A q-ANALOG OF FOULKES CONJECTURE

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ABSTRACT. We propose a q-analog of classical plethystic conjectures due to Foulkes. In our conjecture, up to a divided differences, the Hall-Littlewood polynomials $H_n(\mathbf{x};q)$ replace the complete homogeneous symmetric functions $h_n(\mathbf{x})$. At $q=0$, we get back the original statement of Foulkes, and we show that our version holds at $q=1$. We also give further supporting evidence for our conjecture, and related results.

CONTENTS

1. INTRODUCTION

The aim of this text is to present and discuss a new q-analog of a conjecture due to Foulkes in his paper of 1949 (see [8]). Recall that this classical conjecture states that the difference of plethysms[1]

$$f_{a,b} := h_b[h_a] - h_a[h_b],$$

of homogeneous symmetric function h_a and h_b with $a \leq b$, expands with positive (integer) coefficients in the Schur basis $\{s_\mu\}_{\mu\vdash n}$ (i.e.: μ runs over the set of partitions of $n = a\,b$). For instance, we have

$$f_{2,4} = s_{422} + s_{2222}, \qquad \text{and} \qquad f_{3,4} = s_{732} + s_{5421} + s_{6222}.$$

Here, symmetric functions are in a denumerable set of variables $\mathbf{x} = x_1, x_2, x_3, \ldots$ (likewise later for $\mathbf{y} = y_1, y_2, y_3, \ldots$), which are often not explicitly mentioned. Equivalently, Foulkes conjecture says that there is a monomorphism of $\mathrm{GL}(V)$-module going between the compositions of symmetric power $S^a(S^b(V))$ and $S^b(S^a(V))$, hence each $\mathrm{GL}(V)$-irreducible occurs with smaller multiplicity in $S^a(S^b(V))$ than it does in $S^b(S^a(V))$. Although many partial and related results have been obtained (see [2, 7, 4, 12]), the conjecture is still open in the general case. A recent survey can be found in [6]. Brion [3] has shown that it holds if b is

Date: **January, 2016**. This work was supported by NSERC.

[1]This operation is due to Littlewood [10], and its definition is recalled in an upcoming section.

large enough with respect to a. Furthermore Mckay [14] has obtained an interesting propagation theorem which would be nice to extend to our context. Another that seems to afford a q-version is that of [18]. We explore this somewhat in section 4,

Our q-analog replaces the relevant complete homogeneous symmetric function h_n by the Hall-Littlewood (or Macdonald) polynomial

$$H_n(\mathbf{x};q) := \sum_{\mu\vdash n} K_\mu(q) s_\mu(\mathbf{x}),$$

where

$$K_\lambda(q) = \sum_\tau q^{c(\tau)},$$

with τ running through the set of standard tableaux of shape λ, and $c(\tau)$ standing for the charge statistic. These are special cases[2] of Kostka-Foulkes polynomials. To see how our upcoming statement corresponds to a q-analog of Foulkes conjecture, we recall that $H_n(\mathbf{x};0) = h_n(\mathbf{x})$. Hence we consider here a slightly different q-analog notion than the usual one, since the relevant specialization is at $q = 0$ rather than $q = 1$.

Conjecture 1 (q-Foulkes). *For any integers $0 < a \le b$, the Schur function expansion of the divided difference*

$$F_{a,b}(\mathbf{x};q) := \frac{1}{1-q^2}\left(H_b[H_a] - H_a[H_b]\right) \tag{1.1}$$

has coefficients in $\mathbb{N}[q]$, with the evident specialization $F_{a,b}(\mathbf{x};0) = f_{a,b} = h_b[h_a] - h_a[h_b]$.

We will explain later why it makes sense to divide by $1 - q^2$. For instance, with $a = 2$ and $b = 3$, we find after calculation that expression (1.4) expands as

$$\begin{aligned} F_{2,3}(\mathbf{x};q) \;\; = \;\; & s_{222} + q\,\big((q^2+q)\,s_{33} + (q^3+q^2+q+1)\,s_{321} \\ & +(q^2+q)\,s_{3111} + q\,s_{222} \\ & +(q^4+q^3+2\,q^2+q+1)\,s_{2211} \\ & +(2\,q^3+q^2+q)\,s_{21111} + (q^4+q^2)\,s_{111111}\big). \end{aligned}$$

This does specialize, at $q = 0$, to the corresponding case of Foulkes conjecture:

$$f_{2,3} = h_3[h_2] - h_2[h_3] = s_{222} \tag{1.2}$$

Incidentally, the classical Hilbert's reciprocity law [9] is equivalent to the fact that all Schur functions occurring in the difference $h_b[h_a] - h_a[h_b]$ are indexed par partitions having at least 3 parts. However, this does not generalize to our q-version.

