_{i=1}^{r}\mathfrak{h}'_i$ where $\mathfrak{h}'_i$ is isomorphic to an induced Hoffman subgraph of some Hoffman graph in $\mathfrak{G}$ for $i = 1, \ldots, r$, such that $\mathfrak{h}'$ has the same slim graph as $\mathfrak{h}$.

3 Cospectral graphs with the 2-clique extension of the $(t+1) \times (t+1)$-grid

In this section, we study some consequences of Theorem 3. As mentioned in Section 1, the main goal of this paper is to show Theorem 5. Therefore, from now on we shall prepare the proof for Theorem 5.

Let $t > 0$ and for the rest of this paper, let G be a graph cospectral with the 2-clique extension of the $(t+1) \times (t+1)$-grid with adjacency matrix A. Since G has the same spectrum as the 2-clique extension of the $(t+1) \times (t+1)$-grid, G is a regular graph with valency $k = 4t + 1$ and spectrum

$$\left\{\eta_0^{m_0}, \eta_1^{m_1}, \eta_2^{m_2}, \eta_3^{m_3}\right\} = \left\{(4t+1)^1,\ (2t-1)^{2t},\ (-1)^{(t+1)^2},\ (-3)^{t^2}\right\}.$$

Using (3) we obtain

$$A^3 + (5-2t)A^2 + (7-8t)A + (3-6t)I = (16t+8)J.$$

Thus, we have

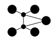
$$A^3_{(x,y)} = \begin{cases} 8t^2 + 4t, & \text{if } x = y; \\ 24t + 1 - (5-2t)\lambda_{x,y}, & \text{if } x \sim y; \\ 16t + 8 - (5-2t)\mu_{x,y}, & \text{if } x \not\sim y. \end{cases} \tag{5}$$

If G is the slim graph of a 2-fat $\{\ ,\ ,\ \}$-line Hoffman graph, then there exists a 2-fat Hoffman graph $\mathfrak{h}$, such that $\mathfrak{h} = \bigoplus_{i=1}^{s} \mathfrak{h}_i$ with slim graph G, and $\mathfrak{h}_i$ is isomorphic to one of the Hoffman graphs in the set $\mathfrak{G} = \{\ ,\ ,\ ,\ ,\ ,\ ,\ \}$ for $i = 1, \ldots, s$.

We will now exclude two Hoffman graphs from the set $\mathfrak{G}$. To do so, we note the following remark:

Remark 15.

(*i*) The Hoffman graph  has the same slim graph as .

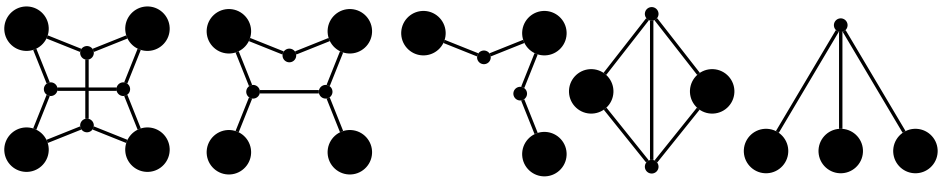
(*ii*) The Hoffman graph  has the same slim graph as , which is the direct sum of two Hoffman graphs isomorphic to  with one common fat neighbor (see Ex. 12).

Remark 15 implies that we may assume that the 2-fat Hoffman graph $\mathfrak{h}$, introduced before Remark 15, satisfies the following property, by adding fat vertices, if necessary.

Property 16.

(*i*) $\mathfrak{h}$ *has* G *as slim graph;*

(*ii*) $\mathfrak{h} = \bigoplus_{i=1}^{s'} \mathfrak{h}'_i$, *where* $\mathfrak{h}'_i$ *isomorphic to one of the Hoffman graphs shown in Figure 3, for* $i = 1, \ldots, s'$.

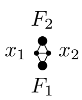


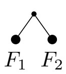
$\mathfrak{g}_1$ $\mathfrak{g}_2$ $\mathfrak{g}_3$ $\mathfrak{g}_4$ $\mathfrak{g}_5$

Figure 3

Using Property 16 and the definition of direct sum, we obtain the following lemma:

Lemma 17.

(*i*) *Any two distinct fat vertices* F_1 *and* F_2 *of* $\mathfrak{h}$ *have at most two common slim neighbors, i.e.,* $|N^s_{\mathfrak{h}}(F_1, F_2)| \leqslant 2$, *and if* F_1 *and* F_2 *have exactly two common slim neighbors* x_1 *and* x_2, *then* x_1 *and* x_2 *are adjacent. In particular, this means that in this case,*  *is an indecomposable factor of* $\mathfrak{h}$.

(*ii*) *If*  *is an induced Hoffman subgraph of one of the* $\mathfrak{h}_i$ *of Figure 3, and* $\mathfrak{h}_i \not\simeq$ , *then* F_1 *and* F_2 *have exactly one slim common neighbor in* $\mathfrak{h}$.

Proof. (*i*) Suppose that $N^s_{\mathfrak{h}}(F_1, F_2) = \{x_1, x_2, \ldots, x_p\}$. By Definition 11 (*iv*), we find that these p distinct slim vertices and two fat vertices should be in the same indecomposable factor of $\mathfrak{h}$. By Figure 3, we see that if $p \geqslant 2$, then $p = 2$ and the only indecomposable factor is (isomorphic to) $\mathfrak{g}_4$.

(*ii*) This follows from (*i*). □

4 Forbidding factors $\mathfrak{g}_1$ and $\mathfrak{g}_2$

In this section, we will show that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ can not occur as an indecomposable factor of $\mathfrak{h}$. For this, we first need the following lemma:

Lemma 18. *Any two distinct nonadjacent vertices x and y in G have at most $2t+2$ common neighbors, that is, $\mu_{x,y} \leqslant 2t+2$.*

Proof. Define a matrix M as follows:

$$\begin{aligned} M &= (A - \eta_1 I)(A - \eta_2 I) \\ &= (A - (2t-1)I)(A + I) \\ &= A^2 - 2(t-1)A - (2t-1)I. \end{aligned} \tag{6}$$

Then M is positive semidefinite (as A has no eigenvalues between η_1 and η_2), and we have

$$M_{(x,y)} = \begin{cases} k - (2t-1) = 2t+2, & \text{if } x = y; \\ -2(t-1) + \lambda_{x,y}, & \text{if } x \sim y; \\ \mu_{x,y}, & \text{if } x \not\sim y. \end{cases} \tag{7}$$

Since M is positive semidefinite, all its principal submatrices are positive semidefinite. Let x and y be two distinct nonadjacent vertices of G. Then

$$\begin{pmatrix} 2t+2 & \mu_{x,y} \\ \mu_{x,y} & 2t+2 \end{pmatrix}$$

is positive semidefinite and hence $\mu_{x,y} \leqslant 2t+2$ holds. □

Using Lemma 18, we obtain the following result:

Lemma 19.

