(Total) Domination in Prisms

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Abstract

Using hypergraph transversals it is proved that $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$, where $\gamma_t(G)$ and $\gamma(G)$ denote the total domination number and the domination number of G, respectively, and Q_n is the n-dimensional hypercube. More generally, it is shown that if G is a bipartite graph, then $\gamma_t(G \Box K_2) = 2\gamma(G)$. Further, we show that the bipartiteness condition is essential by constructing, for any $k \geqslant 1$, a (non-bipartite) graph G such that $\gamma_t(G \Box K_2) = 2\gamma(G) - k$. Along the way several domination-type identities for hypercubes are also obtained.

Keywords: domination; total domination; hypercube; Cartesian product of graphs; covering codes; hypergraph transversal

1 Introduction

Domination and total domination in graphs are very well studied in the literature, here we study these concepts in prisms of graphs, in particular in hypercubes. To determine the domination number γ of the n-dimensional hypercube Q_n, is a fundamental problem in coding theory, computer science, and of course in graph theory. In coding theory, the problem equivalent to the determination of $\gamma(Q_n)$ is to find the size of a minimal covering code of length n and covering radius 1. In computer science, different distribution type problems on interconnection networks can be modelled by domination invariants, where hypercubes in turn form a central model for interconnection networks.

To determine $\gamma(Q_n)$ turns out to be an intrinsically difficult problem. To date, exact values are only known for $n \leqslant 9$. These results are summarized in Table 1.

n	1	2	3	4	5	6	7	8	9	10
$\gamma(Q_n)$	1	2	2	4	7	12	16	32	62	107-120

Table 1: Domination numbers of hypercubes up to dimension 10

We have checked these values by formulating an integer linear program and solving it with CPLEX. The result $\gamma(Q_9) = 62$ due to Östergård and Blass [20] actually presented a breakthrough back in 2001. The value of $\gamma(Q_{10})$ is currently unknown, see [1] for the present best lower bound as given in Table 1 and [16] for the present best upper bound.

Total domination γ_t is, besides classical domination, among the most fundamental concepts in domination theory. It has in particular been extensively investigated on Cartesian product graphs (cf. [4, 12, 18]), which was in a great part motivated by the famous Vizing's conjecture (see the survey [3] and recent papers [2, 7]). Specifically, $\gamma_t(Q_n)$ was recently investigated in the thesis [22] under the notion of a *binary covering code of empty spheres of length n and radius* 1. In particular, values $\gamma_t(Q_n)$ for $n \leqslant 10$ were computed and some bounds established. These exact values intrigued us to wonder whether there exists some general relation between the domination number and the total domination number in hypercubes.

From our perspective it is utmost important that Q_n can be represented as the n^{th} power of K_2 with respect to the Cartesian product operation $\square$, that is, $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geqslant 2$. Our immediate aim in this paper is to prove that $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$ holds for all $n \geqslant 1$. For this purpose, we prove the following much more general result that the total domination number of a bipartite prism of a graph G is equal to twice the domination number of G.

Theorem 1. *If G is a bipartite graph, then*

$$\gamma_t(G \square K_2) = 2\gamma(G)\,.$$

Since Q_n, $n \geqslant 1$, is a bipartite graph, as a special case of Theorem 1 we note that $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$. Our second aim is to show that the bipartiteness condition in the statement of Theorem 1 is essential. For this purpose, we prove the following result.

Theorem 2. *For each integer $k \geqslant 1$, there exists a connected graph G_k satisfying*

$$2\gamma(G_k) - \gamma_t(G_k \,\square\, K_2) = k.$$

We proceed as follows. In the next section concepts used throughout the paper are introduced and known facts and results needed are recalled. In particular, the state of the art on $\gamma(Q_n)$ is surveyed. In Section 3, Theorem 1 is proved and several of its consequences listed. A proof of Theorem 2 is given in Section 4. We conclude the paper with some open problems. In particular we conjecture that the equality in Theorem 1 holds for almost all graphs.

