Colorful Subhypergraphs in Uniform Hypergraphs

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Submitted: May 22, 2016; Accepted: Jan 25, 2017; Published: Feb 3, 2017
Mathematics Subject Classifications: 05C15

Abstract

There are several topological results ensuring in any properly colored graph the existence of a colorful complete bipartite subgraph, whose order is bounded from below by some topological invariants of some topological spaces associated to the graph. Meunier [*Electron. J. Combin.,* 2014] presented the first colorful type result for uniform hypergraphs. In this paper, we give some new generalizations of the $\mathbb{Z}_p$-Tucker Lemma and by use of them, we improve Meunier's result and some other colorful results by Simonyi, Tardif, and Zsbán [*Electron. J. Combin.,* 2014] and by Simonyi and Tardos [*Electron. J. Combin.,* 2007] to uniform hypergraphs. Also, we introduce some new lower bounds for the chromatic number and local chromatic number of uniform hypergraphs. A hierarchy between these lower bounds is presented as well.

Keywords: chromatic number of hypergraphs, $\mathbb{Z}_p$-Tucker-Ky Fan lemma, colorful complete hypergraph, $\mathbb{Z}_p$-box-complex, $\mathbb{Z}_p$-Hom-complex.

1 Introduction

1.1 Background and Motivations

In 1955, Kneser [15] posed a conjecture about the chromatic number of Kneser graphs. In 1978, Lovász [17] proved this conjecture by using algebraic topology. The Lovász proof marked the beginning of the history of topological combinatorics. Nowadays, it is an active stream of research to study the coloring properties of graphs by using algebraic topology. There are several lower bounds for the chromatic number of graphs related to the indices of some topological spaces defined based on the structure of graphs. However, for hypergraphs, there are few such lower bounds, see [3, 7, 13, 16, 24].

A *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite nonempty set, called the vertex set of $\mathcal{H}$, and $E(\mathcal{H})$ is a family of nonempty subsets of $V(\mathcal{H})$, called the edge set of $\mathcal{H}$. Throughout the paper, by a nonempty hypergraph, we mean a hypergraph with at least one edge. The number of vertices of a hypergraph is called its *order*. If any edge $e \in E(\mathcal{H})$ has cardinality r, then the hypergraph $\mathcal{H}$ is called r*-uniform.* For a set $U \subseteq V(\mathcal{H})$, the *induced subhypergraph by* U*,* denoted $\mathcal{H}[U]$, is a hypergraph with the vertex set U and the edge set $\{e \in E(\mathcal{H}) :\ e \subseteq U\}$. Throughout the paper, by a *graph,* we mean a 2-uniform hypergraph. Let r and q be a positive integers, where $q \geqslant r \geqslant 2$. An r-uniform hypergraph $\mathcal{H}$ is called q*-partite* if its vertex set can be partitioned into subsets $U_1, \ldots, U_q$ such that each edge of $\mathcal{H}$ intersects each part U_i in at most one vertex. A *complete* r*-uniform* q*-partite hypergraph* is an r-uniform q-partite hypergraph containing all possible edges. Also, the hypergraph $\mathcal{H}$ is said to be *balanced* if the values of the $|U_j|$'s for $j = 1, \ldots, q$ differ by at most one, i.e., $|U_i| - |U_j| \leqslant 1$ for each $i, j \in [q]$.

Let $\mathcal{H}$ be an r-uniform hypergraph and $U_1, \ldots, U_q$ be q pairwise disjoint subsets of $V(\mathcal{H})$. The hypergraph $\mathcal{H}[U_1, \ldots, U_q]$ is a subhypergraph of $\mathcal{H}$ with the vertex set $\cup_{i=1}^{q} U_i$ and the edge set

$$E(\mathcal{H}[U_1, \ldots, U_q]) = \left\{ e \in E(\mathcal{H}) :\ e \subseteq \bigcup_{i=1}^{q} U_i \text{ and } |e \cap U_i| \leqslant 1 \text{ for each } i \in [q] \right\}.$$

Note that $\mathcal{H}[U_1, \ldots, U_q]$ is an r-uniform q-partite hypergraph with parts $U_1, \ldots, U_q$. By the symbols $[n]$ and $\binom{[n]}{r}$, we respectively mean the set $\{1, \ldots, n\}$ and the family of all r-subsets of $[n]$. The hypergraph $K_n^r = \left([n], \binom{[n]}{r}\right)$ is called the *complete* r*-uniform hypergraph* with n vertices. For $r = 2$, we would rather use K_n instead of K_n^2. The largest possible integer n such that $\mathcal{H}$ contains K_n^r as a subhypergraph is called the *clique number of* $\mathcal{H}$, denoted $\omega(\mathcal{H})$.

Let L be a positive integer. A *proper* L*-coloring* of a hypergraph $\mathcal{H}$ is a map $c : V(\mathcal{H}) \longrightarrow [L]$ such that $\mathcal{H}$ has no monochromatic edge, that is, there is no edge $e \in E(\mathcal{H})$ with $|c(e)| = 1$. A hypergraph $\mathcal{H}$ is called L-colorable if it admits a proper L-coloring. The minimum possible L for which $\mathcal{H}$ is L-colorable is called *the chromatic number of* $\mathcal{H}$, denoted $\chi(\mathcal{H})$. If there is no such an L, we define the chromatic number to be infinite. Let c be a proper coloring of $\mathcal{H}$ and $U_1, \ldots, U_q$ be q pairwise disjoint subsets of $V(\mathcal{H})$. The hypergraph $\mathcal{H}[U_1, \ldots, U_q]$ is said to be *colorful* if for each $j \in [q]$, the vertices in U_j get pairwise distinct colors. For a properly colored graph G, a subgraph is called *multicolored* if its vertices get pairwise distinct colors.

For a hypergraph $\mathcal{H}$, *the Kneser hypergraph* $\mathrm{KG}^r(\mathcal{H})$ is an r-uniform hypergraph with the vertex set $E(\mathcal{H})$ and whose edges are formed by r pairwise vertex-disjoint edges of $\mathcal{H}$, i.e.,

$$E(\mathrm{KG}^r(\mathcal{H})) = \left\{ \{e_1, \ldots, e_r\} :\ e_i \cap e_j = \varnothing \text{ for each } i \neq j \in [r] \right\}.$$

The Kneser hypergraph $\mathrm{KG}^r\left(K_n^k\right)$ is called the "usual" Kneser hypergraph, which is denoted by $\mathrm{KG}^r(n, k)$. Coloring properties of Kneser hypergraphs $\mathrm{KG}^r(n, k)$ have been extensively studied in the literature. Lovász [17] (for $r = 2$) and Alon, Frankl, and

Lovász [4] determined the chromatic number of $\mathrm{KG}^r(n,k)$. For an integer $r \geqslant 2$, they proved that

$$\chi\left(\mathrm{KG}^r(n,k)\right) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil.$$

For a hypergraph $\mathcal{H}$, *the r-colorability defect of $\mathcal{H}$,* denoted $\mathrm{cd}_r(\mathcal{H})$, is the minimum number of vertices that should be removed so that the induced subhypergraph by the remaining vertices is r-colorable, i.e.,

$$\mathrm{cd}_r(\mathcal{H}) = \min\left\{|U| :\ \mathcal{H}[V(\mathcal{H}) \setminus U] \text{ is } r\text{-colorable}\right\}.$$

For a hypergraph $\mathcal{H}$, Dol'nikov [7] (for $r=2$) and Kříž [16] proved that

$$\chi(\mathrm{KG}^r(\mathcal{H})) \geqslant \left\lceil \frac{\mathrm{cd}_r(\mathcal{H})}{r-1} \right\rceil,$$

which is a generalization of the results by Lovász [17] and Alon, Frankl and Lovász [4].

For a positive integer r, let $\mathbb{Z}_r = \{\omega, \omega^2 \ldots, \omega^r\}$ be a cyclic group of order r with generator ω. Consider a vector $X = (x_1, x_2, \ldots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n$. *An alternating subsequence of* X is a sequence $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ of nonzero terms of X such that $i_1 < \cdots < i_m$ and $x_{i_j} \neq x_{i_{j+1}}$ for each $j \in [m-1]$. We denote by $\mathrm{alt}(x)$ the maximum possible length of an alternating subsequence of X. For a vector $X = (x_1, x_2, \ldots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n$ and for an $\varepsilon \in \mathbb{Z}_p$, set $X^\varepsilon = \{i \in [n] :\ x_i = \varepsilon\}$. Note that, by abuse of notation, we can write $X = (X^\varepsilon)_{\varepsilon \in \mathbb{Z}_r}$. For two vectors $X, Y \in (\mathbb{Z}_r \cup \{0\})^n$, by $X \subseteq Y$, we mean $X^\varepsilon \subseteq Y^\varepsilon$ for each $\varepsilon \in \mathbb{Z}_r$.

For a hypergraph $\mathcal{H}$ and a bijection $\sigma : [n] \longrightarrow V(\mathcal{H})$, define

$$\mathrm{alt}_r(\mathcal{H},\sigma) = \max\left\{\mathrm{alt}(X) : X \in (\mathbb{Z}_r \cup \{0\})^n \text{ s.t. } E(\mathcal{H}[\sigma(X^\varepsilon)]) = \varnothing \text{ for each } \varepsilon \in \mathbb{Z}_r\right\}.$$

Also, let

$$\mathrm{alt}_r(\mathcal{H}) = \min_{\sigma} \mathrm{alt}_r(\mathcal{H},\sigma),$$

where the minimum is taken over all bijections $\sigma : [n] \longrightarrow V(\mathcal{H})$. One can readily check that for any hypergraph $\mathcal{H}$, we have $|V(\mathcal{H})| - \mathrm{alt}_r(\mathcal{H}) \geqslant \mathrm{cd}_r(\mathcal{H})$ and the inequality is often strict, see [3]. The present author and Hajiabolhassan [3] improved Dol'nikov-Kříž result by proving that for any hypergraph $\mathcal{H}$ and for any integer $r \geqslant 2$, the quantity $\left\lceil \frac{|V(\mathcal{H})| - \mathrm{alt}_r(\mathcal{H})}{r-1} \right\rceil$ is a lower bound for the chromatic number of $\mathrm{KG}^r(\mathcal{H})$. There are some other lower bounds for the chromatic number of graphs, being sometimes better than the former discussed lower bounds, for instance, see [1, 23, 24]. They are based on some topological indices of some topological spaces connected to the structure of graphs. In spite of these lower bounds being better, they are not combinatorial and most of the times they are difficult to compute.

The existence of large colorful bipartite subgraphs in a properly colored graph has been extensively studied in the literature, see [3, 5, 6, 23, 24, 25]. To be more specific, there are several theorems ensuring the existence of a colorful bipartite subgraph in any properly colored graph, whose order is bounded form below by a function of some topological parameters connected to the graph. In this regard, Simonyi and Tardos [25] improved Dol'nikov's lower bound and proved that in any proper coloring of a Kneser graph $\mathrm{KG}^2(\mathcal{H})$, there is a multicolored complete bipartite graph $K_{\left\lceil\frac{\mathrm{cd}_2(\mathcal{H})}{2}\right\rceil,\left\lfloor\frac{\mathrm{cd}_2(\mathcal{H})}{2}\right\rfloor}$ such that the $\mathrm{cd}_2(\mathcal{H})$ different colors occur alternating on the two parts of the bipartite graph with respect to their natural order. By a combinatorial proof, the present author and Hajiabolhassan [3] improved this result. They proved that the result remains true if we replace $\mathrm{cd}_2(\mathcal{H})$ by $|V(\mathcal{H})| - \mathrm{alt}_2(\mathcal{H})$. Also, a stronger result is proved by Simonyi, Tardif, and Zsbán [23].

Theorem 1. (Zig-zag Theorem [23]). *Let G be a nonempty graph, which is properly colored with arbitrary number of colors. Then G contains a multicolored complete bipartite subgraph $K_{\lceil\frac{t}{2}\rceil,\lfloor\frac{t}{2}\rfloor}$, where $\mathrm{Xind}(\mathrm{Hom}(K_2, G)) + 2 = t$. Moreover, colors appear alternating on the two sides of the bipartite subgraph with respect to their natural ordering.*

The quantity $\mathrm{Xind}(\mathrm{Hom}(K_2, G))$ appearing in the statement of previous theorem is the cross-index of the Hom-complex $\mathrm{Hom}(K_2, G)$, which will be defined in Subsection 2.2. We should mention that there are some other weaker similar results in terms of some other topological parameters, see [24, 25].

Note that all the prior mentioned results concern the existence of colorful bipartite subgraphs in properly colored graphs (2-uniform hypergraphs). In 2014, Meunier [20] found the first colorful type result for uniform hypergraphs, see Theorem 15. He proved that for any prime number p, any properly colored Kneser hypergraph $\mathrm{KG}^p(\mathcal{H})$ must contain a colorful balanced complete p-uniform p-partite subhypergraph, whose order is bounded from below by $|V(\mathcal{H})| - \mathrm{alt}_p(\mathcal{H})$, see Theorem 15.