(*i*) *The Hoffman graph $\mathfrak{g}_1$ can not be an indecomposable factor of $\mathfrak{h}$ when $t > 1$.*

(*ii*) *The Hoffman graph $\mathfrak{g}_2$ can not be an indecomposable factor of $\mathfrak{h}$ when $t > 1$.*

Proof. (*i*) Suppose that $\mathfrak{g}_1$ is an indecomposable factor of $\mathfrak{h}$, where $a_i = |V(Q_{\mathfrak{h}}(F_i))|$, as shown in Figure 4.

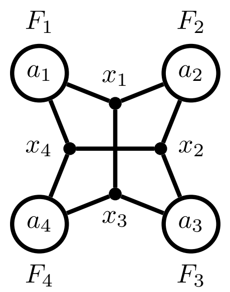


Figure 4: $\mathfrak{g}_1$

By Lemma 17, we find that $N^s_{\mathfrak{h}}(F_1, F_2) = \{x_1\}, N^s_{\mathfrak{h}}(F_2, F_3) = \{x_2\},\ N^s_{\mathfrak{h}}(F_1, F_4) = \{x_4\},\ |N^s_{\mathfrak{h}}(F_1, F_3)| \leqslant 2$, and $|N^s_{\mathfrak{h}}(F_2, F_4)| \leqslant 2$. By the definition of direct sum, we know that if a vertex x ($x \neq x_3$) is adjacent to x_1 in G, then $x \in N^s_{\mathfrak{h}}(F_1)$ or $x \in N^s_{\mathfrak{h}}(F_2)$. So $a_1 + a_2 - 3 = a_1 - 2 + a_2 - 2 + 1 = |N_G(x_1)| = k = 4t + 1$, that is $a_1 + a_2 = 4t + 4$. (In a similar way, we obtain that $a_2 + a_3 = a_3 + a_4 = a_1 + a_4 = 4t + 4$, so $a_1 = a_3$ and $a_2 = a_4$.) Note that $\mu_{x_1,x_2} = a_2 - 2 + |N^s_{\mathfrak{h}}(F_1, F_3)|,\ \mu_{x_1,x_4} = a_1 - 2 + |N^s_{\mathfrak{h}}(F_2, F_4)|$, $\lambda_{x_2,x_4} = |N^s_{\mathfrak{h}}(F_1, F_3)| + |N^s_{\mathfrak{h}}(F_2, F_4)|$. We obtain

$$\mu_{x_1,x_2} + \mu_{x_1,x_4} = 4t + \lambda_{x_2,x_4}. \tag{8}$$

Without loss of generality, we may assume that $\mu_{x_1,x_4} \leqslant \mu_{x_1,x_2}$. From Lemma 18 and Equation (8), we obtain

$$\begin{aligned} &0 \leqslant \lambda_{x_2,x_4} \leqslant 4, \quad \mu_{x_1,x_4} \leqslant \mu_{x_1,x_2} \leqslant 2t + 2, \\ &\text{and } 2t - 2 + \lambda_{x_2,x_4} \leqslant \mu_{x_1,x_4} \leqslant 2t + \left\lfloor \frac{\lambda_{x_2,x_4}}{2} \right\rfloor. \end{aligned} \tag{9}$$

Take the positive semidefinite principal submatrix M_1 of M, corresponding to the vertices $\{x_1, x_2, x_4\}$. Then, we obtain (by using (7)):

$$M_1 = \begin{pmatrix} 2t+2 & \mu_{x_1,x_2} & \mu_{x_1,x_4} \\ \mu_{x_1,x_2} & 2t+2 & -2(t-1) + \lambda_{x_2,x_4} \\ \mu_{x_1,x_4} & -2(t-1) + \lambda_{x_2,x_4} & 2t+2 \end{pmatrix}.$$

Replacing μ_{x_1,x_4} by μ and λ_{x_2,x_4} by λ and using (8), we have

$$M_1 = \begin{pmatrix} 2t+2 & 4t+\lambda-\mu & \mu \\ 4t+\lambda-\mu & 2t+2 & -2(t-1)+\lambda \\ \mu & -2(t-1)+\lambda & 2t+2 \end{pmatrix}.$$

The above matrix M_1 has determinant

$$\begin{aligned} \det(M_1) = &-32t^3 - 8\lambda t^2 + \big((8\lambda + 32)\mu - 4\lambda^2 - 16\lambda + 32\big)t \\ &- (2\lambda + 8)\mu^2 + (2\lambda^2 + 8\lambda)\mu - 4\lambda^2 - 8\lambda, \end{aligned}$$

where $0 \leqslant \lambda \leqslant 4,\ 2t - 2 + \lambda \leqslant \mu \leqslant 2t + \lfloor \frac{\lambda}{2} \rfloor$ (by (9)).

If $t > 1$, by checking all the possible values of λ and μ, we obtain that $\det(M_1) < 0$ and this is impossible since M_1 is positive semidefinite.

(*ii*) can be shown in a similar way. Suppose that $\mathfrak{g}_2$ is an indecomposable factor of $\mathfrak{h}$, where $a_i = |V(Q_{\mathfrak{h}}(F_i))|$, as shown in Figure 5.

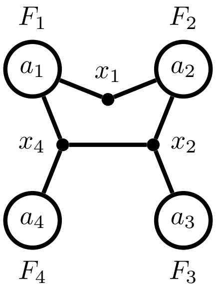


Figure 5: $\mathfrak{g}_2$

Then the submatrix M_1 is replaced by

$$M_2 = \begin{pmatrix} 2t+2 & 4t+1+\lambda-\mu & \mu \\ 4t+1+\lambda-\mu & 2t+2 & -2(t-1)+\lambda \\ \mu & -2(t-1)+\lambda & 2t+2 \end{pmatrix},$$

with determinant:

$$\begin{aligned}\det(M_2) = & -32t^3 - (8\lambda+16)t^2 + \big((8\lambda+32)\mu - 4\lambda^2 - 20\lambda + 14\big)t \\ & - (2\lambda+8)\mu^2 + (2\lambda^2+10\lambda+8)\mu - 4\lambda^2 - 12\lambda - 2,\end{aligned}$$

where $0 \leqslant \lambda = \lambda_{x_2,x_4} \leqslant 4$ and $2t-1+\lambda \leqslant \mu = \mu_{x_1,x_4} \leqslant 2t + \lfloor \frac{1+\lambda}{2} \rfloor$.

If $t > 1$, by checking all the possible values of λ and μ, we obtain that $\det(M_2) < 0$ and the result follows, as this gives a contradiction. □

5 The order of quasi-cliques

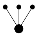
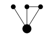
5.1 An upper bound on the order of quasi-cliques

From the above section, we find that the only possible indecomposable factors of $\mathfrak{h}$ are $\mathfrak{g}_3$, $\mathfrak{g}_4$ and $\mathfrak{g}_5$.

