The maximal order of hyper-(b-ary)-expansions

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Submitted: Jul 23, 2015; Accepted: Jan 20, 2017; Published: Feb 3, 2017
Mathematics Subject Classifications: 05A16, 11B37, 11B39

Abstract

Using methods developed by Coons and Tyler, we give a new proof of a recent result of Defant, by determining the maximal order of the number of hyper-(b-ary)-expansions of a nonnegative integer n for general integral bases $b \geqslant 2$.

Keywords: Stern's diatomic sequence; hyper base expansions; maximal order

1 Introduction

If $b \geqslant 2$ is a positive integer and $n \geqslant 0$ is a nonnegative integer, then a *hyper-(b-ary)-expansion of* n is an expansion

$$n = \sum_{i \geqslant 0} a_i b^i,$$

where $a_i \in \{0, 1, \ldots, b\}$ and $a_i = 0$ for all but finitely many indices i. In contrast, the standard base-b expansion requires $a_i \in \{0, 1, \ldots, b-1\}$.

*The research of M. Coons was supported by Australian Research Council grant DE140100223.
†Lukas Spiegelhofer was supported by the Austrian Science Fund (FWF), projects I 1751-N26 and F 5502-N26, the latter of which is a part of the Special Research Program "Quasi Monte Carlo Methods: Theory and Applications".

For $n \geqslant 1$, we denote by $s_b(n)$ the number of hyper-(b-ary)-expansions of $n-1$; moreover, we define $s_b(0)=0$. For example, $\{s_2(n)\}_{n\geqslant 0}$ is the classical diatomic sequence of Stern. This sequence satisfies $s_2(0)=0$, $s_2(1)=1$, and for $n \geqslant 0$,

$$s_2(2n) = s_2(n) \qquad \text{and} \qquad s_2(2n+1) = s_2(n) + s_2(n+1).$$

Similarly, for the general sequence $\{s_b(n)\}_{n\geqslant 0}$, it is straightforward to verify that $s_b(0)=0$, $s_b(1)=1$, and

$$s_b(bn) = s_b(n), \quad s_b(bn+1) = s_b(n) + s_b(n+1) \quad \text{and} \quad s_b(bn+i) = s_b(n+1)$$

for $n \geqslant 0$ and $i = 2, \dots, b-1$.

Recently, answering a question of Calkin and Wilf [3], Coons and Tyler [4] determined the maximal order of Stern's diatomic sequence. Very recently, Defant [5] gave an extension of their result to the functions $s_b(n)$. In this short paper, we use a slight variant of the method of Coons and Tyler, to give a new (and much shorter) proof of Defant's result.

Theorem 1 (Defant). *If $b \geqslant 2$ is an integer, then*

$$\limsup_{m\to\infty} \frac{s_b(m)}{m^{\log_b \varphi}} = \frac{\varphi^{\log_b(b^2-1)}}{\sqrt{5}},$$

where $\varphi = (\sqrt{5}+1)/2$ is the golden ratio.

2 Preliminaries

Let $\{F_k\}_{k\geqslant 0} = 0, 1, 1, 2, \dots$ be the sequence of Fibonacci numbers.

Lemma 2 (Defant). *Let $k \geqslant 2$ and n be positive integers. Then*

$$\max_{b^{k-2}\leqslant n<b^{k-1}} s_b(n) = F_k.$$

Moreover, if a_k denotes the smallest n in the interval $[b^{k-2}, b^{k-1})$ for which this maximum is attained, then

$$a_k = \frac{b^k-1}{b^2-1} + \left(\frac{1-(-1)^k}{2}\right)\frac{b}{b+1}. \tag{1}$$

Thus by definition

$$s_b(a_k) = F_k.$$

We note that a_k has the base b-expansion $((10)^{\ell-1}1)_b$ for $k=2\ell$ and $((10)^{\ell-1}11)_b$ for $k = 2\ell+1$, therefore this Lemma follows from a result of Defant [5, Proposition 2.1], however, this lemma can also be proven directly from the corresponding statement for Stern's sequence (see Lehmer [6] and Lind [7]). To do so, one need only notice that the

