Symmetric Graphs with respect to Graph Entropy

Seyed Saeed Changiz Rezaei

Department of Mathematics
Simon Fraser University
Burnaby, BC, Canada

sschangi@sfu.ca

Ehsan Chiniforooshan

Google Inc.
Waterloo, ON, Canada

chiniforooshan@google.com

Abstract

Let $F_G(P)$ be a functional defined on the set of all the probability distributions on the vertex set of a graph G. We say that G is *symmetric with respect to $F_G(P)$* if the uniform distribution on $V(G)$ maximizes $F_G(P)$. Using the combinatorial definition of the entropy of a graph in terms of its vertex packing polytope and the relationship between the graph entropy and fractional chromatic number, we characterize all graphs which are symmetric with respect to graph entropy. We show that a graph is symmetric with respect to graph entropy if and only if its vertex set can be uniformly covered by its maximum size independent sets. This is also equivalent to saying that the fractional chromatic number of G, $\chi_f(G)$, is equal to $\frac{n}{\alpha(G)}$, where $n = |V(G)|$ and $\alpha(G)$ is the independence number of G. Furthermore, given any strictly positive probability distribution P on the vertex set of a graph G, we show that P is a maximizer of the entropy of the graph G if and only if its vertex set can be uniformly covered by its maximum weighted independent sets. We also show that the problem of deciding if a graph is symmetric with respect to graph entropy, where the weight of the vertices is given by probability distribution P, is co-NP-hard.

1 Introduction

The entropy of a graph is an information theoretic functional which is defined on a graph with a probability distribution on its vertex set. This functional was originally proposed by J. Körner in 1973 to study the minimum number of codewords required for representing an information source [7].

Let $VP(G)$ be the *vertex packing polytope* of a given graph G which is the convex hull of the characteristic vectors of its independent sets. Let $n := |V(G)|$ and P be a probability distribution on $V(G)$. Then the *entropy of G with respect to the probability distribution P* is defined as

$$H(G, P) = \min_{\mathbf{a} \in VP(G)} \sum_{v \in V(G)} p_v \lg(1/a_v),$$

where lg denotes the base 2 logarithm. J. Körner investigated the basic properties of the graph entropy in several papers from 1973 till 1992 [7, 8, 10, 9, 12, 13, 11].

Let F and G be two graphs on the same vertex set V. Then the union of graphs F and G is the graph $F \cup G$ with vertex set V and its edge set is the union of the edge set of graph F and the edge set of graph G. That is

$$V(F \cup G) = V,$$
$$E(F \cup G) = E(F) \cup E(G).$$

Perhaps, the most important property of the entropy of a graph is that it is sub-additive with respect to the union of graphs, that is

$$H(F \cup G, P) \leqslant H(F, P) + H(G, P).$$

This leads to the application of graph entropy for graph covering problem as well as the problem of perfect hashing.

The graph covering problem can be described as follows. Given a graph G and a family of graphs $\mathcal{G}$ where each graph $G_i \in \mathcal{G}$ has the same vertex set as G, we want to cover the edge set of G with the minimum number of graphs from $\mathcal{G}$. Using the sub-additivity of graph entropy one can obtain lower bounds on this number.

Graph entropy was used implicitly in a paper by Fredman and Komlós for the minimum number of perfect hash functions of a given range that hash all k-element subsets of a set of a given size [4].

G. Simonyi showed that the maximum of the graph entropy of a given graph over the probability distribution on its vertex set is equal to the logarithm of its fractional chromatic number [15]. For general information on fractional graph theory we refer the reader to [16]. In this paper, we characterize all strictly positive probability distributions which maximize the entropy of a given graph.

Let $\mathcal{S}$ be a multi-set of independent sets of a graph G. We say $\mathcal{S}$ is *uniform* over a subset of vertices W of the vertex set of G if each vertex $v \in W$ is covered by a constant number of independent sets in $\mathcal{S}$. We prove the following theorem in section 3.