A second part of Foulkes conjecture, shown to be true by Brion [3], concerns the stability of coefficients as b grows while a remains fixed. To simplify its statement, we consider the linear operator which sends a Schur function $s_\mu(\mathbf{x})$ to $s_{\overline{\mu}}(\mathbf{x})$, where $\overline{\mu}$ is the partition obtained by removing the largest part in μ. Let us write $\overline{f}$ for the effect of this operator on a symmetric function f. For example, we get

$$\overline{s_{622} + s_{442} + s_{4222} + s_{22222}} = s_{22} + s_{42} + s_{222} + s_{2222},$$

[2] Because the second index is the one-part partition (n).

which is clearly not homogeneous. Using this notation convention, the second part of Foulkes conjecture states that, for all $a \le b$, the Schur expansion of

$$\overline{f_{a,b+1}} - \overline{f_{a,b}} = \overline{(h_{b+1}[h_a] - h_a[h_{b+1}])} - \overline{(h_b[h_a] - h_a[h_b])} \tag{1.3}$$

also affords positive integers polynomials as coefficients. Observe that the "Bar" operator allows the comparison homogeneous functions of different degrees, namely $f_{a,b+1}$ of degree $a(b+1)$ with $f_{a,b}$ of degree ab. Instances of (1.3) are

$$\begin{aligned}
\overline{f_{2,4}} - \overline{f_{2,3}} &= \overline{(s_{422} + s_{2222})} - \overline{s_{222}}, \qquad \text{(where } \overline{s_{422}} \text{ cancels with } \overline{s_{222}}\text{)},\\
&= s_{222},\\
\overline{f_{2,5}} - \overline{f_{2,4}} &= \overline{(s_{622} + s_{442} + s_{4222} + s_{22222})} - \overline{(s_{422} + s_{2222})},\\
&= s_{42} + s_{2222},\\
\overline{f_{2,6}} - \overline{f_{2,5}} &= s_{44} + s_{422} + s_{22222}.
\end{aligned}$$

A similar phenomenon also seems to hold in our context, leading us to state the following.

Conjecture 2 (q-stability). *For any integers $0 < a \le b$, and any Schur function s_λ, we have*

$$\langle \overline{F_{a,b+1}(\mathbf{x};q)} - \overline{F_{a,b}(\mathbf{x};q)}, s_\lambda(\mathbf{x})\rangle \in \mathbb{N}[q]. \tag{1.4}$$

Here we use the usual scalar product $\langle -,- \rangle$ on symmetric function, for which the Schur functions form an orthonormal basis. The smallest non-trivial example is:

