The three colour hat guessing game on cycle graphs

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Abstract

We study a cooperative game in which each member of a team of N players, wearing coloured hats and situated at the vertices of the cycle graph C_N, is guessing their own hat colour merely on the basis of observing the hats worn by their two neighbours without exchanging the information. Each hat can have one of three colours. A predetermined guessing strategy is winning if it guarantees at least one correct individual guess for every assignment of colours. We prove that a winning strategy exists if and only if N is divisible by 3 or $N = 4$. This asymmetric game is an example of relational system using incomplete information about an unpredictable situation, where at least one participant has to act properly.

Keywords: cooperative games, games on graphs, hat puzzle, hat guessing problems, deterministic strategy, system reliability, information network, graph colouring

1 Introduction

N ladies wearing white hats are sitting around the table and discussing a tricky task which is going to be presented to them by the Wizard. They know he will suddenly paint each hat one of three colours (green, orange or purple) in an unpredictable way and then ask each of them to independently guess her own hat colour. The light is so dim that everyone will only see the hat colours of her two neighbours. If at least one of the ladies guesses right, they will all win; if they all guess wrong, they will lose; and they want to be absolutely certain of winning. However, can they devise a winning strategy before inviting the Wizard? The answer, depending on N, is presented in this paper.

Motivation

This asymmetric random-cooperative game is an example of information network using an incomplete flow of information about an unpredictable situation, where at least one link has to act or work properly, so the strategy has to be deterministic. Problems of this kind have become popular in recent years both as mathematical puzzlers and research subjects, cf. [1], [2], [3], [17].

Basic results so far have mainly concerned two colours or probabilistic variants (the expected number of correct guesses) or estimates. Exact results were obtained for certain kinds of the visibility graph (which is C_N in the present paper) such as a complete graph or a tree. For overviews, see [6], [10], [15], [17] and Section 4.4. The statistical variants were studied, in particular, in [5], [7], [12], [16], [13], and the deterministic ones in [6] and [15]. See also [4], [8], [9], [11].

While the round-table game of three colours is simple and natural to consider, it had not been solved for any $N > 4$ until the complete solution by the present author at the beginning of 2010. An exposition [14] was subsequently published on ArXiV.

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2 Coordinates and strategies

2.1 The setting

Configuration. The team players are seeing each other along the edges of the cyclic graph C_N. Let the set $V_k = \{v_0^k, v_1^k, v_2^k\}$ represent the three different appearances of the k-th hat. We let

$$v_i^k = v^k(i) = (i,k) \quad \text{for} \quad i \in \mathbf{Z}_3,\ k \in \mathbf{Z}_N$$

to make the sets V_k pairwise disjoint within this index range, and we extend the indices periodically to $\mathbf{Z}$ by putting

$$v_i^k = v^k(i) = v^k(j) = v_j^m \quad \text{and} \quad V_k = V_m$$
$$\text{if} \quad i \equiv j \pmod 3 \quad \text{and} \quad k \equiv m \pmod N \qquad (i,j,k,m \in \mathbf{Z})$$

Index k will be counted in the positive direction (to the right). The correspondence $v_i^k \sim i$ will basically be used, but the sets V_k can be conveniently re-parametrized in certain situations.

The configuration graph. The flow of information is represented by the *configuration graph*, denoted $G = G_N = G(N) = 3 * C_N$, whose $3N$-element set of vertices is $V = V(G) = \bigcup_{k=1}^N V_k$ and $9N$-element set of edges is

$$E = E(G_N) = \bigcup_{k \in \mathbf{Z}_N} E_{k,k+1}$$

where

$$E_{k,k+1} = E_{k,k+1}^G = V_k \times V_{k+1} = \{\overline{xy} : x \in V_k \text{ and } y \in V_{k+1}\}$$

is the set of nine edges between V_k and V_{k+1}.

Strategies. Wizard's strategy S is a cycle of length N in graph G_N. Equivalently, S is an infinite colour-assigning sequence $k \mapsto s_k \in V_k$ with $k \in \mathbf{Z}$, having period N.

An *individual* guessing strategy of Player k is represented by a function

$$F_k : V_{k-1} \times V_{k+1} \to V_k.$$

A *composite* or *collective* strategy is a sequence

$$F = (F_1, \dots, F_N),$$

or equivalently, an infinite sequence $\mathbf{Z} \ni k \mapsto F_k$, satisfying $F_{k+N} = F_k$. Strategy S defeats F if

$$s_k \neq F_k(s_{k-1}, s_{k+1}) \quad \text{for all} \quad k, \tag{1}$$

otherwise F wins over S. Strategy F is called *winning* if it wins over every strategy S, i.e., there is no N-periodic sequence (s_k) satisfying (1). If there exists such a sequence, F will be a *losing* strategy.

The main result of this paper is:

Theorem 1. *In the three colour hat guessing game on the cycle of length N a winning strategy exists if and only if $N = 4$ or N is divisible by three.*

Any path which bypasses al the guesses will be called admissible.

Definition 2. Let J be any set of consecutive integers. A path $\omega = (s_k)_{k\in J}$ in graph G_N, where $s_k \in V_k$ for $k \in J$, will be called *admissible* by F if

$$s_k \neq F_k(s_{k-1}, s_{k+1}) \quad \text{whenever} \quad k-1, k+1 \in J$$

The family of all the admissible paths (or sequences) will be denoted $\mathcal{A} = \mathcal{A}_F = \mathcal{A}(F)$.

A path ω is in $\mathcal{A}$ iff all its 2-edged subpaths (3-element subsequences) are in $\mathcal{A}$. If ω is periodic, the period must be a multiple of N, and period N defeats F. For all $(a, c) \in V_{k-1} \times V_{k+1}$ the relation $\overline{axc} \in \mathcal{A}_F$ (i.e., $f_k(a, c) \neq x$) has exactly two solutions, while $\overline{abx} \in \mathcal{A}_F$ and $\overline{xbc} \in \mathcal{A}_F$ may have anywhere from 0 to 3 solutions each.

2.2 Linear coordinates

The method introduced here is essential for the construction of certain strategies, plays some role in the non-existence proofs, and will arise from a different perspective in Section 3.

The covering graph. The configuration graph G_N has an infinite lifting $\Gamma = \Gamma_3 = 3 * \mathbf{Z}$ independent of N, with vertices (i, k) and edges $\overline{(i, k)(j, k+1)}$ whenever $i, j \in \mathbf{Z}_3$ and $k \in \mathbf{Z}$. A covering $\Phi : \Gamma \to G_N$ is induced by any sequence of bijective maps $(i, k) \mapsto \varphi_k(i) \in V_k$. A Φ-lifting of any admissible path will also be called *admissible* (or (F, Φ)-admissible).

