On the multicolor Ramsey number for 3-paths of length three

Tomasz Łuczak* and Joanna Polcyn

A. Mickiewicz University
Faculty of Mathematics and Computer Science
ul. Umultowska 87,
61-614 Poznań, Poland

tomasz@amu.edu.pl

Submitted: Dec 6, 2016; Accepted: Jan 30, 2017; Published: Feb 17, 2017
Mathematics Subject Classifications: 05D10, 05C38, 05C55, 05C65

Abstract

We show that if we color the hyperedges of the complete 3-uniform hypergraph on $2n + \sqrt{18n+1} + 2$ vertices with n colors, then one of the color classes contains a loose path of length three.

Let P denote the 3-uniform path of length three by which we mean the only connected 3-uniform hypergraph on seven vertices with the degree sequence $(2, 2, 1, 1, 1, 1, 1)$. By $R(P; n)$ we denote the multicolored Ramsey number for P defined as the smallest number N such that each coloring of the hyperedges of the complete 3-uniform hypergraph $K_N^{(3)}$ with n colors leads to a monochromatic copy of P. It is easy to check that $R(P; n) \geqslant n+6$ (see [2, 5]), and it is believed that in fact equality holds, i.e.

$$R(P; n) = n + 6.$$

Gyárfás and Raeisi [2] proved, among many other results, that $R(P; 2) = 8$. Their theorem was extended by Omidi and Shahsiah [9] to loose paths of arbitrary lengths, but still only for the case of two colors. On the other hand, in a series of papers [5, 7, 12, 10] it was verified that $R(P; n) = n + 6$ for all $3 \leqslant n \leqslant 10$.

Note that from the fact that for $N \geqslant 8$ the largest P-free 3-uniform hypergraph on N vertices contains at most $\binom{N-1}{2}$ hyperedges (see [6]), it follows that for $n \geqslant 3$ we have (see [2])

$$R(P; n) \leqslant 3n + 1.$$

Our main goal is to improve the above bound.

---

*Partially supported by NCN grant 2012/06/A/ST1/00261.

Theorem. $R(P;n) \leqslant 2n + \sqrt{18n+1} + 2$.

Let C denote the (loose) 3-uniform 3-cycle, i.e. the only 3-uniform linear hypergraph with six vertices and three hyperedges. Furthermore, let F be the 3-uniform hypergraph on vertices v_1, v_2, v_3, v_4, v_5 such that the first four of these vertices span a clique, and v_5 is contained in the following three hyperedges: $v_1v_2v_5$, $v_2v_3v_5$, and $v_3v_4v_5$. The following fact will be crucial for our argument.

Lemma. *Let H be a 3-uniform P-free hypergraph on $n \geqslant 5$ vertices. Then we can delete from H fewer than $3n$ hyperedges in such a way that the resulting hypergraph contains no copies of C and F.*

Proof. Let us first consider components containing C. Jackowska, Polcyn and Ruciński [7] showed that each such component of H on n_i vertices has at most $3n_i - 8 < 3n_i$ hyperedges, provided $n_i \geqslant 7$. Furthermore, from the complete 3-uniform hypergraph on $n_i = 6$ vertices it is enough to delete $10 < 3n_i$ hyperedges to get a star, which clearly contains no copies of C and F. Hence, to get rid of all copies of C (and F in components containing C) it is enough to remove fewer than $3n'$ hyperedges from components containing them, where n' denotes the number of vertices in these components combined. Now let us consider components containing F but not C. It is easy to check by direct inspection that any hyperedge e which shares with F just one vertex would create a copy of P. Moreover, any hyperedge e' which shares with F two vertices would create a copy of C. Consequently, each copy of F in a P-free, C-free 3-uniform hypergraph is contained in a component on 5 vertices. Note that each such component has at most $\binom{5}{3} = 10$ hyperedges and we can destroy F by removing just $4 < 3 \cdot 5$ of them. Thus, one can delete from H fewer than $3n$ hyperedges to destroy all copies of C and F. □

Proof of Theorem. Consider a coloring of the hyperedges of the complete 3-uniform hypergraph on $2n+m$ vertices with n colors. Assume that no color class contains a copy of P. Then, by the lemma, we can mark as 'blank' fewer than $r = 3(2n+m)n$ hyperedges of the hypergraph in such a way that when we ignore blank hyperedges no color class contains a monochromatic copy of C and F.

