Permanent index of matrices associated with graphs

Tsai-Lien Wong*
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan
tlwong@math.nsysu.edu.tw

Xuding Zhu†
Department of Mathematics
Zhejiang Normal University
China
xudingzhu@gmail.com

Submitted: Aug 17, 2015; Accepted: Jan 27, 2017; Published: Feb 17, 2017
Mathematics Subject Classifications: 05C50, 05C78

Abstract

A total weighting of a graph G is a mapping f which assigns to each element $z \in V(G) \cup E(G)$ a real number $f(z)$ as its weight. The vertex sum of v with respect to f is $\phi_f(v) = \sum_{e \in E(v)} f(e) + f(v)$. A total weighting is proper if $\phi_f(u) \neq \phi_f(v)$ for any edge uv of G. A (k, k')-list assignment is a mapping L which assigns to each vertex v a set $L(v)$ of k permissible weights, and assigns to each edge e a set $L(e)$ of k' permissible weights. We say G is (k, k')-choosable if for any (k, k')-list assignment L, there is a proper total weighting f of G with $f(z) \in L(z)$ for each $z \in V(G) \cup E(G)$. It was conjectured in [T. Wong and X. Zhu, Total weight choosability of graphs, J. Graph Theory 66 (2011), 198–212] that every graph is $(2, 2)$-choosable and every graph with no isolated edge is $(1, 3)$-choosable. A promising tool in the study of these conjectures is Combinatorial Nullstellensatz. This approach leads to conjectures on the permanent indices of matrices A_G and B_G associated to a graph G. In this paper, we establish a method that reduces the study of permanent of matrices associated to a graph G to the study of permanent of matrices associated to induced subgraphs of G. Using this reduction method, we show that if G is a subcubic graph, or a 2-tree, or a Halin graph, or a grid, then A_G has permanent index 1. As a consequence, these graphs are $(2, 2)$-choosable.

Keywords: Permanent index, matrix, total weighting

1 Introduction

A *total weighting* of a graph G is a mapping f which assigns to each element $z \in V(G) \cup E(G)$ a real number $f(z)$ as its weight. Given a total weighting f of G, for a vertex v

*Supported by Grant number: MOST 104-2115-M-110 -001 -MY2

†Supported by Grant numbers: NSF11171310 and ZJNSF Z6110786

of G, the *vertex sum* of v with respect to f is defined as $\phi_f(v) = \sum_{e \in E(v)} f(e) + f(v)$. A total weighting is proper if ϕ_f is a proper colouring of G, i.e., for any edge uv of G, $\phi_f(u) \neq \phi_f(v)$. A total weighting ϕ with $\phi(v) = 0$ for all vertices v is also called an *edge weighting*. A proper edge weighting ϕ with $\phi(e) \in \{1, 2, \ldots, k\}$ for all edges e is called a *vertex colouring k-edge weighting* of G. Karonski, Łuczak and Thomason [7] first studied edge weighting of graphs. They conjectured that every graph with no isolated edges has a vertex colouring 3-edge weighting. This conjecture received considerable attention, and is called the 1-2-3 conjecture. Addario-Berry, Dalal, McDiarmid, Reed and Thomason [2] proved that every graph with no isolated edges has a vertex colouring k-edge weighting for $k = 30$. The bound k was improved to $k = 16$ by Addario-Berry, Dalal and Reed in [1] and to $k = 13$ by Wang and Yu in [10], and to $k = 5$ by Kalkowski [8].

Total weighting of graphs was first studied by Przybyło and Woźniak in [11], where they defined $\tau(G)$ to be the least integer k such that G has a proper total weighting ϕ with $\phi(z) \in \{1, 2, \ldots, k\}$ for $z \in V(G) \cup E(G)$. They proved that $\tau(G) \leqslant 11$ for all graphs G, and conjectured that $\tau(G) = 2$ for all graphs G. This conjecture is called the 1-2 conjecture. A breakthrough on 1-2 conjecture was obtained by Kalkowski, Karoński and Pfender in [9], where it was proved that every graph G has a proper total weighting ϕ with $\phi(v) \in \{1, 2\}$ for $v \in V(G)$ and $\phi(e) \in \{1, 2, 3\}$ for $e \in E(G)$.