Notation. Technically, any individual strategy F_k $k \in \mathbf{Z}$ can be described as a function $f = f_k : \mathbf{Z}_3 \times \mathbf{Z}_3 \to \mathbf{Z}_3$ or a 3×3 matrix $M = M_k = M(f_k) = [f_k(i, j)]$. These representations may be simpler if the elements of the neighbouring sets V_{k-1}, V_k, V_{k+1} are conveniently ordered. However, it may turn out that V_k and V_{k+N} should be ordered differently. Still, certain functions f (such as symmetric or cyclic) are insensitive to (a certain kind of) change of ordering as long as it is the same for V_{k-1}, V_k, V_{k+1}.

For example, the function $f(i, j) = -i-j = (i+j)/2$ satisfies $f(i-1, j-1) = f(i, j)-1$ and $f(-i, -j) = -f(i, j)$, which implies $f(\sigma(i), \sigma(j)) = \sigma(f(i, j))$ for each permutation σ.

Any covering $\Phi : \Gamma \to G_N$ defines a system of coordinates $\Phi \sim (\varphi_k : \ k \in \mathbf{Z})$ and corresponds to a certain matrix arrangement of $V(G_N)$:

$$\Phi \sim [\varphi_k(l)] \sim \begin{bmatrix} \cdots & u_0^{-N} & \cdots & u_0^0 & u_0^1 & u_0^2 & \cdots & u_0^N & u_0^{N+1} & \cdots \\ \cdots & u_1^{-N} & \cdots & u_1^0 & u_1^1 & u_1^2 & \cdots & u_1^N & u_1^{N+1} & \cdots \\ \cdots & u_2^{-N} & \cdots & u_2^0 & u_2^1 & u_2^2 & \cdots & u_2^N & u_2^{N+1} & \cdots \end{bmatrix}$$

Here we have $u_l^k = \varphi_k(l)$. Then f_k will operate on line numbers l (with $l = 0, 1, 2$ here).

Coordinates may be simply defined by a $3 \times \infty$ matrix with entries in $\mathbf{Z}_3$ if we assume $i \sim v^k(i)$, if we know the position of column 0 and the line order (which is usually $l = 0, 1, 2$

or $(1,2,3)$ or $(1,0,-1)$) and the value of N. For example, for $N=3$ we can define (if the column numbers are specified):

$$\Phi \sim \begin{bmatrix} \cdots & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & \cdots \\ \cdots & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & \cdots \\ \cdots & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & \cdots \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cdots & 0 & 2 & 1 & 2 & 1 & 0\cdots \\ \cdots & 1 & 0 & 2 & 1 & 0 & 2\cdots \\ \cdots & 2 & 1 & 0 & 0 & 2 & 1\cdots \end{bmatrix} \tag{2}$$

Regularity above means that column $k+3$ is obtained from column k by rotation upward or transposition $(1,-1)$, respectively. In this way all individual strategies may be defined by one invariant function, such as $f(i,j) = -i-j+1$ or $f(i,j) = i-j$, respectively. Matrix Φ may be also vertically infinite ($\mathbf{Z} \times \mathbf{Z}$) if index l is periodically extended to $\mathbf{Z}$.

Representation of strategy. Basic parametrizations are given by $\varphi_k(i) = v_i^k$. The basic represantation of F_k is g_k, where $F_k(v_i^{k-1}, v_j^{k+1}) = v^k(g_k(i,j))$. In general, the relation between F_k and f_k is: $F_k(\varphi_{k-1}(i), \varphi_{k+1}(j)) = \varphi_k(f_k(i,j))$ or equivalently,

$$V_k \ni F_k(x,y) = \varphi_k \circ f_k(\varphi_{k-1}^{-1}(x), \varphi_{k+1}^{-1}(y)) \quad \text{for} \quad (x,y) \in V_{k-1} \times V_{k+1} \tag{3}$$

In addition, consistency requires $F_k = F_{k+N}$ or, in Φ-coordinates,

$$f_k(\sigma_{k-1}(i), \sigma_{k+1}(j)) = \sigma_k(f_{k+N}(i,j)), \quad \text{where} \quad \sigma_k = \varphi_k^{-1} \circ \varphi_{k+N}.$$

The permutation $\sigma_k : \mathbf{Z}_3 \to \mathbf{Z}_3$ may be regarded as a *transition* function and represents the identity between V_k and V_{k+N} in the following way: $\varphi_k(i') = \varphi_{k+N}(i)$ iff $i' = \sigma_k(i)$. If $\sigma_k = \sigma$ and $f_k = f$ are independent of k, consistency reduces to the σ-*invariance* of f:

$$f(\sigma(i), \sigma(j)) = \sigma(f(i,j)) \qquad (\text{if } \sigma = \varphi_k^{-1} \circ \varphi_{k+N} \ (k \in \mathbf{Z})) \tag{4}$$

In particular, we have *twisted* coordinates if σ is a rotation: $\sigma(i) = i \pm 1$, and *transposed* coordinates if σ is a transposition. In the examples (2) we have $\sigma(l) = l-1$ or $\sigma(l) = -l$.

2.3 Short cycles

The case $N = 2$. Let us take a closer look at the simplest puzzle (outside our main problem) with just two players and two possible hat colours. One person should guess that their hats have the same colour and the other person sholud guess the opposite. If we interprete the assigned colours as elements $A, B \in \mathbf{Z}_2$, we can write the effect as a valid alternative: $A = B$ or $B = A+1$.

In fact, this strategy has an even simpler formula in transposed coordinates (adapted for $\mathbf{Z}_2$): $f_k(i) = i$ for all k. Here, the adapted guessing functions have one argument, $f_k : V_{k+1} \to V_k$, and the coordinate matrix has the following form:

$$\begin{matrix} \cdots & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 1 & 1 & 0 & 0 & 1 & 1 & \cdots \end{matrix}$$

Admissible paths are those without horizontal edges. They have period 2 in coordinates, but in the graph G_2 they become one path of period 4, which may be viewed as the boundary of a Möbious band.

bf 2+3 colours. Next, suppose Player 1 hat can still have two colours, but Player 2 hat can have three colours. Then there are six possible colour assignments. With any strategy, Player 1 guesses right for three assignments, Player 2 for two. Since the number of assignments is $6 > 3 + 2$, they can both be wrong and they have no winning strategy.