Proposition 20. *Let q be the order of a quasi-clique Q corresponding to a fat vertex F in $\mathfrak{h}$. Then $q \leqslant 2t+2$ when $t > 1$.*

Proof. We show the following three claims from which the proposition follows.

Claim 21. *In the quasi-clique Q, every vertex has valency at least $q-2$.*

Proof. If there exists a vertex that has two nonneighbors in Q, then in $\mathfrak{h}$, these three slim vertices should be in the same indecomposable factor by Definition 11 (iv). But neither  nor  is an induced Hoffman subgraph of $\mathfrak{g}_3$, $\mathfrak{g}_4$ or $\mathfrak{g}_5$. Hence the claim holds. □

Claim 22. *The order q of the quasi-clique Q is at most $2t+3$ when $t > 1$, and if $q = 2t+3$, then Q has exactly a vertex of valency $2t+2$.*

Proof. Let us consider the partition $\pi = \{V(Q), V(G) - V(Q)\}$ of $V(G)$. The quotient matrix $\widetilde{B}$ of A with respect to the partition π is

$$\widetilde{B} = \begin{pmatrix} q-2+\epsilon & 4t+1-(q-2+\epsilon) \\ \frac{(4t+1-(q-2+\epsilon))q}{2(t+1)^2-q} & 4t+1-\frac{(4t+1-(q-2+\epsilon))q}{2(t+1)^2-q} \end{pmatrix} \tag{10}$$

with eigenvalues $k(=4t+1)$ and $q-2+\epsilon-\frac{(4t+1-(q-2+\epsilon))q}{2(t+1)^2-q}$, where $0 \leqslant \epsilon \leqslant 1$ (by Claim 21). By interlacing (Lemma 7 (i)), we obtain that, the second eigenvalue of the quotient matrix $\widetilde{B}$ is at most $2t-1$, hence $q-2+\epsilon-\frac{(4t+1-(q-2+\epsilon))q}{2(t+1)^2-q} \leqslant 2t-1$ holds.

If $q = 2t+4$, then $2t+2+\epsilon-\frac{(2t-1-\epsilon)(t+2)}{t^2+t-1} = q-2+\epsilon-\frac{(4t+1-(q-2+\epsilon))q}{2(t+1)^2-q} \leqslant 2t-1$. But this is not possible when $t > 1$.

If $q = 2t+3$, then (10) becomes

$$\widetilde{B} = \begin{pmatrix} 2t+1+\epsilon & 2t-\epsilon \\ \frac{(2t-\epsilon)(2t+3)}{2t^2+2t-1} & 4t+1-\frac{(2t-\epsilon)(2t+3)}{2t^2+2t-1} \end{pmatrix}$$

and $2t+1+\epsilon-\frac{(2t-\epsilon)(2t+3)}{2t^2+2t-1} = q-2+\epsilon-\frac{(4t+1-(q-2+\epsilon))q}{2(t+1)^2-q} \leqslant 2t-1$. By solving this inequality, we have $0 \leqslant \epsilon \leqslant \frac{1}{1+t}$. Suppose that there are m_1 vertices with valency $2t+1$ and m_2 vertices with valency $2t+2$ in Q. Then

$$m_1 + m_2 = 2t+3,$$

$$\frac{(2t+1)m_1 + (2t+2)m_2}{m_1+m_2} = 2t+1+\frac{m_2}{m_1+m_2} \leqslant 2t+1+\frac{1}{1+t}.$$

Since m_1 is an even number by the handshaking lemma, it follows that the only possible solution is $m_1 = 2t+2,\ m_2 = 1$. So the claim holds. □

Finally, we show the following:

Claim 23. *There are no quasi-cliques of order* $2t+3$ *when* $t > 1$.

Proof. Assume that there exists a quasi-clique Q' with order $2t+3$, corresponding to fat vertex F in $\mathfrak{h}$. Then, from Claim 22, we obtain that, in Q', there exist two distinct vertices which are not adjacent, say x_1 and x_2. Now consider the factor containing the slim vertices $x_1,\ x_2$ and fat vertex F. Then we see that F should be the fat vertex F_2 in $\mathfrak{g}_3$ (in Figure 6) and $Q' = Q_{\mathfrak{h}}(F_2)$ with order $a_2 = 2t+3$.

Moreover, we obtain that $a_1 - 1 + a_2 - 2 = |N_G(x_1)| = k = 4t+1$ and $a_2 - 2 + a_3 - 1 = |N_G(x_2)| = k = 4t+1$, where $a_1 = |V(Q_{\mathfrak{h}}(F_1))|$ and $a_3 = |V(Q_{\mathfrak{h}}(F_3))|$. Then $|V(Q_{\mathfrak{h}}(F_1))| = |V(Q_{\mathfrak{h}}(F_3))| = 2t+1$ and

$$V(Q_{\mathfrak{h}}(F_1)) \bigcap V(Q_{\mathfrak{h}}(F_2)) = \{x_1\},$$
$$V(Q_{\mathfrak{h}}(F_3)) \bigcap V(Q_{\mathfrak{h}}(F_2)) = \{x_2\}$$

by Lemma 17 (ii). By using Claim 22 again, we find that there are exactly $t+1$ pairs of non adjacent vertices in Q' and for each such a pair of vertices, we find two quasi-cliques with order $2t+1$ containing exactly one of them as a vertex, respectively. It means that there are at least $2t+2$ distinct quasi-cliques with order $2t+1$.
Now let us estimate the cardinality of the set $W = \big\{(x, Q'') \mid x \in V(Q'')$ and Q'' is a quasi-clique corresponding to some fat vertex in $\mathfrak{h}$ with order $2t+1$ or $2t+3\big\}$ by double counting. On the one hand, $|W| \leqslant 2\cdot 2(t+1)^2$, since every vertex can only be a member of at most 2 such quasi-cliques considering its valency is $4t+1$. On the other hand, we know there are at least $2t+2$ quasi-cliques with order $2t+1$ and at least 1 quasi-clique with order $2t+3$. So $|W| \geqslant (2t+2)(2t+1)+1\cdot(2t+3)$. Hence $4(t+1)^2 \geqslant (2t+2)(2t+1)+(2t+3)$, a contradiction. This shows the claim. □

The proposition follows from Claims 21, 22 and 23. □

In addition, we will give a lemma about cliques of G that will be used in next sections.

Lemma 24. *Let c be the order of a clique C in G, then $c \leqslant 2t+2$. If equality holds, then every vertex $x \in V(G) - V(C)$ has exactly 2 neighbors in C.*

Proof. For the inequality case, exactly the same argument applies by replacing ϵ by 1 in the proof of Claim 22. If equality holds, then we have tight interlacing, since $\widetilde{B}$ has k and $2t-1$ as its eigenvalues, which are also the largest and second largest eigenvalues of A. So by Lemma 7 (ii), the partition $\pi = \big\{V(C), V(G) - V(C)\big\}$ is equitable and by (10) ($q = 2t+2$), we obtain that every vertex in $V(G) - V(C)$ has exactly 2 neighbors in C. □

5.2 Determining the order of the quasi-cliques for $\mathfrak{g}_3$ and $\mathfrak{g}_4$

In this subsection, we will determine the order of quasi-cliques for each of the remaining indecomposable factors $\mathfrak{g}_3$ and $\mathfrak{g}_4$. First we consider the indecomposable factor $\mathfrak{g}_3$.