Stern sequence is a subsequence of $\{s_b(n)\}_{n\geqslant 0}$ and that the maximal values occur in this subsequence first. Explicitly, if we define $\psi_b : \mathbb{N} \to \mathbb{N}$ by

$$\psi_b : (\varepsilon_d \dots \varepsilon_0)_2 \mapsto (\varepsilon_d \dots \varepsilon_0)_b,$$

then setting

$$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad A_2 = \cdots = A_{b-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

we easily see that for $n = (\varepsilon_d \dots \varepsilon_0)_b$, we have

$$s_b(n) = \begin{pmatrix} 1 & 0 \end{pmatrix} A_{\varepsilon_0} \cdots A_{\varepsilon_d} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

From this identity it follows at once that

$$s_b(\psi_b(n)) = s_2(n). \tag{3}$$

Moreover, the matrices A_i are nonnegative and A_i is bounded componentwise by A_1 for $i \geqslant 2$. Replacing any such matrix A_i with $i \geqslant 2$ by A_1 therefore does not decrease the value of the product in (2). This proves that

$$s_b((\varepsilon_d \dots \varepsilon_0)_b) \leqslant s_2((\tilde{\varepsilon}_d \dots \tilde{\varepsilon}_0)_2),$$

where $\tilde{\varepsilon}_i = \min(\varepsilon_i, 1)$. Therefore numbers with only zeros and ones in their base-b expansions dominate the sequence $s_b(n)$, that is, for every n there is an $m \leqslant n$ with only zeros and ones in its base-b expansion such that $s_b(m) \geqslant s_b(n)$.

Definition 3. For $k \geqslant 0$ set

$$\tilde{a}_k = \frac{b^k - 1}{b^2 - 1},$$

and let $h(x) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ denote the piecewise linear function connecting the sequence of points $\{(\tilde{a}_k, F_k)\}_{k\geqslant 0} = (0,0), (1/(b+1), 1), (1,1), \dots$.

In the following lemma, we collect some necessary properties of $h(x)$.

Lemma 4. *The function $h(x)$ is continuous in $[0,\infty)$, monotonically increasing, and differentiable in the intervals $(\tilde{a}_k, \tilde{a}_{k+1})$. Moreover, for all $x \geqslant 0$, we have*

$$h(x) + h(bx + 1/(b+1)) = h(b^2 x + 1). \tag{4}$$

For $x \in [\tilde{a}_k, \tilde{a}_{k+1}]$ we have

$$h(x) = F_{k-1}\frac{b+1}{b^k}x - F_{k-1}\frac{b^k - 1}{b^k(b-1)} + F_k. \tag{5}$$

Moreover the sequence $\{\gamma_k\}_{k\geqslant 2} = F_2(b+1)/b^2, F_3(b+1)/b^3, \dots$ of slopes of the line segments is nonincreasing. In particular, the function h is concave in $\mathbb{R}_{\geqslant 1}$.

Proof. The statements in the first sentence follow directly from the definition.

To establish the validity of Equation (4), it is enough to realise that on the interval $[\tilde{a}_k, \tilde{a}_{k+1}]$, both sides are linear functions, which coincide at the endpoints, since $h(\tilde{a}_k) + h(\tilde{a}_{k+1}) = F_k + F_{k+1} = F_{k+2} = h(\tilde{a}_{k+2})$.

The proof of (5) is by inserting the definition of $\tilde{a}_k$ into the equation $h(x) = (x - \tilde{a}_k)(F_{k+1} - F_k)/(\tilde{a}_{k+1} - \tilde{a}_k) + F_k$. Finally, it is an easy consequence of Binet's formula that the sequence of slopes is nonincreasing. $\square$

The piecewise linear function $h(x)$ thus satisfies a recurrence relation resembling that of $s_b(n)$. Using this recurrence, we show that $h(n)$ is, in fact, an upper bound for $s_b(n)$, which is the main tool of our proof.

