New feasibility conditions for directed strongly regular graphs

Sylvia A. Hobart Jason Williford *

Department of Mathematics
University of Wyoming
Laramie, Wyoming, U.S.A

sahobart@uwyo.edu, jwillif1@uwyo.edu

Submitted: Aug 18, 2015; Accepted: Jan 14, 2017; Published: Feb 3, 2017
Mathematics Subject Classifications: 05E30, 05B25

Abstract

We prove two results for directed strongly regular graphs that have an eigenvalue of multiplicity less than k, the common out-degree of each vertex. The first bounds the size of an independent set, and the second determines an eigenvalue of the subgraph on the out-neighborhood of a vertex. Both lead to new nonexistence results for parameter sets.

Keywords: directed strongly regular graph

1 Introduction

Directed strongly regular graphs were defined by Art Duval [4] in 1988, as a directed version of strongly regular graphs. Starting with results of Klin et al in 2004 [8], there has been a lot of work on both constructions and nonexistence results. The website [3] maintains a table of feasible parameters.

In particular, many properties of strongly regular graphs extend to directed strongly regular graphs: the adjacency matrix is diagonalizable, the eigenvalues are integers ([4]), and there is a version of the absolute bound ([7], [5]).

In this paper, we find two nonexistence conditions for directed strongly regular graphs which are related to bounds for strongly regular graphs. One gives a bound on the size of an independent set, and the second gives information about an eigenvalue of the induced subgraph of the out-neighborhood of a vertex. For each, we give examples of feasible parameter sets ruled out by the condition.

*Supported by NSF grant 1400281.

We start with definitions and standard results. For a directed graph Γ and vertices i and j, write $i \to j$ if there is an edge from i to j (and this will include the case that there is also an edge from j to i). If there are edges both from i to j and from j to i, we say that i and j are **adjacent** vertices and write $i \sim j$. As usual, the **adjacency matrix** of Γ is the matrix A whose i, j entry is 1 if $i \to j$ and 0 otherwise.

A directed graph Γ on v vertices is a **directed strongly regular graph** with parameters (v, k, t, λ, μ) if $0 < t < k$ and A satisfies the matrix equations

$$\begin{aligned} JA &= AJ = kJ \\ A^2 &= tI + \lambda A + \mu(J - I - A) \end{aligned}$$

where J is the all 1's matrix. In this case, we will say that Γ is a DSRG(v, k, t, λ, μ). Note that if $k = t$ and these matrix conditions are satisfied, then Γ is an (undirected) strongly regular graph, while if $t = 0$ then Γ is a doubly regular tournament.

The matrix conditions are equivalent to the following structural conditions (which are often given as the definition). A digraph Γ is a DSRG(v, k, t, λ, μ) if and only if

(a) Every vertex has in-degree and out-degree k, and is adjacent to t vertices.

(b) Let i and j be distinct vertices. The number of vertices x such that $i \to x \to j$ is λ if $i \to j$ and μ if $i \not\to j$.

The adjacency matrix of a directed strongly regular graph is diagonalizable but not unitarily diagonalizable. However the all 1's vector $\mathbf{j}$ is an eigenvector with eigenvalue k, and all eigenvectors for other eigenvalues are orthogonal to $\mathbf{j}$. The following theorem gives basic information about eigenvalues.

Theorem 1. *([4]) Let Γ be a DSRG(v, k, t, λ, μ) with adjacency matrix A. Then the eigenvalues of A are integers $\theta_0 = k, \theta_1, \theta_2$ with multiplicities $m_0 = 1, m_1, m_2$, given by the following formulas.*

$$\begin{aligned} \theta_1 &= \frac{1}{2}\left(\lambda - \mu + \sqrt{(\mu - \lambda)^2 + 4(t - \mu)}\right) \\ \theta_2 &= \frac{1}{2}\left(\lambda - \mu - \sqrt{(\mu - \lambda)^2 + 4(t - \mu)}\right) \\ m_1 &= -\frac{k + \theta_2(v-1)}{\theta_1 - \theta_2} \\ m_2 &= \frac{k + \theta_1(v-1)}{\theta_1 - \theta_2} \end{aligned}$$

We will use θ, τ for the eigenvalues when we don't want to order them, and m_θ, m_τ for the corresponding multiplicities.

We will also need the following technical lemma about general digraphs. Recall that an **independent set** is a set of vertices such that the induced subgraph has no directed edges.