2 Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is denoted by $n(G) = |V(G)|$. The *open neighborhood* of a vertex v in G is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. The path and the cycle on n vertices are denoted by P_n and C_n, respectively.

For graphs G and H, the *Cartesian product* $G \,\square\, H$ is the graph with vertex set $V(G) \times V(H)$ where vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. If $(u, v) \in V(G \,\square\, H)$, then the subgraph of $G \,\square\, H$ induced by the vertices of the form (u, x), $x \in V(H)$, is isomorphic to H; it is called the *H-layer* (through (u, v)). Analogously G-layers are defined. The *prism* of a graph G is the graph $G \,\square\, K_2$. Note that $G \,\square\, K_2$ contains precisely two G-layers. Further, if G is a bipartite graph, then we call the prism $G \,\square\, K_2$ the *bipartite prism* of G. As already mentioned in the introduction, Q_n is a (bipartite) prism because $Q_n = Q_{n-1} \,\square\, K_2$.

A *dominating set* of a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S, while a *total dominating set* of G is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G and the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. We refer to the books [10, 14] for more information on the domination number and the total domination number, respectively.

The values $\gamma(Q_7) = 16$ and $\gamma(Q_8) = 32$ also follow from the following result which gives exact values for two infinite families of hypercubes.

Theorem 3. ([9, 25]) *If $k \geqslant 1$, then $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$ and $\gamma(Q_{2^k}) = 2^{2^k-k}$.*

The first assertion of Theorem 3 is based on the fact that hypercubes Q_{2^k-1} contain perfect codes, cf. [9]. Since the domination number of a graph with a perfect code is equal to the size of such a code, the assertion follows. Knowing the existence of such codes, by the divisibility condition one immediately infers that Q_n contains a perfect code if and only if $n = 2^k - 1$ for some $k \geqslant 1$. Lee [17, Theorem 3] further proved that this is equivalent to the fact that Q_n is a regular covering of the complete graph K_{n+1}. The second assertion of Theorem 3 is due to van Wee [25]. Related aspects of domination in hypercubes were investigated in [24].

A set S of vertices in G is a *paired-dominating set* if every vertex of G is adjacent to a vertex in S and the subgraph induced by S contains a perfect matching (not necessarily as an induced subgraph). The minimum cardinality of a paired-dominating set of G is the *paired-domination number* of G, denoted $\gamma_{\rm pr}(G)$. A survey on paired-domination in graphs can be found in [5]. By definition every paired-dominating set is a total dominating set, and every total dominating set is a dominating set. Hence we have the following result first observed by Haynes and Slater [11].

Observation 4. ([11]) *For every isolate-free graph G, $\gamma(G) \leqslant \gamma_t(G) \leqslant \gamma_{\rm pr}(G)$.*

A *total restrained dominating set* of G is a total dominating set S of G with the additional property that every vertex outside S has a neighbor outside S; that is, $G[V(G) \setminus S]$ contains no isolated vertex. The *total restrained domination number* of G, denoted $\gamma_{\rm tr}(G)$, is the minimum cardinality of a total restrained dominating set. The concept of total restrained domination in graphs was introduced by Telle and Proskurowksi [21] as a vertex partitioning problem. By definition every total restrained dominating set if a total dominating set, implying the following observation.

Observation 5. ([11]) *For every isolate-free graph G, $\gamma_t(G) \leqslant \gamma_{\rm tr}(G)$.*

The *open neighborhood hypergraph*, abbreviated ONH, of G is the hypergraph H_G with vertex set $V(H_G) = V(G)$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V(G)\}$ consisting of the open neighborhoods of vertices in G. The *closed neighborhood hypergraph*, abbreviated CNH, of G is the hypergraph H_G^c with vertex set $V(H_G^c) = V(G)$ and with edge set $E(H_G) = \{N_G[x] \mid x \in V(G)\}$ consisting of the closed neighborhoods of vertices in G.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set*) if T has a nonempty intersection with every hyperedge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A transversal of size $\tau(H)$ is called a $\tau(H)$-set.