Lemma 5. *Assume that $b \geqslant 2$ is an integer. For all $m \geqslant 0$, we have*

$$s_b(m) \leqslant h(m). \tag{6}$$

Moreover,

$$\limsup_{m \to \infty} \frac{s_b(m)}{h(m)} = 1. \tag{7}$$

Proof. The limit result (7) follows from the first statement, which yields the inequality "$\leqslant$", together with the equality $s_b(a_{2k}) = h(a_{2k})$ that holds for all $k \geqslant 0$. It is sufficient to prove the assertion (6) for such m whose b-ary expansion consists only of zeros and ones, as for each n there is such an m with $m \leqslant n$ and $s_b(n) \leqslant s_b(m)$; see the comments just before Definition 3 for details.

We prove the lemma using induction, however, we have to strengthen the induction hypothesis by the additional property that

$$s_b(m) \leqslant h(m - b/(b+1)) \quad \text{for} \quad m \equiv b + 1 \bmod b^2. \tag{8}$$

(Remark: in fact we are working with a strictly weaker bound than Coons and Tyler [4] did for the case $b = 2$. They defined the function h by just connecting the maxima at the positions a_k. Our function h hits only those maxima (a_k, F_k) where $2 \mid k$, and the others lie to the right of h, shifted by $b/(b+1)$. Therefore our induction hypothesis has to be stronger, but as an exchange for this additional difficulty the tedious proof of Lemma 2.1 in the cited paper can be dispensed with, since our bound h is easier to work with.)

To get the proof started, we first verify the statement for the sixteen integers m given by the b-ary expansions $(\varepsilon_3\varepsilon_2\varepsilon_1\varepsilon_0)_b$, where $\varepsilon_i \in \{0, 1\}$. The case that $\varepsilon_i = 0$ for all $i \in \{0, 1, 2, 3\}$ is trivial. Concerning the other fifteen indices, by monotonicity of h we only have to consider those positions where a new maximum is attained: we have $s_b((1)_b) = 1$, $s_b((11)_b) = 2$, $s_b((101)_b) = 3$, $s_b((1001)_b) = 4$, $s_b((1011)_b) = 5$ and these values do not appear before. Clearly $s_b(1) = 1 = h(\tilde{a}_2) = h(1)$ and $s_b(b^2 + 1) = 3 = h(\tilde{a}_4) = h(b^2 + 1)$. Moreover, $s_b(b + 1) = 2 = h(\tilde{a}_3) = h(b + 1 - b/(b + 1))$ and $s_b(b^3 + b + 1) = 5 = h(\tilde{a}_5) = h(b^3 + b + 1 - b/(b + 1))$, so that (8) is satisfied for these latter two positions. Finally, we have $s_b(b^3 + 1) = 4$ and by (5) it is not difficult to show that $h(b^3 + 1) = 5 - 2/b^2 \geqslant 4$.

For the induction step, let $k \geqslant 5$ and assume that (6) and (8) hold for all $m < b^{k-1}$ having only zeros and ones as b-ary digits. Let $b^{k-1} \leqslant m < b^k$, where m satisfies the same digit restriction. We distinguish between six cases. If $m = bj$, where $j \geqslant 1$, then

$$s_b(bj) = s_b(j) \leqslant h(j) \leqslant h(bj),$$

where the last inequality is true by the monotonicity of $h(x)$. The case for $m = bn + 1$ for some n is more involved, and it is this case for which we need the special form of the function h. We start with the case $m \equiv 1 \bmod b^3$, that is, $m = b^3 j + 1$, where $j \geqslant b$ since $m \geqslant b^4$. Using the recurrence for s_b, the induction hypothesis, monotonicity of h and the identity (4), in this order, we obtain

$$\begin{aligned} s_b(b^3 j + 1) &= s_b(b^2 j) + s_b(b^2 j + 1) \\ &= s_b(j) + s_b(bj) + s_b(bj + 1) \\ &= 2s_b(j) + s_b(bj + 1) \\ &\leqslant 2h(j) + h(bj + 1) \\ &\leqslant h(j) + h(j + 1/(b + 1)) + h(bj + 1) \\ &= h(j) + h(b^2 j + b + 1/(b + 1)). \end{aligned}$$