Theorem 1. *For every graph G and every probability distribution P over $V(G)$, we have $H(G, P) = \lg \chi_f(G[\{v \in V(G) : p_v > 0\}])$ if and only if there exists a multi-set of independent sets $\mathcal{S}$ such that*

1. *$\mathcal{S}$ is uniform over $\{v \in V(G) : p_v > 0\}$, and*

2. *every independent set $I \in \mathcal{S}$ is a maximum weighted independent set with respect to P.*

We say a graph is *symmetric with respect to graph entropy* if the uniform probability distribution maximizes its entropy. It is worth noting that the notion of a symmetric graph with respect to a functional was already defined by G. Greco [6]. Furthermore, S.S. C. Rezaei and C. Godsil studied some classes of graphs which are symmetric with respect to graph entropy [1, 2]. Our theorem 1 gives a characterization of symmetric graphs, i.e, a graph is symmetric with respect to graph entropy if and only if its vertex set can be uniformly covered by its maximum size independent sets. We combine this characterization with results from [16] to get another characterization for symmetric graphs:

Corollary 1. *A graph G is symmetric if and only if $\chi_f(G) = \frac{n}{\alpha(G)}$.*

Finally we consider the complexity of deciding whether a graph is symmetric with respect to its entropy by proving the following theorem.

Theorem 2. *It is co-NP-hard to decide whether a given graph G is symmetric.*

2 Preliminaries

Here we recall some properties of entropy of graphs. The following lemma shows the monotonicity for graph entropy.

Lemma 2.1. (J. Körner). *Let F be a spanning subgraph of a graph G. Then for any probability distribution P we have $H(F, P) \leqslant H(G, P)$.*

The notion of substitution is defined as follows. Let F and G be two vertex disjoint graphs and v be a vertex of G. We substitute F for v by deleting v and joining all vertices of F to those vertices of G which have been adjacent with v. Let $G_{v \leftarrow F}$ be the resulting graph.

We extend the notion of substitution to distributions. Let P and Q be the probability distributions on $V(G)$ and $V(F)$, respectively. Then the probability distribution $P_{v \leftarrow Q}$ on $V(G_{v \leftarrow F})$ is given by $P_{v \leftarrow Q}(x) = P(x)$ if $x \in V(G) \setminus \{v\}$ and $P_{v \leftarrow Q}(x) = P(v)Q(x)$ if $x \in V(F)$.

Now we state the following lemma which was proved by J. Körner, et. al. in [11].

Lemma 2.2. (J. Körner, G. Simonyi, and Zs. Tuza). *Let F and G be two vertex disjoint graphs, v a vertex of G, while P and Q are probability distributions on $V(G)$ and $V(F)$, respectively. Then we have*

$$H(G_{v \leftarrow F}, P_{v \leftarrow Q}) = H(G, P) + P(v)H(F, Q).$$

Notice that the entropy of an empty graph (a graph with no edges) is always zero (regardless of the distribution on its vertices).

3 Graph entropy and fractional chromatic number

G. Simonyi in [14, 15] established the relationship between the entropy of a graph $H(G, P)$ with its fractional chromatic number $\chi_f(G)$ by showing

$$\max_P H(G, P) = \lg \chi_f(G) \tag{1}$$

Here we characterize strictly positive probability distributions which maximize $H(G, P)$.

First we recall a characterization of uniform independent set covers using fractional chromatic number. A b-fold colouring of the vertices of a graph G is an assignment of b-subsets of a set with a elements such that adjacent vertices get disjoint b-subsets. The least a such that G admits a b-fold colouring is called the b-fold chromatic number of G and is denoted by $\chi_b(G)$.

Theorem 3.1. *([16]) For every graph G and positive integer b*

$$\chi_f(G) \leqslant \frac{\chi_b(G)}{b}.$$

Furthermore, there exists an integer that realizes the equality.