The transversal number of the ONH of a graph is precisely the total domination number of the graph, while the transversal number of the CNH of a graph is precisely the domination number of the graph. We state this formally as follows.

Observation 6. *If G is a graph, then $\gamma_t(G) = \tau(H_G)$ and $\gamma(G) = \tau(H_G^c)$.*

We shall also need the following result from [13] (see also [14]).

Theorem 7. ([13]) *The ONH of a connected bipartite graph consists of two components (which are induced by the two partite sets of the graph), while the ONH of a connected graph that is not bipartite is connected.*

3 Proof of Theorem 1 and its Consequences

In this section, we first present a proof of Theorem 1. Recall its statement.

Theorem 1 *If G is a bipartite graph, then $\gamma_t(G \,\square\, K_2) = 2\gamma(G)$.*

Proof. Note first that $K_1 \square K_2 = K_2$, hence the assertion of the theorem holds for $G = K_1$. Since we can apply the result to each component of the bipartite graph G, we may assume that G is connected. Hence in the rest of the proof, let G be a connected bipartite graph of order at least 2. Further, for notational convenience let F denote the prism $G \square K_2$ throughout the proof; that is, $F = G \square K_2$.

Let G_1 and G_2 be the G-layers of F, and let $V_i = V(G_i)$ for $i \in [2]$. For notational convenience, for each vertex v in G_1 we denote the corresponding vertex in G_2 that is adjacent to v in F by v'. Thus, the set $\cup_{v \in V_1}\{vv'\}$ of edges between V_1 and V_2 in F forms a perfect matching in F.

Since G is a bipartite graph, F is bipartite as well. Let X and Y be the partite sets of F. If $w \in \{v, v'\}$ for some vertex $v \in V_1$, then we define the *complement* of the vertex w to be the vertex $\overline{w} \in \{v, v'\} \setminus \{w\}$. We note that if $w \in V_{3-i}$, then $\overline{w} \in V_i$ for $i \in [2]$. Further, we note that w and $\overline{w}$ belong to different partite sets of F.

Let H be the ONH of F. By Theorem 7, H consists of two components that are induced by the two partite sets, X and Y, of F. Let H_X and H_Y be the two components of H, where $V(H_X) = X$ and $V(H_Y) = Y$. We note that each hyperedge in H_X and H_Y corresponds to the open neighborhood of some vertex in Y and some vertex in X, respectively, in F. For each vertex w in F, let e_w be the associated hyperedge in H; that is, $e_w = N_F(w)$.

We proceed further with the following series of claims.

Claim 8. *The hypergraphs H_X and H_Y are isomorphic.*

Proof. Let $f : X \to Y$ be the function that assigns to each vertex $x \in X$ the vertex $\overline{x} \in Y$. Then, f is a bijection between the vertex set of H_X and H_Y. We show that the bijection f is an isomorphism between the hypergraphs H_X and H_Y.

We first show that for every hyperedge e in H_X, the image $f(e) := \{f(z) \mid z \in e\}$ of e in f is a hyperedge in H_Y, and conversely. Let e be an arbitrary hyperedge in H_X. Thus, $e = e_w = N_F(w)$ for some vertex $w \in Y$. We claim that the image of e is $N_F(\overline{w})$; that is, we claim that the image of $e = e_w$ is precisely the hyperedge in H_Y associated with the vertex $\overline{w} \in X$. Without loss of generality, we may assume that $w \in V_1 \cap Y$. For an arbitrary vertex $z \in e$, either $z = \overline{w}$ or $z \in N_{G_1}(w)$. If $z = \overline{w}$, then $f(z) = w \in N_F(\overline{w})$. If $z \in N_{G_1}(w)$, then $f(z) = \overline{z} \in N_{G_2}(\overline{w}) \subset N_F(\overline{w})$. In both cases, $z \in N_F(\overline{w})$, implying that $f(e) \subseteq N_F(\overline{w})$. Conversely, interchanging the role of X and Y, we have that $f(e) \supseteq N_F(\overline{w})$. Thus, $f(e) = N_F(\overline{w})$.