The list version of edge weighting of graphs was introduced by Bartnicki, Grytczuk and Niwczyk in [6], and the list version of total weighting of graphs was introduced independently by Wong and Zhu in [13] and by Przybyło and Woźniak [12]. Suppose $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, \}$ is a mapping which assigns to each vertex and each edge of G a positive integer. A ψ-list assignment of G is a mapping L which assigns to $z \in V(G) \cup E(G)$ a set $L(z)$ of $\psi(z)$ real numbers. Given a total list assignment L, a proper L-total weighting is a proper total weighting ϕ with $\phi(z) \in L(z)$ for all $z \in V(G) \cup E(G)$. We say G is *total weight ψ-choosable* if for any ψ-list assignment L, there is a proper L-total weighting of G. We say G is (k, k')-choosable if G is ψ-total weight choosable, where $\psi(v) = k$ for $v \in V(G)$ and $\psi(e) = k'$ for $e \in E(G)$.

As strengthening of the 1-2-3 conjecture and the 1-2 conjecture, it was conjectured in [13] that every graph with no isolated edges is $(1, 3)$-choosable and every graph is $(2, 2)$-choosable. Thes two conjectures received a lot of attention and are verified for some special classes of graphs. In particular, it was shown in [14] that every graph is $(2, 3)$-choosable. A promising tool in the study of these conjectures is Combinatorial Nullstellensatz. For each $z \in V(G) \cup E(G)$, let x_z be a variable associated to z. Fix an orientation D of G. Consider the polynomial

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e=uv \in E(D)} \left(\left(\sum_{e \in E(v)} x_e + x_v \right) - \left(\sum_{e \in E(u)} x_e + x_u \right) \right).$$

Assign a real number $\phi(z)$ to the variable x_z, and view $\phi(z)$ as the weight of z. Let $P_G(\phi)$ be the evaluation of the polynomial at $x_z = \phi(z)$. Then ϕ is a proper total weighting of G if and only if $P_G(\phi) \neq 0$. Note that P_G has degree $|E(G)|$.

An *index function* of G is a mapping η which assigns to each vertex or edge z of G a non-negative integer $\eta(z)$ and an index function η is *valid* if $\sum_{z \in V(G) \cup E(G)} \eta(z) = |E(G)|$. For a valid index function η, let c_η be the coefficient of the monomial $\prod_{z \in V \cup E} x_z^{\eta(z)}$ in the expansion of P_G. It follows from Combinatorial Nullstellensatz [3, 5] that if $c_\eta \neq 0$, and L is a list assignment which assigns to each $z \in V(G) \cup E(G)$ a set $L(z)$ of $\eta(z)+1$ real numbers, then there exists a mapping ϕ with $\phi(z) \in L(z)$ such that

$$P_G(\phi) \neq 0.$$

So to prove a graph G is (k, k')-choosable, it suffices to show that there is a valid index function η with $\eta(v) \leqslant k-1$ for $v \in V(G)$, $\eta(e) \leqslant k'-1$ for $e \in E(G)$ and $c_\eta \neq 0$.

We write the polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$ as

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e \in E(D)} \sum_{z \in V(G) \cup E(G)} A_G[e, z] x_z.$$

It is straightforward to verify that for $e \in E(G)$ and $z \in V(G) \cup E(G)$, if $e = (u, v)$ (oriented from u to v), then

$$A_G[e, z] = \begin{cases} 1 & \text{if } z = v, \text{ or } z \neq e \text{ is an edge incident to } v, \\ -1 & \text{if } z = u, \text{ or } z \neq e \text{ is an edge incident to } u, \\ 0 & \text{otherwise.} \end{cases}$$

Now A_G is a matrix, whose rows are indexed by the edges of G and the columns are indexed by edges and vertices of G. Let B_G be the submatrix of A_G consisting of those columns of A_G indexed by edges. It turns out that (k, k')-choosability of a graph G is related to the permanent indices of A_G and B_G.

For an $m \times m$ matrix A (whose entries are reals), the *permanent* of A is defined as

$$\text{per}(A) = \sum_{\sigma \in S_m} \prod_{i=1}^{m} A[i, \sigma(i)]$$

where S_m is the symmetric group of order m, i.e., the summation is taken over all the permutations σ over $\{1, 2, \ldots, m\}$. The *permanent index* of a matrix A, denoted by $\text{pind}(A)$, is the minimum integer k such that there is a matrix A' such that $\text{per}(A') \neq 0$, each column of A' is a column of A and each column of A occurs in A' at most k times (if such an integer k does not exist, then $\text{pind}(A) = \infty$).

