

Permutations that Destroy Arithmetic Progressions in Elementary p -Groups

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Abstract

Given an abelian group G , it is natural to ask whether there exists a permutation π of G that “destroys” all nontrivial 3-term arithmetic progressions (APs), in the sense that $\pi(b) - \pi(a) \neq \pi(c) - \pi(b)$ for every ordered triple $(a, b, c) \in G^3$ satisfying $b - a = c - b \neq 0$. This question was resolved for infinite groups G by Hegarty, who showed that there exists an AP-destroying permutation of G if and only if $G/\Omega_2(G)$ has the same cardinality as G , where $\Omega_2(G)$ denotes the subgroup of all elements in G whose order divides 2. In the case when G is finite, however, only partial results have been obtained thus far. Hegarty has conjectured that an AP-destroying permutation of G exists if $G = \mathbb{Z}/n\mathbb{Z}$ for all $n \neq 2, 3, 5, 7$, and together with Martinsson, he has proven the conjecture for all $n > 1.4 \times 10^{14}$. In this paper, we show that if p is a prime and k is a positive integer, then there is an AP-destroying permutation of the elementary p -group $(\mathbb{Z}/p\mathbb{Z})^k$ if and only if p is odd and $(p, k) \notin \{(3, 1), (5, 1), (7, 1)\}$.

1 Introduction

Let G be an abelian group, and let $\pi : G \rightarrow G$ be any permutation. Following the terminology of Hegarty and Martinsson (see [3]), we say that π *destroys* all nonconstant arithmetic progressions (henceforth, APs) in G if there is no ordered triple $(a, b, c) \in G^3$ such that $b - a = c - b \neq 0$ and $\pi(b) - \pi(a) = \pi(c) - \pi(b)$ (i.e., (a, b, c) and $(\pi(a), \pi(b), \pi(c))$ are never both APs; note that this condition holds for π if and only if it holds for π^{-1}). It is natural to seek a complete classification of abelian groups G that have such an AP-destroying permutation.

For G infinite, it was shown by Hegarty in [2] that there exists an AP-destroying permutation of G if and only if $G/\Omega_2(G)$ has the same cardinality as G , where $\Omega_2(G)$

denotes the subgroup of all elements in G whose order divides 2. On the other hand, in the case when G is finite, such a classification has not yet been obtained. Hegarty conjectured in [2] that there exists an AP-destroying permutation of $\mathbb{Z}/n\mathbb{Z}$ for all $n \neq 2, 3, 5, 7$ (in these four cases, one readily checks that there is no AP-destroying permutation of $\mathbb{Z}/n\mathbb{Z}$). It was shown by Hegarty and Martinsson in [3] that there exists an AP-destroying permutation of $\mathbb{Z}/n\mathbb{Z}$ for all $n \geq n_0 = (9 \cdot 11 \cdot 16 \cdot 17 \cdot 19 \cdot 23)^2 \approx 1.4 \times 10^{14}$. Moreover, the following lemma, proved by Hegarty in [2], shows that one can find AP-destroying permutations of larger groups given AP-destroying permutations of smaller groups:

Lemma 1 (Hegarty). *Let G be an abelian group and $H \subset G$ a subgroup. If there exists an AP-destroying permutation of H and an AP-destroying permutation of G/H , then there exists an AP-destroying permutation of G .*

It follows from Lemma 1 that the set of all finite abelian groups G that have AP-destroying permutations is closed under taking direct sums, and that the set of all positive integers n for which $\mathbb{Z}/n\mathbb{Z}$ has an AP-destroying permutation is closed under multiplication. This last implication motivates a closer study of the case when $n = p$ is a prime, and in this regard, it was shown in [3] that there is an AP-destroying permutation of $\mathbb{Z}/p\mathbb{Z}$ for all primes $p > 3$ such that $p \equiv 3 \pmod{8}$. In this paper, we prove a result that includes Hegarty's conjecture for $\mathbb{Z}/p\mathbb{Z}$ where $p > 7$ is any prime. Our main theorem is stated as follows:

Theorem 2. *Let p be a prime and k be a positive integer. Then there is an AP-destroying permutation of $(\mathbb{Z}/p\mathbb{Z})^k$ if and only if p is odd and $(p, k) \notin \{(3, 1), (5, 1), (7, 1)\}$.*

Remark. Although one can use Lemma 1 to show that there is an AP-destroying permutation of $(\mathbb{Z}/p\mathbb{Z})^k$ for all $k > 3$ given the existence of such a permutation for $(\mathbb{Z}/p\mathbb{Z})^k$ where $k = 1$ or where $k \in \{2, 3\}$, our approach yields such a permutation directly for all elementary p -groups of odd order greater than 7. Using this result, together with Lemma 1 and the result of [3], we then see that to prove Hegarty's conjecture for finite cyclic groups, it suffices to find AP-destroying permutations of $\mathbb{Z}/n\mathbb{Z}$ for all n such that $n \in \{2p, 3p, 5p, 7p : p \text{ prime}\}$ and $n < n_0$.