Suppose that e_Y is a hyperedge of H_Y. Thus, $e_Y = e_x = N_F(x)$ for some vertex $x \in X$. Analogously as before, $f(e_Y) = N_F(\overline{x})$ is precisely the hyperedge in H_X associated with the vertex $\overline{x} \in Y$. Thus, the bijective function f preserves adjacency, implying that H_X and H_Y are isomorphic. (□)

Claim 9. $\gamma_t(F) = 2\tau(H_X)$.

Proof. By Observation 6, $\gamma_t(F) = \tau(H) = \tau(H_X) + \tau(H_Y)$. By Claim 8, $\tau(H_X) = \tau(H_Y)$, and so $\gamma_t(F) = 2\tau(H_X)$. (□)

Claim 10. $\gamma_t(F) \leqslant 2\gamma(G)$.

Proof. Let D be a minimum dominating set in G, and let D_1 and D_2 be the copies of G in G-layers G_1 and G_2, respectively. Clearly, $v \in D_1$ if and only if $v' \in D_2$. The set $D_1 \cup D_2$ is a total dominating set of F, and so $\gamma_t(F) \leqslant |D_1 \cup D_2| = 2|D| = 2\gamma(G)$. (□)

Claim 11. $\gamma(G) \leqslant \tau(H_X)$.

Proof. Let H^c be the CNH of G. By Observation 6, $\gamma(G) = \tau(H^c)$. We show that $\tau(H^c) \leqslant \tau(H_X)$. Let T_X be a minimum transversal in H_X, and so $|T_X| = \tau(H_X)$. We now define the set T_X^c as follows. For each vertex $v \in T_X$, we add v to T_X^c if $v \in V_1$, otherwise if $v \in V_2$, we add $\overline{v}$ to T_X^c. We show that T_X^c is a transversal in H^c. Let e be an arbitrary hyperedge in H^c. Thus, $e = N_F[w]$ for some vertex w in F. We may assume without loss of generality that $w \in V_1$. Thus, $\overline{w} = w' \in V_2$.

Suppose that $w \in Y$. In this case, the hyperedge $e_w = N_F(w) = e \setminus \{w\}$ is a hyperedge of H_X and is therefore covered by some vertex, say z, of T_X. If $z = \overline{w}$, then $\overline{w} \in V_2$ and $w \in T_X^c$. If $z \neq \overline{w}$, then $z \in e_w \setminus \{\overline{w}\} \subset V_1$ and $z \in T_X^c$. In both cases, the hyperedge e is covered by a vertex in T_X^c.

Suppose that $w \in X$, and so $\overline{w} \in Y \cap V_2$. In this case, the hyperedge $e_{\overline{w}} = N_F(\overline{w})$ is a hyperedge of H_X and is therefore covered by some vertex, say z, of T_X. If $z = w$, then since $w \in V_1$, the vertex $w \in T_X^c$. If $z \neq \overline{w}$, then $z \in e_{\overline{w}} \setminus \{w\} \subset V_2$ and $\overline{z} \in T_X^c$. However, since $z \in e_{\overline{w}}$, we note that $\overline{z} \in e_w$. Thus in both cases, the hyperedge e is covered by a vertex in T_X^c.

Thus, whenever $w \in X$ or $w \in Y$, the hyperedge e is covered by a vertex in T_X^c. Since e is an arbitrary hyperedge of H^c, this implies that T_X^c is a transversal of H^c, and therefore that $\tau(H^c) \leqslant |T_X^c| = |T_X| = \tau(H_X)$. (□)

We now return to the proof of Theorem 1 one final time. By Claims 9, 10, and 11, the following holds.

$$2\tau(H_X) \overset{\text{Claim 9}}{=} \gamma_t(F) \overset{\text{Claim 10}}{\leqslant} 2\gamma(G) \overset{\text{Claim 11}}{\leqslant} 2\tau(H_X)\,.$$

Consequently, we must have equality throughout the above inequality chain. In particular, $\gamma_t(F) = 2\gamma(G)$. This completes the proof of Theorem 1. □