Consider the matrix A_G defined above. Given a vertex or edge z of G, let $A_G(z)$ be the column of A_G indexed by z. For an index function η of G, let $A_G(\eta)$ be the matrix, each of its column is a column of A_G, and each column $A_G(z)$ of A_G occurs $\eta(z)$ times as a column of $A_G(\eta)$. It is known [4, 13] and easy to verify that for a valid index function η of G, $c_\eta \neq 0$ if and only if $\text{per}(A_G(\eta)) \neq 0$. Thus if $\text{pind}(A_G) = 1$, then G is $(2, 2)$-choosable; if $\text{pind}(B_G) \leqslant 2$, then G is $(1, 3)$-choosable. The following two conjectures are proposed in [13]:

Conjecture 1. [6] For any graph G with no isolated edges, $\text{pind}(B_G) \leqslant 2$.

Conjecture 2. [13] For any graph G, $\text{pind}(A_G) = 1$.

The discussion above shows that Conjecture 1 implies that any graph without isolated edges is $(1,3)$-choosable, and Conjecture 2 implies that every graph is $(2,2)$-choosable.

We say an index function η is *non-singular* if there is a valid index function $\eta' \leqslant \eta$ with $\text{per}(A_G(\eta')) \neq 0$. In this paper, we are interested in non-singularity of index functions η for which $\eta(e) = 1$ for every edge e and $\eta(v)$ can be any non-negative integers for any every vertex v. Assume η is such an index function of G. We delete a vertex v, and construct an index function η' for $G - v$ from the restriction of η to $G - v$ by doing the following modification: $\eta(v)$ of the neighbours u of v have $\eta'(u) = \eta(u) + 1$, and all the other neighbours u of v (if any) have $\eta'(u) = \eta(u) - 1$. We prove that if η' is a non-singular index function of $G - v$, then η is a non-singular index function of G. Applying this reduction method, we prove that Conjecture 2 holds for subcubic graphs, 2-trees, Halin graphs and grids. Consequently, subcubic graphs, 2-trees, Halin graphs and grids are $(2,2)$-choosable.

2 Reduction to induced subgraphs

To study non-singularity of index functions of G, we shall consider matrices whose columns are linear combinations of columns of A_G. Assume A is a square matrix whose columns are linear combinations of columns of A_G. Define an index function $\eta_A : V(G) \cup E(G) \to \{0, 1, \ldots, \}$ as follows:

For $z \in V(G) \cup E(G)$, $\eta_A(z)$ is the number of columns of A in which $A_G(z)$ appears with nonzero coefficient.

It is known [13] that columns of A_G are not linearly independent. In particular, if $e = uv$ is an edge of G, then

$$A_G(e) = A_G(u) + A_G(v) \tag{1}$$

Thus a column of A may have different ways to be expressed as linear combinations of columns of A_G. So the index function η_A is not uniquely determined by A. Instead, it is determined by the way we choose to express the columns of A as linear combinations of columns of A_G. For simplicity, we use the notation η_A, however, whenever the function η_A is used, an explicit expression of the columns of A as linear combinations of columns of A_G is given, and we refer to that specific expression.

It is well-known (and follows easily from the definition) that the permanent of a matrix is multi-linear on its column vectors and row vectors: If a column C of A is a linear combination of two columns vectors $C = \alpha C' + \beta C''$, and A' (respectively, A'') is obtained from A by replacing the column C with C' (respectively, with C''), then

$$\text{per}(A) = \alpha \ \text{per}(A') + \beta \ \text{per}(A''). \tag{2}$$

By using (2) repeatedly, one can find matrices $A_1, A_2, \ldots, A_q$ and real numbers $a_1, a_2, \ldots, a_q$ such that

$$\text{per}(A) = \sum_{j=1}^{q} a_j \text{per}(A_j)$$

where each A_j is a square matrix consisting of columns of A_G, with each column $A_G(z)$ appears at most $\eta(z)$ times. Thus if $\text{per}(A) \neq 0$, then one of the $\text{per}(A_j) \neq 0$. Thus if $\text{per}(A) \neq 0$, then η_A is a non-singular index function of G.

Theorem 3. *Suppose G is a graph, η is an index function of G for which $\eta(e) = 1$ for every edge e. Let v be a vertex of G. Let η' be obtained from the restriction of η to $G - v$ by the following modification: Choose $d_G(v) - \eta(v)$ neighbours u of v with $\eta(u) \geqslant 1$, and let $\eta'(u) = \eta(u) - 1$. For the other $\eta(v)$ neighbours u of v, let $\eta'(u) = \eta(u) + 1$. If η' is a non-singular index function of $G - v$, then η is a non-singular index function of G.*

Theorem 3 follows from the following more general statement.