As a corollary to the above theorem we have

Corollary 3.2. *Let $\mathcal{S}$ be a multi-set of independent sets of G. Then each element of $\mathcal{S}$ induces a maximum independent set of G with size $\alpha(G)$ and $\mathcal{S}$ is a uniform cover over the vertices of G if and only if $\chi_f(G) = \frac{n}{\alpha(G)}$.*

Proof. First, assuming $\chi_f = \frac{n}{\alpha}$, we prove that there exists a uniform independent set cover whose elements are maximum independent sets. From Theorem 3.1, there exists a b-fold colouring such that $\chi_b(G) = a$ and we have

$$\chi_f(G) = \frac{n}{\alpha} = \frac{a}{b}. \tag{2}$$

Now we construct a uniform maximum independent cover $\mathcal{S}$ as follows. Let A be the set of colours of size a and B be the function that assigns a b-subset of A to every vertex. For every $x \in A$, $I_x = \{v \in V(G) : x \in B(v)\}$ is an independent set. So, $\mathcal{S} = \{I_x : x \in A\}$ is a uniform independent set cover of size a. Equation (2) tells us that the average size of independent sets in $\mathcal{S}$ is α and so they all must be maximum independent sets.

Conversely, assume that G admits a uniform maximum independent set cover $\mathcal{S}$ such that each vertex $v \in V(G)$ lies in exactly b elements of $\mathcal{S}$, and so $\frac{|\mathcal{S}|}{b} = \frac{n}{\alpha}$. Then, $B(v) = \{S \in \mathcal{S} : v \in S\}$ is a b-fold colouring. Let b' be an integer such that $\chi_f(G) = \frac{\chi_{b'}(G)}{b'}$ and B' be the b'-fold colouring that achieves this. Then

$$\frac{n}{\alpha} \leqslant \frac{\chi_{b'}(G)}{b'} = \chi_f(G) \leqslant \frac{\chi_b(G)}{b} \leqslant \frac{|\mathcal{S}|}{b} = \frac{n}{\alpha}.$$

The left-most inequality comes from the fact that $\alpha\chi_{b'}(G)$ is an over-estimation of $\sum_{v \in V(G)} |B'(v)| = nb'$. $\qquad\square$

We prove Theorem 1 in two steps, i.e., using Theorems 3.3 and 3.5 below.

Theorem 3.3. *For every graph G and every probability distribution P over $V(G)$, if $H(G, P) = \lg \chi_f(G[\{v \in V(G)|p_v > 0\}])$, then there exists a multi-set of independent sets $\mathcal{S}$ such that*

1. *$\mathcal{S}$ is uniform over $\{v \in V(G) : p_v > 0\}$, and*

2. *every independent set $I \in \mathcal{S}$ is a maximum weighted independent set with respect to P.*

Proof. Consider a fractional colouring $(\mathcal{I}, w)$ of G, where $\mathcal{I}$ is the family of independent sets of G and $w : \mathcal{I} \to \mathbb{Q}^+$ is a weight function such that

$$\sum_{I \in \mathcal{I}} w_I = \chi_f(G), \tag{3}$$

and

$$\sum_{I \in \mathcal{I}, v \in I} w_I \geqslant 1, \tag{4}$$

for all $v \in V(G)$. We define $\mathbf{x}^* = \sum_{I \in \mathcal{I}} \frac{w_I}{\chi_f(G)} \cdot \mathbf{I}$, where $\mathbf{I}$ is the characteristic vector of the independent set I. That is $\mathbf{I} \in \{0,1\}^{|V(G)|}$, and $\mathbf{I}_v = 1$ if $v \in I$ and $\mathbf{I}_v = 0$ otherwise. Clearly, $\mathbf{x}^* \in \mathrm{VP}(G)$ due to (3), and $x_v^* \geqslant \frac{1}{\chi_f(G)}$, for all $v \in V(G)$, due to (4).