1.2 Main Results

For a given graph G, there are several complexes defined based on the structure of G, for instance, the box-complex of G, denoted $\mathrm{B}_0(G)$, and the Hom-complex of G, denoted $\mathrm{Hom}(K_2, G)$, see [18, 23, 24]. In this paper, we naturally generalize the definitions of box-complex and Hom-complex of graphs to uniform hypergraphs. Also, the definition of $\mathbb{Z}_p$-cross-index of $\mathbb{Z}_p$-posets will be introduced. Using these complexes, as a first main result of this paper, we generalize Meunier's theorem [20] (Theorem 15) to the following theorem.

Theorem 2. *Let r and p be two positive integers, where p is prime and $p \geqslant r \geqslant 2$. Also, let $\mathcal{H}$ be an r-uniform hypergraph and $c : V(\mathcal{H}) \longrightarrow [L]$ be a proper coloring of $\mathcal{H}$ (L arbitrary). Then we have the following assertions.*

(i) *There is a colorful balanced complete r-uniform p-partite subhypergraph in $\mathcal{H}$ with $\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)) + 1$ vertices. In particular,*

$$\chi(\mathcal{H}) \geqslant \frac{\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)) + 1}{r - 1}.$$

(ii) *If $p \leqslant \omega(\mathcal{H})$, then there is a colorful balanced complete r-uniform p-partite subhypergraph in $\mathcal{H}$ with $\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r,\mathcal{H}))+p$ vertices. In particular,*

$$\chi(\mathcal{H}) \geqslant \frac{\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r,\mathcal{H}))+p}{r-1}.$$

The quantities $\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H},\mathbb{Z}_p))$ and $\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r,\mathcal{H}))$ appearing in the statement of Theorem 2 are respectively the $\mathbb{Z}_p$-index and the $\mathbb{Z}_p$-cross-index of the $\mathbb{Z}_p$-box-complex $\mathrm{B}_0(\mathcal{H},\mathbb{Z}_p)$ and the $\mathbb{Z}_p$-Hom-complex $\mathrm{Hom}(K_p^r,\mathcal{H})$, which will be defined in Subsection 2.2. It is worth mentioning that these quantities are defined in a way that if $p=2$, then for any graph G, we have $\mathrm{B}_0(G)=\mathrm{B}_0(G,\mathbb{Z}_2)$, $\mathrm{Hom}_{\mathbb{Z}_2}(K_2,G)=\mathrm{Hom}(K_2,G)$, $\mathrm{ind}(-)=\mathrm{ind}_{\mathbb{Z}_2}(-)$, and $\mathrm{Xind}(-)=\mathrm{Xind}_{\mathbb{Z}_2}(-)$. In other words, they are true generalizations of box-complex, Hom-complex and their indices to the case of hypergraphs. The assumption $p \leqslant \omega(\mathcal{H})$ in the statement of Theorem 2 is required to grantee that the ground set of the $\mathbb{Z}_p$-Hom-complex $\mathrm{Hom}(K_p^r,\mathcal{H})$ is not empty, see Section 2.2.

Let $\mathcal{H}$ be a properly colored r-uniform hypergraph with L colors. Clearly, the existence of a colorful balanced complete p-partite subhypergraph $\mathcal{H}[U_1,\ldots,U_p]$ in $\mathcal{H}$ provides a lower bound on L. To see this, note that any color appears in at most $r-1$ vertices of each edge of $\mathcal{H}$ and consequently, in at most $r-1$ number of U_i's. Clearly, this implies

$$L \geqslant \frac{1}{r-1}\sum_{i=1}^{p}|V_i|.$$

This observation shows that the inequalities appearing in the statement of Theorem 2 are immediate consequences of the existence of the claimed subhypergraphs. Therefore, for the proof of this theorem, we just need to prove the existence of such subhypergraphs.

As we said before, there are several topological lower bounds for the chromatic number of graphs. The following chain of inequalities provides a hierarchy between some of these lower bounds;

$$\begin{array}{rcl}\chi(G) & \geqslant & \mathrm{Xind}(\mathrm{Hom}(K_2,G))+2 \geqslant \mathrm{ind}(\mathrm{B}_0(G))+1 \\ & \geqslant & \mathrm{coind}(\mathrm{B}_0(G))+1 \geqslant |V(\mathcal{F})|-\mathrm{alt}_2(\mathcal{F}) \geqslant \mathrm{cd}_2(\mathcal{F}),\end{array} \tag{1}$$

where $\mathcal{F}$ is any hypergraph such that $\mathrm{KG}^2(\mathcal{F})$ and G are isomorphic, see [1, 3, 23, 24]. It should be mentioned that for any graph G, there are several hypergraphs $\mathcal{F}$ such that $\mathrm{KG}^2(\mathcal{F})$ and G are isomorphic. In the next theorem, we compare the lower bounds for the chromatic number of r-uniform hypergraphs introduced in Theorem 2, providing a hierarchy between these lower bounds. Note that, in view of the discussion right after Theorem 2, next theorem somehow generalizes Chain 1 to the case of hypergraphs.

Theorem 3. *Let r be a positive integer and p be a prime number, where $2 \leqslant r \leqslant p$. For any r-uniform hypergraph $\mathcal{H}$, we have the followings.*
(i) *If $p \leqslant \omega(\mathcal{H})$, then*

$$\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r,\mathcal{H}))+p \geqslant \mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H},\mathbb{Z}_p))+1.$$

(ii) *If* $\mathcal{H} = \mathrm{KG}^r(\mathcal{F})$ *for some hypergraph* $\mathcal{F}$*, then*

$$\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)) + 1 \geqslant |V(\mathcal{F})| - \mathrm{alt}_p(\mathcal{F}) \geqslant \mathrm{cd}_p(\mathcal{F}).$$

In view of Theorem 3, Theorem 2 is a common extension of Theorem 1 and Theorem 15 (Meunier's colorful theorem). Furthermore, for $r = 2$, Theorem 2 implies the next corollary, which is a generalization of Theorem 1.

Corollary 4. *Let* p *be a prime number and* G *be a nonempty graph, which is properly colored with arbitrary number of colors. Then there is a multicolored complete* p*-partite subgraph* $K_{n_1,n_2,\ldots,n_p}$ *of* G *such that*

- $\displaystyle\sum_{i=1}^{p} n_i = \mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(G, \mathbb{Z}_p)) + 1,$
- $|n_i - n_j| \leqslant 1$ *for each* $i, j \in [p]$.

Moreover, if $p \leqslant \omega(G)$*, then* $\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(G, \mathbb{Z}_p))+1$ *can be replaced by* $\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p, G))+p$.

In view of the prior mentioned results, the following question naturally arises.

Question 5. Do Theorem 2 and Theorem 3 remain true for non-prime p?

1.3 Applications to Local Chromatic Number of Uniform Hypergraphs

For a graph G and a vertex $v \in V(G)$, the *closed neighborhood of* v, denoted $N[v]$, is the set $\{v\} \cup \{u : uv \in E(G)\}$. The *local chromatic number of* G, denoted $\chi_l(G)$, is defined in [8] as follows:

$$\chi_l(G) = \min_c \max\{|c(N[v])| : v \in V(G)\},$$

where the minimum is taken over all proper colorings c of G. Theorem 1 gives the following lower bound for the local chromatic number of a nonempty graph G:

$$\chi_l(G) \geqslant \left\lceil \frac{\mathrm{Xind}(\mathrm{Hom}(K_2, G))}{2} \right\rceil + 2. \tag{2}$$

Note that for a Kneser hypergraph $\mathrm{KG}^2(\mathcal{H})$, by using the Simonyi–Tardos colorful result [25] or its extension given by the present author and Hajiabolhassan [3], there are two similar lower bounds for $\chi_l(\mathrm{KG}^2(\mathcal{H}))$, which respectively use $\mathrm{cd}_2(\mathcal{H}) - 2$ and $|V(\mathcal{H})| - \mathrm{alt}_2(\mathcal{H}) - 2$ instead of $\mathrm{Xind}(\mathrm{Hom}(K_2, G))$ in Inequality 2. However, as it is stated in Theorem 3, the lower bound in terms of $\mathrm{Xind}(\mathrm{Hom}(K_2, G)) + 2$ is better than these two last mentioned lower bounds. Using Corollary 4, we have the following lower bound for the local chromatic number of graphs, which is an improvement of Inequality 2.

Corollary 6. *Let G be a nonempty graph and p be a prime number. Then*

$$\chi_l(G) \geqslant t - \left\lfloor \frac{t}{p} \right\rfloor + 1,$$

where $t = \mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(G, \mathbb{Z}_p)) + 1$. Moreover, if $p \leqslant \omega(G)$, then $\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(G, \mathbb{Z}_p)) + 1$ can be replaced with $\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p, G)) + p$.

Note that if we set $p = 2$, then previous theorem implies the Simonyi–Tardos lower bound for the local chromatic number. Moreover, in general, this lower bound might be even better than the Simonyi–Tardos lower bound. To see this, let k be a fixed integer, where $k \geqslant 2$. Consider the Kneser graph $\mathrm{KG}^2(n, k)$ and let $p(n)$ be a prime number such that $p = p(n) \in O(\ln n)$. By Theorem 3, for $n \geqslant pk$, we have

$$\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathrm{KG}^2(n, k), \mathbb{Z}_p)) + 1 \geqslant \mathrm{cd}_p(K_n^k) = n - p(k - 1).$$

Note that the lower bound for $\chi_l(\mathrm{KG}^2(n, k))$ coming form Inequality 2 is

$$1 + \left\lceil \frac{n - 2k + 2}{2} \right\rceil = \frac{n}{2} - o(1), \tag{3}$$

while, in view of Corollary 6, we have

$$\chi_l(\mathrm{KG}^2(n, k)) \geqslant n - p(k - 1) - \left\lfloor \frac{n - p(k - 1)}{p} \right\rfloor + 1 = n - o(n),$$

which is clearly better than the quantity in Equation 3 provided that n is sufficiently large. However, since the induced subgraph by the neighbors of any vertex of $\mathrm{KG}(n, k)$ is isomorphic to $\mathrm{KG}(n - k, k)$, we have $\chi_l(\mathrm{KG}(n, k)) \geqslant n - 3(k - 1)$, which is better than the above-mentioned lower bounds.

Before stating the next result, we remind the reader that for a hypergraph $\mathcal{H}$, the maximum number of vertices of $\mathcal{H}$ containing no edge is called *the independence number of* $\mathcal{H}$, which is denoted by $\alpha(\mathcal{H})$.

Corollary 7. *Let $\mathcal{F}$ be a hypergraph and $\alpha(\mathcal{F})$ be its independence number. Then for any prime number p, we have*

$$\chi_l(\mathrm{KG}^2(\mathcal{F})) \geqslant \left\lceil \frac{(p - 1)|V(\mathcal{F})|}{p} \right\rceil - (p - 1) \cdot \alpha(\mathcal{F}) + 1.$$

Proof. In view of Theorem 3, we have

$$\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathrm{KG}^2(\mathcal{F}), \mathbb{Z}_p)) + 1 \geqslant \mathrm{cd}_p(\mathcal{F}) \geqslant |V(\mathcal{F})| - p \cdot \alpha(\mathcal{F}).$$

Now, Corollary 6 implies the assertion. ∎

Meunier [20] naturally generalized the definition of local chromatic number of graphs to uniform hypergraphs as follows. Let $\mathcal{H}$ be a uniform hypergraph. For a set $X \subseteq V(\mathcal{H})$, the neighborhood of X, denoted $\mathcal{N}(X)$, is defined as follows;

$$\mathcal{N}(X) = \{v \in V(\mathcal{H}) : \exists\, e \in E(\mathcal{H}) \text{ such that } e \setminus X = \{v\}\}.$$

The *closed neighborhood of* X, denoted $\mathcal{N}[X]$, is the set $X \cup \mathcal{N}(X)$. The *local chromatic number of* $\mathcal{H}$ is defined as follows:

$$\chi_l(\mathcal{H}) = \min_c \max\{|c(\mathcal{N}[e \setminus \{v\}])| : e \in E(\mathcal{H}) \text{ and } v \in e\},$$

where the minimum is taken over all proper colorings c of $\mathcal{H}$.

Meunier [20], by using his colorful theorem (Theorem 15), generalized the Simonyi–Tardos Lower bound [25] for the local chromatic number of Kneser graphs to the local chromatic number of Kneser hypergraphs. He proved that for any hypergraph $\mathcal{F}$ and for any prime number p, we have

$$\chi_l(\mathrm{KG}^p(\mathcal{F})) \geqslant \min\left(\left\lceil\frac{|V(\mathcal{F})| - \mathrm{alt}_p(\mathcal{F})}{p}\right\rceil + 1, \left\lceil\frac{|V(\mathcal{F})| - \mathrm{alt}_p(\mathcal{F})}{p-1}\right\rceil\right).$$

In what follows, we generalize this result.