$$\begin{aligned}
\overline{F_{2,4}(\mathbf{x};q)} - \overline{F_{2,3}(\mathbf{x};q)} = \ & \left(q^3+2\,q^4+2\,q^5+q^6\right) s_3 + \left(q^2+2\,q^3+3\,q^4+3\,q^5+2\,q^6+q^7\right) s_{21}\\
&+ \left(q^3+2\,q^4+2\,q^5+q^6\right) s_{111} + \left(q^2+2\,q^4+q^6\right) s_4\\
&+ \left(q+2\,q^2+6\,q^3+7\,q^4+9\,q^5+6\,q^6+4\,q^7+q^8\right) s_{31}\\
&+ \left(q+3\,q^2+4\,q^3+7\,q^4+5\,q^5+6\,q^6+2\,q^7+2\,q^8\right) s_{22}\\
&+ \left(2\,q^2+6\,q^3+10\,q^4+13\,q^5+11\,q^6+8\,q^7+3\,q^8+q^9\right) s_{211}\\
&+ \left(q^3+4\,q^4+6\,q^5+7\,q^6+4\,q^7+2\,q^8\right) s_{1111}\\
&+ \left(q+2\,q^2+5\,q^3+6\,q^4+8\,q^5+6\,q^6+5\,q^7+2\,q^8+q^9\right) s_{32}\\
&+ \left(3\,q^2+4\,q^3+10\,q^4+9\,q^5+11\,q^6+6\,q^7+4\,q^8+q^9\right) s_{311}\\
&+ \left(2\,q+4\,q^2+9\,q^3+12\,q^4+15\,q^5+13\,q^6+11\,q^7+6\,q^8+3\,q^9+q^{10}\right) s_{221}\\
&+ \left(2\,q^2+6\,q^3+11\,q^4+16\,q^5+17\,q^6+14\,q^7+9\,q^8+4\,q^9+q^{10}\right) s_{2111}\\
&+ \left(q^3+3\,q^4+6\,q^5+7\,q^6+8\,q^7+5\,q^8+3\,q^9+q^{10}\right) s_{11111}\\
&+ \left(1+2\,q^2+q^3+4\,q^4+2\,q^5+5\,q^6+q^7+3\,q^8+q^{10}\right) s_{222}\\
&+ \left(q+q^2+4\,q^3+5\,q^4+9\,q^5+8\,q^6+9\,q^7+5\,q^8+4\,q^9+q^{10}+q^{11}\right) s_{2211}\\
&+ \left(q^2+q^3+5\,q^4+5\,q^5+9\,q^6+7\,q^7+7\,q^8+3\,q^9+2\,q^{10}\right) s_{21111}\\
&+ \left(q^3+q^4+3\,q^5+3\,q^6+4\,q^7+3\,q^8+3\,q^9+q^{10}+q^{11}\right) s_{111111}\\
&+ \left(q^4+q^6+q^8+q^{10}\right) s_{1111111},
\end{aligned}$$

and we do observe that this specializes to the (much simpler) classical Foulkes case when we set $q = 0$. Another stability in the vein of Manivel [13], that seems to hold in our q-context, is that

$$(\overline{F_{a+1,b+1}(\mathbf{x};q)} - \overline{F_{a+1,b}(\mathbf{x};q)}) - (\overline{F_{a,b+1}(\mathbf{x};q)} - \overline{F_{a,b}(\mathbf{x};q)}),$$

is Schur positive, for all $a < b$.

2. Definitions and background

Trying to make this text self contained, we rapidly recall most of the necessary background on symmetric functions. As is usual, we often write symmetric functions without explicit mention of the variables. Thus, we denote by p_k (as in [11]) the power-sum symmetric functions

$$p_k = p_k(x_1, x_2, x_3, \ldots) := x_1^k + x_2^k + x_3^k + \ldots,$$

using which, we can expand the complete homogeneous symmetric functions as

$$h_n = \sum_{\mu \vdash n} \frac{p_\mu}{z_\mu}, \qquad \text{with} \qquad p_\mu := p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}, \tag{2.1}$$

where $d_k = d_k(\mu)$ is the multiplicity of the part k in the partition μ of n, and z_μ stands for the integer

$$z_\mu := \prod j^{d_j}\, d_j!.$$

For instance, we have the very classical expansions

$$h_2 = \frac{p_1^2}{2} + \frac{p_2}{2}, \qquad h_3 = \frac{p_1^3}{6} + \frac{p_1 p_2}{2} + \frac{p_3}{3}.$$

As is also very well known, the homogeneous degree n component λ_n of the graded ring Λ of symmetric functions, affords as a linear basis the set of Schur functions $\{s_\mu\}_{\mu \vdash n}$, indexed by partitions of n. Among the manifold interesting formulas regarding these, we will need the Cauchy-kernel identity.

$$\begin{aligned} h_n(\mathbf{x}\mathbf{y}) &= \sum_{\mu \vdash n} s_\mu(\mathbf{x}) s_\mu(\mathbf{y}) & (2.2)\\ &= \sum_{\mu \vdash n} \frac{p_\mu(\mathbf{x}) p_\mu(\mathbf{y})}{z_\mu}, & (2.3)\end{aligned}$$

with $h_n(\mathbf{x}\mathbf{y}) = h_n(..., x_i y_j, ...)$ corresponding to the evaluation of h_n in the "variables" $x_i y_j$. Otherwise stated, we may express this by the generating function identity

$$\sum_{n \geq 0} h_n(\mathbf{x}\mathbf{y})\, z^n = \prod_{i,j} \frac{1}{1 - x_i y_j\, z}.$$