The proof of Theorem 2 occupies the remainder of this paper. We first construct a permutation f that destroys all but $O(1)$ APs in $(\mathbb{Z}/p\mathbb{Z})^k$. We then show that if p^k is large enough, say $p^k > n_1$, then a small modification of f destroys all APs. This n_1 , unlike the bound n_0 of [3], is small enough that we can deal with the remaining cases $p^k \leq n_1$ by exhibiting an AP-destroying permutation in each case. This concludes the proof.

2 Proof of Theorem 2

Let p be any prime, let k be any positive integer, and let G denote the elementary p -group $(\mathbb{Z}/p\mathbb{Z})^k$. If π is a permutation of G , then for any $a, r \in G$ with $r \neq 0$, the permutation π destroys the AP $(a - r, a, a + r)$ if and only if π destroys the reversed AP $(a + r, a, a - r)$. Moreover, if $p = 3$, then all six permutations of the AP $(a - r, a, a + r)$ are APs, and if

π destroys one of them then π destroys them all. Thus, in the remainder of our proof, we will somewhat loosely use the notation “ $(a - r, a, a + r)$ ” to refer to both the AP $(a - r, a, a + r)$ and the reversed AP $(a + r, a, a - r)$, and when $p = 3$, the notation “ $(a - r, a, a + r)$ ” will refer to any permutation of this AP.

If $p = 2$ then no permutation of G destroys any AP, so we need only consider the case when p is odd. We identify $(\mathbb{Z}/p\mathbb{Z})^k$ with the additive group of the finite field $\mathbb{F}_q$ of order $q = p^k$. We shall construct an AP-destroying permutation of $(\mathbb{Z}/p\mathbb{Z})^k$ as a permutation of $\mathbb{F}_q$, by applying small modifications to the fixed permutation $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1, \\ \frac{1}{x} & \text{else.} \end{cases} \quad (1)$$

The next lemma shows that f is indeed very close to being an AP-destroying permutation:

Lemma 3. *The permutation f destroys all APs in $\mathbb{F}_q$ other than $(-1, 0, 1)$ when $p = 3$ and $(0, \frac{3}{2}, 3)$, $(\frac{1}{3}, \frac{2}{3}, 1)$ when $p > 3$.*

Proof. Let $(a - r, a, a + r)$ be an AP in $\mathbb{F}_q$ such that $\{a - r, a, a + r\} \cap \{0, 1\} = \emptyset$. Then f sends the AP $(a - r, a, a + r)$ to $(\frac{1}{a-r}, \frac{1}{a}, \frac{1}{a+r})$, which is an AP when

$$\frac{2}{a} = \frac{1}{a-r} + \frac{1}{a+r} \implies 2(a^2 - r^2) = 2a^2 \implies 2r^2 = 0,$$

but this cannot hold since $p > 2$. Thus, all APs disjoint from $\{0, 1\}$ are destroyed by f . The remaining cases are handled as follows:

- (a) First, consider APs of the form $(-r, 0, r)$. If $r \neq \pm 1$, then f sends $(-r, 0, r)$ to $(-\frac{1}{r}, 1, \frac{1}{r})$, which is not an AP because $p > 2$. However, f sends $(-1, 0, 1)$ to $(-1, 1, 0)$, which is an AP if and only if $p = 3$.
- (b) Next, consider APs of the form $(0, r, 2r)$. If $\{r, 2r\} \cap \{1\} = \emptyset$, then f sends $(0, r, 2r)$ to $(1, \frac{1}{r}, \frac{1}{2r})$, which is an AP if and only if $\frac{2}{r} = 1 + \frac{1}{2r}$, and this happens if and only if $p > 3$ and $r = \frac{3}{2}$. If $r = 1$, then f sends $(0, r, 2r)$ to $(1, 0, \frac{1}{2})$, which is an AP if and only if $p = 3$. If $2r = 1$, then f sends $(0, r, 2r)$ to $(1, 2, 0)$, which is again an AP if and only if $p = 3$.

We may now restrict our attention to APs containing 1 but not 0.

- (c) Consider APs of the form $(1 - r, 1, 1 + r)$, where $r \neq \pm 1$. The permutation f sends this AP to $(\frac{1}{1-r}, 0, \frac{1}{1+r})$, which is not an AP because $p > 2$.
- (d) Finally, consider APs of the form $(1, 1 + r, 1 + 2r)$ with $r \notin \{-1, -\frac{1}{2}\}$. Then f sends $(1, 1 + r, 1 + 2r)$ to $(0, \frac{1}{1+r}, \frac{1}{1+2r})$, which is an AP if and only if $\frac{2}{1+r} = \frac{1}{1+2r} + 0$, and this happens if and only if $p > 3$ and $r = -\frac{1}{3}$. $\square$

Remark. Because f is an involution, it also acts as an involution on the set of APs that are *not* destroyed by f .