Theorem 4. *Suppose G is a graph, v is a vertex of G and $E(v) = \{e_1, e_2, \ldots, e_k\}$, with $e_i = vv_i$ for $i = 1, 2, \ldots, k$. Assume η is an index function of G. Here $\eta(e)$ can be any non-negative integer. Choose a subset J of $\{1, 2, \ldots, k\}$ and integers $1 \leqslant k_i \leqslant \min\{\eta(e_i), \eta(v_i)\}$ such that $\eta(v) + \sum_{i \in J} k_i = k$. Let η' be the index function of $G' = G - v$ which is equal to the restriction of η to $G - v$, except that*

1. *For $i \in J$, $\eta'(v_i) = \eta(v_i) - k_i$.*
2. *For $i \in \{1, 2, \ldots, k\} \setminus J$, $\eta'(v_i) = \eta(v_i) + \eta(e_i)$.*

If η' is a non-singular index function for G', then η is a non-singular index function for G.

Proof. Assume that η' is non-singular. Let $\eta'' \leqslant \eta'$ be a valid index function with $\text{per}(A_{G'}(\eta'')) \neq 0$.

Assume $|E(G)| = m$ and $|E(G')| = m' = m - k$. By viewing each vertex and each edge of G' as a vertex and an edge of G, $A_G(\eta'')$ is an $m \times m'$ matrix, consisting m' columns of A_G. First we extend $A_G(\eta'')$ into an $m \times m$ matrix A by adding k copies of the column $A_G(v)$. The added k columns has k rows (the rows indexed by edges incident to v) that are all 1's (with all these edges oriented towards v), and all the other entries of these k columns are 0. Therefore $\text{per}(M) = \text{per}(A_{G'}(\eta''))k!$, and hence $\text{per}(M) \neq 0$.

Starting from the matrix M, for each $i \in \{1, 2, \ldots, k\} \setminus J$, remove $\min\{\eta(e_i), \eta''(v_i)\}$ copies of the column $A_G(v_i)$ and add $\min\{\eta(e_i), \eta''(v_i)\}$ copies of the column $A_G(e_i)$. Denote by M' the resulting matrix.

Claim 5. *For the matrix M' constructed above, we have* $\text{per}(M') = \text{per}(M)$.

Proof. Since by (1), $A_G(e_i) = A_G(v_i) + A_G(v)$, we re-write $\min\{\eta(e_i), \eta''(v_i)\}$ copies of the column $A_G(e_i)$ of M' as $A_G(v) + A_G(v_i)$. Then we expand the permanent using its multilinear property (i.e. using (2) repeatedly), to obtain the following equation:

$$\text{per}(M') = \text{per}(M) + \sum_{M''} \text{per}(M'')$$

where M'' are those matrices which contain at least $k+1$ copies of the column $A_G(v)$. Since these $k + 1$ columns has all 1's in k rows and 0 in all other entries, we have $\text{per}(M'') = 0$ for all M'', and so $\text{per}(M') = \text{per}(M)$. □

For each $i \in J$, write k_i copies of $A_G(v)$ in M' as $A_G(e_i) - A_G(v_i)$. Note that this step does not change the matrix, since $A_G(v) = A_G(e_i) - A_G(v_i)$ (by (1)). Now each column of M' is a linear combination of columns of A_G.

We shall show that, with the linear combination of columns of M' given in the above paragraph, $\eta_{M'}(z) \leqslant \eta(z)$ for $z \in V(G) \cup E(G)$.

If $z \notin \{e_i, v_i : i = 1, 2, \ldots, k\} \cup \{v\}$, $\eta_{M'}(z) = \eta_M(z) \leqslant \eta''(z) \leqslant \eta'(z) = \eta(z)$. If $i \in \{1, 2, \ldots, k\} - J$, then $\eta_{M'}(e_i) = \min\{\eta(e_i), \eta''(v_i)\} \leqslant \eta(e_i)$, and $\eta_{M'}(v_i) = \eta_M(v_i) - \min\{\eta(e_i), \eta''(v_i)\} \leqslant \max\{0, \eta''(v_i) - \eta(e_i)\} \leqslant \eta'(v_i) - \eta(e_i) = \eta(v_i)$. If $i \in J$, then $\eta_{M'}(e_i) = k_i \leqslant \eta(e_i)$ and $\eta_{M'}(v_i) = \eta''(v_i) + k_i \leqslant \eta'(v_i) + k_i = \eta(v_i)$. Finally, $\eta_{M'}(v) = k - \sum_{i \in J} k_i = \eta(v)$. As $\mathrm{per}(M') \neq 0$, we conclude that η is a non-singular index function for G. This completes the proof of Theorem 4. □

Theorem 3 follows from Theorem 4 by choosing $k_i = 1$ and $|J| = d(v) - \eta(v)$. By definition, if η'' is non-singular and $\eta' \geqslant \eta''$, then η' is also non-singular. So the following is equivalent to Theorem 3.