The case $N = 3$. If $A, B, C \in \mathbf{Z}_3$ represent the appearances of hats, a winning strategy can be based on the alternative:

$$A = -B - C \quad \text{or} \quad B = -C - A - 1 \quad \text{or} \quad C = -A - B + 1 \tag{5}$$

clearly valid in $\mathbf{Z}_3$.

It turns out that the same can be expressed with a single formula in twisted coordinates which are given by $\varphi_k(s) = s$ for $k = 1, 2, 3$ and $\varphi_0(s) = s - 1$, $\varphi_4(s) = s + 1$. This corresponds to the matrix:

$$\begin{matrix} \dots & 2 & 0 & 0 & 0 & 1 & \dots \\ \dots & 0 & 1 & 1 & 1 & 2 & \dots \\ \dots & 1 & 2 & 2 & 2 & 0 & \dots \end{matrix} \tag{6}$$

In this system we define $f_k(i, j) = f(i, j) = -i - j = (i + j)/2$, which is consistent since $\sigma(s) = s + 1$ and $f(i + 1, j + 1) = f(i, j) + 1$. By (3), we convert to basic coordinates:

$$g_2(i, j) = f(i, j) = -i - j,$$
$$g_1(i, j) = f(i + 1, j) = -i - j - 1, \quad g_3(i, j) = f(i, j - 1) = -i - j + 1$$

obtaining equations (5).

A strategy defined by $f(i, j) = -i - j$ in any twisted coordinates on G_N may be denoted $\boldsymbol{C0(N)}$. However, for $N = 4$ it is defeated by the sequence

$$\dots(0100)(2022)(1211)(0100)\dots$$

(coordinates like above) and loses also for all $N > 4$ (cf. Proposition 3). Nevertheless, this method can be generalized (outside our main problem) to the classic complete visibility graph K_N with N colours $X_k \in Z_N$ by using the alternative:

$$X_k = k - \sum_{j \neq k} X_j \qquad (k \in \mathbf{Z}_N).$$

The case $N = 4$. Let a strategy F on G_4 have the form $f_k(i, j) = f(i, j) = j - i$ in the transposed coordinates given by $\sigma(s) = -s$. This is consistent since $f(-i, -j) = -f(i, j)$. Let $A, B, C, D \in \mathbf{Z}_3$ represent elements of V_1, V_2, V_3, V_4, respectively. Then the basic form of F is:

$$\begin{aligned} g_1(D, B) = f(-D, B) = D + B, \quad & g_2(A, C) = f(A, C) = -A + C, \\ g_3(B, D) = f(B, D) = -B + D, \quad & g_4(C, A) = f(C, -A) = -C - A \end{aligned} \tag{7}$$

Let us verify the alternative:

$$\begin{aligned} A = D + B \quad & B = -A + C, \\ C = -B + D \quad & D = -C - A \end{aligned} \tag{8}$$

If neither equality in the left column is satisfied, then their sum or difference will be satisfied in $\mathbf{Z}_3$. By adding them we get $D = -C - A$, by subtracting we get $B = -A + C$. It follows that F wins. We also note that exactly one player guesses right in 72 cases and all guess right in nine.

The analogous strategy on G_N will be denoted $\boldsymbol{S1(N)}$, but for $N = 5$ it is defeated by the sequence $\ldots(00101)(00202)(00101)\ldots$ and it loses for all $N > 4$ as well (cf. next section 2.4).

2.4 Characteristic 0

The strategies $\boldsymbol{S0(N)}$ and $\boldsymbol{S1(N)}$ have the following property: every edge $\beta \in E(G_N)$ has exactly two left-incident edges α satisfying $\overline{\alpha\beta} \in \mathcal{A}_F$ and exactly two right-incident edges γ satisfying $\overline{\beta\gamma} \in \mathcal{A}_F$. That is, every edge of G_N has exactly four admissible immediate continuations to the left and right. Any strategy F with this property will be called *symmetric* or of *characteristic* 0, which will be denoted $\chi(F) = 0$. (The characteristic $\chi(F)$ will be defined in general in Section 3.)

For $\boldsymbol{S1}$, the admissible paths starting with $\overline{00}$ are shown in the diagram (Fig. 1). None of them has period 4 over G_4, but some have period 5 over G_5 or period 6 over G_6.

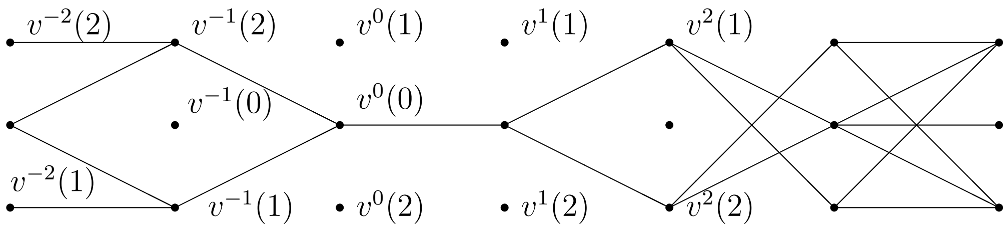

Figure 1: Bifurcated admissible paths over G_N (strategy $\mathbf{S1}$, $N = 4, 5, \ldots$).

Proposition 3. *If $N > 4$ and a strategy F for C_N has characteristic 0 ($\chi(F) = 0$), then F is losing.*

Proof. Take any edge $\beta \in E_{0,1} \in V(G_N)$. The admissible extensions of β to the right of length $m \geqslant 2$ have at least two possible endpoints in V_m. If $m \geqslant 3$, these extensions end in a set of at least four edges in $E_{m-1,m}$. Likewise for the left continuations of β. Consider all extensions of β to the right of length 3 and to the left of length $N - 2 \geqslant 3$, both meeting in V_3. Their endpoints form the sets $A, B \subset V_3$ and their final edges form the sets $X \subset E_{2,3}$ and $Y \subset E_{3,4}$, respectively.

First, suppose A has three elements. Then there are at least six admissible right continuations into $E_{3,4}$ (the example in Fig. 1 allows seven) while $|Y| \geqslant 4$. Since $6+4 > 9$, an edge in $A \cap B$ closes the path and F is losing.