We turn the set $\mathcal{I}$ to a multi-set $\mathcal{S}$ by setting the multiplicity of any independent set I to $r \cdot w_I$, where r is an integer such that $r \cdot w_I \in \mathbb{N}$ for all $I \in \mathcal{I}$. If for some $v \in D_P = \{v \in V(G) : p_v > 0\}$, $x_v^* > \frac{1}{\chi_f(G)}$, then $H(G,P) \leqslant -\sum_{v \in V(G)} p_v \lg x_v^* < \lg \chi_f(G)$, which is a contradiction. So, every $v \in D_P$ is in exactly $r \cdot x_v^* \cdot \chi_f(G) = r$ independent sets of $\mathcal{S}$, which means $\mathcal{S}$ is uniform over D_P.

We also claim that $P_I = \alpha_P$ for all $I \in \mathcal{I}$, where α_P is the weight of maximum weighted independent set of G with respect to P. Indeed, if there exists an $I_1 \in \mathcal{I}$ such that $P_{I_1} < \alpha_P$, then we define $\mathbf{x}^\epsilon = \mathbf{x}^* - \epsilon \mathbf{I}_1 + \epsilon \mathbf{M}$, where $\mathbf{M}$ is the characteristic vector of an arbitrary maximum weighted independent set M. Note that $\mathbf{x}^\epsilon \in \mathrm{VP}(G)$ for all $0 \leqslant \epsilon \leqslant \frac{w_{I_1}}{\chi_f(G)}$.

Now consider

$$d(\epsilon) = \sum_{v \in I_1 \setminus M} p_v \lg \frac{x_v^\epsilon}{x_v^*} + \sum_{v \in M \setminus I_1} p_v \lg \frac{x_v^\epsilon}{x_v^*} = \lg(1 - \epsilon \chi_f(G)) \sum_{v \in I_1 \setminus M} p_v + \lg(1 + \epsilon \chi_f(G)) \sum_{v \in M \setminus I_1} p_v.$$

We have $d(0) = 0$, d is differentiable at zero, and

$$d'(0) = (\sum_{v \in M \setminus I_1} p_v - \sum_{v \in I_1 \setminus M} p_v) \chi_f(G) > 0.$$

So, there exists a positive number ϵ^+ such that $d(\epsilon)$ is positive for all ϵ in $(0, \epsilon^+)$. Then, for an arbitrary $\epsilon \in (0, \min\left\{\frac{w_{I_1}}{\chi_f(G)}, \epsilon^+\right\})$,

$$H(G,P) \leqslant -\sum_{v \in V(G)} p_v \lg x_v^\epsilon = -\sum_{v \in V(G)} p_v \lg x_v^* - d(\epsilon) < -\sum_{v \in V(G)} p_v \lg x_v^* = \lg \chi_f(G),$$

which is a contradiction. $\qquad \square$

It is easy to see that the converse of Theorem 3.3 holds for any uniform distribution. We call a graph G *symmetric* if $H(G,U) \geqslant H(G,P)$ for all probability distributions P on $V(G)$, where U is the uniform distribution.

Corollary 3.4. *A graph G is symmetric if and only if $\chi_f(G) = \frac{n}{\alpha(G)}$.*

Proof. For every $\mathbf{x} \in \mathrm{VP}(G)$ we have $\sum_{v \in V(G)} x_v \leqslant \alpha$. If $\chi_f(G) = \frac{n}{\alpha}$, then

$$\prod_{v \in V(G)} x_v \leqslant \left(\frac{\alpha}{n}\right)^n \Rightarrow -\frac{1}{n} \sum_{v \in V(G)} \lg x_v \geqslant -\frac{1}{n} \lg \left(\frac{\alpha}{n}\right)^n = -\lg \frac{\alpha}{n} = \lg \chi_f(G).$$

This along with equation (1) implies $H(G,U) = \lg \chi_f(G)$, and so G is symmetric.

On the other hand, if G is symmetric, then, according to Theorem 3.3, there exists a uniform maximum independent set cover. Thus from Corollary 3.2, we have $\chi_f(G) = \frac{n}{\alpha}$. $\qquad \square$

Now we prove the converse of Theorem 3.3 for every distribution.