Theorem 6. *Suppose G is a graph, η is an index function of G for which $\eta(e) = 1$ for every edge e. Let v be a vertex of G. Let η' be obtained from the restriction of η to $G - v$ by the following modification: Choose at least $d_G(v) - \eta(v)$ neighbours u of v with $\eta(u) \geqslant 1$, and let $\eta'(u) = \eta(u) - 1$. For the other neighbours u of v, let $\eta'(u) = \eta(u) + 1$. If η' is a non-singular index function of $G - v$, then η is a non-singular index function of G.*

We shall apply Theorem 6 repeatedly and delete a sequence of vertices in order. We need to record which vertices are deleted, and when a vertex is deleted, for which neighbours u we have $\eta'(u) = \eta(u) + 1$. For this purpose, instead of really removing the deleted vertices, we indicate the deletion of v by orient all the edges incident to v from v to its neighbours, and then choose a subset of these oriented edges (to indicate those neighbours u for which $\eta'(u) = \eta(u) + 1$).

The index function η is changing in the process of the deletion. For convenience, we denote by η_i the index function after the deletion of the ith vertex. In particular, $\eta_0 = \eta$.

Assume a vertex v is deleted in the ith step, for each neighbour u of v (at the time v is deleted), orient the edge as an arc from v to u. After a sequence of vertices are deleted, we obtain a digraph D formed by edges incident to the "deleted" vertices. Let D' be the sub-digraph of D formed by those arcs (v, u) with u be the neighbour of v (at the time v is deleted) and for which we have $\eta'(u) = \eta(u) + 1$.

If u is deleted in the ith step, then $d^+_{D'}(u) \leqslant \eta_{i-1}(u)$. After the ith step, all edges incident to u are oriented. On the other hand, $d^-_{D'}(u)$ is the number of indices $j < i$ for which $\eta_j(u) = \eta_{j-1}(u) + 1$, and $d^-_D(u) - d^-_{D'}(u)$ is the number of indices $j < i$ for which $\eta_j(u) = \eta_{j-1}(u) - 1$. Thus $d^+_{D'}(u) \leqslant \eta(u) + d^-_{D'}(u) - (d^-_D(u) - d^-_{D'}(u))$.

If after the ith step, u is not deleted, then $d^+_{D'}(u) = 0$ and $\eta_i(u) = \eta(u) + d^-_{D'}(u) - (d^-_D(u) - d^-_{D'}(u)) \geqslant 0$.

The following corollary summarize the final effect of the repeated application of Theorem 3.

Corollary 7. *Suppose G is a graph, η is an index function of G with $\eta(e) = 1$ for all edges e, and X is a subset of $V(G)$. Let $G' = G - E[X]$ be obtained from G by deleting edges in $G[X]$. Let D be an acyclic orientation of G', in which each vertex $v \in X$ is a sink. Assume D' is a sub-digraph of D such that for all $v \in V(D)$,*

$$\eta(v) + 2d^-_{D'}(v) - d^-_D(v) \geqslant d^+_{D'}(v), \qquad (*)$$

Let η' be the index function defined as $\eta'(e) = 1$ for every edge e of $G[X]$ and $\eta'(v) = \eta(v) + 2d^-_{D'}(v) - d^-_D(v)$ for $v \in X$. If η' is a non-singular index function for $G[X]$, then η is a non-singular index function for G.

Proof. Assume η' is non-singular for $G[X]$. We shall prove that η is non-singular for G. We prove this by induction on $|V - X|$. If $V - X = \emptyset$, then $\eta = \eta'$ and there is nothing to prove.

Assume $V - X \neq \emptyset$. Since the orientation D is acyclic, there is a source vertex $v \notin X$. Let $e_1, e_2, \ldots, e_k$ be the set of edges incident to v and $e_i = vv_i$.

Consider the graph $G - v$. Let η'' be the index function on $G - v$ defined as $\eta'' = \eta$ on $G - v$, except that for $i = 1, 2, \ldots, k$, if $e_i \notin D'$, then $\eta''(v_i) = \eta(v_i) - 1$, and if $e_i \in D'$, then $\eta''(v_i) = \eta(v_i) + 1$.