Theorem 8. *Let $\mathcal{H}$ be a nonempty r-uniform hypergraph and p be a prime number, where $2 \leqslant r \leqslant p \leqslant \omega(\mathcal{H})$. Let $t = \mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r, \mathcal{H})) + p$. If $t = ap + b$, where a and b are nonnegative integers and $0 \leqslant b \leqslant p-1$, then*

$$\chi_l(\mathcal{H}) \geqslant \min\left(\left\lceil\frac{(p-r+1)a + \min(p-r+1, b)}{\min(p-r+1, r-1)}\right\rceil + 1, \left\lceil\frac{t}{r-1}\right\rceil\right).$$

Proof. Let c be an arbitrary proper coloring of $\mathcal{H}$ and let $\mathcal{H}[U_1, \ldots, U_p]$ be the balanced colorful complete r-uniform p-partite subhypergraph of $\mathcal{H}$, whose existence is ensured by Part (ii) of Theorem 2. Note that b of the subsets U_i's, say $U_1, \ldots, U_b$, have the cardinality $\lceil\frac{t}{p}\rceil$, while the others have the cardinality $\lfloor\frac{t}{p}\rfloor \geqslant 1$. Consider $U_1, \ldots, U_{p-r+1}$. Since any color appears in at most $r-1$ vertices in $\bigcup\limits_{i=1}^{p} U_i$, we have $\left|\bigcup\limits_{i=1}^{p} c(U_i)\right| \geqslant \lceil\frac{t}{r-1}\rceil$. Two different cases will be distinguished.

1. If $\left|\bigcup\limits_{i=1}^{p-r+1} c(U_i)\right| < \lceil\frac{t}{r-1}\rceil$, then there is a vertex $v \in \bigcup\limits_{i=p-r+2}^{p} U_i$, whose color is not in $\bigcup\limits_{i=1}^{p-r+1} c(U_i)$. Consider an edge e of $\mathcal{H}[U_1, \ldots, U_p]$ containing v and such that $e \cap U_i = \varnothing$ for $i = 1, \ldots, p-r$ and $|e \cap U_i| = 1$ for $i = p-r+1, \ldots, p$. Let $e \cap U_{p-r+1} = \{u\}$. Clearly, since $v \in e \setminus \{u\}$ and $\bigcup\limits_{i=1}^{p-r+1} U_i \subseteq \mathcal{N}(e \setminus \{u\})$, we have

$$\{v\} \cup \left(\bigcup_{i=1}^{p-r+1} U_i\right) \subseteq (e \setminus \{u\}) \cup \mathcal{N}(e \setminus \{u\}) = \mathcal{N}[e \setminus \{u\}].$$

Note that since any color appears in at most $\min(p-r+1, r-1)$ vertices in $\bigcup_{i=1}^{p-r+1} U_i$, we have

$$\left|\bigcup_{i=1}^{p-r+1} c(U_i)\right| \geqslant \left\lceil \frac{\sum_{i=1}^{p-r+1} |U_i|}{\min(p-r+1, r-1)} \right\rceil.$$

On the other hand, since $c(v) \notin \bigcup_{i=1}^{p-r+1} c(U_i)$ and $\sum_{i=1}^{p-r+1} |U_i| = (p-r+1)a + \min(p-r+1, b)$, we have

$$\begin{aligned}
|c(\mathcal{N}[e \setminus \{u\}])| &\geqslant \left|\{c(v)\} \cup \left(\bigcup_{i=1}^{p-r+1} c(U_i)\right)\right| \\
&\geqslant 1 + \left\lceil \frac{\sum_{i=1}^{p-r+1} |U_i|}{\min(p-r+1, r-1)} \right\rceil \\
&= 1 + \left\lceil \frac{(p-r+1)a + \min(p-r+1, b)}{\min(p-r+1, r-1)} \right\rceil,
\end{aligned}$$

which completes the proof in Case 1.

2. If $\left|\bigcup_{i=1}^{p-r+1} c(U_i)\right| \geqslant \lceil \frac{t}{r-1} \rceil$, then consider an edge e of $\mathcal{H}[U_1, \ldots, U_p]$ such that $e \cap U_i = \varnothing$ for $i = 1, \ldots, p-r$ and $|e \cap U_i| = 1$ for $i = p-r+1, \ldots, p$. Let $e \cap U_{p-r+1} = \{u\}$. One can see that
$$\bigcup_{i=1}^{p-r+1} c(U_i) \subseteq c(\mathcal{N}(e \setminus \{u\})),$$
which completes the proof in Case 2. □

Corollary 9. *Let $\mathcal{H}$ be a nonempty p-uniform hypergraph, where p is a prime number. Then*

$$\chi_l(\mathcal{H}) \geqslant \min\left(\left\lceil \frac{\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^p, \mathcal{H}))}{p} \right\rceil + 2, \left\lceil \frac{\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^p, \mathcal{H})) + 1}{p-1} \right\rceil + 1 \right).$$

Proof. Since $\mathcal{H}$ has at least one edge, we have $\omega(\mathcal{H}) \geqslant p$. Therefore, if we set $r = p$ in Theorem 8, then the assertion follows immediately. □

Note that if $\mathcal{H} = \mathrm{KG}^p(\mathcal{F})$ is a nonempty hypergraph, then, in view of Theorem 3, we have
$$\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^p, \mathcal{H})) + p \geqslant |V(\mathcal{F})| - \mathrm{alt}_p(\mathcal{F}).$$
This implies that the previous corollary is an improvement of Meunier's lower bound for the local chromatic number of $\mathrm{KG}^p(\mathcal{F})$

1.4 Plan

Section 2 contains some backgrounds and essential definitions used elsewhere in the paper. In Section 3, we present some new topological tools, which help us for the proofs of main results. Section 4 is devoted to the proofs of Theorem 2 and Theorem 3.

2 Preliminaries

2.1 Some Topological Indices

We assume basic knowledge in combinatorial algebraic topology. Here, we are going to bring a brief review of some essential notations and definitions, which will be needed throughout the paper. For more, one can see the book written by Matoušek [18]. Also, the definitions of box-complex, Hom-complex, and cross-index will be generalized to $\mathbb{Z}_p$-box-complex, $\mathbb{Z}_p$-Hom-complex, and $\mathbb{Z}_p$-cross-index, respectively.

Let $\mathbb{G}$ be a finite nontrivial group acting on a topological space X. We call X a *topological* $\mathbb{G}$*-space* if for each $g \in \mathbb{G}$, the map $g : X \longrightarrow X$ with $x \mapsto g \cdot x$ is continuous. A *free topological* $\mathbb{G}$*-space* X is a topological $\mathbb{G}$-space for which $\mathbb{G}$ acts on it freely, i.e., for each $g \in \mathbb{G} \setminus \{\mathbf{1}\}$, the map $g : X \longrightarrow X$ has no fixed point. Here, by $\mathbf{1}$, we mean the identity element of the group $\mathbb{G}$. For two topological $\mathbb{G}$-spaces X and Y, a map $f : X \longrightarrow Y$ is called a $\mathbb{G}$*-equivariant map*, if $f(g\cdot x) = g\cdot f(x)$ for each $g \in \mathbb{G}$ and $x \in X$. Also, any continuous $\mathbb{G}$-map simply is called a $\mathbb{G}$*-map*. We write $X \xrightarrow{\mathbb{G}} Y$ to mention that there is a $\mathbb{G}$-map from X to Y.

Simplicial complexes provide a bridge between combinatorics and topology. A simplicial complex can be viewed as a combinatorial object, called abstract simplicial complex, or as a topological space, called geometric simplicial complex. Here, we just remind the definition of an abstract simplicial complex. However, we assume that the reader is familiar with the concept of how an abstract simplicial complex and its geometric realization are connected to each other. A *simplicial complex* is a pair (V, K), where V is a finite set and K is a family of subsets of V such that if $F \in K$ and $F' \subseteq F$, then $F' \in K$. Any set in K is called a *simplex of* K. Since we may assume that $V = \bigcup\limits_{F\in K} F$, we can write K instead of (V, K). The *dimension of* K is defined as follows:

$$\dim(K) = \max\{|F| - 1 :\ F \in K\}.$$

For two simplicial complexes C and K, by a *simplicial map* $f : C \longrightarrow K$, we mean a map from $V(C)$ to $V(K)$ such that the image of any simplex of C is a simplex of K. For a nontrivial finite group $\mathbb{G}$, a *simplicial* $\mathbb{G}$*-complex* K is a simplicial complex with a $\mathbb{G}$-action on its vertices such that each $g \in \mathbb{G}$ induces a simplicial map from K to K, that is the map which maps v to $g\cdot v$ for each $v \in V(K)$. If for each $g \in \mathbb{G}\setminus\{\mathbf{1}\}$, there is no fixed simplex under the simplicial map made by g, then K is called a *free simplicial* $\mathbb{G}$*-complex.* A map $f : C \longrightarrow K$ is called $\mathbb{G}$*-equivariant*, if $f(g \cdot v) = g \cdot f(v)$ for each $g \in \mathbb{G}$ and $v \in V(C)$. For two simplicial $\mathbb{G}$-complexes C and K, a *simplicial* $\mathbb{G}$*-map* is a $\mathbb{G}$-equivariant simplicial map form C to K. By Im(f), we mean a simplicial $\mathbb{G}$-subcomplex of K, whose vertex set is $f(V(C))$ and whose simplex set is $\{\tau \in K \,:\, \exists\, \sigma \in C \,\text{ such that }\, f(\sigma) = \tau\}$.

For an integer $n \geqslant 0$ and a nontrivial finite group $\mathbb{G}$, an $E_n\mathbb{G}$ *space* is a free $(n-1)$-connected n-dimensional simplicial $\mathbb{G}$-complex. A concrete example of an $E_n\mathbb{G}$ space is the $(n+1)$-fold join $\mathbb{G}^{*(n+1)}$. As a topological space $\mathbb{G}^{*(n+1)}$ is a $(n+1)$-fold join of an $(n+1)$-point discrete space. This is known that for any two $E_n\mathbb{G}$ space X and Y, there is a $\mathbb{G}$-map from X to Y.

For a $\mathbb{G}$-space X, define

$$\mathrm{ind}_{\mathbb{G}}(X) = \min\{n :\; X \xrightarrow{\mathbb{G}} E_n\mathbb{G}\}.$$

Note that here $E_n\mathbb{G}$ can be any $E_n\mathbb{G}$ space since there is a $\mathbb{G}$-map between any two $E_n\mathbb{G}$ spaces, see [18]. Also, for a simplicial complex K, by $\mathrm{ind}_{\mathbb{G}}(K)$, we mean $\mathrm{ind}_{\mathbb{G}}(||K||)$, where $||K||$ denotes the geometric realization of K. One must note that any simplicial $\mathbb{G}$-map from C to K induces a $\mathbb{G}$-map from $||C||$ to $||K||$. Throughout the paper, for $\mathbb{G} = \mathbb{Z}_2$, we would rather use $\mathrm{ind}(-)$ instead of $\mathrm{ind}_{\mathbb{Z}_2}(-)$. In the following, we remind some of the properties of the $\mathbb{G}$-index, which will be used throughout the paper.

Properties of the $\mathbb{G}$-index. [18] Let $\mathbb{G}$ be a finite nontrivial group.

(i) $\mathrm{ind}_{\mathbb{G}}(X) > \mathrm{ind}_{\mathbb{G}}(Y)$ implies $X \overset{\mathbb{G}}{\not\to} Y$.

(ii) $\mathrm{ind}_{\mathbb{G}}(E_n\mathbb{G}) = n$ for any $E_n\mathbb{G}$ space.

(iii) $\mathrm{ind}_{\mathbb{G}}(X * Y) \leqslant \mathrm{ind}_{\mathbb{G}}(X) + \mathrm{ind}_{\mathbb{G}}(Y) + 1$.

(iv) If K is a free simplicial $\mathbb{G}$-complex of dimension n, then $\mathrm{ind}_{\mathbb{G}}(K) \leqslant n$.

Note that by the second property, since for each n, the simplicial $\mathbb{G}$-complex $\mathbb{G}^{*n}$ is an $E_{n-1}\mathbb{G}$ space, we have $\mathrm{ind}_{\mathbb{G}}(\mathbb{G}^{*n}) = n-1$. We will use this fact throughout the paper without any further discussion.