As an immediate consequence of Theorem 1 we state that the problems of determining the domination number and the total domination number of hypercubes are equivalent in the following sense:

Corollary 12. *If $n \geqslant 1$, then $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$.*

Combining Corollary 12 with Theorem 3 we also deduce the following result:

Corollary 13. *If $k \geqslant 1$, then $\gamma_t(Q_{2^k+1}) = 2^{2^k-k+1}$ and $\gamma_t(Q_{2^k}) = 2^{2^k-k}$.*

While the first assertion of Corollary 13 appears to be new, the second assertion goes back to Johnson [15], see also [23, Theorem 1(b)].

As another consequence of Theorem 1, we have the following result.

Corollary 14. *If G is a bipartite graph, then*

$$\gamma_t(G \,\square\, K_2) = \gamma_{\rm pr}(G \,\square\, K_2) = \gamma_{\rm tr}(G \,\square\, K_2)\,.$$

Proof. For notational convenience, we let $F = G \,\square\, K_2$. As shown in the proof of Claim 10 in Theorem 1, if D_1 is a minimum dominating set in G_1, and $D_2 = \{v' \mid v \in D_1\}$, then the set $D^* = D_1 \cup D_2$ is a total dominating set of F. We note that D^* is also a paired-dominating set of F. Further, $|D^*| = 2\gamma(G)$. By Observation 4 and Theorem 1, this implies that

$$\gamma_t(F) \leqslant \gamma_{\rm pr}(F) \leqslant |D^*| = 2\gamma(G) = \gamma_t(F)\,.$$

Consequently, we must have equality throughout the above inequality chain. In particular, $\gamma_t(F) = \gamma_{\rm pr}(F)$. We note that D^* is also a total restrained dominating set of F. Thus, by Observation 5, $\gamma_t(F) \leqslant \gamma_{\rm tr}(F) \leqslant |D^*| = 2\gamma(G) = \gamma_t(F)$, implying that $\gamma_t(F) = \gamma_{\rm tr}(F)$. □

As a special case of Theorem 1 and Corollary 14, we have the following result.

Corollary 15. *If $n \geqslant 1$, then $\gamma_t(Q_n) = \gamma_{\rm pr}(Q_n) = \gamma_{\rm tr}(Q_n)$.*

4 Proof of Theorem 2

In this section, we consider general prisms and show that the bipartiteness condition in the statement of Theorem 1 is essential. First we recall the trivial lower bound on the total domination number of a graph in terms of the maximum degree of the graph: If G is a graph of order n and maximum degree Δ with no isolated vertex, then $\gamma_t(G) \geqslant n/\Delta$, cf. [14, Theorem 2.11].

Proposition 16. *If $\ell \geqslant 1$, then $\gamma_t(C_{6\ell+1} \,\square\, K_2) = 2\gamma(C_{6\ell+1}) - 1$.*

Proof. Let $G \cong C_{6\ell+1}$ for some integer $\ell \geqslant 1$. Then, $\gamma(G) = \lceil n(G)/3 \rceil = 2\ell + 1$. For notational convenience, we let $F = G \,\square\, K_2$. We show that $\gamma_t(F) = 4\ell + 1$. Let G_1 and G_2 be the G-layers of F, where G_1 is the cycle $u_1u_2 \ldots u_{6\ell+1}u_1$ and G_2 is the cycle $v_1v_2 \ldots v_{6\ell+1}v_1$, and where $u_iv_i \in E(G)$. The set

$$S = \left(\bigcup_{i=0}^{\ell-1} \{u_{6i+1}, u_{6i+2}, v_{6i+4}, v_{6i+5}\} \right) \cup \{u_{6\ell+1}\}$$

is a total dominating set of F, implying that $\gamma_t(F) \leqslant |S| = 4\ell + 1$. Conversely, since F is a cubic graph of order $12\ell + 2$, the trivial lower bound on the total domination number of F is given by $\gamma_t(F) \geqslant (12\ell+2)/3$, implying that $\gamma_t(F) \geqslant 4\ell+1$. Consequently, $\gamma_t(F) = 4\ell + 1$. As observed earlier, $\gamma(G) = 2\ell + 1$. Therefore, $\gamma_t(F) = 2\gamma(G) - 1$. □

We show next that there are connected, non-bipartite graphs G for which the difference $\gamma_t(G \,\square\, K_2) - 2\gamma(G)$ can be arbitrarily large. Recall the statement of Theorem 2.