Next, suppose A has only two elements. Then each $p \in A$ is the rigt endpoint of exactly two edges: $\overline{ap}, \overline{bp} \in X$. Take a point $p \in A \cap B$ with $\overline{pq} \in Y$ and the left admissible continuations $\overline{cpq}$ and $\overline{dpq}$. There is $e \in \{a, b\} \cap \{c, d\}$ making the edge $\overline{ep}$ close the path, so F is losing. □

2.5 The spiral strategy

Let us introduce a new strategy, denoted $F = \mathbf{S3(N)}$, which exists for all $N > 2$ and wins for all N divisible by 3 (and will naturally arise in section 3.3). In the twisted coordinates satisfying

$$\varphi_k^{-1} \circ \varphi_{k+N}(i) = \sigma_k(i) = \sigma(i) = i + 1 \quad \text{for all} \quad k$$

it is defined by the single rotation-invariant function

$$f_k(i,j) = f(i,j) = \begin{cases} i+1 & \text{if} \quad i = j \\ i & \text{if} \quad i \neq j \end{cases} \tag{9}$$

whose matrix form is

$$M = M_f = [f(i,j)]_{i,j=0,1,2} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix} \sim [f(i,j)]_{i,j=1,2,3} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 1 \end{bmatrix}.$$

Alternatively, we can use $\sigma(i) = i - 1$.

Proposition 4. *Strategy $\boldsymbol{C3(N)}$ is winning if and only if N is divisible by 3.*

Proof. By definition (formula (9)), there are at least nine forms of admissible infinite paths $\omega = (u_k)$ in graph G_N, given in twisted coordinates by $u_k = \varphi_k(t_k)$ $(k \in \mathbf{Z})$, where:

$$t_k = t_k(s, \lambda) = s + k\lambda \in \mathbf{Z}_3 \quad (s, \lambda \in \mathbf{Z}_3), \tag{10}$$

that is: $\omega \sim (\ldots s, s, s, \ldots)$, $(\ldots s, s+1, s+2, \ldots)$, $(\ldots s, s-1, s-2, \ldots)$. If the values of t_k are extended to $\mathbf{Z}$, these paths can be represented by straight lines through the lattice points $(k, s + k\lambda)$.

Conversely, let us observe that for every periodic (of any period) admissible sequence (t_k) the difference $d_k = t_k - t_{k-1}$ remains constant. Indeed:

- If $\overline{ttx} \in \mathcal{A}$ (i.e., $f(t,x) \neq t$), then $x = t$. If $\overline{x, t+1, t} \in \mathcal{A}$ (i.e., $f(x,t) \neq t+1$), then $x = t + 2$.
- Hence, if $d_k = 0$ or $d_k = -1$ for some $k = k_0$, then d_k is costant for all $k \geqslant k_0$ (resp. $k \leqslant k_0$) and, by periodicity, for all $k \in \mathbf{Z}$.
- The only remaining value $d_k = 1$ must also be constant.

Consequently, all the admissible paths are given by (10). If $3|N$, they all have period N in twisted coordinates, since $t_N = s + N\lambda = s = t_0$ (in $\mathbf{Z}_3$). However, none has period N in the graph G_N:

$$u_0 = \varphi_0(s) \neq \varphi_N(s) = u_N$$

since $\sigma = \varphi_0^{-1} \circ \varphi_N$ is a rotation and has no fixed point s. This proves the winning case; moreover, nine forms of (10) correspond to three admissible paths in G_N, each with period $3N$.

If N is not divisible by 3, then three admissible paths (10) with the same starting point $t_0 = s$ have distinct values of $t_N = s + N\lambda$. Hence, for some λ, the path ω in G_N returns to the starting point:

$$u_0 = \varphi_0(t_0) = \varphi_N(t_N) = u_N$$

that is, $t_0 = \sigma(t_N)$. It follows that $t_k = \sigma(t_{k+N})$ for all k, since any rotation satisfies $\sigma(t + d) = \sigma(t) + d$ (where $t = t_N$, $d = k\lambda$). Consequently, $u_k = u_{k+N}$ for all k and the path has period N over G_N (which is also evident from the straight line representation of ω) and F is losing. Note that three admissible loops in G_N have period N and two have period $3N$. □

The case $\boldsymbol{N = 3}$ in basic coordinates. To illustrate the advantage of our method, let us also examine the simple strategy $\boldsymbol{S3(3)}$ using the basic coordinates g. F is given in twisted coordinates by (6). The conversion formula (3) yields:

$$g_1(C, B) = f(C + 1, B), \quad g_2(A, C) = f(A, C), \quad g_3(B, A) = f(B, A - 1).$$

By the definition of f (9), the corresponding Players 1, 2 and 3 guess right in the following cases:

$$\begin{array}{llll} (A1) & C + 1 = A - 1 = B & (A2) & C + 1 = A \neq B \\ (B1) & A = B - 1 = C & (B2) & A = B \neq C \\ (C1) & B = C - 1 = A - 1 & (C2) & B = C \neq A - 1. \end{array}$$

Each of the 27 variations (A, B, C) implies some of the above cases in the following way:

$$\left\{ \begin{array}{lll} A = B \neq C & \Rightarrow & (B2) \\ B = C \neq A & \Rightarrow & (A2) \text{ or } (C2) \\ C = A \neq B & \Rightarrow & (B1) \text{ or } (C1) \end{array} \right. \qquad \left\{ \begin{array}{lll} A = B = C & \Rightarrow & (C2) \\ B = A - 1 = C + 1 & \Rightarrow & (A1) \\ A = B - 1 = C + 1 & \Rightarrow & (A2) \end{array} \right.$$

reflecting the fact that $\boldsymbol{S3(3)}$ is a winning strategy. We also note that each player guesses right in 9 cases and two players cannot guess right simultaneously. For every N each player guesses right in 1/3 of all cases, although for $N > 3$ not disjointly.

3 General structure of strategies

The proof of Theorem 1 was started in Section 2. By the analysis of strategy $\boldsymbol{S1(4)}$ and $\boldsymbol{S3(3n)}$ in Proposition 4, only the non-existence remains to be proved. Particular cases of non-existence have already been resolved above by Propositions 3 and 4.

The remaining part of the proof will be carried out in this section by using an interplay of relatively simple local and global combinatorial methods. The assumption $N > 4$ will be unnecessary for certain claims.

3.1 Admissible extensions

Definition 5. Let F be a fixed (collective) strategy. For any edge

$$\overline{bc} \in E_{k,k+1} = V_k \times V_{k+1},$$

define

$$\ell_+(\overline{bc}) = \#\{d \in V_{k+2} : \ \overline{bcd} \in \mathcal{A}(F)\},$$

i.e., the number of the immediate admissible continuations of $\overline{bc}$ to the right, and

$$\ell_-(\overline{bc}) = \#\{a \in V_{k-1} : \ \overline{abc} \in \mathcal{A}(F)\},$$

i.e., the number of the analogous continuations to the left.