Our strategy is to modify f by composing it with a permutation τ that is a simple transposition if $p = 3$ and a product of two transpositions τ_1, τ_2 if $p > 3$, with each transposition moving exactly one term in each of the APs not destroyed by f . The resulting permutation f' then destroys those APs but may restore others. However, if we choose τ at random then the expected number of restored APs is $O(1)$, so once q is at all large there should be some choices of τ for which no AP is restored and thus f' destroys all APs. We will prove this by counting how many τ or τ_i move a given AP term and introduce no new APs, and showing that the count is positive. Because our f is given by an algebraic rule, each of the needed enumerations reduces to estimating the number of points on certain algebraic curves over $\mathbb{F}_q$. The estimates suffice with few enough small exceptions (each with $p > 3$) that we can dispose of each remaining q computationally. In most cases we find some τ_i that works (even though the estimate was not strong enough to guarantee its existence). In the remaining cases, q is prime and small enough that an AP-destroying permutation of $\mathbb{Z}/p\mathbb{Z}$ was already exhibited by Hegarty in [2]; we also construct such permutations by starting from the τ_i that come closest to destroying all APs and then composing with further transpositions until the number of surviving APs drops to zero.

In what follows, we consider the cases of $p = 3$ and $p > 3$ separately, because we saw in the proof of Lemma 3 that the APs not destroyed by f are different in each case.

2.1 The Case $p = 3$

By hypothesis $k > 1$, so $q = 3^k > 3$; hence there exists y in $\mathbb{F}_q \setminus \{0, \pm 1\}$. Consider the permutation f' obtained by switching the images of -1 and y under f (i.e. $f'(-1) = 1/y$, $f'(y) = -1$, and $f'(x) = f(x)$ otherwise). The AP $(-1, 0, 1)$ is clearly destroyed by f' , because $y \neq -1$. Thus, by Lemma 3, if an AP is not destroyed by f' , it must either (A) contain -1 but not y or (B) contain y . We treat the cases (A) and (B) separately as follows.

- (A) If $\{-1 + r, -1 + 2r\} \cap \{0, \pm 1, y\} = \emptyset$, then f' sends the AP $(-1, -1 + r, -1 + 2r)$ to $(\frac{1}{y}, \frac{1}{-1+r}, \frac{1}{-1+2r})$, which is an AP when

$$\frac{2}{-1+r} = \frac{1}{y} + \frac{1}{-1+2r} \implies r^2 = y + 1.$$

Otherwise, if $\{-1 + r, -1 + 2r\} \cap \{0, \pm 1\} \neq \emptyset$, then we observe that f' destroys the AP $(-1, -1 + r, -1 + 2r)$ if and only if f' destroys the AP $(-1, 0, 1)$, but this holds by the construction of f' . (We intersected with $\{0, \pm 1\}$, not $\{0, \pm 1, y\}$, because $y \in \{-1 + r, -1 + 2r\}$ belongs in case (B).)

- (B) If $\{y + r, y + 2r\} \cap \{0, \pm 1, y\} = \emptyset$, then f' sends the AP $(y, y + r, y + 2r)$ to

$\left(-1, \frac{1}{y+r}, \frac{1}{y+2r}\right)$, which is an AP when

$$\frac{2}{y+r} = -1 + \frac{1}{y+2r} \implies r^2 = y^2 + y.$$

Also, one readily checks that if $\{y+r, y+2r\} \cap \{0, \pm 1\} \neq \emptyset$, the AP $(y, y+r, y+2r)$ is destroyed by f' . To verify this claim, note that there are 6 cases to consider, depending on which of $y+r$ and $y+2r$ belongs to the set $\{0, \pm 1\}$. For the sake of clarity, we shall work out the case where $y+r=0$; the remaining five cases may be handled analogously. If $y+r=0$, then because $y \notin \{0, \pm 1\}$, we have that f' sends the AP $(y, y+r, y+2r) = (-r, 0, r)$ to $(-1, 1, \frac{1}{r})$, which is an AP if and only if $\frac{1}{r} = 3$, but this is of course impossible modulo 3. Thus, f' destroys the AP $(y, y+r, y+2r)$ when $y+r=0$.

Now, let χ denote the Legendre symbol over $\mathbb{F}_q$. It follows from the above case analysis that if $y \notin \{0, \pm 1\}$ is chosen so that

$$(1 - \chi(y+1)) \cdot (1 - \chi(y(y+1))) > 0,$$

then f' destroys all APs in $\mathbb{F}_q$. Such a y exists if and only if the sum

$$A_q(y) := \sum_{y \in \mathbb{F}_q \setminus \{0, \pm 1\}} (1 - \chi(y+1)) \cdot (1 - \chi(y(y+1))) \quad (2)$$

is positive. To compute $A_q(y)$, we use the following well-known elementary formula:

Lemma 4. *Let $\mathbb{F}_q$ be a finite field of odd characteristic, and let χ be the Legendre symbol on $\mathbb{F}_q$. If q is odd and $g \in \mathbb{F}_q[x]$ is a polynomial of degree at most 2 such that $g \neq c \cdot h^2$ for any $c \in \mathbb{F}_q$ and $h \in \mathbb{F}_q[x]$, then*

$$\sum_{y \in \mathbb{F}_q} \chi(g(y)) = -\chi(a), \quad (3)$$

where a is the coefficient of the degree-2 term in g .