2.2 $\mathbb{Z}_p$-Box-Complex, $\mathbb{Z}_p$-Poset, and $\mathbb{Z}_p$-Hom-Complex

In this subsection, for any r-uniform hypergraph $\mathcal{H}$, we shall define two objects namely $\mathbb{Z}_p$-box-complex of $\mathcal{H}$ and $\mathbb{Z}_p$-Hom-complex of $\mathcal{H}$, which the first one is a simplicial $\mathbb{Z}_p$-complex and the second one is a $\mathbb{Z}_p$-poset. Moreover, for any $\mathbb{Z}_p$-poset P, we assign a combinatorial index to P called the $\mathbb{Z}_p$-cross-index of P. However, as one can see in [4, 13, 14], plenty of simplicial complexes have been associated to graphs and hypergraphs, used for studying the coloring properties of graphs and hypergraphs. Before defining $\mathbb{Z}_p$-box-complex and $\mathbb{Z}_p$-Hom-complex, let us review some of these simplicial complexes and briefly describe their relations with our definitions of $\mathbb{Z}_p$-box-complex and $\mathbb{Z}_p$-Hom-complex in this paper.

For any r-uniform hypergraph $\mathcal{H}$, Alon, Frankl, and Lovász [4] defined a kind of $\mathbb{Z}_p$-box-complex, denoted $C(\mathcal{H})$, and by using the connectivity of this simplicial complex, they presented a lower bound for the chromatic number of $\mathcal{H}$ provided that r is odd.

They used their lower bound for finding the chromatic number of Kneser hypergraphs $\mathrm{KG}^r(n,k)$, solving a conjecture posed by Erdős [9]. Inspired by the works of Lovász [17] and Alon, Frankl, and Lovász [4], Kříž [16] introduced a $\mathbb{Z}_r$-poset called the resolution of $\mathcal{H}$. He used this $\mathbb{Z}_r$-poset for introducing another version of box complexes, which is called "resolution complex". Using this box complex, he presented some lower bounds for the chromatic number of r-uniform hypergraphs. Although, the definition of this $\mathbb{Z}_r$-poset is the same as our definition of $\mathbb{Z}_r$-Hom-complex of $\mathcal{H}$, we will use this $\mathbb{Z}_r$-poset for a completely different purpose from what Kříž did in his paper.

Moreover, for a graph G, there is another famous simplicial complex $\mathrm{Hom}(K_2, G)$, which is introduced by Lovász [17]. Iriye and Kishimoto [13] extended Lovász's definition to r-uniform hypergraphs. For any r-uniform hypergraph $\mathcal{H}$, they introduced a simplicial complex $\mathrm{Hom}(K_r^{(r)}, \mathcal{H})$, and used it for providing some lower bounds for the chromatic number of r-uniform hypergraphs provided that r is prime. One must note that Lovász's definition and its extension by Iriye and Kishimoto are completely different from the definition of $\mathbb{Z}_r$-Hom-complex $\mathrm{Hom}(K_r^r, \mathcal{H})$ in this paper. Also, Thansri [26] compared the homotopy type of the simplicial complex $\mathrm{Hom}(K_r^{(r)}, \mathcal{H})$ defined by Iriye and Kishimoto and the box-complex $C(\mathcal{H})$ defined by Alon, Frankl, and Lovász [4].

$\mathbb{Z}_p$-Box-Complex. Let r be a positive integer and p be a prime number, where $2 \leqslant r \leqslant p$. For an r-uniform hypergraph $\mathcal{H}$, define the *$\mathbb{Z}_p$-box-complex of $\mathcal{H}$,* denoted $\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)$, to be a simplicial complex with the vertex set $\biguplus_{i=1}^{p} V(\mathcal{H}) = \mathbb{Z}_p \times V(\mathcal{H})$ and the simplex set consisting of all $\{\omega^1\} \times U_1 \cup \cdots \cup \{\omega^p\} \times U_p$, where

- $U_1, \ldots, U_p$ are pairwise disjoint subsets of $V(\mathcal{H})$ and
- the hypergraph $\mathcal{H}[U_1, U_2, \ldots, U_p]$ is a complete r-uniform p-partite hypergraph.

Note that some of the sets U_i's might be empty. In fact, if $U_1, \ldots, U_p$ are pairwise disjoint subsets of $V(\mathcal{H})$ and the number of nonempty U_i's is less than r, then $\mathcal{H}[U_1, U_2, \ldots, U_p]$ is a complete r-uniform p-partite hypergraph and thus $\{\omega^1\} \times U_1 \cup \cdots \cup \{\omega^p\} \times U_p \in \mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)$. For each $\varepsilon \in \mathbb{Z}_p$ and each $(\varepsilon', v) \in V(\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p))$, define $\varepsilon \cdot (\varepsilon', v) = (\varepsilon \cdot \varepsilon', v)$. One can see that this action makes $\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)$ a free simplicial $\mathbb{Z}_p$-complex. Whenever $\mathcal{H} = G$ is a graph (2-uniform hypergraph), the $\mathbb{Z}_2$-box-complex $\mathrm{B}_0(G, \mathbb{Z}_2)$ is extensively studied in the literature, where it is known as the box complex of G, denoted $\mathrm{B}_0(G)$, for instance, see [24, 25]. This simplicial complex is used for introducing some lower bounds for the chromatic number of graphs, see [24].

$\mathbb{Z}_p$-Poset. A partially ordered set, or simply a *poset*, is defined as an ordered pair $P = (V(P), \preceq)$, where $V(P)$ is a nonempty set called the ground set of P and $\preceq$ is a partial order on $V(P)$. For two posets P and Q, by an order-preserving map $\phi : P \longrightarrow Q$, we mean a map ϕ from $V(P)$ to $V(Q)$ such that for each $u, v \in V(P)$, if $u \preceq v$, then $\phi(u) \preceq \phi(v)$. A poset P is called a *$\mathbb{Z}_p$-poset*, if $\mathbb{Z}_p$ acts on $V(P)$ and furthermore, for each $\varepsilon \in \mathbb{Z}_p$, the map $\varepsilon : V(P) \longrightarrow V(P)$ with $v \mapsto \varepsilon \cdot v$ is an automorphism of P (order-preserving bijective map). If for each $\varepsilon \in \mathbb{Z}_p \setminus \{\mathbf{1}\}$, this map has no fixed point, then P

is called a *free $\mathbb{Z}_p$-poset*. Here, by 1, we mean the identity element of $\mathbb{Z}_p$, i.e., $1 = \omega^0$. For two $\mathbb{Z}_p$-posets P and Q, by an order-preserving $\mathbb{Z}_p$-map $\phi : P \longrightarrow Q$, we mean an order-preserving map from $V(P)$ to $V(Q)$ such that for each $v \in V(P)$ and $\varepsilon \in \mathbb{Z}_p$, we have $\phi(\varepsilon \cdot v) = \varepsilon \cdot \phi(v)$.

For a nonnegative integer n and a prime number p, let $Q_{n,p}$ be a free $\mathbb{Z}_p$-poset with the ground set $\mathbb{Z}_p \times [n+1]$ such that for each two elements $(\varepsilon, i), (\varepsilon', j) \in Q_{n,p}$, we have $(\varepsilon, i) <_{Q_{n,p}} (\varepsilon', j)$ whenever $i < j$. Clearly, $Q_{n,p}$ is a free $\mathbb{Z}_p$-poset with the action $\varepsilon \cdot (\varepsilon', j) = (\varepsilon \cdot \varepsilon', j)$ for each $\varepsilon \in \mathbb{Z}_p$ and $(\varepsilon', j) \in Q_{n,p}$. For a $\mathbb{Z}_p$-poset P, the *$\mathbb{Z}_p$-cross-index of P*, denoted $\mathrm{Xind}_{\mathbb{Z}_p}(P)$, is the least integer n such that there is a $\mathbb{Z}_p$-map from P to $Q_{n,p}$. Throughout the paper, for $p = 2$, we speak about $\mathrm{Xind}(-)$ rather than $\mathrm{Xind}_{\mathbb{Z}_2}(-)$. It should be mentioned that $\mathrm{Xind}(-)$ is first defined in [23].

Let P be a poset. We can define an *order-complex* ΔP with the vertex set same as the ground set of P and simplex set consisting of all chains in P. One can see that if P is a free $\mathbb{Z}_p$-poset, then ΔP is a free simplicial $\mathbb{Z}_p$-complex. Moreover, any order-preserving $\mathbb{Z}_p$-map $\phi : P \longrightarrow Q$ can be lifted to a simplicial $\mathbb{Z}_p$-map from ΔP to ΔQ. Clearly, there is a simplicial $\mathbb{Z}_p$-map from $\Delta Q_{n,p}$ to $\mathbb{Z}_p^{*(n+1)}$ (identity map). Therefore, if $\mathrm{Xind}_{\mathbb{Z}_p}(P) = n$, then we have a simplicial $\mathbb{Z}_p$-map from ΔP to $\mathbb{Z}_p^{*(n+1)}$. This implies that $\mathrm{Xind}_{\mathbb{Z}_p}(P) \geqslant \mathrm{ind}_{\mathbb{Z}_p}(\Delta P)$.

Theorem 10. [2] *Let P be a free $\mathbb{Z}_2$-poset and $\phi : P \longrightarrow Q_{s,2}$ be an order-preserving $\mathbb{Z}_2$-map. Then P contains a chain $p_1 \prec_P \cdots \prec_P p_k$ such that $k = \mathrm{Xind}(P) + 1$ and the signs of $\phi(p_i)$ and $\phi(p_{i+1})$ differ for each $i \in [k-1]$. Moreover, if $s = \mathrm{Xind}(P)$, then for any $(s+1)$-tuple $(\varepsilon_1, \ldots, \varepsilon_{s+1}) \in \mathbb{Z}_2^{s+1}$, there is at least one chain $p_1 \prec_P \cdots \prec_P p_{s+1}$ such that $\phi(p_i) = (\varepsilon_i, i)$ for each $i \in [s+1]$.*

$\mathbb{Z}_p$-Hom-Complex. Let $\mathcal{H}$ be an r-uniform hypergraph. Also, let p be a prime number such that $2 \leqslant r \leqslant p \leqslant \omega(\mathcal{H})$. The *$\mathbb{Z}_p$-Hom-complex* $\mathrm{Hom}(K_p^r, \mathcal{H})$ is a free $\mathbb{Z}_p$-poset with the ground set consisting of all ordered p-tuples $(U_1, \ldots, U_p)$, where U_i's are nonempty pairwise disjoint subsets of V and $\mathcal{H}[U_1, \ldots, U_p]$ is a complete r-uniform p-partite hypergraph. For two p-tuples $(U_1, \ldots, U_p)$ and $(U_1', \ldots, U_p')$ in $\mathrm{Hom}(K_p^r, \mathcal{H})$, we define $(U_1, \ldots, U_p) \preceq (U_1', \ldots, U_p')$ if $U_i \subseteq U_i'$ for each $i \in [p]$. Also, for each $\omega^i \in \mathbb{Z}_p = \{\omega^1, \ldots, \omega^p\}$, let $\omega^i \cdot (U_1, \ldots, U_p) = (U_{1+i}, \ldots, U_{p+i})$, where $U_j = U_{j-p}$ for $j > p$. Clearly, this action is a free $\mathbb{Z}_p$-action on $\mathrm{Hom}(K_p^r, \mathcal{H})$. Consequently, $\mathrm{Hom}(K_p^r, \mathcal{H})$ is a free $\mathbb{Z}_p$-poset with this $\mathbb{Z}_p$-action. Note that since $p \leqslant \omega(\mathcal{H})$, the ground set of $\mathrm{Hom}(K_p^r, \mathcal{H})$ is not empty.

For a nonempty graph G and $p = 2$, we would rather use $\mathrm{Hom}(K_2, G)$ instead of $\mathrm{Hom}(K_2^2, G)$. Also, it should be mentioned that $\mathrm{Hom}(K_2, G)$ is first defined in [23], known as the Hom-complex of G.

3 Notations and Tools

For a simplicial complex K, by $\mathrm{sd}\, K$, we mean *the first barycentric subdivision of K*. It is the order-complex obtained from the poset consisting of all nonempty simplices in K

ordered by inclusion. Throughout the paper, by σ^{r-1}_{t-1}, we mean the $(t-1)$-dimensional simplicial complex with vertex set $\mathbb{Z}_r$, containing all t-subsets of $\mathbb{Z}_r$ as its maximal simplices. The join of two simplicial complexes C and K, denoted $C * K$, is a simplicial complex with the vertex set $V(C) \uplus V(K)$ and such that the set of its simplices is $\{F_1 \biguplus F_2 : \ F_1 \in C \text{ and } F_2 \in K\}$. Clearly, we can see $\mathbb{Z}_r$ as a 0-dimensional simplicial complex. Note that the vertex set of simplicial complex $\mathrm{sd}\, \mathbb{Z}_r^{*\alpha}$ can be identified with $(\mathbb{Z}_r \cup \{0\})^\alpha \setminus \{\mathbf{0}\}$ and the vertex set of $(\sigma^{r-1}_{t-1})^{*n}$ is the set of all pairs (ε, i), where $\varepsilon \in \mathbb{Z}_r$ and $i \in [n]$. Also, for each simplex $\tau \in (\sigma^{p-1}_{p-2})^{*m}$ and for each $\varepsilon \in \mathbb{Z}_p$, define $\tau^\varepsilon = \{(\varepsilon, j) : \ (\varepsilon, j) \in \tau\}$.