Let us color a pair of vertices vw with a color s, $s = 1, 2, \ldots, n$, if there exist at least three hyperedges of color s, $s = 1, 2, \ldots, n$, which contain this pair. If there are many such colors we choose any of them; if there are none we leave vw uncolored. Note that every uncolored pair must be contained in at least $m-2$ blank hyperedges. Consequently, fewer than $3r/(m-2)$ pairs remain uncolored. But then there exists a color t, $t = 1, 2, \ldots, n$, such that there are more than

$$\left[\binom{2n+m}{2} - \frac{3r}{m-2}\right] \Big/ n = 2n + m + (2n+m)\left[\frac{m-1}{2n} - \frac{9}{m-2}\right]$$

pairs colored with t. If $m \geqslant \sqrt{18n+1} + 2$, then

$$\frac{m-1}{2n} - \frac{9}{m-2} > 0\,,$$

and the graph G_t spanned by these pairs has more edges than vertices. But this means that G_t contains a path of length 3, i.e. there are vertices v_1, v_2, v_3, v_4 and a color t such that each of the three pairs v_1v_2, v_2v_3, v_3v_4 is contained in at least three different hyperedges colored with t. We shall show that this leads to a contradiction.

Indeed, let H_t be a hypergraph spanned by hyperedges colored with the tth color. Observe first that since v_2v_3 is contained in three different hyperedges of H_t there must be one which is different from $v_1v_2v_3$ and $v_2v_3v_4$; let us call it $v_2v_3v_5$ where $v_5 \neq v_1, v_2, v_3, v_4$. Furthermore, v_1v_2 must be contained in a hyperedge v_1v_2w of H_t where $w \neq v_3, v_5$, while v_3v_4 is contained in some v_3v_4u, where $u \neq v_2, v_5$. Note now that if $w \neq v_4$ and $u \neq w, v_1$, then H_t contains a copy of P which contradicts the fact that it is P-free. The case $w = u \neq v_1, v_4$, as well as the cases $w = v_4$, $u \neq v_1$, and $u = v_1$, $w \neq v_4$, would lead to a cycle C. Finally, if the only possible choices for w and u are $w = v_4$ and $u = v_1$, then the vertices $v_1, \ldots, v_5$ span a copy of F, so we arrive at a contradiction again. □

Remark The bound $3n$ given in the lemma is rather crude. Using the fact that each component of H on n_i vertices containing C and two disjoint hyperedges has at most n_i+5 hyperedges, provided $n_i \geqslant 7$ (see [10]) and by careful analysis of 3-uniform intersecting families (see [1, 3, 4, 8, 11]) one can improve it by a constant factor and, consequently, improve by a constant factor the second order term in the estimate for $R(P; n)$. Nonetheless our method, based on the reduction of the hypergraph problem to the analogous problem for graphs, clearly cannot be used to produce an upper bound better than $2n$. However, we still believe that $n + 6$ could be the correct value for $R(P; n)$.

Acknowledgements

We wish to thank Andrzej Ruciński for stimulating conversations and the anonymous referee for his/her comments.

References

[1] P. Erdős, C. Ko, R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2), 12 (1961), 313–320.

[2] A. Gyárfás, G. Raeisi, *The Ramsey number of loose triangles and quadrangles in hypergraphs*, Electron. J. Combin., 19(2) (2012), #R30.

[3] J. Han, Y. Kohayakawa, *Maximum size of a non-trivial intersecting uniform family which is not a subfamily of the Hilton-Milner family*, Proc. Amer. Math. Soc., 145(1) (2017), 73–87.

[4] A. J. W. Hilton, E. C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2), 18 (1967), 369–384.

[5] E. Jackowska, *The 3-color Ramsey number for a 3-uniform loose path of length 3*, Australas. J. Combin, 63(2) (2015), 314–320.

[6] E. Jackowska, J. Polcyn, A. Ruciński, *Turán numbers for 3-uniform linear paths of length 3*, Electron. J. Combin., 23(2) (2016), #P2.30

[7] E. Jackowska, J. Polcyn, A. Ruciński, *Multicolor Ramsey numbers and restricted Turán numbers for the loose 3-uniform path of length three*, arXiv:1506.03759v1.

[8] A. Kostochka, D. Mubayi, *The structure of large intersecting families*, Proc. Amer. Math. Soc., to appear, arXiv:1602.01391.

[9] G. R. Omidi, M. Shahsiah, *Ramsey Numbers of 3-Uniform Loose Paths and Loose Cycles*, J. Combin. Theory Ser. A, 121 (2014), 64–73.

[10] J. Polcyn, *One more Turán number and Ramsey number for the loose 3-uniform path of length three*, Discuss. Math. Graph Theory, to appear, arXiv:1511.09073.

[11] J. Polcyn, A. Ruciński, *A hierarchy of maximal intersecting triple systems*, Opuscula Math., to appear, arXiv:1608.06114.

[12] J. Polcyn, A. Ruciński, *Refined Turán numbers and Ramsey numbers for the loose 3-uniform path of length three*, Discrete Math., 340 (2017), 107–118.