Hence, in general, we have $\ell_+(\overline{bc}), \ell_-(\overline{bc}) \in \{0, 1, 2, 3\}$. The following lemma holds for all $N > 2$.

Lemma 6. *Consider a fixed strategy F.*

(a) The average value of ℓ_- (resp. ℓ_+) over any three right-adjacent (resp. left-adjacent) edges of G equals 2. That is, for any vertex $b \in V_k$ we have

$$\sum_{a \in V_{k-1}} \ell_-(\overline{ab}) = \sum_{c \in V_{k+1}} \ell_+(\overline{bc}) = 6$$

(b) If two edges of G have the same left (resp. right) endpoint then one of them has at least two admissible immediate continuations to the right (resp. left). That is, for any vertex $b \in V_k$ and any distinct vertices $c_1, c_2 \in V_{k+1}$ there is a choice of $i \in \{1, 2\}$ and distinct vertices $d_1, d_2 \in V_{k+2}$ such that $\overline{bc_i d_j} \in \mathcal{A}(F)$ for $j = 1, 2$; likewise for passages to the left.

(c) If the graph G contains an F-admissible path $\overline{s_1 \dots s_n}$ such that $2 \leqslant n \leqslant N - 1$ and

$$\ell_-(\overline{s_1 s_2}) + \ell_+(\overline{s_{n-1} s_n}) \geqslant 5,$$

then F is a losing strategy.

(d) If F is a winning strategy, then for every edge $\beta \in E(G)$ we have

$$\ell_+(\beta) + \ell_-(\beta) = 4.$$

Proof. (a): Consider ℓ_+. For any vertex $d \in V_{k+2}$, the set V_{k+1} contains two vertices different from $f_{k+1}(b, d)$, defining two admissible connections of b with each of the three choices of d. The situation with ℓ_- is analogous.

(b): For each of three points $d \in V_{k+2}$ we can choose an $i \in \{1, 2\}$ such that $c_i \neq F_{k+1}(b, d)$. For some two points $d = d_1, d_2$ the choice of i must be the same.

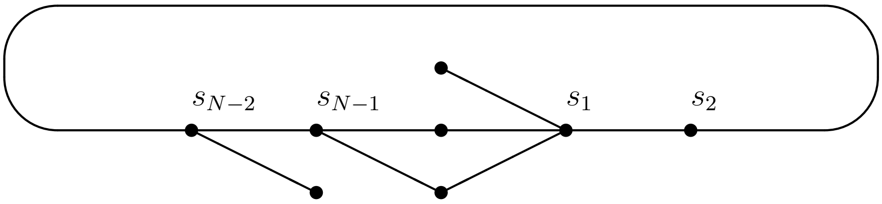

Figure 2: Closing the path, claim (c). At least one connection between s_{N-1} and s_1 is admissible.

(c): We may assume $\ell_-(\overline{s_1 s_2}) = 3$ and $\ell_+(\overline{s_{n-1} s_n}) \geqslant 2$, with $s_1 \in V_1$. By (b), the path can be extended to the right until $n = N-1$. Then the paths of the form $\overline{x s_1 s_2 \ldots s_{N-1} y}$ are in $\mathcal{A}(f)$ for all three values of $x \in V_0$ and at least two values of $y \in V_N = V_0$. Now it is enough to choose $y \neq f_0(s_{N-1} s_1)$ to make the ends meet, obtaining an N-periodic F-admissible path $\overline{y s_1 s_2 \ldots s_{N-1} y s_1 \ldots}$. (This argument is illustrated in Figure 2.)

(d): Denote $\ell(\gamma) = \ell_+(\gamma) + \ell_-(\gamma)$ for all $\gamma \in E(G)$. If $\ell(\beta) > 4$ for some $\beta \in E(G)$, then F is losing by (c) applied to the single edge β. However, (a) implies that the average value of $\ell(\gamma)$ over $\gamma \in E_{k,k+1}$ equals 4. Hence, if there was an edge $\alpha \in E_{k,k+1}$ with $\ell(\alpha) < 4$, there would also be an edge $\beta \in E_{k,k+1}$ with $\ell(\beta) > 4$, the case already excluded. ☐

The three categories of edges. Every winning strategy F satisfies the conclusion of Lemma 6(d):

$$\ell_+(\gamma) + \ell_-(\gamma) = 4 \quad \text{for all} \quad \gamma \in E(G). \tag{11}$$

Definition 7. If (11) is satisfied, all the edges $\gamma \in E(G)$ are divided into three categories:

- If $\ell_-(\gamma) = 3$ and $\ell_+(\gamma) = 1$, let us paint γ *yellow* and direct it *right.*
- If $\ell_-(\gamma) = 1$ and $\ell_+(\gamma) = 3$, let us paint γ *red* and direct it *left.*
- If $\ell_-(\gamma) = \ell_+(\gamma) = 2$, let us paint γ *blue* and leave it *undirected.*

Any strategy F satisfying (11) will be called *balanced* or *colourable*.

The three patterns can thus be shown as in Figure 3. The arrow is pointing to the unique admissible continuation (cf. also Lemma 8(b)).

By Lemma 6(d) every winning strategy is balanced, but not vice versa. By Proposition 3, the graph G_N for every winning strategy with $N > 4$ contains some directed edge(s).

3.2 The directed cycles

The sets of all the yellow, red, and blue edges in $E(G)$ (or in $E_{k,k+1}$) will be denoted E^+, E^-, and E^0 (or $E^+_{k,k+1}$, $E^-_{k,k+1}$, and $E^0_{k,k+1}$), respectively.

Lemma 8. *Let F be a colourable strategy for C_N.*

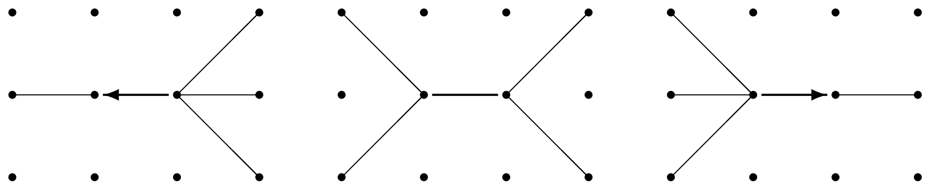

Figure 3: Examples of edges (red, blue, yellow) with their admissible continuations.

(a) In each set $E_{k,k+1}$ any three left-adjacent or right-adjacent edges either have three different colours or all are blue. In particular, two edges of the same direction are either disjoint or following one another.