The first case $m \equiv 1 \bmod b^3$ is finished as soon as we prove that this last expression is bounded from above by

$$h(bj) + h(b^2 j + 1/(b + 1)) = h(b^3 j + 1).$$

This estimate follows from the concavity of h in $\mathbb{R}_{\geqslant 2}$, comparing the arguments of the function h: we have $j \geqslant b \geqslant 2$, and therefore $bj - j \geqslant b = b^2 j + b + 1/(b+1) - (b^2 j + 1/(b+1))$, which yields the statement.

By a very similar argument we treat the case $m = b^4 j + b + 1$, however we consider the case $j = 1$ separately. Using (3) and (5), we get in this case

$$s_b(b^4 + b + 1) = s_2(19) = 7 \leqslant 8 - 3/b^2 = h(b^4 + b + 1/(b + 1)),$$

as required by (8). Assume now that $j \geqslant 2$. We have (note that the argument leading to the second row works equally well for $b = 2$)

$$\begin{aligned} s_b(b^4 j + b + 1) &= s_b(b^3 j + 1) + s_b(b^3 j + 2) \\ &= s_b(b^3 j + 1) + s_b(b^2 j + 1) \\ &= s_b(j) + 2s_b(b^2 j + 1) \\ &\leqslant h(j) + 2h(b^2 j + 1) \end{aligned}$$

and an analogous argument as before, using the concavity of h, shows that this can be estimated by

$$\begin{aligned} h(bj) + 2h(b^2 j + 1/(b + 1)) &= h(b^2 j + 1/(b + 1)) + h(b^3 j + 1) \\ &= h(b^4 j + b + 1 - b/(b + 1)). \end{aligned}$$

The remaining three cases are simpler as we do not need the concavity of h for them. We have

$$\begin{aligned} s_b(b^4j+b^3+b+1) &= s_b(b^3j+b^2+1)+s_b(b^3j+b^2+2) \\ &= s_b(b^3j+b^2+1)+s_b(b^2j+b+1) \\ &\leqslant h(b^3j+b^2+1)+h(b^2j+b+1/(b+1)) \\ &= h(b^4j+b^3+b+1-b/(b+1)), \end{aligned}$$

$$\begin{aligned} s_b(b^3j+b^2+1) &= s_b(bj+1)+s_b(b^2j+b+1) \\ &\leqslant h(bj+1)+h(b^2j+b+1/(b+1)) \\ &= h(b^3j+b^2+1) \end{aligned}$$

and

$$\begin{aligned} s_b(b^3j+b^2+b+1) &= s_b(b^2j+b+1)+s_b(b^2j+b+2) \\ &= s_b(b^2j+b+1)+s_b(bj+2) \\ &= s_b(b^2j+b+1)+s_b(j+1) \\ &\leqslant h(b^2j+b+1/(b+1))+h(j+1) \\ &\leqslant h(b^2j+b+1/(b+1))+h(bj+1) \\ &= h(b^3j+b^2+1) \\ &\leqslant h(b^3j+b^2+b+1-b/(b+1)), \end{aligned}$$

which settles the cases $m \equiv b^3+b+1 \bmod b^4$, $m \equiv b^2+1 \bmod b^3$ and $m \equiv b^2+b+1 \bmod b^3$ respectively. Note that all of these calculations are also valid for $b=2$.

The residue classes considered cover the set of nonnegative integers with b-ary expansions containing only zeros and ones, and therefore

$$s_b(m) \leqslant h(m)$$

for all $m \geqslant 0$. $\square$

Lemma 6. *If*

$$H(x) = \frac{(b^2-1)^{\log_b \varphi}}{\sqrt{5}} x^{\log_b \varphi},$$

then

$$\limsup_{x\to\infty} (h-H)(x) = 0,$$

and specifically,

$$\lim_{k\to\infty} |(h-H)(a_k)| = 0.$$

Proof. Let $\tilde{h}(x)$ be the piecewise linear function satisfying $\tilde{h}(\tilde{a}_k) = \varphi^k/\sqrt{5}$ for $k \geqslant 0$, and set

$$\tilde{H}(x) = \frac{((b^2-1)x+1)^{\log_b \varphi}}{\sqrt{5}} = H(x + 1/(b^2-1)).$$