The famous Borsuk–Ulam theorem has many interesting generalizations, which have been used extensively for investigating graphs coloring properties. For examples, Tucker's lemma [27], the Z_p-Tucker Lemma [28], and Ky Fan's lemma [10] are some of these interesting generalizations. For more details about the Borsuk–Ulam theorem and its generalizations, we refer the reader to [18].

Indeed, Tucker's lemma is a combinatorial counterpart of the Borsuk–Ulam theorem. There are several interesting and surprising applications of Tucker's lemma in combinatorics, including a combinatorial proof of the Lovász–Kneser theorem by Matoušek [19].

Lemma 11. (Tucker's lemma [27]). *Let m and n be positive integers and $\lambda : \{-1, 0, +1\}^n \setminus \{\mathbf{0}\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm m\}$ be a map satisfying the following properties:*

- *for any $X \in \{-1, 0, +1\}^n \setminus \{\mathbf{0}\}$, we have $\lambda(-X) = -\lambda(X)$ (a Z_2-equivariant map),*
- *no two signed vectors X and Y are such that $X \subseteq Y$ and $\lambda(X) = -\lambda(Y)$.*

Then, we have $m \geqslant n$.

Another interesting generalization of the Borsuk–Ulam Theorem is the well-known Ky Fan's lemma [10]. This generalization ensures that with the same hypotheses as in Lemma 11, there are odd number of chains $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ such that

$$\{\lambda(X_1), \ldots, \lambda(X_n)\} = \{+c_1, -c_2, \ldots, (-1)^{n-1} c_n\},$$

where $1 \leqslant c_1 < \cdots < c_n \leqslant m$. Ky Fan's lemma has been used in several articles to study some coloring properties of graphs, see [2, 6, 11]. There are also some other generalizations of Tucker's lemma. A $\mathbb{Z}_p$ version of Tucker's lemma, called the $\mathbb{Z}_p$-Tucker Lemma, is proved by Ziegler [28] and extended by Meunier [22]. In the next subsection, we present a $\mathbb{Z}_p$ version of Ky Fan's lemma, which is called the $\mathbb{Z}_p$-Tucker-Ky Fan Lemma.

3.1 New Generalizations of Tucker's Lemma

This subsection is devoted to introduce some new topological tools, which will be used elsewhere in the paper. Note that if we set $\mathbb{Z}_2 = \{-1, +1\}$, then any map λ satisfying the conditions of Tucker's lemma (Lemma 11) can be considered as a $\mathbb{Z}_2$-simplicial map from $\mathrm{sd}\, \mathbb{Z}_2^{*n}$ to $\mathbb{Z}_2^{*m}$. In this point of view, Ky Fan's lemma says that for any such a map λ, there is at least one $(n-1)$-dimensional simplex $\sigma \in (\mathrm{sd}\, \mathbb{Z}_2^{*n})$ such that $\lambda(\sigma) =$

$\{+c_1, -c_2, \ldots, (-1)^{n-1}c_n\}$, where $1 \leqslant c_1 < \cdots < c_n \leqslant m$. Note that $\mathbb{Z}_2^{*m}$ is a sub-complex of $(\sigma_{p-2}^{p-1})^{*m}$ and $\operatorname{ind}_{\mathbb{Z}_2}(\operatorname{sd}\mathbb{Z}_2^{*n}) = n-1$. Therefore, the next lemma can be considered as a $\mathbb{Z}_p$-generalization of Ky Fan's lemma.

Lemma 12. *Let C be a free simplicial $\mathbb{Z}_p$-complex such that $\operatorname{ind}_{\mathbb{Z}_p}(C) \geqslant t$ and let $\lambda : C \longrightarrow (\sigma_{p-2}^{p-1})^{*m}$ be a simplicial $\mathbb{Z}_p$-map. Then there is at least one t-dimensional simplex $\sigma \in C$ such that $\tau = \lambda(\sigma)$ is a t-dimensional simplex and for each $\varepsilon \in \mathbb{Z}_p$, we have $\lfloor \frac{t+1}{p} \rfloor \leqslant |\tau^\varepsilon| \leqslant \lceil \frac{t+1}{p} \rceil$.*

Before starting the proof of Lemma 12, we need to introduce three functions. These functions are crucial for the rest of the paper and we will use them throughout the paper in several times. Let m be a positive integer and p be a prime number. We have already noted that $(\sigma_{p-2}^{p-1})^{*m}$ is a free simplicial $\mathbb{Z}_p$-complex with the vertex set $\mathbb{Z}_p \times [m]$.

The value function $l(-)$. Let $\tau \in (\sigma_{p-2}^{p-1})^{*m}$ be a simplex. For each $\varepsilon \in \mathbb{Z}_p$, we remind the reader that $\tau^\varepsilon = \{(\varepsilon, j) : (\varepsilon, j) \in \tau\}$. Now, define

$$l(\tau) = \max\left\{ |\bigcup_{\varepsilon\in\mathbb{Z}_p} B^\varepsilon| : \forall\varepsilon \in \mathbb{Z}_p,\ B^\varepsilon \subseteq \tau^\varepsilon \text{ and } \forall\varepsilon_1,\varepsilon_2 \in \mathbb{Z}_p,\ |\,|B^{\varepsilon_1}| - |B^{\varepsilon_2}|\,| \leqslant 1 \right\}.$$

Note that if we set $h(\tau) = \min\limits_{\varepsilon\in\mathbb{Z}_p} |\tau^\varepsilon|$, then

$$l(\tau) = p \cdot h(\tau) + |\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| > h(\tau)\}|.$$

It is clear that the function $l(-)$ is monotone, i.e., if $\tau_1 \subseteq \tau_2$, then $l(\tau_1) \leqslant l(\tau_2)$. Also, the following remark can be readily obtained from the definition of the function $l(-)$.

Remark 13. If there is a simplex $\tau_1 \in (\sigma_{p-2}^{p-1})^{*m}$ such that $l(\tau_1) \geqslant l$, then there is a simplex $\tau \subseteq \tau_1$ with $|\tau| = l(\tau) = l$ and such that for each $\varepsilon \in \mathbb{Z}_p$, we have $\lfloor \frac{l}{p} \rfloor \leqslant |\tau^\varepsilon| \leqslant \lceil \frac{l}{p} \rceil$.

The sign functions $s(-)$ and $s_0(-)$. For an $a \in [m]$, let W_a be the set of all nonempty simplices $\tau \in (\sigma_{p-2}^{p-1})^{*m}$ such that $|\tau^\varepsilon| \in \{0, a\}$ for each $\varepsilon \in \mathbb{Z}_p$. Let $W = \bigcup\limits_{a=1}^{m} W_a$. Choose an arbitrary $\mathbb{Z}_p$-equivariant map $s : W \longrightarrow \mathbb{Z}_p$. Also, consider a $\mathbb{Z}_p$-equivariant map $s_0 : \sigma_{p-2}^{p-1} \longrightarrow \mathbb{Z}_p$. Note that since $\mathbb{Z}_p$ acts freely on both W and σ_{p-2}^{p-1}, these maps can be easily built by choosing one representative in each orbit. We should emphasize that both functions $s(-)$ and $s_0(-)$ are first introduced in [20].

Proof of Lemma 12. For simplicity of notation, let $K = \operatorname{Im}(\lambda)$. In view of Remark 13, to prove the assertion, it is enough to show that there is a t-dimensional simplex $\tau \in K$ such that $l(\tau) \geqslant t+1$. For a contradiction, suppose that there is no such a t-dimensional simplex. Therefore, for each simplex τ of K, we have $l(\tau) \leqslant t$.

Let $\Gamma : \operatorname{sd} K \longrightarrow \mathbb{Z}_p^{*t}$ be a map which will be defined as follows. Let τ be a vertex of $\operatorname{sd} K$. Consider two following cases depending on the value of $h(\tau) = \min\limits_{\varepsilon\in\mathbb{Z}_p} |\tau^\varepsilon|$.

(i) If $h(\tau)=0$, then define $\bar{\tau}=\{\varepsilon\in\mathbb{Z}_p:\ \tau^\varepsilon=\varnothing\}\in\sigma_{p-2}^{p-1}$ and

$$\Gamma(\tau)=(s_0(\bar{\tau}),l(\tau))\,.$$

(ii) If $h(\tau)>0$, then define $\bar{\tau}=\bigcup\limits_{\{\varepsilon\in\mathbb{Z}_p:\ |\tau^\varepsilon|=h(\tau)\}}\tau^\varepsilon\in W$ and

$$\Gamma(\tau)=(s(\bar{\tau}),l(\tau))\,.$$

First, we show that Γ is a simplicial $\mathbb{Z}_p$-map from $\operatorname{sd}K$ to $\mathbb{Z}_p^{*t}$. Since both functions $s(-)$ and $s_0(-)$ are $\mathbb{Z}_p$-equivariant, it is clear that Γ is a $\mathbb{Z}_p$-equivariant map. For a contradiction, suppose that Γ is not a simplicial map. Therefore, there are $\tau,\tau'\in\operatorname{sd}K$ such that $\tau\subsetneq\tau'$, $\Gamma(\tau)=(\varepsilon_1,\beta)$, and $\Gamma(\tau')=(\varepsilon_2,\beta)$, where $\varepsilon_1\neq\varepsilon_2$. Clearly, in view of the definition of Γ, we have $l(\tau)=l(\tau')=\beta$. Now, we consider three different cases.

(i) If $h(\tau)=h(\tau')=0$, then since $\tau\subsetneq\tau'$ and

$$\varepsilon_1=s_0(\{\varepsilon\in\mathbb{Z}_p:\ \tau^\varepsilon=\varnothing\})\neq s_0(\{\varepsilon\in\mathbb{Z}_p:\ {\tau'}^\varepsilon=\varnothing\})=\varepsilon_2,$$

we have $\{\varepsilon\in\mathbb{Z}_p:\ {\tau'}^\varepsilon=\varnothing\}\subsetneq\{\varepsilon\in\mathbb{Z}_p:\ \tau^\varepsilon=\varnothing\}$. This implies that

$$l(\tau')=p-|\{\varepsilon\in\mathbb{Z}_p:\ {\tau'}^\varepsilon=\varnothing\}|>p-|\{\varepsilon\in\mathbb{Z}_p:\ \tau^\varepsilon=\varnothing\}|=l(\tau),$$

a contradiction.

(ii) If $h(\tau)=0$ and $h(\tau')>0$, then $l(\tau)\leqslant p-1$ and $l(\tau')\geqslant p$, contradicting $l(\tau)=l(\tau')$.

(iii) If $h(\tau)>0$ and $h(\tau')>0$, then $l(\tau)=p\cdot h(\tau)+|\{\varepsilon\in\mathbb{Z}_p:|\tau^\varepsilon|>h(\tau)\}|$ and $l(\tau')=p\cdot h(\tau')+|\{\varepsilon\in\mathbb{Z}_p:|{\tau'}^\varepsilon|>h(\tau')\}|$. For this case, two different sub-cases will be distinguished.

(a) $h(\tau)=h(\tau')=h$. Since

$$\varepsilon_1=s(\bigcup_{\{\varepsilon\in\mathbb{Z}_p:\ |\tau^\varepsilon|=h\}}\tau^\varepsilon)\neq s(\bigcup_{\{\varepsilon\in\mathbb{Z}_p:\ |{\tau'}^\varepsilon|=h\}}{\tau'}^\varepsilon)=\varepsilon_2,$$

we must have

$$\bigcup_{\{\varepsilon\in\mathbb{Z}_p:\ |\tau^\varepsilon|=h\}}\tau^\varepsilon\neq\bigcup_{\{\varepsilon\in\mathbb{Z}_p:\ |{\tau'}^\varepsilon|=h\}}{\tau'}^\varepsilon.$$

Note that $\tau\subseteq\tau'$ and $\min\limits_{\varepsilon\in\mathbb{Z}_p}|\tau^\varepsilon|=\min\limits_{\varepsilon\in\mathbb{Z}_p}|{\tau'}^\varepsilon|$. Therefore, we should have

$$\{\varepsilon\in\mathbb{Z}_p:\ |{\tau'}^\varepsilon|=h\}\subsetneq\{\varepsilon\in\mathbb{Z}_p:\ |\tau^\varepsilon|=h\}$$

and consequently $l(\tau)<l(\tau')$, which is not possible.