Theorem 2 *For each integer $k \geqslant 1$, there exists a connected graph G_k satisfying*

$$2\gamma(G_k) - \gamma_t(G_k \,\square\, K_2) = k.$$

Proof. For notational convenience, we let $F_k = G_k \,\square\, K_2$. For $k = 1$, let $G_1 \cong C_7$. By Proposition 16, $\gamma_t(F_1) = 2\gamma(G_1) - 1$. Hence, we assume in what follows that $k \geqslant 2$. For $i \in [k]$ and $\ell := i - 1$, let Z_i be the 5-cycle $v_{5\ell+1}v_{5\ell+2}v_{5\ell+4}v_{5\ell+5}v_{5\ell+3}v_{5\ell+1}$. Let G_k be obtained from the disjoint union of the cycles $Z_1, \ldots, Z_k$ by adding the edges $v_{5j}v_{5j+1}$ for $j \in [k-1]$. By construction, G_k is a connected graph of order $5k$. The following two claims determine the domination number of G_k and total domination numbers of the prism F_k.

Claim A *For $k \geqslant 2$, $\gamma(G_k) = 2k$.*

Proof. Every dominating set of G_k contains at least two vertices from $V(Z_i)$ in order to dominate the vertices in $V(Z_i)$ for each $i \in [k]$, and so $\gamma(G_k) \geqslant 2k$. Conversely, every set consisting of two non-adjacent vertices from each set $V(Z_i)$ forms a dominating set of G_k, and so $\gamma(G_k) \leqslant 2k$. Consequently, $\gamma(G_k) = 2k$. (□)

Claim B *For $k \geqslant 2$, $\gamma_t(F_k) = 3k$.*

Proof. Let G_k^1 and G_k^2 be the two copies of the graph G_k in the prism F_k, where the vertex in G_k^1 and G_k^2 corresponding to the vertex v_j in G_k is labeled x_j and y_j, respectively, for $j \in [5k]$. Thus, the set $\cup_{j=1}^{5k}\{x_jy_j\}$ of edges between $V(G_k^1)$ and $V(G_k^2)$ in F_k forms a perfect matching in F_k. For $i \in [k]$ and $\ell := i - 1$, let

$$V_i = \bigcup_{j=1}^{5} \{x_{5\ell+j}, y_{5\ell+j}\}.$$

When $k = 6$, the prism F_k is illustrated in Figure 1, where the vertices in V_1 are labelled. Let S be an arbitrary total dominating set of F_k. For $i \in [k]$, let $S_i = S \cap V_i$. For $i \in [k]$ and $\ell := i - 1$, let

$$X_i = \bigcup_{j=2}^{4} \{x_{5\ell+j}\} \quad \text{and} \quad Y_i = \bigcup_{j=2}^{4} \{y_{5\ell+j}\}$$

In order to totally dominate the vertices in the set X_i, we note that $|S_i| \geqslant 2$ for all $i \in [k]$. Suppose that $|S_i| = 2$ for some $i \in [k]$. If both vertices in S_i belong to the same copy of G_k, say to G_k^2, then at least one vertex in X_i is not totally dominated by S. If the vertices in S_i belong to different copies of G_k, then at least two vertices in $X_i \cup Y_i$ are not totally dominated by S. Both cases produce a contradiction, implying that $|S_i| \geqslant 3$. Hence,

$$|S| = \sum_{i=1}^{k} |S_i| \geqslant 3k.$$

Since S is an arbitrary total dominating set of F_k, this implies that $\gamma_t(F_k) \geqslant 3k$. To prove the converse, let $\ell := i - 1$ and

$$X = \bigcup_{i=1}^{\lfloor k/2 \rfloor} \{x_{10\ell+1}, x_{10\ell+2}, x_{10i}\} \quad \text{and} \quad Y = \bigcup_{i=1}^{\lfloor k/2 \rfloor} \{y_{10\ell+5}, y_{10\ell+6}, y_{10\ell+7}\}.$$