Lemma 25. *Suppose that $\mathfrak{g}_3$ is an indecomposable factor of $\mathfrak{h}$ with fat vertices F_1, F_2 and F_3. Then for $i = 1, 2, 3$, the quasi-clique $Q_{\mathfrak{h}}(F_i)$ corresponding to F_i has order $2t+2$ when $t > 1$.*

Proof. Let $\mathfrak{g}_3$ be an indecomposable factor of $\mathfrak{h}$ as shown in Figure 6, where $a_i = |V(Q_{\mathfrak{h}}(F_i))|$, for $i = 1, 2, 3$.

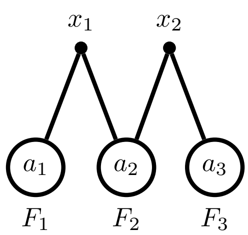


Figure 6: $\mathfrak{g}_3$

It is clear that $a_1 - 1 + a_2 - 2 = |N_G(x_1)| = k = 4t+1$, that is, $a_1 + a_2 = 4t+4$. From Proposition 20, it follows that $a_1 = a_2 = 2t+2$. By interchanging the roles of x_1 and x_2, the result follows. □

Now we consider the indecomposable factor $\mathfrak{g}_4$.

Lemma 26. *Suppose that $\mathfrak{g}_4$ is an indecomposable factor of $\mathfrak{h}$ with fat vertices K_1 and K_2 and slim vertices x and y. Then for $i = 1,2$, the quasi-clique $Q_{\mathfrak{h}}(K_i)$ corresponding to K_i has order $2t+2$ when $t > 1$.*

Moreover, the partition $\pi = \{V_1, V_2, V_3\}$ on $V(G)$ is equitable with quotient matrix

$$\begin{pmatrix} 1 & 4t & 0 \\ 2 & 2t-1 & 2t \\ 0 & 4 & 2t-3 \end{pmatrix},$$

where $V_1 = \{x, y\}$, $V_2 = V(Q_{\mathfrak{h}}(K_1)) \bigcup V(Q_{\mathfrak{h}}(K_2)) - V_1$ and $V_3 = V(G) - V_1 \bigcup V_2$.

Proof. Consider $\mathfrak{g}_1$ in Figure 7, where d_i is the order of quasi-clique $Q_{\mathfrak{h}}(K_i)$, for $i = 1,2$.

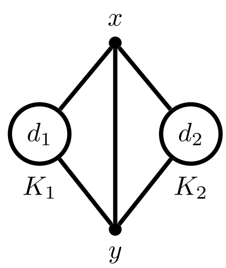


Figure 7: $\mathfrak{g}_4$

Then by definition of direct sum and Lemma 17 (i), we obtain that $d_1 - 2 + d_2 - 2 + 1 = |N_G(x)| = 4t+1$, that is, $d_1 + d_2 = 4t+4$. By using Proposition 20 again, it is easy to see that $d_1 = d_2 = 2t+2$.

Now we will show that the partition is equitable. Suppose that α is the average number of edges leading from a vertex in V_3 to vertices in V_2. Then the quotient matrix $\widetilde{B}$ of A with respect to π is:

$$\widetilde{B} = \begin{pmatrix} 1 & d_1 + d_2 - 4 & 0 \\ 2 & k - 2 - \frac{(|V(G)| - 2 - (d_1 + d_2 - 4))\alpha}{d_1 + d_2 - 4} & \frac{(|V(G)| - 2 - (d_1 + d_2 - 4))\alpha}{d_1 + d_2 - 4} \\ 0 & \alpha & k - \alpha \end{pmatrix},$$

that is,

$$\widetilde{B} = \begin{pmatrix} 1 & 4t & 0 \\ 2 & 4t - 1 - \frac{\alpha t}{2} & \frac{\alpha t}{2} \\ 0 & \alpha & k - \alpha \end{pmatrix} \tag{11}$$

with eigenvalues k, θ_1 and θ_2, where $\theta_1 + \theta_2 = 4t - \frac{\alpha t}{2} - \alpha, \theta_1\theta_2 = -4t - \frac{\alpha t}{2} + \alpha - 1$. From Lemma 7 (i), the eigenvalues of (11) interlace the eigenvalues of A, that is, $-3 \leqslant \theta_1, \theta_2 \leqslant 2t - 1$, and we obtain the following inequalities:

$$(-3)^2 - (4t - \frac{\alpha t}{2} - \alpha)(-3) - 4t - \frac{\alpha t}{2} + \alpha - 1 \geqslant 0, \tag{12}$$

$$(2t-1)^2 - (4t - \frac{\alpha t}{2} - \alpha)(2t-1) - 4t - \frac{\alpha t}{2} + \alpha - 1 \geqslant 0. \tag{13}$$

Inequalities (12) and (13) are only satisfied for $\alpha = 4$, and for this value of α, they become equalities. This means that (11) becomes

$$\widetilde{B} = \begin{pmatrix} 1 & 4t & 0 \\ 2 & 2t-1 & 2t \\ 0 & 4 & 4t-3 \end{pmatrix} \tag{14}$$

with eigenvalues $k, 2t-1$ and -3. So we have tight interlacing and Lemma 7 (ii) implies that this is an equitable partition. $\square$

5.3 Determining the order of the quasi-cliques for $\mathfrak{g}_5$

In this subsection, we will determine the order of the quasi-cliques corresponding to an indecomposable factor isomorphic to $\mathfrak{g}_5$. For the rest of this subsection, we will assume that $\mathfrak{g}_5$ is an indecomposable factor of $\mathfrak{h}$ and that $\mathfrak{g}_5$ is as in Figure 8, where the slim vertex x has fat neighbors I_1, I_2 and I_3. Let $Q_{\mathfrak{h}}(I_j)$ be the quasi-clique corresponding to the fat vertex I_j and $b_j = |V(Q_{\mathfrak{h}}(I_j))|$ for $j = 1, 2, 3$. Without loss of generality, we may assume that $b_1 \geqslant b_2 \geqslant b_3$.