Theorem 3.5. *Let G be a graph with probability distribution P on its vertex set. Suppose that there exists a multi-set of independent sets $\mathcal{S}$ such that*

1. *$\mathcal{S}$ is uniform over $\{v \in V(G) : p_v > 0\}$, and*

2. *every independent set $I \in \mathcal{S}$ is a maximum weighted independent set with respect to P.*

Then $H(G, P) = \lg \chi_f(G[\{v \in V(G) : p_v > 0\}])$.

Proof. First we assume that the probability of every vertex $v \in V(G)$, i.e., p_v is equal to $\frac{n_v}{m}$ for some n_v and $m \in \mathbb{N}$. We then construct the graph G' by blowing each vertex v up to a set of size n_v and making the corresponding vertices adjacent to the neighbours of v. We consider each set of vertices substituted for each vertex v as an independent set F_v of size n_v with uniform distribution $\frac{1}{n_v}$. Then G' is a probabilistic graph with uniform distribution $\frac{1}{m}$ on its vertex set, and repeated application of Lemma 2.2 leads to

$$H(G', \frac{1}{m}) = H(G, P) + \sum_v \frac{n_v}{m} H(F_v, \frac{1}{n_v}).$$

Noting that F_v is an independent set, we have

$$H(F_v, \frac{1}{n_v}) = 0, \ \forall \ v \in V(G),$$

and consequently,

$$H(G', \frac{1}{m}) = H(G, P). \tag{5}$$

Note that the vertex set of G is uniformly covered by maximum weighted independent set with respect to P. Therefore, due to the construction of G', graph G' is uniformly covered by its maximum independent sets. Thus using Corollary 3.2 and Corollary 3.4 and equation (5), we have

$$H(G', \frac{1}{m}) = \lg \chi_f(G') = H(G, P) \leqslant \lg \chi_f(G). \tag{6}$$

Noting that there is a homomorphism from G to G', we get

$$\chi_f(G) \leqslant \chi_f(G').$$

This along with (6) implies that $\chi_f(G) = \chi_f(G')$, and hence, probability distribution P over the vertex set of G maximizes $H(G, P)$.

Now, suppose that P is a real probability distribution over the vertex set of G. Next we will show that graph entropy $H(G, P)$ is a continuous function of P. Let P be a strictly positive probability distribution over $V(G)$. Then for every $\epsilon > 0$ we set

$$\delta = \frac{1}{2} \min_{v \in V} p_v \times \min\left\{1, \frac{\epsilon}{|V(G)|H(G, P)}\right\}.$$

Therefore, we have

$$\|P - P'\| < \delta \rightarrow |H(G, P') - H(G, P)| < \epsilon.$$

First we show $H(G, P') < H(G, P) + \epsilon$. Let $\mathbf{a}^* \in VP(G)$ achieve $H(G, P)$, that is

$$H(G, P) = \sum_v p_v \lg \frac{1}{a_v^*}.$$

Thus

$$\left(\min_v p_v \right) \times \sum_v \lg \frac{1}{a_v^*} \leqslant H(G, P)$$

$$\Rightarrow \sum \lg \frac{1}{a_v^*} \leqslant \frac{H(G, P)}{\min_v p_v} \Rightarrow \lg \frac{1}{a_v^*} \leqslant \frac{H(G, P)}{\min_v p_v}, \ \forall v \in V. \tag{7}$$

On the other hand, setting $\delta_v = p_v' - p_v$, we have

$$\begin{aligned} H(G, P') &= \min_{\mathbf{a} \in VP(G)} \sum_v (p_v + \delta_v) \lg \frac{1}{a_v} \\ &\leqslant \sum_v (p_v + \delta_v) \lg \frac{1}{a_v^*} = H(G, P) + \sum_v \delta_v \lg \frac{1}{a_v^*} \\ &\leqslant H(G, P) + |V(G)| \times \delta \times \frac{H(G, P)}{\min_v p_v} \leqslant H(G, P) + \frac{\epsilon}{2}. \end{aligned} \tag{8}$$