Now, taking the sum in (2) over all of $\mathbb{F}_q$ and applying the result of Lemma 4, we find that

$$\sum_{y \in \mathbb{F}_q} (1 + \chi(y) - \chi(y+1) - \chi(y(y+1))) = q + 1.$$

Because

$$\sum_{y \in \{0, \pm 1\}} (1 + \chi(y) - \chi(y+1) - \chi(y(y+1))) = 3 - \chi(-1) \leq 4,$$

we conclude that $A_q(y) > 0$ provided $q + 1 > 4$, which happens once $k \geq 2$. Thus, there exists $y \in \mathbb{F}_q \setminus \{0, \pm 1\}$ such that f' destroys all APs in $\mathbb{F}_q$ for $q = 3^k$ and $k \geq 2$.

Remark. Alternatively, note that we can handle the case $p = 3$ by simply exhibiting AP-destroying permutations of $\mathbb{F}_9$ and $\mathbb{F}_{27}$, for it would then follow by Lemma 1 that there is an AP-destroying permutation of $\mathbb{F}_{3^k}$ for each $k > 1$. Making the identification $\mathbb{F}_9 \simeq \mathbb{F}_3[\alpha]/(\alpha^2 + 2\alpha + 2)$, one readily checks that the permutation f' of $\mathbb{F}_9$ obtained by taking $y = \alpha + 1$ destroys all APs in $\mathbb{F}_9$. Similarly, making the identification $\mathbb{F}_{27} \simeq \mathbb{F}_3[\beta]/(\beta^3 + 2\beta + 1)$, one readily checks that the permutation f' of $\mathbb{F}_{27}$ obtained by taking $y = \beta^2$ destroys all APs in $\mathbb{F}_{27}$. Nonetheless, this *ad hoc* argument does not readily generalize to primes $p > 3$, while the proof provided prior to the present remark extends quite naturally to primes $p > 3$, as we demonstrate in Sections 2.2 and 2.3.

2.2 Destroying $(0, \frac{3}{2}, 3)$ in the Case $p > 3$

This case takes more work than the case $p = 3$, but the strategy is similar. We begin by constructing a permutation f' of $\mathbb{F}_q$ that destroys all but one AP. Take $y \in \mathbb{F}_q \setminus \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{3}{2}, 3\}$ (note that this already requires $q > 5$), and let f' be the permutation obtained by switching the images of 3 and y under f ; that is, $f'(3) = \frac{1}{y}$, $f'(y) = \frac{1}{3}$, and $f'(x) = f(x)$ otherwise. The AP $(0, \frac{3}{2}, 3)$ is clearly destroyed by f' , because $y \neq 3$. Thus, by Lemma 3, if an AP other than $(\frac{1}{3}, \frac{2}{3}, 1)$ is not destroyed by f' , it must either (A) contain 3 but not y or (B) contain y . The cases (A) and (B) each have two subcases depending on the position of 3 or y in the AP; we study each of these subcases separately as follows.

- (A) (a) If $\{3 + r, 3 + 2r\} \cap \{0, 1, 3, y\} = \emptyset$, then f' sends the AP $(3, 3 + r, 3 + 2r)$ to $(\frac{1}{y}, \frac{1}{3+r}, \frac{1}{3+2r})$, which is an AP when

$$\frac{2}{3+r} = \frac{1}{y} + \frac{1}{3+2r} \implies 2r^2 + (9-3y)r + (9-3y) = 0.$$

If $r = -3$ and $3 + 2r = -3 \neq y$, then f' does not destroy the AP $(3, 3 + r, 3 + 2r) = (3, 0, -3)$ when $y = \frac{3}{7}$. One readily checks that f' destroys all other APs of the form $(3, 3 + r, 3 + 2r)$ that do not contain y and satisfy $\{3 + r, 3 + 2r\} \cap \{0, 1, 3\} \neq \emptyset$. (Note that there are 3 cases left to consider, making a total of four cases in all, according as $3 + r$ or $3 + 2r$ belongs to $\{0, 1\}$.)

- (b) If $\{3 - r, 3 + r\} \cap \{0, 1, 3, y\} = \emptyset$, then f' sends the AP $(3 - r, 3, 3 + r)$ to $(\frac{1}{3-r}, \frac{1}{y}, \frac{1}{3+r})$, which is an AP when

$$\frac{2}{y} = \frac{1}{3-r} + \frac{1}{3+r} \implies r^2 = 9 - 3y.$$

If $r = 3$ and $3 + r = 6 \neq y$, then f' does not destroy the AP $(3 - r, 3, 3 + r) = (0, 3, 6)$ when $y = \frac{12}{7}$. If $r = 2$ and $3 + r = 5 \neq y$, then f' does not destroy the AP $(3 - r, 3, 3 + r) = (1, 3, 5)$ when $y = 10$. There are no cases that remain to be considered for the AP $(3 - r, 3, 3 + r)$.