(b) If F is winning and $N \geqslant 4$, then every directed edge $\beta \in E^-_{k,k+1}$ or $\beta \in E^+_{k,k+1}$ is admissibly followed by an edge with the same direction: $\alpha \in E^-_{k-1,k}$ or $\gamma \in E^+_{k+1,k+2}$, respectively.

(c) In each set $E_{k,k+1}$ there are equal numbers of right- and left-directed edges, i.e., $|E^+_{k,k+1}| = |E^-_{k,k+1}|$. This number is the same for all k.

(d) If F is winning and $N \geqslant 4$, then every admissible cycle (of any length) in G_N has one colour.

Proof. (a): By definition, colours are characterized by the numbers of the corresponding continuations which, by Lemma 6(a), for any vertex $a \in V(G_N)$ sum up to:

$$\sum_x \ell^-(\overline{xa}) = \sum_y \ell_-(\overline{ay}) = 6.$$

The number 6 can be partitioned into three terms equal 1, 2 or 3 in just two ways: $1+2+3$ and $2+2+2$.

(b): If $\beta \in E^+_{k,k+1}$ is yellow and its unique right continuation γ is not, then $\ell_+(\gamma) \geqslant 2$ while $\ell_-(\beta) = 3$. Then Lemma 6(c) applied to the path $\overline{\beta\gamma} \in \mathcal{A}$, with $n = 3 \leqslant N-1$ and $\ell_-(\beta) + \ell^+(\gamma) \geqslant 5$, implies that F is a losing strategy, contrary to the assumption. Likewise for $\beta \in E^-$.

(c): By (a), the number of red and yellow edges is the same (0 or 1) even if we fix their left or right endpoint. By (b), the edges of $E^+_{k-1,k}$ are continued into $E^+_{k,k+1}$, which implies $|E^+_{k-1,k}| \leqslant |E^+_{k,k+1}| \leqslant \ldots \leqslant |E^+_{k-1,k}|$ in cycle.

(d): By definition, every directed edge is uniquely admissibly continued forward, and its continuation has the same direction by (b). Hence, if an admissible path (not necessarily cyclic) contains any directed edge, then all its consecutively following edges have the same direction, so the path becomes eventually cyclic of constant colour (other than blue). For a cyclic path this refers to the whole. □

The situation at a vertex adjacent to some directed edge is illustrated in Fig. 4. The only non-admissible connections are: blue-red, yellow-red, yellow-blue. The other possible situation is six blue edges meeting at a point.

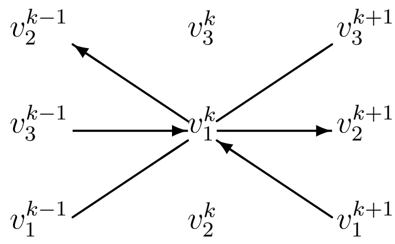

Figure 4: A typical example of the configuration at the head or tail of any directed edge.

Definition 9. For any winning strategy F for C_N ($N > 3$) the number

$$\chi(F) = |E^-_{k,k+1}| = |E^+_{k,k+1}| \in \{0, 2, 3\}$$

(independent of k) will be called *characteristic number* or the *characteristic* of F.

By lemma 8(c), χ does not depend on k. By lemma 8(b), the right-directed (resp. left-directed) edges form a single admissible cycle of length $N\chi(F)$, as the losing N-cycles are absent. By Proposition 3, the value $\chi(F) = 0$ is only possible for $N = 4$. We move on to the analysis of two remaining possibilities: $\chi(F) = 2$ or 3.

3.3 Characteristic 3

By Lemma 8(a)(b), the yellow edges for F are arranged as follows in the graph G_N:

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow u_1^0 & \longrightarrow u_1^1 & \longrightarrow u_1^2 & \longrightarrow & \cdots & \longrightarrow u_1^{N-1} & \longrightarrow u_1^N & \longrightarrow u_1^{N+1} & \longrightarrow & \cdots \\
\cdots & \longrightarrow u_2^0 & \longrightarrow u_2^1 & \longrightarrow u_2^2 & \longrightarrow & \cdots & \longrightarrow u_2^{N-1} & \longrightarrow u_2^N & \longrightarrow u_2^{N+1} & \longrightarrow & \cdots \\
\cdots & \longrightarrow u_3^0 & \longrightarrow u_3^1 & \longrightarrow u_3^2 & \longrightarrow & \cdots & \longrightarrow u_3^{N-1} & \longrightarrow u_3^N & \longrightarrow u_3^{N+1} & \longrightarrow & \cdots
\end{array}$$

where $\{u_1^k, u_2^k, u_3^k\} = V_k$ for all $k \in \mathbf{Z}$. We also write $u_i^k = u^k(i)$.

As already observed, the yellow edges in G_N form one admissible cycle of length $3N$ (since three N-cycles or $2N$-cycle plus N-cycle imply losing). Since $V_0 = V_N$, we have $u^N(i) = u^0(\sigma(i))$ for $i \in \mathbf{Z}_3$, where $\sigma : \mathbf{Z}_3 \to \mathbf{Z}_3$ is a permutation without fixed points, $\sigma(i) = i \pm 1$. Since the yellow edges establish a correspondence between u_i^k and $u_i^{k\pm1}$, we have inductively

$$u^{k+N}(i) = u^k(\sigma(i)) \quad \text{for all} \quad k \in \mathbf{Z}, \; i \in \mathbf{Z}_3.$$

Hence, the diagram shows twisted coordinates. We will use functions $f_k : \mathbf{Z}_3^2 \to \mathbf{Z}_3$ representing individual strategies F_k in the diagram coordinates.

By definition, the yellow edges have unique admissible continuations to the right, namely $\overline{iii}$. Hence we have $f_k(i,j) = i$ whenever $i \neq j$. That is, $f_k(i,j) \neq s$ whenever $s \neq i \neq j$ and equivalently,

$$\overline{u^{k-1}(i)u^k(s)u^{k+1}(j)} \in \mathcal{A} \quad \text{whenever} \quad s \neq i \neq j.$$

This holds, in particular, for $i = s+1 = j+2$ and $i = s-1 = j-2$. It follows that for each fixed $u = u_s^0 \in V_0$ the following two periodic paths (x_k) and (z_k) starting at u in G_N are F-admissible (apart from the path $y_k = u_{(}^{k}s)$)):

$$x_k = u^k(s+k) \quad \text{and} \quad z_k = u^k(s-k). \tag{12}$$

By Lemma 8(a), (x_k) and (z_k) start at u with a red and a blue edge. Then by (d), the path (x_k) is red and (z_k) is blue (or conversely). They pass through the vertices

$$x_N = u^N(s+N) = u^0(\sigma(s+N)) \quad \text{and} \quad z_N = u^N(s-N) = u^0(\sigma(s-N))$$

Now, if N is not divisible by 3, the numbers $s, s+N, s-N$ are distinct $(\mathrm{mod}\, 3)$ while $\sigma(s) \neq s$, which implies that $s = \sigma(s \pm N)$ for some choice of sign. It follows that $x_0 = x_N$ or $z_0 = z_N$. We also have $x_k = x_{k+N}$ for all $k > 0$ (or the same for z_k) since both paths have one colour and there is only one edge to choose from left-incident edges (by Lemma 8(a)). Consequently, F is losing and this contradiction closes the case $\chi(F) = 3$. See also Fig. 5.