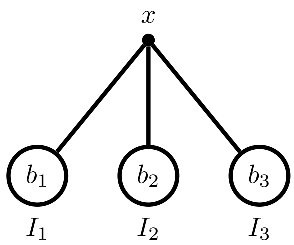


Figure 8: $\mathfrak{g}_5$

It is easy to see that $b_1 - 1 + b_2 - 1 + b_3 - 1 = 4t + 1$, hence

$$b_1 + b_2 + b_3 = 4t + 4. \tag{15}$$

Note that the above implies that there cannot be two quasi-cliques with order $2t + 2$, so it follows that

$$1 \leqslant b_3 \leqslant b_2 \leqslant 2t + 1. \tag{16}$$

Let $e_G(x)$ be the number of edges in the subgraph of G induced by the set of neighbors of x, $N_G(x)$. From (5) it follows:

$$e_G(x) = 4t^2 + 2t = 2\binom{2t+1}{2}. \tag{17}$$

Now we give the following proposition to obtain bounds on $e_G(x)$ (of (17)):

Proposition 27. *Let $1 \leqslant i < j \leqslant 3$. Then any vertex y ($y \neq x$) in $Q_{\mathfrak{h}}(I_j)$ has at most 2 neighbors in $V(Q_{\mathfrak{h}}(I_i)) - \{x\}$.*

Proof. We show it for $i = 1$ and $j = 2$. The other cases follow in a similar way. Suppose y is a vertex in $Q_{\mathfrak{h}}(I_2)$ and $y \neq x$. Since $b_2 = |V(Q_{\mathfrak{h}}(I_2))| \leqslant 2t+1$, and the indecomposable factors $\mathfrak{g}_3$ and $\mathfrak{g}_4$ do not have quasi-clique with order at most $2t + 1$ (Lemma 25 and Lemma 26), the indecomposable factor containing y as slim vertex is isomorphic to $\mathfrak{g}_5$. Now we need the following claim:

Claim 28. *For a fat vertex $F \in N^f_{\mathfrak{h}}(y)$, we have $|N^s_{\mathfrak{h}}(I_1, F)| \leqslant 1$.*

Proof. Clearly, when F is the fat vertex I_2, the result holds. Suppose F is a fat neighbor distinct from I_2. By Lemma 17 (i), we have $|N^s_{\mathfrak{h}}(I_1, F)| \leqslant 2$. Now assume that $|N^s_{\mathfrak{h}}(I_1, F)| = 2$ and $N^s_{\mathfrak{h}}(I_1, F) = \{x', y'\}$. By Lemma 17 ($i$), it follows that the Hoffman subgraph induced by the slim vertices x' and y' and the fat vertices I_1 and F is isomorphic to the indecomposable factor $\mathfrak{g}_4$ and by Lemma 26, we have $b_1 = |V(Q_{\mathfrak{h}}(I_1))| = |V(Q_{\mathfrak{h}}(F))| = 2t + 2$. As $x \notin V(Q_{\mathfrak{h}}(F))$ and $y \notin V(Q_{\mathfrak{h}}(I_1))$, we obtain that $\{x', y'\}$ and $\{x, y\}$ are disjoint. By using Lemma 26 again, we obtain that the partition

$$\{V_1, V_2, V_3\} = \Big\{\{x', y'\}, V(Q_{\mathfrak{h}}(I_1)) \bigcup V(Q_{\mathfrak{h}}(F)) - \{x', y'\}, V(G) - V(Q_{\mathfrak{h}}(I_1)) \bigcup V(Q_{\mathfrak{h}}(F))\Big\}$$

is equitable and x has exactly $2t-1$ neighbors in V_2, since $x \in V(Q_{\mathfrak{h}}(I_1)) - \{x', y'\} \subseteq V_2$. But, on the other hand, x has at least $|(V(Q_{\mathfrak{h}}(I_1)) - \{x', y'\} - \{x\}) \bigcup \{y\}| = 2t$ neighbors in V_2. This gives a contradiction. □

We can finish now the proof of Proposition 27. Note that y is the slim vertex of an indecomposable factor isomorphic to $\mathfrak{g}_5$, see Figure 9, where $I_2(y) = I_2$.

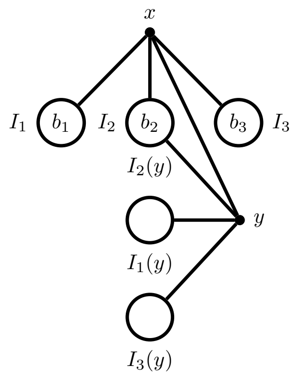


Figure 9

Then from $N^s_{\mathfrak{h}}(I_1, I_2(y)) = \{x\}$ (by Lemma 17 (ii)), $|N^s_{\mathfrak{h}}(I_1, I_1(y))| \leqslant 1$ and $|N^s_{\mathfrak{h}}(I_1, I_3(y))| \leqslant 1$, we find that y has at most 2 neighbors in $V(Q_{\mathfrak{h}}(I_1)) - \{x\}$ and the result holds. $\square$

From Proposition 27, it follows

$$e_G(x) \leqslant \binom{b_1 - 1}{2} + \binom{b_2 - 1}{2} + \binom{b_3 - 1}{2} + 2(b_2 - 1) + 4(b_3 - 1). \tag{18}$$

By using (17) and (18), we obtain

$$\binom{b_1 - 1}{2} + \binom{b_2 - 1}{2} + \binom{b_3 - 1}{2} + 2(b_2 - 1) + 4(b_3 - 1) \geqslant 2\binom{2t + 1}{2}.$$

This means

$$\begin{aligned} 2(2t+1)2t &\leqslant (b_1 - 1)(b_1 - 2) + (b_2 - 1)(b_2 - 2) + (b_3 - 1)(b_3 - 2) + 4(b_2 - 1) \\ &\quad + 8(b_3 - 1) \\ &= b_1^2 - 3b_1 + b_2^2 + b_3^2 + 4b_3 + (b_2 + b_3) - 6 \\ &= b_1^2 - 3b_1 + b_2^2 + b_3^2 + 4b_3 + (4t + 4 - b_1) - 6 \\ &= (b_1 - 2)^2 + b_2^2 + (b_3 + 2)^2 + 4t - 10, \end{aligned}$$

and we obtain $(b_1 - 2)^2 + b_2^2 + (b_3 + 2)^2 \geqslant 8t^2 + 10$, where $1 \leqslant b_3 \leqslant b_2 \leqslant b_1 \leqslant 2t + 2$, and $b_3 + b_2 + b_1 = 4t + 4$.

When $t > 4$ holds, we find that $b_3 \leqslant 2$ and there are three possible cases for the order of the quasi-cliques of $\mathfrak{g}_5$: $(b_1, b_2, b_3) = (2t + 2, 2t + 1, 1)$, $(b_1, b_2, b_3) = (2t + 2, 2t, 2)$, or $(b_1, b_2, b_3) = (2t + 1, 2t + 1, 2)$.