- (B) (a) If $\{y+r, y+2r\} \cap \{0, 1, 3, y\} = \emptyset$, then f' sends the AP $(y, y+r, y+2r)$ to $(\frac{1}{3}, \frac{1}{y+r}, \frac{1}{y+2r})$, which is an AP when

$$\frac{2}{y+r} = \frac{1}{3} + \frac{1}{y+2r} \implies 2r^2 + (3y-9)r + (y^2-3y) = 0.$$

If $y+r=0$ and $y+2r \notin \{0, 1, 3\}$, then f' does not destroy the AP $(y, y+r, y+2r) = (y, 0, -y)$ when $5y = -3$. If $y+r=1$ and $y+2r \notin \{0, 1, 3\}$, then f' does not destroy the AP $(y, y+r, y+2r) = (y, 1, 2-y)$ when $y=5$. If $y+r=3$ and $y+2r \notin \{0, 1, 3\}$, then f' does not destroy the AP $(y, y+r, y+2r) = (y, 3, 6-y)$ when $y=12$. If $y+2r=1$ and $y+r \notin \{0, 1, 3\}$, then f' does not destroy the AP $(y, y+r, y+2r) = (y, \frac{y+1}{2}, 1)$ when $y=11$. One readily checks that f' destroys all other APs of the form $(y, y+r, y+2r)$ that satisfy $\{y+r, y+2r\} \cap \{0, 1, 3\} \neq \emptyset$. (Note that the remaining cases here are when $y+2r \in \{0, 3\}$ and $y+r \notin \{0, 1, 3\}$ and when both $y+r$ and $y+2r$ are in $\{0, 1, 3\}$.)

- (b) If $\{y-r, y+r\} \cap \{0, 1, 3, y\} = \emptyset$, then f' sends the AP $(y-r, y, y+r)$ to $(\frac{1}{y-r}, \frac{1}{3}, \frac{1}{y+r})$, which is an AP when

$$\frac{2}{3} = \frac{1}{y-r} + \frac{1}{y+r} \implies r^2 = y^2 - 3y.$$

If $y-r=0$ and $y+r \notin \{0, 1, 3\}$, then f' does not destroy the AP $(y-r, y, y+r) = (0, y, 2y)$ when $y = -\frac{3}{2}$. If $y-r=1$ and $y+r \notin \{0, 1, 3\}$, then f' does not destroy the AP $(y-r, y, y+r) = (1, y, 2y-1)$ when $y = \frac{5}{4}$. If $y-r=3$ and $y+r \notin \{0, 1, 3\}$, then f' does not destroy the AP $(y-r, y, y+r) = (3, y, 2y-3)$ when $y = \frac{3}{4}$. The only remaining AP of the form $(y-r, y, y+r)$ that satisfies $\{y-r, y+r\} \cap \{0, 1, 3\} \neq \emptyset$ is given by $(y-r, y, y+r) = (1, 2, 3)$, but f' evidently destroys this AP.

Now, taking χ to be the Legendre symbol over $\mathbb{F}_q$ as before, it follows from the above case analysis that if

$$y \notin S := \left\{ -\frac{3}{2}, -\frac{3}{5}, 0, \frac{1}{3}, \frac{3}{7}, \frac{2}{3}, \frac{3}{4}, 1, \frac{5}{4}, \frac{12}{7}, \frac{3}{2}, 3, 5, 10, 11, 12 \right\}$$

is chosen so that

$$(1-\chi((3-y)(3-9y))) \cdot (1-\chi(3(3-y))) \cdot (1-\chi((3-y)(27-y))) \cdot (1-\chi(-y(3-y))) > 0, \quad (4)$$

then f' destroys all APs other than the AP $(\frac{1}{3}, \frac{2}{3}, 1)$ in $\mathbb{F}_q$. Expanding the product on the left-hand side (LHS) of (4) under the assumption that $y \neq 3$ (so that $\chi((3-y)^2) = 1$), we obtain a lengthy expression that we denote by $B_q(y)$ for the sake of readability:

$$\begin{aligned} B_q(y) := & [1 - \chi((3-y)(\frac{1}{3}-y)) - \chi(9-3y) - \chi((3-y)(27-y)) - \chi(y(y-3)) + \chi(1-3y) + \\ & \chi((27-y)(\frac{1}{3}-y)) + \chi(y(y-\frac{1}{3})) + \chi(3(27-y)) + \chi(-3y) + \chi(y(y-27)) - \chi((3-y)(27-y)(1-3y)) - \\ & \chi(y(3-y)(3y-1)) - \chi(y(3-y)(27-y)(y-\frac{1}{3})) - \chi(3y(3-y)(y-27)) + \chi(y(27-y)(3y-1))]. \end{aligned}$$

Clearly, there exists $y \in \mathbb{F}_q \setminus S$ satisfying (4) if and only if

$$\sum_{y \in \mathbb{F}_q \setminus S} B_q(y) > 0. \quad (5)$$

To estimate the LHS of (5), we first estimate $\sum_{y \in \mathbb{F}_q} B_q(y)$, for which must invoke not only Lemma 4 but also the Hasse bound (see [1] for the original paper and Corollary 1.4 of [4] for a more modern reference):

Theorem 5. [Hasse] *Let $\mathbb{F}_q$ be a finite field of odd characteristic, and let χ be the Legendre symbol on $\mathbb{F}_q$. If $g \in \mathbb{F}_q[x]$ is a polynomial of degree 3 or 4 such that $g \neq c \cdot h^2$ for any $c \in \mathbb{F}_q$ and $h \in \mathbb{F}_q[x]$, then*

$$\left| \chi(a) + \sum_{y \in \mathbb{F}_q} \chi(g(y)) \right| \leq 2\sqrt{q}, \quad (6)$$

where a is the coefficient of the degree-4 term in g .