Let $H = D - v$ and $H' = D' - v$. We shall show that

$$\eta''(u) + 2d^-_{H'}(u) - d^-_H(u) \geqslant d^+_{H'}(u) \text{ for all } u \in V(H) \qquad (**)$$

If $u \notin \{v_1, v_2, \ldots, v_k\}$, then $(**)$ is the same as $(*)$. If $u = v_i$ and $e_i \in D'$, then $\eta''(v_i) = \eta(v_i) + 1, d^-_{H'}(v_i) = d^-_{D'}(v_i) - 1, d^-_H(v_i) = d^-_D(v_i) - 1$ and $d^+_H(v_i) = d^+_D(v_i)$. So $(**)$ follows from $(*)$. If $u = v_i$ and $e_i \notin D'$, then $\eta''(v_i) = \eta(v_i) - 1, d^-_{H'}(v_i) = d^-_{D'}(v_i), d^-_H(v_i) = d^-_D(v_i) - 1$ and $d^+_H(v_i) = d^+_D(v_i)$. Again $(**)$ follows from $(*)$.

Therefore, by induction hypothesis, η'' is non-singular for $G - v$. Apply Theorem 3 to η'' and η, with $J = \{i : 1 \leqslant i \leqslant k, e_i \notin D'\}$ and $k_i = 1$ for $i \in J$, we conclude that η is non-singular for G. □

3 Application of the reduction method

Lemma 8. *Suppose G is a connected graph, and η is an index function with $\eta(e) = 1$ for all $e \in E(G)$. Assume one of the following holds:*

- *$\eta(v) \geqslant \max\{1, d_G(v) - 2\}$ for every vertex v.*
- *Each vertex v has $\eta(v) \geqslant d_G(v) - 2$ and at least one vertex v has $\eta(v) \geqslant d_G(v)$.*

Then η is a non-singular index function of G.

Proof. Assume the lemma is not true and G is a counterexample with minimum number of vertices.

Assume first that $\eta(v) \geqslant \max\{1, d_G(v) - 2\}$ for all v. By reducing the value of η if needed, we may assume that $\eta(v) = \max\{1, d_G(v) - 2\}$. Let v be a non-cut vertex of G

and let $v_1, \dots, v_k$ be the neighbours of v. Consider the graph $G - v$. Let η' be the index function of $G - v$ defined as $\eta' = \eta$, except that $\eta'(v_i) = \eta(v_i) - 1$ for $i = 1, 2, \dots, k-1$ and $\eta'(v_k) = \eta(v_k) + 1$. For each $i \in \{1, 2, \dots, k-1\}$, we have $\eta'(v_i) \geqslant d_{G-v}(v_i) - 2$, and $\eta'(v_k) \geqslant d_{G-v}(v_k)$. As $G - v$ is connected, the condition of the lemma is satisfied by $G - v$ and η'. By the minimality of G, η' is a non-singular index function for $G - v$. By Theorem 3, η is a non-singular index function for G.

Assume each vertex u has $\eta(u) \geqslant d_G(u) - 2$ and one vertex v has $\eta(v) \geqslant d_G(v)$. Let η' be the index function of $G - v$ defined as $\eta' = \eta$ except that $\eta'(u) = \eta(u) + 1$ for all neighbours u of v. Note that for all the neighbours u of v, $\eta'(u) \geqslant d_{G-v}(u)$. Thus each component of $G - v$, together with η', satisfies the condition of the lemma. By the minimality of G, η' is a non-singular index function for $G - v$. Apply Theorem 3 again, we conclude that η is a non-singular index function for G. □

A graph G is called *subcubic* if G has maximum degree at most 3.

Corollary 9. *Conjecture 2 holds for subcubic graphs, i.e., if G is a subcubic graph, then* $\mathrm{pind}(A_G) = 1$. *As a consequence, subcubic graphs are* $(2, 2)$*-choosable.*

Proof. If G has maximum degree at most 3, then it follows from Lemma 8 that $\eta(z) = 1$ for all $z \in V(G) \cup E(G)$ is a non-singular index function. □

A graph G is a 2*-tree* if there is an acyclic orientation of G (also denoted by G) such that the following hold: (1) there are two adjacent vertices v_0, v_1 with $d^+_G(v_i) = i$ $(i = 0, 1)$. (2) every other vertex v has $d^+_G(v) = 2$, and the two out-neighbours of v are adjacent. If $N^+_G(v) = \{u, w\}$ and (u, w) is an arc, then v is called a *son* of the arc $e = (u, w)$. For an acyclic oriented graph G, for $v \in V(G)$, let $\rho_G(v)$ be the length of the longest directed path ending at v. So if v is a source, then $\rho_G(v) = 0$.