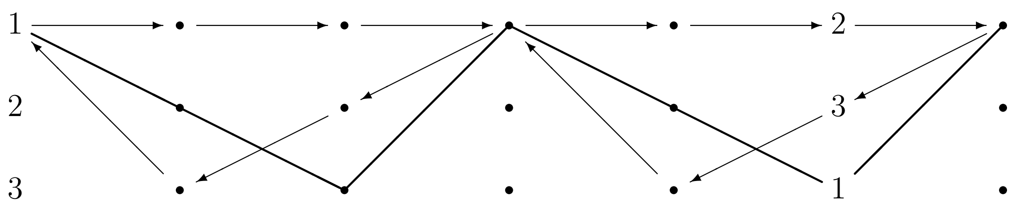

Figure 5: Part of a hypothetical strategy of characteristic 3 on C_5. However, the undirected path has period 5 in G_5.

3.4 Characteristic 2

Lemma 8(a)(b) implies the following arrangement of all the yellow edges and some blue edges in graph G_N:

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & u_1^0 & \longrightarrow & u_1^1 & \longrightarrow & u_1^2 & \longrightarrow & \cdots & \longrightarrow & u_1^{N-1} & \longrightarrow & u_1^N & \longrightarrow & u_1^{N+1} \longrightarrow \cdots \\
\cdots & \text{——} & u_0^0 & \text{——} & u_0^1 & \text{——} & u_0^2 & \text{——} & \cdots & \text{——} & u_0^{N-1} & \text{——} & u_0^N & \text{——} & u_0^{N+1} \text{——} \cdots \\
\cdots & \longrightarrow & u_2^0 & \longrightarrow & u_2^1 & \longrightarrow & u_2^2 & \longrightarrow & \cdots & \longrightarrow & u_2^{N-1} & \longrightarrow & u_2^N & \longrightarrow & u_2^{N+1} \longrightarrow \cdots
\end{array}$$

where $\{u_1^k, u_0^k, u_2^k\} = V_k$ for all $k \in \mathbf{Z}$. We also write $u_i^k = u^k(i)$.

Since $V_k = V_{k+N}$ and since the yellow edges establish a correspondence between V_k and $V_{k\pm1}$, we have $u^{k+N}(i) = u^k(\sigma(i))$ for all k and i, where σ is a permutation independent of k. Since F is winning, G_N contains one yellow cycle with period $2N$ and σ is a transposition, $\sigma(i) = -i$. Consequently, the diagram shows transposed coordinates.

By the incidence property (Lemma 8(a)), the red edges are precisely $\overline{u^k(i)u^{k+1}(-i)}$ with $i = \pm1$ (connecting the yellow paths). Since F is winning, these left-directed edges also form one cycle of length $2N$ in G_N, which implies that N is even. The roles of red and yellow edges are analogous.

These conditions determine the matrix $M_k = [f_k(i,j)]_{i,j=1,0,2}$ of each individual strategy F_k in the diagram coordinates. Indeed, the admissible paths $\overline{i\,i\,i}$, $\overline{i\,-i\,i} \in \mathcal{A}_F$ $(i,j = 1,-1)$ are directed, which implies two conclusions: $M_k = \begin{bmatrix} * & 1 & 1 \\ 2 & * & 1 \\ 2 & 2 & * \end{bmatrix}$ (since the remaining forward continuations $\overline{i\,i\,x}$ and $\overline{x\,-i\,i}$ are non-admissible), and $f(i,i) \neq \pm1$ for $i \in \{1,-1\}$. Finally, since $\overline{00} \in E^0$, the equation $f_k(0,x) = 0$ has a solution, $x = 0$. It follows that

$$M_k = [f_k(i,j)]_{i,j=1,0,2} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix} \qquad (k \in \mathbf{Z}) \tag{13}$$

and that (exactly) eight kinds of undirected paths are admissible:

$$\overline{\mp1,0,\pm1},\ \overline{0,\pm1,0},\ \overline{0,0\pm1},\ \overline{\pm1,0,0}\ \in \mathcal{A} \tag{14}$$

This suffices to construct admissible blue cycles of length N:

- For $N = 4n+2$, the path with period $\overline{0102}$ in Γ is admissible and has period N over G_N.
- For $N = 4n$ $(n > 1)$, the path with period $\overline{[(0010)X^{n-1}][(0020)X^{n-1}]}$ in Γ, where $X = \overline{0102}$ or $X = \overline{0201}$, is admissible and has period N over G_N.

This contradiction concludes the case $\chi(F) = 2$ and completes the proof of Theorem 1.

The situation for $N = 8$ is illustrated in the diagram (Fig. 6). Any strategy given by (13) in some transposed coordinates where $i \mapsto -i$ will be denoted $\boldsymbol{S2(N)}$. Using (13) or (14), it is easy to check that $\boldsymbol{S2(4)}$ is winning on C_4.

4 Conclusions and remarks

4.1 Uniqueness of winning strategies

Proposition 10. *For $n > 1$, strategy $S3(3n)$ is unique as a winning strategy up to permutations of colours for each player.*

Proof. Section 3.3 presents the unique configuration (up to permutations) where the transition $\sigma(i) = \pm1$ may be assumed $\sigma(i) = i + 1$ as we can change the direction of all cycles

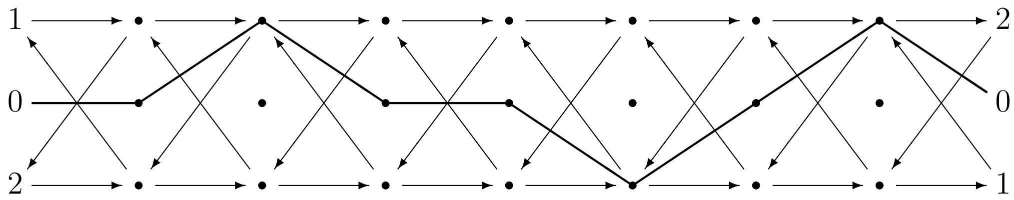

Figure 6: Part of a hypothetical strategy of characteristic 2 on C_8, showing all the directed edges. However, the undirected path (whose next segment of length 8 will look reversed in the diagram) has period 8 in the graph G_8.