The constant term 1 in $B_q(y)$ yields a contribution of q to the sum. Each of the other terms in $B_q(y)$ is of the form $\chi(g(y))$ for some polynomial $g \in \mathbb{F}_q[x]$. Applying Lemma 4 to the terms of the form $\chi(g(y))$ where g has degree at most 2, we obtain a total contribution of 0 from such terms. Then, applying Theorem 5 to the remaining terms, which are of the form $\chi(g(y))$ where g has degree 3 or 4, yields

$$\sum_{y \in \mathbb{F}_q} B_q(y) \geq q - 10\sqrt{q} - 1. \quad (7)$$

Estimating $\sum_{y \in S} B_q(y)$ by using the trivial bound $|\chi(g(y))| \leq 1$ for each $y \in S$ unless $g(y) = 0$ (which can occur when $y \in \{0, \frac{1}{3}, 3\}$), we find that

$$\sum_{y \in S} B_q(y) \leq 13 \cdot 16 + 0 + 8 + 8 = 224, \quad (8)$$

where the term $13 \cdot 16$ bounds the contributions of $y \in S \setminus \{0, \frac{1}{3}, 3\}$, the term 0 is the contribution of $y = 0$, and the terms $8 + 8$ bound the contributions of $y \in \{\frac{1}{3}, 3\}$. Combining the estimates (7) and (8), we deduce that (5) holds if $q - 10\sqrt{q} \geq 225$, which happens when $q \geq 434$. Thus, there exists $y \in \mathbb{F}_q \setminus S$ such that f' destroys all APs other than the AP $(\frac{1}{3}, \frac{2}{3}, 1)$ in $\mathbb{F}_q$ for $q \geq 434$ a prime power.

2.3 Destroying $(\frac{1}{3}, \frac{2}{3}, 1)$ in the Case $p > 3$

We now perform an analogous maneuver to construct a permutation that destroys *all* APs in $\mathbb{F}_q$ for sufficiently large q . With $q \geq 434$, take $y \in \mathbb{F}_q \setminus S$ so that the permutation f' destroys all APs other than the AP $(\frac{1}{3}, \frac{2}{3}, 1)$, and for $z \in \mathbb{F}_q \setminus \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{3}{2}, 3, y\}$, consider the permutation f'' obtained by switching the images of $\frac{1}{3}$ and z under f' (i.e. $f''(\frac{1}{3}) = \frac{1}{z}$, $f''(z) = 3$, and $f''(x) = f'(x)$ otherwise). The AP $(\frac{1}{3}, \frac{2}{3}, 1)$ is clearly destroyed by f'' , as $z \neq \frac{1}{3}$. Thus, by Lemma 3, if an AP is not destroyed by f'' , it must either (A) contain $\frac{1}{3}$ but not z or (B) contain z . The cases (A) and (B) each have two subcases depending on the position of $\frac{1}{3}$ or z in the AP; we study each of these subcases separately as follows:

- (A) (a) If $\{\frac{1}{3} + r, \frac{1}{3} + 2r\} \cap \{0, \frac{1}{3}, 1, 3, y, z\} = \emptyset$, then f'' sends the AP $(\frac{1}{3}, \frac{1}{3} + r, \frac{1}{3} + 2r)$ to $(\frac{1}{z}, \frac{1}{\frac{1}{3} + r}, \frac{1}{\frac{1}{3} + 2r})$, which is an AP when

$$\frac{2}{\frac{1}{3} + r} = \frac{1}{z} + \frac{1}{\frac{1}{3} + 2r} \implies 2r^2 + (1 - 3z)r + (\frac{1}{9} - \frac{z}{3}) = 0.$$

We must now deal with the cases where the AP does not contain z but $\{\frac{1}{3} + r, \frac{1}{3} + 2r\} \cap \{0, \frac{1}{3}, 1, 3, y\} \neq \emptyset$; as the computations are more complicated and numerous in the present situation, we will not be as explicit as we were in Section 2.2. If $\frac{1}{3} + r = 0$ and $\frac{1}{3} + 2r = -\frac{1}{3} \neq z$, then there is at most one value of z , call it a_1 , such that f'' does not destroy the AP $(\frac{1}{3}, 0, -\frac{1}{3})$. (The value a_1 , if it exists, is uniquely determined by the particular field $\mathbb{F}_q$ with which we are working.) Similarly, from the cases where $\frac{1}{3} + r \in \{1, 3, y\}$, there are at most three additional values of z , call them a_2, a_3, a_4 , such that f'' does not destroy the AP $(\frac{1}{3}, \frac{1}{3} + r, \frac{1}{3} + 2r)$. From the cases where $\frac{1}{3} + 2r \in \{0, 3, y\}$, there are at most three values of z , call them a_5, a_6, a_7 , such that f'' does not destroy the AP $(\frac{1}{3}, \frac{1}{3} + r, \frac{1}{3} + 2r)$. The only remaining case to consider is when $\frac{1}{3} + 2r = 1$, but the corresponding AP $(\frac{1}{3}, \frac{1}{3} + r, \frac{1}{3} + 2r) = (\frac{1}{3}, \frac{2}{3}, 1)$ is, as mentioned before, destroyed by f' .