This shows the following lemma:

Lemma 29. *Suppose that $\mathfrak{g}_5$ is an indecomposable factor of $\mathfrak{h}$ with fat vertices I_1, I_2 and I_3. For $i = 1, 2, 3$, let b_i be the order of the quasi-clique $Q_{\mathfrak{h}}(I_i)$ corresponding to the fat vertex I_i in $\mathfrak{g}_5$ with $b_1 \geqslant b_2 \geqslant b_3$. If $t > 4$, then one of the following holds:*

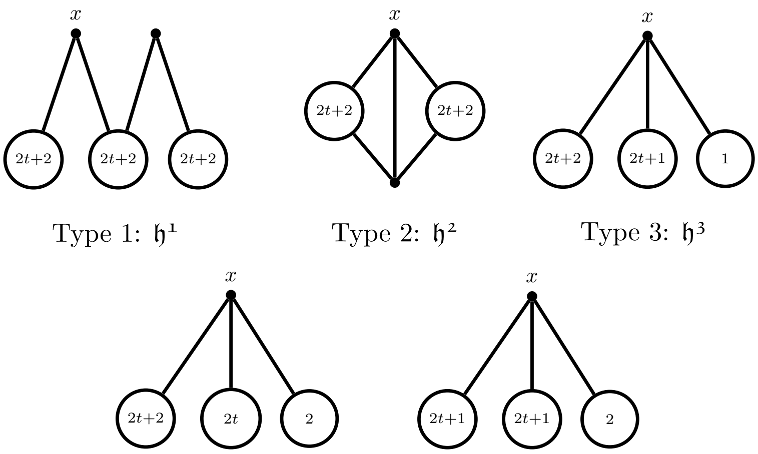
(1) $(b_1, b_2, b_3) = (2t + 2, 2t + 1, 1)$;

(2) $(b_1, b_2, b_3) = (2t + 2, 2t, 2)$;

(3) $(b_1, b_2, b_3) = (2t + 1, 2t + 1, 2)$.

6 Finishing the proof of Theorem 5

In Figure 10, we summarize what we have shown until now. We give the possible indecomposable factors together with the order of their quasi-cliques under the condition $t > 4$. We will refer to a slim vertex x having Type i ($i = 1, 2, 3, 4, 5$) if the indecomposable factor which contains x is of Type i.



Type 1: $\mathfrak{h}^1$ Type 2: $\mathfrak{h}^2$ Type 3: $\mathfrak{h}^3$

Type 4: $\mathfrak{h}^4$ Type 5: $\mathfrak{h}^5$

Figure 10

Suppose that there are n_i vertices of Type i and q_j quasi-cliques of order j in G, where $i = 1, 2, 3, 4, 5$ and $j = 2t, 2t+1, 2t+2$. Consider the sets $W_i = \{(x, Q) \mid x \in V(Q)$, where Q is a quasi-clique of order $2t - 1 + i$ corresponding to some fat vertex in $\mathfrak{h}\}$, $i = 1, 2, 3$. Then, by double counting the cardinalities of the sets W_1, W_2 and W_3, we obtain

$$n_4 = 2tq_{2t}, \tag{19}$$
$$n_3 + 2n_5 = (2t + 1)q_{2t+1}, \tag{20}$$
$$2n_1 + 2n_2 + n_3 + n_4 = (2t + 2)q_{2t+2}, \tag{21}$$
$$n_1 + n_2 + n_3 + n_4 + n_5 = |V(G)| = 2(t + 1)^2. \tag{22}$$

From (19),(20),(21) and (22), we obtain

$$2tq_{2t} + (2t + 1)q_{2t+1} + (2t + 2)q_{2t+2} = (2t + 2)^2, \tag{23}$$

which implies

$$-q_{2t} + q_{2t+2} \equiv 1 \pmod{2t + 1}. \tag{24}$$

From (23), it is easy to see that

$$q_{2t+2} \leqslant 2t + 2. \tag{25}$$

From (19) and (21), it follows that $n_4 = 2tq_{2t} \leqslant (2t+2)q_{2t+2}$, hence $q_{2t} \leqslant \lfloor (1+1/t)q_{2t+2} \rfloor \leqslant q_{2t+2} + 2$. This shows that the only possible solutions of (24) are the following:

Case 1: $q_{2t+2} = q_{2t} + 2t + 1 + 1$.

By (23) and (25), it follows that $q_{2t+2} = 2t + 2$ and $q_{2t} = q_{2t+1} = 0$.

Case 2: $q_{2t+2} = q_{2t} + 1$.

In this case (23) becomes

$$2q_{2t} + q_{2t+1} = 2t + 2$$

So $q_{2t} \leqslant t + 1$. If there exists a quasi-clique Q with order $2t$, then every vertex in this quasi-clique belongs to Type 4 and we can find a quasi-clique Q' with order $2t + 2$ exactly sharing this vertex with Q by Lemma 17 (i). This means that $q_{2t+2} \geqslant |V(Q)| = 2t$. So $t + 1 \geqslant q_{2t} = q_{2t+2} - 1 \geqslant 2t - 1$, but this is not possible. Hence $q_{2t} = 0$, and this implies $q_{2t+2} = 1$ and $q_{2t+1} = 2t + 2$.

Summarizing, we only have the following two cases:
Case 1: $q_{2t} = 0,\ q_{2t+1} = 0,\ q_{2t+2} = 2t + 2$;
Case 2: $q_{2t} = 0,\ q_{2t+1} = 2t + 2,\ q_{2t+2} = 1$.

Now we are going to determine the n_i's for $i = 1, 2, 3, 4, 5$. Observe that $q_{2t} = 0$ holds for both cases, which implies that $n_4 = 0$ holds in both cases by using (19).

Proposition 30. *If $q_{2t} = q_{2t+1} = 0$, $q_{2t+2} = 2t + 2$ and $t > 4$, then G is the 2-clique extension of the $(t + 1) \times (t + 1)$-grid.*

Proof. Since $q_{2t+1} = 0$, we find $n_3 = n_5 = 0$ from (20). Hence all vertices of G are of Type 1 or Type 2 and every vertex of G has exactly two fat neighbors. We want to show that $n_1 = 0$. Suppose this is not the case. Then there exists a vertex x belonging to Type 1 and the Hoffman graph shown in Figure 11 is an indecomposable factor of $\mathfrak{h}$, where $x, x' \in N^s_{\mathfrak{h}}(F_2)$ and $x \not\sim x'$.

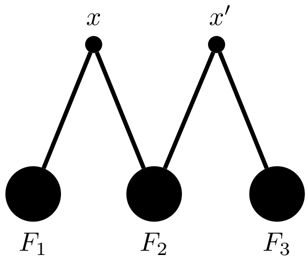


Figure 11

In a similar way as in Claim 28, we can show that, for any neighbor y of x in the quasi-clique $Q_{\mathfrak{h}}(F_2)$ and for any fat vertex $F \in N^f_{\mathfrak{h}}(y)$, it follows that $|N^s_{\mathfrak{h}}(F_1, F)| \leqslant 1$. Observing that y has only one fat neighbor besides the fat vertex F_2, this implies that y has at most one neighbor in $Q_{\mathfrak{h}}(F_1)$ besides x. Suppose that $a_1 = |V(Q_{\mathfrak{h}}(F_1))|,\ a_2 = |V(Q_{\mathfrak{h}}(F_2))|$. Since x' has no neighbor in the quasi-clique $Q_{\mathfrak{h}}(F_1)$, it implies that $Q_{\mathfrak{h}}(F_1)$ cannot be a clique by Lemma 24. Therefore, the subgraph of G induced by $V(Q_{\mathfrak{h}}(F_1)) - \{x\}$ is not a clique. By counting the number of triangles throught x we obtain

$$\begin{aligned} A^3_{(x,x)} &\leqslant 2\left(\binom{a_1-1}{2}-1\right)+2\binom{a_2-2}{2}+2(a_2-2) \\ &\leqslant 2\left(\binom{2t+1}{2}-1\right)+2\binom{2t}{2}+2\cdot 2t \\ &= 8t^2+4t-2. \end{aligned}$$

But, as G has the same spectrum as the 2-clique extension of the $(t+1)\times(t+1)$-grid, we obtain that $A^3_{(x,x)} = 8t^2+4t$ by (5). This gives a contradiction. Hence, we just showed that all the vertices of G are of Type 2.