(b) $h(\tau) < h(\tau')$. Then, one can see that

$$l(\tau) \leqslant p \cdot h(\tau) + p - 1 < p \cdot (h(\tau) + 1) \leqslant l(\tau'),$$

which is a contradiction.

Therefore, Γ is a simplicial $\mathbb{Z}_p$-map from $\operatorname{sd} K$ to $\mathbb{Z}_p^{*t}$. On the other hand, λ can naturally be lifted to a simplicial $\mathbb{Z}_p$-map $\bar{\lambda} : \operatorname{sd} C \longrightarrow \operatorname{sd} K$. Thus $\Gamma \circ \bar{\lambda}$ is a simplicial $\mathbb{Z}_p$-map from $\operatorname{sd} C$ to $\mathbb{Z}_p^{*t}$. In view of Property (i) in Properties of the $\mathbb{G}$-index, it implies that $\operatorname{ind}_{\mathbb{Z}_p}(C) = \operatorname{ind}_{\mathbb{Z}_p}(\operatorname{sd} C) \leqslant t - 1$, which is not possible. □

The $\mathbb{Z}_p$-Tucker lemma [22, 28] is a famous generalization of Tucker's lemma, having many applications in Kneser hypergraph coloring, for instance see [2, 3, 5, 6, 11, 19]. Although Lemma 12 can be considered as a $\mathbb{Z}_p$ version of Ky Fan's lemma, it is not stated in the simple form as Ky Fan's lemma did, which makes Lemma 12 difficult to use for non-familiars with algebraic topology. In the next result, we present a generalization of the $\mathbb{Z}_p$-Tucker lemma, called $\mathbb{Z}_p$-Tucker-Ky Fan lemma, in a form of combinatorial language. As an application of this result, we give a simple proof of Meunier's theorem (Theorem 15). Even though, the only contribution of the $\mathbb{Z}_p$-Tucker-Ky Fan lemma in this paper is to simplify the original proof of Meunier's theorem, this lemma is interesting in its own right since it simultaneously generalizes Tucker's lemma, Ky Fan's lemma, and the $\mathbb{Z}_p$-Tucker Lemma.

Lemma 14. ($\mathbb{Z}_p$-Tucker-Ky Fan lemma). *Let m, n, p and α be nonnegative integers, where $m, n \geqslant 1$, $m \geqslant \alpha \geqslant 0$, and p is prime. Let*

$$\begin{array}{rrcl} \lambda : & (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\} & \longrightarrow & \mathbb{Z}_p \times [m] \\ & X & \longmapsto & (\lambda_1(X), \lambda_2(X)) \end{array}$$

be a $\mathbb{Z}_p$-equivariant map satisfying the following conditions.

- *For $X_1 \subseteq X_2 \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, if $\lambda_2(X_1) = \lambda_2(X_2) \leqslant \alpha$, then $\lambda_1(X_1) = \lambda_1(X_2)$.*
- *For $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_p \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, if $\lambda_2(X_1) = \lambda_2(X_2) = \cdots = \lambda_2(X_p) \geqslant \alpha + 1$, then*
$$|\{\lambda_1(X_1), \lambda_1(X_2), \ldots, \lambda_1(X_p)\}| < p.$$

Then there is a chain

$$Z_1 \subset Z_2 \subset \cdots \subset Z_{n-\alpha} \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$$

such that

1. *for each $i \in [n - \alpha]$, $\lambda_2(Z_i) \geqslant \alpha + 1$,*
2. *for each $i \neq j \in [n - \alpha]$, $\lambda(Z_i) \neq \lambda(Z_j)$, and*

3. for each $\varepsilon \in \mathbb{Z}_p$,

$$\left\lfloor \frac{n-\alpha}{p} \right\rfloor \leqslant |\{j :\ \lambda_1(Z_j) = \varepsilon\}| \leqslant \left\lceil \frac{n-\alpha}{p} \right\rceil.$$

In particular, $n-\alpha \leqslant (p-1)(m-\alpha)$.

Proof. Clearly, the map λ can be considered as a simplicial $\mathbb{Z}_p$-map from $\operatorname{sd}\mathbb{Z}_p^{*n}$ to $(\mathbb{Z}_p^{*\alpha}) * ((\sigma_{p-2}^{p-1})^{*(m-\alpha)})$. Let $K = \operatorname{Im}(\lambda)$. Note that each simplex in K can be represented in a unique form $\sigma \cup \tau$ such that $\sigma \in \mathbb{Z}_p^{*\alpha}$ and $\tau \in (\sigma_{p-2}^{p-1})^{*m-\alpha}$. In view of Remark 13, to prove the assertion, it suffices to show that there is a simplex $\sigma \cup \tau \in K$ such that $l(\tau) \geqslant n-\alpha$. For a contradiction, suppose that for each $\sigma \cup \tau \in K$, we have $l(\tau) \leqslant n-\alpha-1$.

Define the map $\Gamma : \operatorname{sd} K \longrightarrow \mathbb{Z}_p^{*(n-1)}$ such that for each vertex $\sigma \cup \tau \in V(\operatorname{sd} K)$, $\Gamma(\sigma \cup \tau)$ is defined as follows.

- If $\tau = \varnothing$, then $\Gamma(\sigma \cup \tau) = (\varepsilon, j)$, where j is the maximum possible value for which $(\varepsilon, j) \in \sigma$. Note that since $\sigma \in \mathbb{Z}_p^{*\alpha}$, there is only one $\varepsilon \in \mathbb{Z}_p$ for which the maximum is attained. Therefore, in this case, the function Γ is well-defined.

- If $\tau \neq \varnothing$. Define $h(\tau) = \min\limits_{\varepsilon \in \mathbb{Z}_p} |\tau^\varepsilon|$.

  (i) If $h(\tau) = 0$, then define $\bar{\tau} = \{\varepsilon \in \mathbb{Z}_p :\ \tau^\varepsilon = \varnothing\} \in \sigma_{p-2}^{p-1}$ and
  $$\Gamma(\sigma \cup \tau) = (s_0(\bar{\tau}), \alpha + l(\tau)).$$

  (ii) If $h(\tau) > 0$, then define $\bar{\tau} = \bigcup\limits_{\{\varepsilon \in \mathbb{Z}_p :\ |\tau^\varepsilon| = h(\tau)\}} \tau^\varepsilon \in W$ and
  $$\Gamma(\sigma \cup \tau) = (s(\bar{\tau}), \alpha + l(\tau)).$$

We first show that Γ is a simplicial $\mathbb{Z}_p$-map from $\operatorname{sd} K$ to $\mathbb{Z}_p^{*(n-1)}$. It is clear that Γ is a $\mathbb{Z}_p$-equivariant map. For a contradiction, suppose that there are $\sigma \cup \tau, \sigma' \cup \tau' \in \operatorname{sd} K$ such that $\sigma \subseteq \sigma'$, $\tau \subseteq \tau'$, $\Gamma(\sigma \cup \tau) = (\varepsilon_1, \beta)$, and $\Gamma(\sigma' \cup \tau') = (\varepsilon_2, \beta)$, where $\varepsilon_1 \neq \varepsilon_2$. First note that in view of the definition of Γ and the assumption $\Gamma(\sigma \cup \tau) = (\varepsilon_1, \beta)$ and $\Gamma(\sigma' \cup \tau') = (\varepsilon_2, \beta)$, the case $\tau = \varnothing$ and $\tau' \neq \varnothing$ is not possible. If $\tau' = \varnothing$, then $\tau = \tau' = \varnothing$ and we should have $(\varepsilon_1, \beta), (\varepsilon_2, \beta) \in \sigma' \in \mathbb{Z}_p^{*\alpha}$, which implies that $\varepsilon_1 = \varepsilon_2$, a contradiction. If $\varnothing \neq \tau \subseteq \tau'$, then in view of definition of Γ, we should have $l(\tau) = l(\tau') = \beta - \alpha$. Now, similar to the proof of Lemma 12, we can consider three different cases, each of them resulting in a contradiction.

Therefore, Γ is a simplicial $\mathbb{Z}_p$-map from $\operatorname{sd} K$ to $\mathbb{Z}_p^{*(n-1)}$. Naturally, λ can be lifted to a simplicial $\mathbb{Z}_p$-map $\bar{\lambda} : \operatorname{sd}^2 \mathbb{Z}_p^{*n} \longrightarrow \operatorname{sd} K$. Thus $\Gamma \circ \bar{\lambda}$ is a simplicial $\mathbb{Z}_p$-map from $\operatorname{sd}^2 \mathbb{Z}_p^{*n}$ to $\mathbb{Z}_p^{*(n-1)}$. Therefore, by Property (i) in Properties of the $\mathbb{G}$-index, we must have

$$n - 1 = \operatorname{ind}_{\mathbb{Z}_p}(\operatorname{sd}^2 \mathbb{Z}_p^{*n}) \leqslant \operatorname{ind}_{\mathbb{Z}_p}(\mathbb{Z}_p^{*(n-1)}) = n-2,$$

which is not possible. □

As an application of Lemma 14, we present a short simple proof of Meunier's colorful result.

Theorem 15. (Meunier's theorem [20]) *Let $\mathcal{H}$ be a hypergraph and let p be a prime number. Then any proper coloring $c : V(\mathrm{KG}^p(\mathcal{H})) \longrightarrow [L]$ (L arbitrary) must contain a colorful balanced complete p-uniform p-partite hypergraph with $|V(\mathcal{H})| - \mathrm{alt}_p(\mathcal{H})$ vertices.*

Proof. Consider a bijection $\pi : [n] \longrightarrow V(\mathcal{H})$ such that $\mathrm{alt}_p(\mathcal{H},\pi) = \mathrm{alt}_p(\mathcal{H})$. We are going to define a map

$$\begin{array}{rcccc} \lambda: & (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\} & \longrightarrow & \mathbb{Z}_p \times [m] \\ & X & \longmapsto & (\lambda_1(X), \lambda_2(X)) \end{array}$$

satisfying the conditions of Lemma 14 with parameters $n = |V(\mathcal{H})|$, $m = \mathrm{alt}_p(\mathcal{H}) + L$, and $\alpha = \mathrm{alt}_p(\mathcal{H})$. Assume that $2^{[n]}$ is equipped with a total ordering $\preceq$. For each $X \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, define $\lambda(X)$ as follows.

- If $\mathrm{alt}(X) \leqslant \mathrm{alt}_p(\mathcal{H},\pi)$, then let $\lambda_1(X)$ be the first nonzero coordinate of X and $\lambda_2(X) = \mathrm{alt}(X)$.

- If $\mathrm{alt}(X) \geqslant \mathrm{alt}_p(\mathcal{H},\pi) + 1$, then in view of the definition of $\mathrm{alt}_p(\mathcal{H},\pi)$, there is some $\varepsilon \in \mathbb{Z}_p$ such that $E(\pi(X^\varepsilon)) \neq \varnothing$. Define

  $$c(X) = \max\{c(e) : \exists \varepsilon \in \mathbb{Z}_p \text{ such that } e \subseteq \pi(X^\varepsilon)\}$$

  and $\lambda_2(X) = \mathrm{alt}_p(\mathcal{H},\pi) + c(X)$. Choose X^ε so that there is at least one edge $e \in \pi(X^\varepsilon)$ with $c(e) = c(X)$ and such that X^ε is the maximum one having this property. By maximum, we mean maximum according to the total ordering $\preceq$. It is clear that ε is defined uniquely. Now, let $\lambda_1(X) = \varepsilon$.

One can check that λ satisfies the conditions of Lemma 14. Consider the chain $Z_1 \subset Z_2 \subset \cdots \subset Z_{n-\mathrm{alt}_p(\mathcal{H},\pi)}$, whose existence is ensured by Lemma 14. Note that for each $i \in [n - \mathrm{alt}_p(\mathcal{H},\pi)]$, we have $\lambda_2(Z_i) > \mathrm{alt}_p(\mathcal{H},\pi)$. Consequently, $\lambda_2(Z_i) = \mathrm{alt}_p(\mathcal{H},\pi) + c(Z_i)$. Let $\lambda(Z_i) = (\varepsilon_i, j_i)$. Note that for each i, there is at least one edge $e_{i,\varepsilon_i} \subseteq \pi(Z_i^{\varepsilon_i}) \subseteq \pi(Z^{\varepsilon_i}_{n-\mathrm{alt}_p(\mathcal{H},\pi)})$ such that $c(e_{i,\varepsilon_i}) = j_i - \mathrm{alt}_p(\mathcal{H},\pi)$. For each $\varepsilon \in \mathbb{Z}_p$, define $U_\varepsilon = \{e_{i,\varepsilon_i} : \varepsilon_i = \varepsilon\}$. We have the following three properties for U_ε's.