- (b) If $\{\frac{1}{3} - r, \frac{1}{3} + r\} \cap \{0, \frac{1}{3}, 1, 3, y, z\} = \emptyset$, then f'' sends the AP $(\frac{1}{3} - r, \frac{1}{3}, \frac{1}{3} + r)$ to $(\frac{1}{\frac{1}{3} - r}, \frac{1}{z}, \frac{1}{\frac{1}{3} + r})$, which is an AP when

$$\frac{2}{z} = \frac{1}{\frac{1}{3} - r} + \frac{1}{\frac{1}{3} + r} \implies r^2 = \frac{1}{9} - \frac{z}{3}.$$

From the cases where $\frac{1}{3} - r \in \{0, 1, 3, y\}$, there are at most four values of z , call them a_8, a_9, a_{10}, a_{11} , such that f'' does not destroy the AP $(\frac{1}{3} - r, \frac{1}{3}, \frac{1}{3} + r)$. By symmetry, we have also taken care of the cases where $\frac{1}{3} + r \in \{0, 1, 3, y\}$. There are no cases that remain to be considered for the AP $(\frac{1}{3} - r, \frac{1}{3}, \frac{1}{3} + r)$.

- (B) (a) If $\{z + r, z + 2r\} \cap \{0, \frac{1}{3}, 1, 3, y, z\} = \emptyset$, then the AP $(z, z + r, z + 2r)$ is sent under f'' to $(3, \frac{1}{z + r}, \frac{1}{z + 2r})$, which is an AP when

$$\frac{2}{z + r} = 3 + \frac{1}{z + 2r} \implies 6r^2 + (9z - 3)r + (3z^2 - z) = 0.$$

The cases where $\{z + r, z + 2r\} \cap \{0, \frac{1}{3}, 1, 3, y\} \neq \emptyset$ give at most 29 values of z , call them $a_{12}, \dots, a_{40}$, such that f'' does not destroy the AP $(z, z + r, z + 2r)$. To see why, observe that there are at most five values of z arising from the possibility that $z + r \in \{0, \frac{1}{3}, 1, 3, y\} \not\supset z + 2r$, because for each value of $z + r \neq \frac{1}{3}$ we obtain a linear equation in z , and when $z + r = \frac{1}{3}$, we obtain a quadratic equation one of whose solutions is $z = \frac{1}{3}$ and must therefore be discarded. Similarly, we obtain at most five values of z arising from the possibility that $z + 2r \in \{0, \frac{1}{3}, 1, 3, y\} \not\supset z + r$. In the cases where $z + r, z + 2r \in \{0, \frac{1}{3}, 1, 3, y\}$,

we can solve for z immediately (without even imposing the condition that f'' destroys the AP $(z, z+r, z+2r)$); we obtain at most 20 values of z , one corresponding to each of the $5 \cdot 4 = 20$ different ordered pairs of distinct elements of $\{0, \frac{1}{3}, 1, 3, y\}$. Nevertheless, it is clear that we should discard the case where $z+r = \frac{1}{3}$ and $z+2r = 0$, which would imply that $z = \frac{2}{3}$, contradicting our restriction on the value of z . We therefore end up with at most $5+5+20-1 = 29$ values of z . There are no cases that remain to be considered for the AP $(z, z+r, z+2r)$.

- (b) If $\{z-r, z+r\} \cap \{0, \frac{1}{3}, 1, 3, y, z\} = \emptyset$, then the AP $(z-r, z, z+r)$ is sent under f'' to $(\frac{1}{z-r}, 3, \frac{1}{z+r})$, which is an AP when

$$6 = \frac{1}{z-r} + \frac{1}{z+r} \implies r^2 = z^2 - \frac{z}{3}.$$

The cases where $\{z-r, z+r\} \cap \{0, \frac{1}{3}, 1, 3, y\} \neq \emptyset$ give at most 14 values of z , call them $a_{41}, \dots, a_{54}$, such that f'' does not destroy the AP $(z-r, z, z+r)$. To see why, observe that there are at most five values of z arising from the possibility that $z-r \in \{0, \frac{1}{3}, 1, 3, y\} \not\equiv z+r$, because for each value of $z-r \neq \frac{1}{3}$ we obtain a linear equation in z , and when $z-r = \frac{1}{3}$, we obtain a quadratic equation one of whose solutions is $z = \frac{1}{3}$ and must therefore be discarded. By symmetry, we have also taken care of the possibility that $z+r \in \{0, \frac{1}{3}, 1, 3, y\} \not\equiv z-r$. In the cases where $z-r, z+r \in \{0, \frac{1}{3}, 1, 3, y\}$, we can solve for z immediately (without even imposing the condition that f'' destroys the AP $(z-r, z, z+r)$); we obtain at most 10 values of z , one corresponding to each of the $\binom{5}{2} = 10$ different pairs of distinct elements of $\{0, \frac{1}{3}, 1, 3, y\}$. Nevertheless, it is clear that we should discard the case where $z-r = 0$ and $z+r = 3$, which would imply that $z = \frac{3}{2}$, contradicting our restriction on the value of z . We therefore end up with at most $5 + 10 - 1 = 14$ values of z . There are no cases that remain to be considered for the AP $(z-r, z, z+r)$.