If k is even, let

$$D = (X \cup Y \cup \{x_{5k-1}\}) \setminus \{y_{5k-3}\}.$$

If k is odd, let

$$D = X \cup Y \cup \{x_{5k-4}, y_{5k-1}, y_{5k}\}.$$

For $k = 6$, the set D is illustrated by the darkened vertices in Figure 1. In both cases, D is a total dominating set of F_k, and $|D \cap V_i| = 3$ for each $i \in [k]$, implying that

$$\gamma_t(F_k) \leqslant |D| = \sum_{i=1}^{k} |D \cap V_i| = 3k.$$

Consequently, $\gamma_t(F_k) = 3k$. $(\Box)$

By Claim A and Claim B, for $k \geqslant 2$, $\gamma(G_k) = 2k$ and $\gamma_t(F_k) = 3k$. This completes the proof of Theorem 2. $\Box$

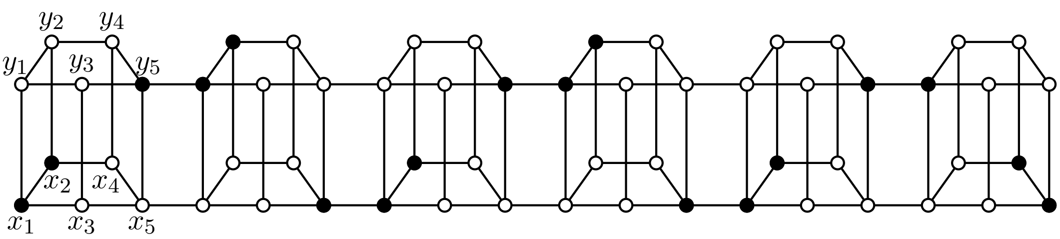

Figure 1: The prism $G_6 \,\Box\, K_2$

5 Concluding Remarks

Let us say that a graph G is *γ_t-prism perfect* if $\gamma_t(G \,\Box\, K_2) = 2\gamma(G)$. We have seen that all bipartite graphs are γ_t-prism perfect. It would certainly be interesting to characterize γ_t-prism perfect graphs in general, but this appears to be a challenging problem. Instead, one could try to characterize γ_t-prism perfect graphs within some interesting families of graphs, say triangle-free graphs.

A computation shows that among the 11.117 connected graphs of order 8, precisely 297 graphs are not γ_t-prism perfect. Similarly, there are 79.638 graphs that are not γ_t-prism perfect among the 11.716.571 connected graphs of order 9. These computations led us to conjecture the following conjecture.

Conjecture 17. Almost all graphs are γ_t-prism perfect.

With respect to the conjecture we refer to [6] for the investigation of the behavior of the domination number in random graphs.

Motivated by the construction presented in the proof of Theorem 2 we wonder whether the following lower bound on the total domination number of prisms holds true. If so, then the construction implies that the bound is sharp.

Problem 18. Is it true that for any graph G, $\gamma_t(G \,\Box\, K_2) \geqslant \frac{3}{2}\gamma(G)$?

One may be tempted to try to extend the presented results to additional Cartesian product graphs. We note that $\gamma(P_3) = 2$ and an easy computation gives $\gamma_t(P_3 \Box K_3) = \gamma_t(P_3 \Box P_3) = 4$. Similarly, $\gamma_t(P_3 \Box K_4) = 4$ and $\gamma_t(P_3 \Box P_4) = 6$, indicating that our result cannot be extended by a matter of parity. Moreover for all listed Cartesian products we were able to find pairs of bipartite graphs with the same domination number so that the total domination number of the respective Cartesian product differs. These examples give a strong evidence that the identity of Theorem 1 cannot be generalized in "obvious" directions.

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