Then $\tilde{H}(x)$ is a concave function that passes through the points $\left(\tilde{a}_k, \varphi^k/\sqrt{5}\right)$. It follows that $\tilde{h}(x) \leqslant \tilde{H}(x)$ for $x \geqslant 0$, where we have equality when $x = \tilde{a}_k$. Moreover, if $\tilde{a}_k \leqslant x \leqslant \tilde{a}_{k+1}$, by Binet's formula $|\tilde{h}(x) - h(x)| \leqslant \varphi^{-k}/\sqrt{5}$. Furthermore, the mean value theorem gives $|\tilde{H}(x) - H(x)| \leqslant Cx^{-\eta}$ for some C and some $\eta > 0$. Combining these three estimates yields the first statement.

We easily get $\lim_{k\to\infty} |(h - H)(\tilde{a}_k)| = 0$ by a similar argument. Combining this with $|a_k - \tilde{a}_k| \leqslant 1$, $|h(x+t) - h(x)| \to 0$, and $|H(x+t) - H(x)| \to 0$ for $x \to \infty$ gives the last statement of the lemma. □

3 Proof of the main result

Proof of Theorem 1. For brevity, we write $c_b = \varphi^{\log_b(b^2-1)}/\sqrt{5}$. By Lemma 6, noting that $h(x)$ increases to infinity, and applying Lemma 5, we get

$$\limsup_{m\to\infty} \frac{s_b(m)}{c_b m^{\log_b \varphi}} = \limsup_{m\to\infty} \frac{s_b(m)}{H(m)} \leqslant \limsup_{m\to\infty} \frac{s_b(m)}{h(m)} \leqslant 1.$$

Similarly, using the second part of Lemma 6 and again applying Lemma 5, we have

$$\limsup_{m\to\infty} \frac{s_b(m)}{H(m)} \geqslant \limsup_{k\to\infty} \frac{s_b(a_k)}{H(a_k)} = \limsup_{k\to\infty} \frac{s_b(a_k)}{h(a_k)} = 1,$$

which proves the theorem. □

4 Concluding remarks

In this paper, we gave a short proof determining the maximal order of the number $s_b(n)$ of hyper-(b-ary)-expansions of a nonnegative integer $n-1$ for general integral bases $b \geqslant 2$. Our proof was based on considering the finite number of specific recurrences satisfied by $s_b(n)$ over arithmetic progressions $bn + i$, and constructing a piecewise linear function approximating those recurrences.

Functions satisfying recurrences like those satisfied by $s_b(n)$ are plentiful in the literature, and often given special attention by number theorists and theoretical computer scientists; see Allouche and Shallit's work on b-regular sequences [1, 2] for a general treatment and several specific examples. The recurrences satisfied by b-regular sequences lead to a matrix classification, which has proven to be very useful; here the usefulness comes by recognising that the Stern sequence $s_2(n)$ is the dominating subsequence of $s_b(n)$ (for each $b \geqslant 2$), and that the growth properties of the Stern sequence determine the growth properties of $s_b(n)$.

We end by pointing the interested reader to the extremely useful result of Allouche and Shallit [1, Theorem 2.2], which formalises this relationship.

Theorem 7 (Allouche and Shallit)**.** *The integer sequence* $\{f(n)\}_{n\geqslant 0}$ *is* b*-regular if and only if there exist positive integers* m *and* d*, matrices* $\mathbf{A}_0, \ldots, \mathbf{A}_{b-1} \in \mathbb{Z}^{d\times d}$*, and vectors* $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$ *such that*

$$f(n) = \mathbf{w}^T \mathbf{A}_{i_0} \cdots \mathbf{A}_{i_s} \mathbf{v},$$

where $n = (i_s \cdots i_0)_b$*.*

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