As in Section 2.2, we can use the above case analysis to write down a condition on when f'' destroys all APs in $\mathbb{F}_q$. Indeed, if

$$z \notin S' := \{a_i : 1 \leq i \leq 54\} \cup \left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{3}{2}, 3, y\right\}$$

is chosen so that we have

$$(1 - \chi((1-27z)(1-3z))) \cdot (1 - \chi(1-3z)) \cdot (1 - \chi((1-3z)(1-\frac{z}{3}))) \cdot (1 - \chi(-3z(1-3z))) > 0, \quad (9)$$

then f'' destroys all APs in $\mathbb{F}_q$. Let $w := \frac{1}{z}$ for $z \neq 0$, and define $S'' := \{\frac{1}{x} : x \in S' \setminus \{0\}\} \cup \{0\}$. Then, rewriting the above condition in terms of w and S'' , we obtain the following “new condition”: if $w \notin S''$ is chosen so that

$$(1 - \chi((3-w)(3-9w))) \cdot (1 - \chi(3(3-w))) \cdot (1 - \chi((3-w)(27-w))) \cdot (1 - \chi(-w(3-w))) > 0, \quad (10)$$

then f'' destroys all APs in $\mathbb{F}_q$. But upon making the replacements $y \rightsquigarrow w$ and $S \rightsquigarrow S''$, one readily observes that this “new condition” is the same as the analogous condition obtained in Section 2.2, namely (4). Therefore, there exists $w \in \mathbb{F}_q \setminus S''$ satisfying (9) if and only if

$$\sum_{w \in \mathbb{F}_q \setminus S''} B_q(w) > 0. \quad (11)$$

To estimate the LHS of (11), we first recall the bound (7):

$$\sum_{w \in \mathbb{F}_q} B_q(w) \geq q - 10\sqrt{q} - 1. \quad (12)$$

Next, estimating $\sum_{w \in S''} B_q(w)$ by using the trivial bound $|\chi(g(w))| \leq 1$ for each $w \in S''$ unless $g(w) = 0$ (which can occur when $w \in \{0, \frac{1}{3}, 3\}$), we find that

$$\sum_{w \in S''} B_q(w) \leq 58 \cdot 16 + 0 + 8 + 8 = 944, \quad (13)$$

where the term $58 \cdot 16$ bounds the contributions of $w \in S'' \setminus \{0, \frac{1}{3}, 3\}$, the term 0 is the contribution of $w = 0$, and the terms $8 + 8$ bound the contributions of $w \in \{\frac{1}{3}, 3\}$. Combining the estimates (12) and (13), we deduce that (11) holds if $q - 10\sqrt{q} \geq 945$, which happens when $q \geq 1307$. Thus, there exists $w \in \mathbb{F}_q \setminus S''$ such that f'' destroys all APs in $\mathbb{F}_q$ for $q \geq 1307$ a prime power.

2.4 Remaining Cases

We have now shown that there exists an AP-destroying permutation of $(\mathbb{Z}/p\mathbb{Z})^k$ if $p = 3$ and $k \geq 2$ and if $p^k \geq 1307$. To complete the proof of Theorem 2, it remains to check the finitely many remaining cases, and we do this by resorting to a computer program. By Lemma 1, it suffices to check that $(\mathbb{Z}/p\mathbb{Z})^k$ has an AP-destroying permutation for the following cases: $(p, k) \in \{(p, 1) : 7 < p < 1307 \text{ is prime}\} \cup \{(5, 2), (5, 3), (7, 2), (7, 3)\}$. In each of these cases other than $p = q = 11, 13, 29, 31$, our code relies on the construction used in the argument of Section 2. Indeed, for these cases, the idea of modifying the values of $f(3)$ and $f(\frac{1}{3})$ actually works to yield an AP-destroying permutation of $(\mathbb{Z}/p\mathbb{Z})^k$. For $q \in \{11, 13, 29, 31\}$, explicit AP-destroying permutations of $\mathbb{Z}/p\mathbb{Z}$ were constructed by Hegarty in [2]. The code required to check these cases, as well as a database listing the explicit AP-destroying permutations for all of the above exceptional cases, may be obtained by downloading the source files from the following website: <https://arxiv.org/format/1601.07541v3>.

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References

- [1] Helmut Hasse, *Zur Theorie der abstrakten elliptischen Funktionenkörper. I, II & III*, J. reine angew. Math. (Crelle's J.) **175** (1936).
- [2] Peter Hegarty, *Permutations avoiding arithmetic patterns*, Electron. J. Combin. **11** (2004), no. 1.
- [3] Peter Hegarty and Anders Martinsson, *Permutations destroying arithmetic progressions in finite cyclic groups*, Electron. J. Combin. **22** (2015), no. 4.
- [4] Joseph Silverman, *The Arithmetic of Elliptic Curves*, 2nd Edition, Graduate Texts in Mathematics, Springer, 2009.