Now, consider the following equivalence relation $\mathcal{R}$ on the vertex set $V(G)$:

$$x\mathcal{R}x' \text{ if and only if } \{x\}\cup N_G(x) = \{x'\}\cup N_G(x'), \text{ where } x, x' \in V(G).$$

It means that for each vertex x, there exists an unique distinct vertex x' such that $x\mathcal{R}x'$ and $x' \sim x$. So two vertices in the same equivalent class induce a 2-clique. Let us define a graph $\underline{G}$ whose vertices are the equivalent classes, and such that two classes $\{x, x'\}$ and $\{y, y'\}$ are adjacent in $\underline{G}$ if and only if $x \sim y, x' \sim y, x \sim y', x' \sim y'$. Then $\underline{G}$ is a regular graph with valency $2t$, and G is the 2-clique extension of $\underline{G}$. Note that the spectrum of $\underline{G}$ follows immediately from (1) and (2) and is equal to

$$\left\{(2t)^1, (t-1)^{2t}, (-2)^{t^2}\right\}.$$

Since $\underline{G}$ is a connected regular graph with valency $2t$ with multiplicity 1, and since it has exactly three distinct eigenvalues, it follows that $\underline{G}$ is a strongly regular graph with parameters $\left((t+1)^2, 2t, t-1, 2\right)$. From [15], it follows that if $t \neq 3$, then the graph with these parameters is unique and is the $(t+1)\times(t+1)$-grid. So we obtained that G is the 2-clique extension of the $(t+1)\times(t+1)$-grid when $t > 4$. $\square$

Now let us assume that we are in **Case 2**, that is $q_{2t} = 0,\ q_{2t+1} = 2t+2$, and $q_{2t+2} = 1$. We have already seen that $n_4 = 0$. We will show that this case is impossible. But to show this, we will need a few lemmas.

As a vertex of Type 1 or Type 2 lies in two distinct quasi-cliques of order $2t+2$ and $q_{2t+2} = 1$, we find that there are no vertices of Type 1 or Type 2. So we obtain $n_1 = n_2 = 0$. This implies $n_3 = 2t+2$ and $n_5 = 2t(t+1)$ by (21) and (22). As $n_1 = 0$, all quasi-cliques are actually cliques since every vertex is adjacent to all of the vertices in the same quasi-clique except itself.

Let Q be the unique quasi-clique of order $2t+2$ and let $\mathcal{L} = \{Q' \mid Q' \text{ is a quasi-clique of order } 2t+1\}$. We already noticed that Q and $Q' \in \mathcal{L}$ are actually cliques. Now we will show the following lemma:

Lemma 31.

(i) *For every vertex x in Q, there exists an unique quasi-clique $Q'_x \in \mathcal{L}$ such that $x \in V(Q'_x)$;*

(ii) *For distinct vertices x_1 and x_2 in Q, the quasi-cliques Q'_{x_1} and Q'_{x_2} are distinct;*

(iii) *For every quasi-clique $Q' \in \mathcal{L}$, there exists an unique vertex $x_{Q'}$ such that $x_{Q'} \in V(Q)$;*

(iv) *For distinct quasi-cliques Q'_1 and Q'_2 in $\mathcal{L}$, the vertices $x_{Q'_1}$ and $x_{Q'_2}$ are distinct.*

Proof. (i) It follows from before the fact that, for all $x \in V(Q)$, x is of Type 3.

(ii) By Lemma 17 (ii), we have $|V(Q') \bigcap V(Q)| \leqslant 1$ for any $Q' \in \mathcal{L}$. If Q'_{x_1} and Q'_{x_2} are the same, then Q'_{x_1} shares two common vertices with Q, it is not possible. So the result follows.

(iii) Since $|\mathcal{L}| = q_{2t+1} = 2t+2$ and $|V(Q)| = 2t+2$, it follows from (i) and (ii).
(iv) It follows from (i)-(iii). □

Let $W = V(G) - V(Q)$ and let G' be the induced subgraph of G on W. Let G'' be the spanning subgraph of G' such that the vertices w_1, w_2 are adjacent in G'' if there exists a quasi-clique $Q' \in \mathcal{L}$ such that w_1 and w_2 are in Q'. Now we have the following lemma:

Lemma 32. *The graph G'' is the line graph of the* cocktail-party graph $CP(2t+2)$.

Proof. Define the graph H with vertex set $\mathcal{L}$ and two quasi-clique $Q'_1, Q'_2 \in \mathcal{L}$ are adjacent if they intersect in a unique element. It is easy to see that the graph G'' is the line graph of H. As any quasi-clique Q' of $\mathcal{L}$ has $2t$ vertices in W and any vertex in W lies in two quasi-cliques in $\mathcal{L}$, it follows that H is $2t$-regular. So H is the cocktail-party graph $CP(2t+2)$ as it has $2t+2$ vertices. Hence, the lemma holds. □

Let $\Omega = \{x_1, \ldots, x_{t+1}, x'_1, x'_2, \ldots, x'_{t+1}\}$, and let $\Omega^2 = \{\text{2-subsets of } \Omega\} - \bigcup_{i=1}^{t+1}\{x_i, x'_i\}$. (For convenience, we will use $x_i x_j$ to represent the subset $\{x_i, x_j\}$, and similarly for the other 2-subsets in Ω^2.) We define the graph G^0 with vertex set $\Omega \bigcup \Omega^2$ and three kinds of edges as follows:

(1) the edges of the form $\{x, y\}$, where $x, y \in \Omega$;

(2) the edges of the form $\{x, xy\}$, where $x \in \Omega, xy \in \Omega^2$;

(3) the edges of the form $\{xy, xz\}$, where $xy, xz \in \Omega^2$.

By Lemma 32, Lemma 31 and the definition of Q, we see that G^0 is isomorphic to a spanning subgraph of G, and hence we can identify $V(G)$ with $\Omega \bigcup \Omega^2$.