Theorem 10. *Let G be a* 2*-tree and let η be an index function of G. Assume $\eta(z) \geqslant 1$ for all $z \in E(G) \cup V(G)$, except that possibly there is one arc (u, w) with $\rho_G(u) \leqslant 1$, for which $\eta(w) \geqslant 0$ and $\eta(u) \geqslant 2$. Then η is non-singular for G.*

Proof. Assume the theorem is not true and G is a counterexample with minimum number of vertices. If the special arc (u, w) specified in the theorem does not exist, then let $e = (u, w)$ be an arc which has at least one son, and with $\rho_G(u) = 1$. Note that all the sons of e are sources. Let v be a son of (u, w) and let η' be the index function of $G' = G - v$ which is equal to η, except that $\eta'(u) = \eta(u) + 1 \geqslant 2$ and $\eta'(w) = \eta(w) - 1 \geqslant 0$. Then G' and η' satisfying the condition of the theorem, with e be the special edge (note that $\rho_{G-v}(u) \leqslant \rho_G(u) = 1$). Hence η' is non-singular for G'. It follows from Theorem 3 that η is non-singular for G.

Assume the special arc $e = (u, w)$ exists. If u is a source, then delete u, and let η' be the index function of $G' = G - u$ which is equal to η, except that $\eta'(v) = \eta(v) + 1$ for neighbours v of u. Then $\eta'(v) \geqslant 1$ for each vertex of G', hence G' and η' satisfying the condition of the theorem. So η' is non-singular for G', and it follows from Theorem 3 that η is non-singular for G.

If u is not a source vertex and e has a son v, then v is a source vertex. We delete v and let η' be the index function of $G' = G - v$ which is equal to η, except that $\eta'(u) = \eta(u) - 1$ and $\eta'(w) = \eta(w) + 1$. Then G' and η' satisfying the condition of the theorem, and hence η' is non-singular for G'. It follows from Theorem 3 that η is non-singular for G.

If u is not a source vertex and e has no son, then there is an arc $e' = (u, w')$ which has a son a. Since $\rho_G(u) \leqslant 1$, all the sons of e' are sources. If e' has more than one son, say a, b are both sons of e', then let η' be the restriction of η to $G - \{a, b\}$. By the minimality of G, η' is non-singular for $G - \{a, b\}$. By Corollary 7 (with D consists of the four arcs incident to a, b and D' consists of arcs au, bw'), η is non-singular for G. Assume e' has only one son a. Let η' be the restriction of η to $G - \{a, u\}$, except that $\eta'(w) = 1$. By the minimality of G, η' is non-singular for $G - \{a, u\}$. By Corollary 7 (with D consists of the four arcs incident to a, u and D' consists of arcs aw', uw), η is non-singular for G. □

Corollary 11. *Conjecture 2 holds for 2-trees, i.e., if G is a 2-tree, then* $\mathrm{pind}(A_G) = 1$, *and hence is* $(2, 2)$*-choosable.*

Theorem 12. *If T is a tree with leaves $v_1, v_2, \ldots, v_n$, and G is obtained from T by adding edges v_iv_{i+1} ($i = 1, 2, \ldots, n$, with $v_{n+1} = v_1$), then* $\mathrm{pind}(A_G) = 1$, *and hence G is* $(2, 2)$*-choosable.*

Proof. First we construct an acyclic orientation of G as follows: We choose a non-leaf vertex u of T as the root of T. Orient the edges of the tree from father to son. Then orient the added edges from v_i to v_{i+1} for $i = 1, 2, \ldots, n-1$, and orient the edge v_1v_n from v_1 to v_n. The resulting digraph is D. Now we choose a sub-digraph D' of D as follows: D' consists of a directed path P from the root vertex u to v_1, and all the edges v_iv_{i+1} for $i = 1, 2, \ldots, n-1$, and the edge v_1v_n. Let η be the constant function $\eta \equiv 1$, let $X = \{v_n\}$ and let $\eta'(v_n) = 0$, which is an index function of $G[X]$. Then η' is a non-singular index function of $G[X]$. To prove that $\mathrm{pind}(A_G) = 1$, i.e., η is a non-singular index function of G, it suffices, by Corollary 7, to show that for each vertex v,

$$1 + 2d^-_{D'}(v) - d^-_D(v) \geqslant d^+_{D'}(v).$$

This is a routine check. Assume first that v is not a leaf of T.