Lemma 2. *Let Γ be a digraph with v vertices, with every vertex of in-degree 1. Then Γ has an independent set of size at least $\lceil \frac{v}{3} \rceil$.*

Proof. It is not hard to see that the strong components of Γ are all directed cycles (including those of length 2) or isolated vertices. Each weak component must be an isolated vertex, a directed cycle, a directed tree (an oriented tree with all directions away from the root), or a cycle with directed trees rooted at one or more of its vertices. We can take at least half of the vertices of the directed trees (excluding the root). For a directed cycle of length c, we can take $\lfloor \frac{c}{2} \rfloor$ of the vertices. The worst case occurs when all of the vertices (save 1 or 2, depending on v $(mod\ 3)$) are in directed cycles of length three. $\square$

2 Independent Sets

For a strongly regular graph which is not multipartite, interlacing shows that the size of an independent set is bounded above by the multiplicity of the negative eigenvalue, see [1], Theorem 9.4.1. We can derive a similar result for directed strongly regular graphs using the fact that A is diagonalizable. In fact, the proof also works for strongly regular graphs since the eigenvalues are given by the same formulas.

Theorem 3. *Let Γ be a DSRG(v, k, t, λ, μ) with eigenvalues k, θ, τ and multiplicities $1, m_\theta, m_\tau$. Suppose Γ has an independent set Y of size c. If $\theta \neq 0$, then $c \leqslant m_\tau$.*

Proof. Let $B = J - I - A$ (the adjacency matrix of the complement of Γ), and let

$$E_\tau = \frac{1}{v(\theta - \tau)}\big((k + (v-1)\theta)I + (k - v - \theta)A + (k - \theta)B\big).$$

Using the fact that eigenvectors for τ and θ are orthogonal to $\mathbf{j}$, easy calculations show that E_τ is the projection onto the eigenspace for τ and hence is an idempotent of rank m_τ.

The principal submatrix of E_τ indexed by Y equals

$$\frac{1}{v(\theta - \tau)}\Big((k + (v-1)\theta)I + (k - \theta)(J - I)\Big) = \frac{1}{v(\theta - \tau)}\Big((k - \theta)J + v\theta I\Big).$$

Since $\theta \neq 0$ and $k > 0$, this matrix has full rank, namely rank c. Therefore $c \leqslant m_\tau$. $\square$

We get the following immediate corollary, which surprisingly rules out parameter sets whose existence was open.

Corollary 4. *Let Γ be a DSRG(v, k, t, λ, μ) with $\lambda = 0$ and $\theta \neq 0$. Then $k \leqslant m_\tau$.*

The corollary follows from the fact that in such a graph, the out-neighborhood of a vertex must be an independent set. This result can also be derived from Corollary 7.

We note that this rules out the following parameter sets.

Parameters	**Spectrum**
$(70, 15, 9, 0, 4)$	$15^1, 1^{55}, -5^{14}$
$(78, 19, 13, 0, 6)$	$19^1, 1^{65}, -7^{12}$
$(128, 25, 13, 0, 6)$	$25^1, 1^{108}, -7^{19}$

The first of these is listed as unknown in [3], showing that this is a new nonexistence condition. The second is also ruled out by a condition due to Jørgensen ([7]), and the third is a feasible parameter set which is not ruled out by any other results (but has too many vertices to be listed in [3]).

Corollary 5. *Let* Γ *be a DSRG*(v, k, t, λ, μ) *with* $\lambda = 1$ *and* $\theta \neq 0$*. Then* $\lceil \frac{k}{3} \rceil \leqslant m_\tau$*.*

Proof. The out-neighborhood of a vertex has k vertices, and it is easy to see that $\lambda = 1$ implies that each of these has in-degree 1. By Lemma 2 this neighborhood must contain an independent set of size at least $\lceil \frac{k}{3} \rceil$. The result then follows from Theorem 3. □

It is less clear how to apply Theorem 3 for larger λ. For particular parameters, one may be able to apply the theorem directly.

The smallest parameter set ruled out by Corollary 5 has 585 points, and is thus too large to be listed in [3]. Here's a list of some feasible parameter sets are ruled out by Corollary 5:

Parameters	**Spectrum**
$(585, 119, 92, 1, 30)$	$119^1, 2^{545}, -31^{39}$
$(672, 134, 101, 1, 330)$	$134^1, 2^{630}, -34^{41}$
$(703, 112, 42, 1, 21)$	$112^1, 1^{665}, -21^{37}$

3 Eigenvalue bound

The work in this paper began as an attempt to extend the Krein conditions to directed strongly regular graphs. There are many proofs for strongly regular graphs, and the standard matrix proof using the entrywise matrix product does not extend. However, Godsil and Royle give a proof which begins by determining an eigenvalue of the first subconstituent ([6], section 10.7), and this part of the proof can be generalized.