Now consider the partition $\pi = \{V_1, V_2, V_3, V_4\}$ of $V(G)$, where

$$\begin{gathered}
V_1 = \{x_1, x_1'\},\\
V_2 = \{x_i, x_i' : 2 \leqslant i \leqslant t+1\},\\
V_3 = \{x_1x_i, x_1x_i', x_1'x_i, x_1'x_i' : 2 \leqslant i \leqslant t+1\},\\
V_4 = \{x_ix_j, x_ix_j', x_i'x_j, x_i'x_j', 2 \leqslant i < j \leqslant t+1\}.
\end{gathered}$$

The quotient matrix $\widetilde{B}$ of the adjacency matrix A of G with respect to the above partition π is given as follows:

$$\widetilde{B} = \begin{pmatrix} 1 & 2t & 2t & 0 \\ 2 & 2t-1 & 2 & 2t-2 \\ 1 & 1 & \alpha & 4t-1-\alpha \\ 0 & 2 & \frac{2(4t-1-\alpha)}{t-1} & 4t-1-\frac{2(4t-1-\alpha)}{t-1} \end{pmatrix} \tag{26}$$

with $2t \leqslant \alpha \leqslant 2t+1$.

We will show that $\alpha = 2t+1$, and hence the partition π is an equitable partition of G. To show this, note that by (5), we have

$$\begin{aligned} A^3_{(x_1,x_1')} &= 24t+1-(5-2t)\lambda_{x_1,x_1'} \\ &= 24t+1-(5-2t)\cdot 2t \\ &= 4t^2+14t+1. \end{aligned} \tag{27}$$

On the other hand,

$$A^3_{(x_1,x_1')} = 4t+1+\sum\nolimits_{z\in G_1(x_1)\bigcap G_1(x_1')} \lambda_{x_1,z} + \sum\nolimits_{z\in G_2(x_1)\bigcap G_1(x_1')} \mu_{x_1,z}, \tag{28}$$

where $|G_1(x_1)\bigcap G_1(x_1')| = 2t,\ |G_2(x_1)\bigcap G_1(x_1')| = 2t$ and

$$\begin{gathered} \lambda_{x_1,z} = 2t+1, \text{ for } z \in G_1(x_1)\bigcap G_1(x_1'), \\ 3 \leqslant \mu_{x_1,z} \leqslant 4, \text{ for } z \in G_2(x_1)\bigcap G_1(x_1'). \end{gathered}$$

Then, from (27) and (28), we obtain that

$$\mu_{x_1,z} = 4, \text{for } z \in G_2(x_1)\bigcap G_1(x_1'),$$

which implies that $\alpha = 2t+1$. Therefore, we have an equitable partition with partition diagram as shown in Figure 12.

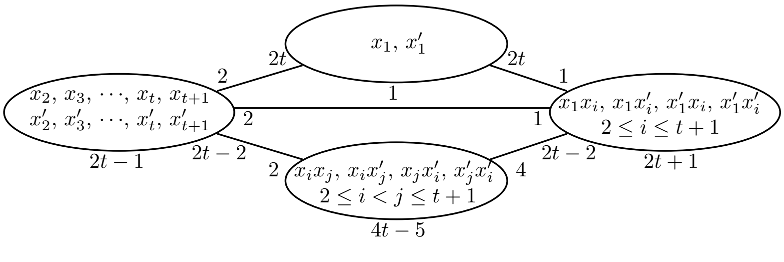


Figure 12: Equitable partition

In this case, the quotient matrix (26) becomes

$$\widetilde{B} = \begin{pmatrix} 1 & 2t & 2t & 0 \\ 2 & 2t-1 & 2 & 2t-2 \\ 1 & 1 & 2t+1 & 2t-2 \\ 0 & 2 & 4 & 4t-5 \end{pmatrix} \tag{29}$$

with eigenvalues $\{4t+1, 2t-1, t-2 \pm \sqrt{t^2-1}\}$.

From Lemma 8, we find that the eigenvalues of $\widetilde{B}$ should be the eigenvalues of A. But B has eigenvalues $t-2 \pm \sqrt{t^2-1}$, which are not the eigenvalues of A. So we obtain a contradiction. This shows that the case $q_{2t} = 0,\ q_{2t+1} = 2t+2,\ q_{2t+2} = 1$ is not possible. This concludes the proof to show that G is the 2-clique extension of the $(t+1) \times (t+1)$ grid.

Remark 33. Note that we used walk-regularity (which follows from the fact that the 2-clique extension of the $(t+1) \times (t+1)$-grid is regular with exactly 4 distinct eigenvalues) to show this result, and therefore it is not so clear how to extend this result to the 2-clique extension of a non-square grid graph.

References

[1] A. Abiad, A.E. Brouwer, and W.H. Haemers, Godsil-McKay switching and isomorphism. *Electr. J. Linear Algebra*, 28:4–11, 2015.

[2] S. Bang, E.R. van Dam, and J.H. Koolen, Spectral characterization of the Hamming graphs. *Linear Algebra Appl,* 429:2678–2686, 2008.

[3] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin-New York, 1989.

[4] E.R. van Dam, Regular graphs with four eigenvalues. *Linear Algebra Appl,* 226-228:139–162, 1995.

[5] E.R. van Dam, Graphs with few eigenvalues – An interplay between combinatorics and algebra, Ph.D. thesis, 1996.

[6] E.R. van Dam and W.H. Haemers, Which graphs are determined by their spectrum? *Linear Algebra Appl.*, 373:241–272, 2003.

[7] E.R. van Dam, J.H. Koolen, and H. Tanaka, Distance-regular graphs. *Electron. J. Combin.* (2016), #DS22.

[8] A. Gavrilyuk and J.H. Koolen, On a characterization of the Grassmann graphs over the binary field, preprint (2016).

[9] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

[10] G. Greaves, J. Koolen, A. Munemasa, Y. Sano, and T. Taniguchi, Edge-signed graphs with smallest eigenvalue greater than −2. *J. Combin. Theory Ser. B*, 110:90-111, 2015.

[11] W.H. Haemers, Interlacing eigenvalues and graphs. *Linear Algebra Appl,* 226-228:593–616, 1995.

[12] A.J. Hoffman, On the polynomial of a graph. *Amer. Math. Monthly*, 70:30–36, 1963.

[13] H.J. Jang, J. Koolen, A. Munemasa, and T. Taniguchi, On fat Hoffman graphs with smallest eigenvalue at least -3. *ARS Mathematica Contemporanea*, 7:105–121, 2014.

[14] J.H. Koolen, J.Y. Yang and Q. Yang, A generalization of a theorem of Hoffman, submitted.

[15] S.S. Shrikhande, The uniqueness of the L_2 association scheme. *Ann. Math. Statist*, 30:781–798.

[16] R. Woo and A. Neumaier, On graphs whose smallest eigenvalue is at least $-1-\sqrt{2}$. *Linear Algebra Appl,* 226-228:577–591, 1995.

[17] H. Yu, On the limit points of the smallest eigenvalues of regular graphs. *Des. Codes Cryptography*, 65:77–88, 2012.