V_k. If f_k denotes the individual strategy in the diagram coordinates, the horizontal position of right-directed edges implies $f_k(i,j) = i$ for $i \neq j$. Each other admissible path of the form $t_k = s + \varepsilon k$ is red or blue depending on $\varepsilon \in \{-1, 1\}$ by Lemma 8(a)(d). If the left-directed edges have slope ε, then $f_k(i,j) = j - \varepsilon$ for $i \neq j - 2\varepsilon$.

It follows that individual strategies f_k with their matrices $M(f_k) = [f_k(i,j)]_{i,j=0,1,2}$ do not depend on k and may have two forms:

$$f_k(i,j) = \begin{cases} i & \text{for} \quad i \neq j \\ i + \varepsilon & \text{for} \quad i = j, \end{cases} \qquad M(f_k) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

where $\varepsilon = \pm 1$. In fact, one form is obtained from the other by reversing the cycle $\mathbf{Z}_3$. $\square$

For $N = 3n$, the three cycles $t_k = \alpha k$ (for $\alpha \in \mathbf{Z}_3$) discussed in section 3.3 may be assigned three colours in six ways. Hence the strategy $\boldsymbol{S3(3n)}$ has six regular forms:

$$f(i,j) = f_{\delta,\varepsilon}(i,j) = \begin{cases} i + \varepsilon + \delta & \text{for} \quad j = i + 2\varepsilon \\ i + \varepsilon & \text{for} \quad j \neq i + 2\varepsilon \end{cases} \tag{15}$$

where $\varepsilon \in \mathbf{Z}_3$ is the slope of the yellow cycle, $\delta \in \{1, -1\}$ and $\varepsilon - \delta$ is the slope of the red cycle. Matrix transposition M_f^T is obtained by the substitution $\varepsilon' = -\varepsilon + \delta$ and corresponds to reversing the orientation of C_N.

The winning strategy $\boldsymbol{S2(4)}$ with matrix $\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$, obtained in section 3.4, is also unique as a winning strategy with $\chi = 2$ for the cycle C_4 up to permutations. In total, we have obtained the following winning strategies: $\boldsymbol{S0(3)}$, $\boldsymbol{S1(4)}$, $\boldsymbol{S2(4)}$ and $\boldsymbol{S3(3n)}$ for $n = 1, 2, 3, \ldots$

4.2 Minimality of assumptions

The configuration of three colours for each player is a maximal solvable one. If we increase the number of colours guessed by some player, there is no winning strategy even if $N = 3n$.

Proposition 11. *If, in the round table hat game on any cycle C_N $N > 4$, the number of colours is increased to 4 for one player and stays 3 for all the other players, then the team has no winning strategy.*

Proof. If F were a winning strategy for this game, then it would also be winning for any of its 3-colour restrictions. It follows that removing any value c of the four colours to be guessed by the distinguished Player k — such that c was among the values actually assumed by F_k — and changing all assignments $F_k(a, b) = c$ into $F_k(a, b) \neq c$ (while F_{k-1} and F_{k+1} simply restict their domains) would result in a winning strategy for the 3-colour game. However, by Proposition 10, such a strategy is unique up to permutations and requires F_k to assume each value exactly three times. Some arbitrary choice of $F_k(a, b) \neq c$ can always violate that requirement, contrary to the fact that F should remain a winning strategy. □

4.3 Stochastic variant

Even if N is not divisible by 3, the probability of winning by using a purely *random* strategy equals $1 - \left(\frac{2}{3}\right)^N$. However, a much more effective strategy can be designed.

Proposition 12. *For the randomized three-colour game on any cycle C_N there exists a strategy whose winning probability is $\geqslant 1 - 3^{-N+1}$.*

Proof. This follows from the fact that strategy $\boldsymbol{S3(N)}$ has at most three admissible N-periodic paths. If the colours are randomly re-labelled by the team, the adversary will have no information about the admissability of any given path. □

4.4 Hat games on graphs

In a more general setting an arbitrary visibility pattern can be assumed. For other presentations of those models, see [6], [5], [10], [15], [13], [17]. We assume that the directed visibility graph C has N vertices corresponding to the players, and edges $\vec{AB} \in E(\Gamma) = E$ wherever player A is seen by player B. For each vertex $v \in V(C) = V$ a nonempty set of 'colours' V_v is known to all. For each assignment of colours, i.e., a selector $S : V \to \bigcup_v V_v$ with $S(v) \in V_v$, each player $u \in V$ tries to guess $S(u)$ by using a function

$$F_u : \prod V_v \to V_u \quad \text{(product taken over } \vec{vu} \in E).$$

as an individual strategy. The combined, or collective, strategy is the collection $F = \{F_u : u \in V\}$. The game is thus played against an opponent assigning colours — whom we may called Wizard, Demon, etc. — or the assignement may be stochastic. In our approach, the notion of winnig or losing refers to the cooperative players. The strategy effectiveness depends only on the numbers of possible colours, i.e., the height function h given by $h(v) = |V_v|$. Let $X_h(F, S)$ denote the number of the correct guesses. For three colour games we have $h(v) = \text{const} = 3$. The deterministic minimax approach defines the *value* of the game as

$$\mu(C, h) = \max_F \min_S X_h(F, S).$$

The mimimal requirement of $\mu > 0$ has been the subject of this paper. The μ-value in general has a distant parallel for the stochastic case as the Riis *guessing number*: see [5], [7], [12], [16].

Question 13. Is it possible to characterize in simple terms all those hat guessing games — particularly three colour ones — for which the team is winning? Are those games algorithmically recognizable in polynomial time?

Question 14. Is it possible to characterize in simple terms some winning strategy for every hat guessing game — particularly a three colour one — assuming the team is winning? Is such a strategy algorithmically constructible in polynomial time?

Acknowledgements

The consideration of this game model was inspired by the classic complete graph hat problem which was communicated to the author by Sławomir Porębski by the end of 2009. The main result of this paper was presented to the Graph Theoretic Seminar at Warsaw Technical School in March 2010, and the Theoretical Computer Science Seminar at Jagiellonian University in April 2010, the latter at the invitation of Jarosław Grytczuk. Marcin Krzywkowski read a shorter version of this paper and the referee's questions inspired a broader exposition.

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