We will use the following notation in the section, for Γ a directed strongly regular graph. Fix a vertex x, and let X_1 be the set of out-neighbors of x. Let A_1 be the adjacency matrix for the induced subgraph on X_1, so A_1 is a principal submatrix of A.

Theorem 6. *Let* Γ *be a DSRG with* $m_\theta < k$*. Then* τ *is an eigenvalue of* A_1*, with geometric multiplicity at least* $k - m_\theta$*.*

Note that this does not depend on the choice of vertex x.

Proof. We partition the vertex set V into $X_0 = \{x\}$, $X_1 = \{w : x \to w\}$, $X_2 = V \setminus (X_0 \cup X_1)$, and order the vertices $x_0, \ldots, x_{v-1}$ of Γ so that $X_1 = \{x_0\}$, $X_1 = \{x_1, \ldots, x_k\}$, and $X_2 = \{x_{k+1}, \ldots, x_{v-1}\}$.

Let M be a matrix whose columns are a basis for the right $\tau-$eigenspace of A, so M has $m_\tau = v - m_\theta - 1$ columns. Let N be the $(v - k - 1) \times v$ matrix whose rows are the elementary basis vectors $e_0, e_{k+1}, \ldots e_{v-2}$. Then NM is a $v - k - 1$ by $v - m_\theta - 1$ matrix; since $m_\theta < k$, NM has right nullity at least $k - m_\theta$.

For any nonzero vector z in the right nullspace of NM, $\mathcal{N}(NM)$, Mz is an eigenvector of A with eigenvalue τ. Let $Mz = w = (w_i)$; we will investigate the entries of w.

Since $Nw = NMz = 0$, we have that $w_i = 0$ for $i \in \{0, k+1, \dots, v-2\}$. Also, $(Aw)_0 = \tau w_0 = 0$, and the first row of A is $\sum_{i=1}^{k} e_i$, implying that $(\sum_{i=1}^{k} e_i)^T w = 0$. Since $\mathbf{j}^T w = 0$, we obtain

$$(\sum_{i=1}^{k} e_i)^T w + w_{v-1} = \mathbf{j}^T w = 0$$

and hence $w_{v-1} = 0$. This shows that the support of w is contained entirely in X_1, and hence restricting w to X_1 gives an eigenvector of A_1 with eigenvalue τ.

If $z_1, \dots, z_t$ is a basis for $\mathcal{N}(NM)$, then $Mz_1, \dots, Mz_t$ are also linearly independent since M has independent columns. Hence the right τ eigenspace of A_1 has dimension greater than or equal to the nullity of NM, which is greater than or equal to $k - m_\theta$. $\square$

Corollary 7. *Let Γ be a DSRG with $m_\theta < k$. Then $|\tau| \leqslant \lambda$.*

Proof. Use the same notation as Theorem 6, and note that the definition of parameter λ implies that the column sums of A_1 are λ, constant for all columns. Since A is a $(0,1)$ matrix, we can apply Perron-Frobenius theory, and hence λ is the spectral radius of A_1. Therefore $|\tau| \leqslant \lambda$. $\square$

Corollary 7 rules out the following feasible parameter sets. The first of these is also ruled out by Corollary 4.

Parameters	**Spectrum**
$(70, 15, 9, 0, 4)$	$15^1, 1^{55}, -1^{14}$
$(145, 31, 26, 1, 8)$	$31^1, 2^{115}, -9^{18}$

Acknowledgements

The software package MAGMA [2] was used to find parameters ruled out by these conditions.

References

[1] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs.* Universitext, Springer, 2012.

[2] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language. *J. Symbolic Comput.* 24:235-265, 1997.

[3] A. E. Brouwer and S. A. Hobart, Parameters of directed strongly regular graphs. http://homepages.cwi.nl/~aeb/math/dsrg/dsrg.html

[4] A. Duval, A directed graph version of strongly regular graphs. *J. Comb. Theory Ser. A* 47:71-100, 1988.

[5] C. D. Godsil, S. A. Hobart, and W. J. Martin, Representations of directed strongly regular graphs. *Europ. J. Comb.* 28:1980-1993, 2007.

[6] C. Godsil and G. Royle, *Algebraic Graph Theory*. Graduate Texts in Mathematics 207, Springer, 2001.

[7] L. K. Jørgensen, Non-existence of directed strongly regular graphs. *Discrete Math.* 264:111-126, 2003.

[8] M. Klin, A. Munemasa, M. Muzychuk, and P.-H. Zieschang, Directed strongly regular graphs obtained from coherent algebras. *Linear Algebra Appl.* 377:83-109, 2004.