- Since the chain $Z_1 \subset Z_2 \subset \cdots \subset Z_{n-\mathrm{alt}_p(\mathcal{H},\pi)}$ is satisfying Condition 3 of Lemma 14, we have $\left\lfloor \frac{n-\mathrm{alt}_p(\mathcal{H},\pi)}{p} \right\rfloor \leqslant |U_\varepsilon| \leqslant \left\lceil \frac{n-\mathrm{alt}_p(\mathcal{H},\pi)}{p} \right\rceil$.

- The edges in U_ε get distinct colors. If there are two edges $e_{i,\varepsilon}$ and $e_{i',\varepsilon}$ in U_ε such that $c(e_{i,\varepsilon}) = c(e_{i',\varepsilon})$, then $\lambda(Z_i) = \lambda(Z_{i'})$, which is not possible.

- If $\varepsilon \neq \varepsilon'$, then for each $e \in U_\varepsilon$ and $f \in U_{\varepsilon'}$, we have $e \cap f = \varnothing$. It is clear because $e \subseteq \pi(Z^{\varepsilon}_{n-\mathrm{alt}_p(\mathcal{H},\pi)})$, $f \subseteq \pi(Z^{\varepsilon'}_{n-\mathrm{alt}_p(\mathcal{H},\pi)})$, and

  $$\pi(Z^{\varepsilon}_{n-\mathrm{alt}_p(\mathcal{H},\pi)}) \cap \pi(Z^{\varepsilon'}_{n-\mathrm{alt}_p(\mathcal{H},\pi)}) = \varnothing.$$

Now, it is clear that the subhypergraph $\mathrm{KG}^p(\mathcal{H})[U_{\omega^1},\ldots,U_{\omega^p}]$ is the desired subhypergraph. $\square$

Next proposition is an extension of Theorem 10. However, we lose some properties by this extension.

Proposition 16. *Let P be a free $\mathbb{Z}_p$-poset and*

$$\begin{array}{rcl}\psi : P & \longrightarrow & Q_{s,p}\\ x & \longmapsto & (\psi_1(x),\psi_2(x))\end{array}$$

be an order-preserving $\mathbb{Z}_p$-map. Then P contains a chain $x_1 \prec_P \cdots \prec_P x_k$ such that

- $k = \mathrm{ind}_{\mathbb{Z}_p}(\Delta P) + 1$,
- *for each $i \in [k-1]$, $\psi_2(x_i) < \psi_2(x_{i+1})$, and*
- *for each $\varepsilon \in \mathbb{Z}_p$,*
$$\left\lfloor \frac{k}{p} \right\rfloor \leqslant |\{j:\ \psi_1(x_j)=\varepsilon\}| \leqslant \left\lceil \frac{k}{p} \right\rceil.$$

Proof. Clearly, the map ψ can be considered as a simplicial $\mathbb{Z}_p$-map from ΔP to $\mathbb{Z}_p^{*n} \subseteq (\sigma_{p-2}^{p-1})^{*n}$. Consider the $(k-1)$-dimensional simplex $x_1 \prec_P \cdots \prec_P x_k$ in ΔP, whose existence is ensured by Lemma 12. Set $\tau = \{\psi(x_1),\ldots,\psi(x_k)\}$. First note that we already know $\left\lfloor \frac{k}{p} \right\rfloor \leqslant |\tau^\varepsilon| \leqslant \left\lceil \frac{k}{p} \right\rceil$ for each $\varepsilon \in \mathbb{Z}_p$. The fact that τ is a $(k-1)$-dimensional simplex in $\mathbb{Z}_p^{*n}$ implies that $\psi(x_i) \neq \psi(x_j)$ for each $i \neq j \in [k]$. On the other hand, since τ is a simplex in $\mathbb{Z}_p^{*n}$ and ψ is an order-preserving $\mathbb{Z}_p$-map, we have $\psi_2(x_i) < \psi_2(x_{i+1})$ for each $i \in [k-1]$. Therefore, we have $|\tau^\varepsilon| = |\{j:\ \psi_1(x_j)=\varepsilon\}|$ for each $\varepsilon \in \mathbb{Z}_p$, completing the proof. $\square$

Note that, for $p = 2$, since $\mathrm{Xind}(P) \geqslant \mathrm{ind}(\Delta P)$, Theorem 10 is better than Proposition 16. However, we were not able to prove that Proposition 16 is valid if we replace $\mathrm{ind}(\Delta P)$ by $\mathrm{Xind}(P)$.

In an unpublished paper, Meunier [21] introduced a generalization of Ky Fan's lemma. He presented a version of the $\mathbb{Z}_q$-Fan lemma, being valid for each odd integer $q \geqslant 3$. To be more specific, he proved that if q is an odd positive integer and $\lambda : V(T) \longrightarrow \mathbb{Z}_q \times [m]$ is a $\mathbb{Z}_q$-equivariant labeling of a $\mathbb{Z}_q$-equivariant triangulation of a $(d-1)$-connected free $\mathbb{Z}_q$-space T such that there is no edge in T, whose vertices are labeled with (ε, j) and (ε', j) with $\varepsilon \neq \varepsilon'$ and $j \in [m]$, then there is at least one n-dimensional simplex in T, whose vertices are labelled with labels $(\varepsilon_0, j_0), (\varepsilon_1, j_1), \ldots, (\varepsilon_n, j_n)$, where $\varepsilon_i \neq \varepsilon_{i+1}$ and $j_i < j_{i+1}$ for all $i \in \{0,1,\ldots,n-1\}$. Also, he asked the question if the result is true for even values of q. This question received a positive answer owing to the work of B. Hanke et al. [12]. In both mentioned works, the proofs of the $\mathbb{Z}_q$-Fan lemma are built in involved construction. Here, we take the opportunity of this paper to propose the following generalization of this result with a short simple proof uisng some similar techniques as we already used in the paper.

Lemma 17. ($\mathbb{G}$-Fan lemma). *Let $\mathbb{G}$ be a nontrivial finite group and let T be a free $\mathbb{G}$-simplicial complex such that $\mathrm{ind}_{\mathbb{G}}(T) = n$. Assume that $\lambda : V(T) \longrightarrow \mathbb{G} \times [m]$ is a $\mathbb{G}$-equivariant labeling such that there is no edge in T, whose vertices are labelled with (g, j) and (g', j) with $g \neq g' \in \mathbb{G}$ and $j \in [m]$. Then there is at least one n-dimensional simplex in T, whose vertices are labelled with labels $(g_0, j_0), (g_1, j_1), \ldots, (g_n, j_n)$, where $g_i \neq g_{i+1}$ and $j_i < j_{i+1}$ for each $i \in \{0, 1, \ldots, n-1\}$. In particular, $m \geqslant n + 1$.*

Proof. Clearly, the map λ can be considered as a $\mathbb{G}$-simplicial map from T to $\mathbb{G}^{*m}$. Note that, naturally each nonempty simplex $\sigma \in \mathbb{G}^{*m}$ can be identified with a vector $X = (x_1, x_2, \ldots, x_m) \in (\mathbb{G} \cup \{0\})^n \setminus \{\mathbf{0}\}$. To prove the assertion, it is enough to show that there is a simplex $\sigma \in T$ such that $\mathrm{alt}(\lambda(\sigma)) \geqslant n + 1$. For a contradiction, suppose that, for each simplex $\sigma \in T$, we have $\mathrm{alt}(\lambda(\sigma)) \leqslant n$. Define

$$\begin{array}{rrcl} \Gamma: & V(\mathrm{sd}\, T) & \longrightarrow & \mathbb{G} \times [n] \\ & \sigma & \longmapsto & (g, \mathrm{alt}(\lambda(\sigma))), \end{array}$$

where g is the first nonzero coordinate of the vector $\lambda(\sigma) \in (\mathbb{G} \cup \{0\})^n \setminus \{\mathbf{0}\}$. One can check that Γ is a simplicial $\mathbb{G}$-map from $\mathrm{sd}\, T$ to $\mathbb{G}^{*n}$. Note $\mathbb{G}^{*n}$ is an $E_{n-1}\mathbb{G}$ space. Consequently, $\mathrm{ind}_{\mathbb{G}}(\mathbb{G}^{*n}) = n - 1$. This implies that $\mathrm{ind}_{\mathbb{G}}(T) \leqslant n - 1$, which is a contradiction. □

3.2 Hierarchy of Indices

The aim of this subsection is to introduce some tools for the proof of Theorem 3.

Let n, α, and p be integers where $n \geqslant 1$, $n \geqslant \alpha \geqslant 0$, and p is prime. Define

$$\Sigma_p(n, \alpha) = \Delta \left\{X \in (\mathbb{Z}_p \cup \{0\})^n : \ \mathrm{alt}(X) \geqslant \alpha + 1\right\}.$$

Note that $\Sigma_p(n, \alpha)$ is a free simplicial $\mathbb{Z}_p$-complex with the vertex set

$$\left\{X \in (\mathbb{Z}_p \cup \{0\})^n : \ \mathrm{alt}(X) \geqslant \alpha + 1\right\}.$$

Lemma 18. *Let n, α, and p be integers where $n \geqslant 1$, $n \geqslant \alpha \geqslant 0$, and p is prime. Then*

$$\mathrm{ind}_{\mathbb{Z}_p}(\Sigma_p(n, \alpha)) \geqslant n - \alpha - 1.$$

Proof. Define

$$\begin{array}{rrcl} \lambda: & \mathrm{sd}\, \mathbb{Z}_p^{*n} & \longrightarrow & (\mathbb{Z}_p^{*\alpha}) * (\Sigma_p(n, \alpha)) \\ & X & \longmapsto & \left\{ \begin{array}{cl} (\varepsilon, \mathrm{alt}(X)) & \text{if } \mathrm{alt}(X) \leqslant \alpha \\ X & \text{if } \mathrm{alt}(X) \geqslant \alpha + 1, \end{array} \right. \end{array}$$

where ε is the first nonzero coordinate of X. Clearly, the map λ is a simplicial $\mathbb{Z}_p$-map. Therefore, in view of Properties (i) and (iii) in Properties of the $\mathbb{G}$-index, we have

$$\begin{array}{rcl} n - 1 = \mathrm{ind}_{\mathbb{Z}_p}(\mathrm{sd}\, \mathbb{Z}_p^{*n}) & \leqslant & \mathrm{ind}_{\mathbb{Z}_p}\left(\mathbb{Z}_p^{*\alpha} * \Sigma_p(n, \alpha)\right) \\ & \leqslant & \mathrm{ind}_{\mathbb{Z}_p}(\mathbb{Z}_p^{*\alpha}) + \mathrm{ind}_{\mathbb{Z}_p}(\Sigma_p(n, \alpha)) + 1 \\ & \leqslant & \alpha + \mathrm{ind}_{\mathbb{Z}_p}(\Sigma_p(n, \alpha)), \end{array}$$

which completes the proof. □

Proposition 19. *Let $\mathcal{F}$ be a hypergraph. For any integer $r \geqslant 2$ and any prime number $p \geqslant r$, we have*

$$\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathrm{KG}^r(\mathcal{F}), \mathbb{Z}_p)) + 1 \geqslant |V(\mathcal{F})| - \mathrm{alt}_p(\mathcal{F}).$$

Proof. For convenience, let $|V(\mathcal{F})| = n$ and $\alpha = \mathrm{alt}_p(\mathcal{F})$. Let $\pi : [n] \longrightarrow V(\mathcal{F})$ be a bijection such that $\mathrm{alt}_p(\mathcal{F}, \pi) = \mathrm{alt}_p(\mathcal{F})$. Define

$$\begin{array}{rrcl} \lambda: & \Sigma_p(n,\alpha) & \longrightarrow & \mathrm{sd}\,\mathrm{B}_0(\mathrm{KG}^r(\mathcal{F}), \mathbb{Z}_p)) \\ & X & \longmapsto & \{\omega^1\} \times U_1 \cup \cdots \cup \{\omega^p\} \times U_p, \end{array}$$

where $U_i = \{e \in E(\mathcal{F}) : \ e \subseteq \pi(X^{\omega^i})\}$. One can see that λ is a simplicial $\mathbb{Z}_p$-map. Consequently,

$$\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathrm{KG}^r(\mathcal{F}), \mathbb{Z}_p)) \geqslant \mathrm{ind}_{\mathbb{Z}_p}(\Sigma_p(n,\alpha)) \geqslant n - \mathrm{alt}_p(\mathcal{F}) - 1. \qquad \square$$

Proposition 20. *Let $\mathcal{H}$ be an r-uniform hypergraph and $p \geqslant r \geqslant 2$ be a prime number. Then*

$$\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r, \mathcal{H})) + p \geqslant \mathrm{ind}_{\mathbb{Z}_p}(\Delta\mathrm{Hom}(K_p^r, \mathcal{H})) + p \geqslant \mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)) + 1.$$

Proof. Since we already know that $\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r, \mathcal{H})) \geqslant \mathrm{ind}_{\mathbb{Z}_p}(\Delta\mathrm{Hom}(K_p^r, \mathcal{H}))$, to prove the assertion, it suffices to show that $\mathrm{ind}_{\mathbb{Z}_p}(\Delta\mathrm{Hom}(K_p^r, \mathcal{H})) + p \geqslant \mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)) + 1$. To this end, let

$$\lambda: \ \mathrm{sd}\,\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p) \ \longrightarrow \ \big(\mathrm{sd}\,\sigma_{p-2}^{p-1}\big) * \big(\Delta\mathrm{Hom}(K_p^r, \mathcal{H})\big)$$

be a map such that for each vertex $\tau = \bigcup\limits_{i=1}^{p} \big(\{\omega^i\} \times U_i\big)$ of $\mathrm{sd}\,\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)$, we define $\lambda(\tau)$ as follows.