1. If v is not on path P, then $d^-_{D'}(v) = 0$, $d^-_D(v) = 1$ and $d^+_{D'}(v) = 0$. So $1 + 2d^-_{D'}(v) - d^-_D(v) = 0 \geqslant d^+_{D'}(v)$.

2. If v is on P, but is not the root u, then $d^-_{D'}(v) = 1$, $d^-_D(v) = 1$ and $d^+_{D'}(v) = 1$. So $1 + 2d^-_{D'}(v) - d^-_D(v) = 2 \geqslant d^+_{D'}(v)$.

3. If $v = u$, then $d^-_{D'}(v) = 0$, $d^-_D(v) = 0$ and $d^+_{D'}(v) = 1$. So $1 + 2d^-_{D'}(v) - d^-_D(v) = 1 \geqslant d^+_{D'}(v)$.

Next, consider the case that v is a leaf of T.

1. If $v = v_1$, then $d^-_{D'}(v) = 1$, $d^-_D(v) = 1$ and $d^+_{D'}(v) = 2$. So $1 + 2d^-_{D'}(v) - d^-_D(v) = 2 \geqslant d^+_{D'}(v)$.

2. If $v = v_i$, for $1 < i < n$, then $d^-_{D'}(v) = 1$, $d^-_D(v) = 2$ and $d^+_{D'}(v) = 1$. So $1 + 2d^-_{D'}(v) - d^-_D(v) = 1 \geqslant d^+_{D'}(v)$.

3. If $v = v_n$, then $d^-_{D'}(v) = 2$, $d^-_D(v) = 3$ and $d^+_{D'}(v) = 0$. So $1 + 2d^-_{D'}(v) - d^-_D(v) = 2 \geqslant d^+_{D'}(v)$. □

A *Halin graph* is a planar graph obtained by taking a plane tree (an embedding of a tree on the plane) without degree 2 vertices by adding a cycle connecting the leaves of the tree cyclically.

Corollary 13. *Conjecture 2 holds for Halin graphs, i.e., if G is a Halin graph, then* $\mathrm{pind}(A_G) = 1$*, and hence is* $(2,2)$*-choosable.*

A *grid* is the Cartesian product of two paths, $P_n \Box P_m$, with vertex set

$$V = \{(i,j) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$$

and edge set

$$E = \{(i,j)(i',j') : i = i', j' = j+1 \text{ or } i' = i+1, j' = j\}.$$

Lemma 14. *Assume $m, n \geqslant 1$. Let η be an index function of $P_n \Box P_m$, with $\eta(e) = 1$ for edges e, and one of the following holds:*

1 $\eta(v) = 1$ for all vertices v.

2 $\eta(v) = 1$ for all vertices v, except that $\eta(n,1) = 0$, and $\eta((n,j)) = 2$ for $2 \leqslant j \leqslant m$.

Then η is non-singular for G.

Proof. We prove it by induction on the number of vertices of G. The case $n = 1$ or $m = 1$ is easy and omitted. Assume $n, m \geqslant 2$. If $\eta(v) = 1$ for all vertices v, then we delete vertices $(n,1), (n,2), \ldots, (n,m)$ in this order. When deleting $(n,1)$, we increase $\eta(n,2)$ by 1 and decrease $\eta(n-1,1)$ by 1. When deleting (n,j) for $j \geqslant 2$, we increase $\eta(n,j+1)$ by 1 and increase $\eta(n-1,j)$ by 1. After all the vertices $(n,1), (n,2), \ldots, (n,m)$ are deleted, we obtain a grid $P_{n-1} \Box P_m$ and an index function η' which satisfies the condition of the lemma and hence is non-singular. By Theorem 3, η is non-singular.

Assume $\eta(n,1) = 0$ and $\eta(n,j) = 2$ for $2 \leqslant j \leqslant m$. We delete vertices $(n,m), (n,m-1), \ldots, (n,1)$ in this order, and need not to change η except for while deleting $(n,2)$, we increase $\eta(n,1)$ by 1. It follows from induction hypothesis that the resulting index function is non-singular for $P_{n-1} \Box P_m$, and by Theorem 3 that the original index function η is non-singular for G. □

Corollary 15. *Conjecture 2 holds for grids, and hence grids are* $(2,2)$*-choosable.*

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