Now we show that $H(G, P') > H(G, P) - \epsilon$. Let $\mathbf{b}^* \in VP(G)$ achieve $H(G, P')$, that is

$$H(G, P') = \sum_v p_v' \lg \frac{1}{b_v^*}. \tag{9}$$

If $H(G, P) \leqslant H(G', P)$, we are done. Thus we may assume $H(G, P) > H(G, P')$. Then using the value for δ defined above we have

$$\lg \left(\frac{1}{b_v^*} \right) \leqslant \frac{H(G, P')}{\min_v p_v'} \leqslant \frac{2H(G, P)}{\min_v p_v}, \ \forall v \in V(G).$$

Now using the above equation and equation (9), we get

$$\begin{aligned} H(G, P') &= \sum_v p_v \lg \frac{1}{b_v^*} + \sum_v \delta_v \lg \frac{1}{b_v^*} \\ &\geqslant H(G, P) + \sum -|\delta_v| \lg \frac{1}{b_v^*} \\ &> H(G, P) - |V(G)| \times 2 \times \frac{H(G, P)}{\min_v p_v} \times \delta \\ &\geqslant H(G, P) - \epsilon. \end{aligned} \tag{10}$$

Thus $H(G, P)$ is a continuous function. Now note that there exists a sequence of rational probability distribution P_k which tends to P as $k \to \infty$. Consequently, there exists a sequence of graphs G_k' with the corresponding sequence of uniform probability distributions

$\frac{1}{m_k}$ constructed as explained above. Then, noting that $H(G, P)$ is a continuous function with respect to P and using (6), we have

$$\lim_{k\to\infty} H(G'_k, \frac{1}{m_k}) = \lim_{k\to\infty} \lg \chi_f(G'_k) = \lim_{k\to\infty} H(G, P_k) = H(G, P) \leqslant \lg \chi_f(G). \qquad (11)$$

Since there is a homomorphism from G to G'_k for every k, we get

$$\chi_f(G) \leqslant \lim_{k\to\infty} \chi_f(G'_k). \qquad (12)$$

Therefore, (11) and (12) imply that probability distribution P over the vertex set of G maximizes $H(G, P)$. $\qquad\square$

3.1 Computational Complexity

In this section, we discuss the complexity of computing graph entropy by proving the following theorem.

Theorem 3.6. *It is co-NP-hard to decide whether a given graph G is symmetric.*

Proof. We use a reduction from the maximum independent set problem. Assume we are given a graph F and an integer k, and let A and B be two disjoint sets of size $k-1$ disjoint from $V(F)$. Then, graph F has an independent set of size at least k if and only if the graph G, defined below, is not symmetric:

$$V(G) = A \cup V(F) \times B,$$

$$\begin{aligned}
E(G) \;=\; & \{\{a, (v, b))\} : a \in A, v \in V(F), b \in B\} \cup \\
& \{\{(v, b), (v', b')\} : v, v' \in V(F), b, b' \in B, v \neq v', b \neq b'\} \cup \\
& \{\{(v, b), (v', b)\} : (v, v') \in E(F), b \in B\}
\end{aligned}$$

To see this, it is enough to note that $\alpha(G) = \max\{\alpha(F), k-1\}$ and G has a uniform independent set cover whose independents sets are all of size $k-1$. If $\alpha(F) \leqslant k-1$, then $\alpha(G) = k-1$ and due to the construction of G the vertex set of G is covered uniformly by its maximum size independent sets. Therefore, using Corollary 3.2 and Corollary 3.4, graph G is symmetric with respect to graph entropy. Conversely, if $\alpha(F) \geqslant k$, then since the vertices of G in A are adjacent to all vertices in $V(G) \setminus A$, the vertex set of G is not covered uniformly by its maximum size independent sets. Consequently, from Corollary 3.2 and Corollary 3.4, graph G is not symmetric with respect to graph entropy. $\qquad\square$

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