- If $U_i \neq \varnothing$ for each $i \in [p]$, then $\lambda(\tau) = \tau$.
- If $U_i = \varnothing$ for some $i \in [p]$, then

$$\lambda(\tau) = \{\omega^i \in \mathbb{Z}_p : \ U_i = \varnothing\}.$$

One can check that the map λ is a simplicial $\mathbb{Z}_p$-map. Also, since σ_{p-2}^{p-1} is a free simplicial $\mathbb{Z}_p$-complex of dimension $p-2$, we have $\mathrm{ind}_{\mathbb{Z}_p}(\sigma_{p-2}^{p-1}) \leqslant p-2$ (see Property (iv) in Properties of the $\mathbb{G}$-index). This implies that

$$\begin{array}{rcl} \mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H}, \mathbb{Z}_p)) & \leqslant & \mathrm{ind}_{\mathbb{Z}_p}\big(\big(\mathrm{sd}\,\sigma_{p-2}^{p-1}\big) * \big(\Delta\mathrm{Hom}(K_p^r, \mathcal{H})\big)\big) \\ & \leqslant & \mathrm{ind}_{\mathbb{Z}_p}(\sigma_{p-2}^{p-1}) + \mathrm{ind}_{\mathbb{Z}_p}(\Delta\mathrm{Hom}(K_p^r, \mathcal{H})) + 1 \\ & \leqslant & p - 1 + \mathrm{ind}_{\mathbb{Z}_p}(\Delta\mathrm{Hom}(K_p^r, \mathcal{H})), \end{array}$$

which completes the proof. $\square$

4 Proofs of Theorem 2 and Theorem 3

Now, we are ready to prove Theorem 2 and Theorem 3.

Proof of Theorem 2. **Part (i).** For convenience, let $\mathrm{ind}_{\mathbb{Z}_p}(\mathrm{B}_0(\mathcal{H},\mathbb{Z}_p)) = t$. One can readily check that the map

$$\begin{array}{rrcl}\Gamma: & \mathbb{Z}_p \times V(\mathcal{H}) & \longrightarrow & \mathbb{Z}_p \times [L] \\ & (\varepsilon, v) & \longmapsto & (\varepsilon, c(v))\end{array}$$

is a simplicial $\mathbb{Z}_p$-map from $\mathrm{B}_0(\mathcal{H},\mathbb{Z}_p)$ to $(\sigma^{p-1}_{r-2})^{*L}$. Therefore, in view of Lemma 12, there is a t-dimensional simplex $\sigma = \bigcup\limits_{i=1}^{p}(\{\omega^i\}\times U_i) \in \mathrm{B}_0(\mathcal{H},\mathbb{Z}_p)$ such that $\tau = \Gamma(\sigma)$ is also t-dimensional and moreover, $\lfloor \frac{t+1}{p} \rfloor \leqslant |\tau^\varepsilon| \leqslant \lceil \frac{t+1}{p} \rceil$ for each $\varepsilon \in \mathbb{Z}_p$. Since σ is a t-dimensional simplex in $\mathrm{B}_0(\mathcal{H},\mathbb{Z}_p)$,

- $\sum\limits_{i=1}^{p} |U_i| = t+1$ and
- $\mathcal{H}[U_1,\ldots,U_p]$ is a complete r-uniform p-partite subhypergraph of $\mathcal{H}$.

In view of the definition of Γ and since $\tau = \Gamma(\sigma)$ is also a t-dimensional simplex, we must have $|U_i| = |c(U_i)| = |\tau^{\omega^i}|$ for each $i \in [p]$. Now, it is clear that $\mathcal{H}[U_1,\ldots,U_p]$ is the desired subhypergraph, completing the proof in this part.

Part (ii). First note that since $p \leqslant \omega(\mathcal{H})$, the $\mathbb{Z}_p$-poset $\mathrm{Hom}(K^r_p,\mathcal{H}))$ is not empty. For convenience, let $\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K^r_p,\mathcal{H})) = t$. Let F be *the face poset of* $(\sigma^{p-1}_{r-2})^{*L}$, i.e., the poset consisting of all nonempty simplices of $(\sigma^{p-1}_{r-2})^{*L}$ ordered by inclusion. Since $(\sigma^{p-1}_{r-2})^{*L}$ is a free simplicial $\mathbb{Z}_p$-complex, one can readily check that F is a free $\mathbb{Z}_p$-poset. Also, it is clear that the ground set of F is the same as the vertex set of $\mathrm{sd}\left((\sigma^{p-1}_{r-2})^{*L}\right)$. Now, define the map

$$\lambda : \mathrm{Hom}(K^r_p,\mathcal{H}) \longrightarrow F$$

such that for each $(U_1,\ldots,U_p) \in \mathrm{Hom}(K^r_p,\mathcal{H})$,

$$\lambda(U_1,\ldots,U_p) = \{\omega^1\}\times c(U_1) \cup \cdots \cup \{\omega^p\}\times c(U_p).$$

Claim. There is a p-tuple $(U_1,\ldots,U_p) \in \mathrm{Hom}(K^r_p,\mathcal{H})$ such that for $\tau = \lambda(U_1,\ldots,U_p)$, we have $l(\tau) \geqslant t+p$.

Proof of Claim. For sake a contradiction, suppose that for each $\tau \in \mathrm{Im}(\lambda)$, we have $l(\tau) \leqslant t+p-1$. One can readily check that λ is an order-preserving $\mathbb{Z}_p$-map. Clearly, for each $\tau \in \mathrm{Im}(\lambda)$, we have $h(\tau) = \min\limits_{\varepsilon\in\mathbb{Z}_p} |\tau^\varepsilon| \geqslant 1$ and consequently, $l(\tau) \geqslant p$. Now, define

$$\begin{array}{rcrcl}\Gamma & : & \mathrm{Im}(\lambda) & \longrightarrow & Q_{t-1,p} \\ & & \tau & \longmapsto & (s(\bar{\tau}), l(\tau)-p+1),\end{array}$$

where $\bar{\tau} = \bigcup\limits_{\{\varepsilon\in\mathbb{Z}_p:\, |\tau^\varepsilon|=h(\tau)\}} \tau^\varepsilon \in W$. Clearly, since $s(-)$ is a $\mathbb{Z}_p$-equivariant map, Γ is a $\mathbb{Z}_p$-equivariant map as well. One can see that the map $\Gamma : \mathrm{Im}(\lambda) \longrightarrow Q_{t-1,p}$ is an order-preserving $\mathbb{Z}_p$-map. To this end, in view of the definition of the ordering in $Q_{t-1,p}$, it suffices to show that if $\Gamma(\tau) = (\varepsilon_1, \beta)$ and $\Gamma(\tau') = (\varepsilon_2, \beta)$ for some $\tau \subsetneq \tau' \in \mathrm{Im}(\lambda)$, then $\varepsilon_1 = \varepsilon_2$. For a contradiction, suppose that there are $\tau, \tau' \in \mathrm{Im}(\lambda)$ such that $\tau \subsetneq \tau'$, $\Gamma(\tau) = (\varepsilon_1, \beta)$, and $\Gamma(\tau') = (\varepsilon_2, \beta)$, where $\varepsilon_1 \neq \varepsilon_2$. Clearly, in view of definition of Γ, we have $l(\tau) = l(\tau') = \beta + p - 1$. On the other hand, we know that

$$l(\tau) = p \cdot h(\tau) + |\{\varepsilon \in \mathbb{Z}_p :\ |\tau^\varepsilon| > h(\tau)\}| \text{ and } l(\tau') = p \cdot h(\tau') + |\{\varepsilon \in \mathbb{Z}_p :\ |\tau'^\varepsilon| > h(\tau')\}|.$$

The facts that $l(\tau) = l(\tau')$ and

$$\max\left\{|\{\varepsilon \in \mathbb{Z}_p :\ |\tau^\varepsilon| > h(\tau)\}|, |\{\varepsilon \in \mathbb{Z}_p :\ |\tau'^\varepsilon| > h(\tau')\}|\right\} \leqslant p - 1$$

imply that $h(\tau) = h(\tau')$ and

$$|\{\varepsilon \in \mathbb{Z}_p :\ |\tau^\varepsilon| > h\}| = |\{\varepsilon \in \mathbb{Z}_p :\ |\tau'^\varepsilon| > h\}|, \tag{4}$$

where we set $h = h(\tau) = h(\tau')$. In view of

$$\varepsilon_1 = s(\bigcup_{\{\varepsilon\in\mathbb{Z}_p:\, |\tau^\varepsilon|=h\}} \tau^\varepsilon) \neq s(\bigcup_{\{\varepsilon\in\mathbb{Z}_p:\, |\tau'^\varepsilon|=h\}} \tau'^\varepsilon) = \varepsilon_2,$$

we must have

$$\bigcup_{\{\varepsilon\in\mathbb{Z}_p:\, |\tau^\varepsilon|=h\}} \tau^\varepsilon \neq \bigcup_{\{\varepsilon\in\mathbb{Z}_p:\, |\tau'^\varepsilon|=h\}} \tau'^\varepsilon.$$

Also, $\tau \subsetneq \tau'$ and $\min\limits_{\varepsilon\in\mathbb{Z}_p} |\tau^\varepsilon| = \min\limits_{\varepsilon\in\mathbb{Z}_p} |\tau'^\varepsilon|$ implies that

$$\left\{\varepsilon \in \mathbb{Z}_p :\ |\tau'^\varepsilon| = h\right\} \subsetneq \{\varepsilon \in \mathbb{Z}_p :\ |\tau^\varepsilon| = h\},$$

which contradicts Equality 4.

Therefore, since both Γ and λ are order-preserving $\mathbb{Z}_p$-maps,

$$\Gamma \circ \lambda : \mathrm{Hom}(K_p^r, \mathcal{H}) \longrightarrow Q_{t-1,p}$$

is an order-preserving $\mathbb{Z}_p$-map as well, contradicting the fact that $\mathrm{Xind}_{\mathbb{Z}_p}(\mathrm{Hom}(K_p^r, \mathcal{H})) = t$. □

Amongst all p-tuples, whose existence are ensured by Claim, choose a minimal one, say $T = (V_1, \ldots, V_p) \in \mathrm{Hom}(K_p^r, \mathcal{H})$. First note that since $(V_1, \ldots, V_p)$ is in $\mathrm{Hom}(K_p^r, \mathcal{H})$, the subhypergraph $\mathcal{H}[V_1, \ldots, V_p]$ is a complete r-uniform p-partite hypergraph. Set $\tau = \lambda(T)$. In view of the minimality of T, we clearly have

$$\sum_{i=1}^{p} |V_i| = |T| = |\tau| = l(\tau) = t + p.$$

In view of the definition of $\lambda(-)$, it implies that $|V_i| = |c(V_i)| = |\tau^{\omega^i}|$ for each $i \in [p]$. Also, the equalities $|\tau| = l(\tau) = t + p$ imply that $\left\lfloor \frac{t+p}{p} \right\rfloor \leqslant |\tau^{\omega^i}| \leqslant \left\lceil \frac{t+p}{p} \right\rceil$. Therefore, $\mathcal{H}[V_1, \ldots, V_p]$ is the desired complete r-uniform p-partite subhypergraph, completing the proof. □

Proof of Theorem 3. It has already be noted that $|V(\mathcal{F})| - \mathrm{alt}_p(\mathcal{F}) \geqslant \mathrm{cd}_p(\mathcal{F})$ for any hypergraph $\mathcal{F}$. Therefore, the proof follows by Proposition 19 and Proposition 20. □

Acknowledgments

This research was in part supported by a grant from IPM (No. 95050014). I would like to acknowledge Professor Frédéric Meunier for interesting discussions about the paper and his invaluable comments. Also, I would like to thank Professor Hossein Hajiabolhassan and Mrs Roya Abyazi Sani for their useful comments. Moreover, the author is grateful to the anonymous referee for carefully reading the paper and giving several constructive comments that helped to improve the presentation of the paper.

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