Plethysm is characterized by the following properties. Let f_1, f_2, g_1 and g_2 be any symmetric functions, and α and β be in $\mathbb{Q}$, then

$$\begin{aligned}
&a) && (\alpha\, f_1 + \beta\, f_2)[g] = \alpha\, f_1[g] + \beta\, f_2[g],\\
&b) && (f_1 \cdot f_2)[g] = f_1[g] \cdot f_2[g],\\
&c) && p_k[\alpha\, g_1 + \beta\, g_2] = \alpha\, p_k[g_1] + \beta\, p_k[g_2], \qquad (2.4)\\
&d) && p_k[g_1 \cdot g_2] = p_k[g_1] \cdot p_k[g_2],\\
&e) && p_k[p_j] = p_{kj}, \qquad \text{and} \qquad p_k[q] = q^k.
\end{aligned}$$

The first four properties reduce any calculation of plethysm to instances of the fifth one. In this context, it is useful to consider variable sets as sums $\mathbf{x} = x_1 + x_2 + x_3 + \ldots$, so that $f[\mathbf{x}]$

corresponds to the evaluation of the symmetric function f in the variables $\mathbf{x}$. In particular, Cauchy's formula gives an explicit expression for the expansion of

$$h_n[\mathbf{x}\mathbf{y}] = h_n[(x_1 + x_2 + x_3 + \ldots)(y_1 + y_2 + y_3 + \ldots)].$$

Likewise $f[1/(1-q)] = f[1+q+q^2+\ldots]$, corresponds to the evaluation $f(1, q, q^2, \ldots)$. With all this at hand, the polynomials $H_n(\mathbf{x}; q)$ can be explicitly defined as

$$H_n(\mathbf{x}; q) := [n]_q!\,(1-q)^n\, h_n\left[\frac{\mathbf{x}}{1-q}\right] \tag{2.5}$$

where $[n]_q!$ stands for classical the q-analog of $n!$:

$$[n]_q! := [1]_q\,[2]_q \cdots [n]_q, \qquad \text{with} \qquad [k]_q = 1 + q + \ldots + q^{k-1}.$$

Calculating with the plethystic rules (2.4), and formula (2.1), we get the explicit power-sum expansion

$$H_n(\mathbf{x}; q) = \sum_{\mu \vdash n} \frac{[n]_q!}{z_\mu\,[\mu_1]_q[\mu_2]_q \cdots [\mu_\ell]_q}\,(1-q)^{n-\ell(\mu)}\; p_\mu(\mathbf{x}), \tag{2.6}$$

where $\ell = \ell(\mu)$ is the number of parts of μ. To get a Schur expansion for $H_n(\mathbf{x}; q)$, we recall the hook length expression

$$\begin{aligned} s_\mu[1/(1-q)] &= s_\mu(1, q, q^2, q^3, \ldots) \\ &= q^{n(\mu)} \prod_{1 \le i \le \mu_j} \frac{1}{1-q^{h_{ij}}}, \end{aligned} \tag{2.7}$$

where $h_{ij} = h_{ij}(\mu)$ is the hook length of a cell (i, j) of the Ferrers diagram of μ, and $n(\mu) := \sum_{(i,j)} j$. Now, using Cauchy's formula (2.2), with $\mathbf{y} = 1 + q + q^2 + \ldots$, we find that

$$H_n(\mathbf{x}; q) = \sum_{\mu \vdash n} \frac{q^{n(\mu)}[n]_q!}{\prod_{1 \le i \le \mu_j}[h_{ij}]_q}\; s_\mu(\mathbf{x}). \tag{2.8}$$

It is well known that the coefficient of $s_\mu(\mathbf{x})$ occurring here is a positive integer polynomial that q-enumerates standard tableaux with respect to the charge statistic. This is the q-hook formula. Thus, we find the two expansions.

$$\begin{aligned} H_3(\mathbf{x}; q) &= \frac{[2]_q[3]_q}{6}\, p_1(\mathbf{x})^3 + \frac{[3]_q}{2}(1-q)\, p_1(\mathbf{x})p_2(\mathbf{x}) + \frac{[2]_q}{3}(1-q)^2\, p_3(\mathbf{x}), \\ &= s_3(\mathbf{x}) + (q+q^2)s_{21}(\mathbf{x}) + q^3 s_{111}(\mathbf{x}). \end{aligned}$$

It is clear that $H_n(\mathbf{x}; 0) = h_n$. The $H_n(\mathbf{x}; q)$ function encodes, as a Frobenius transform, the character of several interesting isomorphic graded $\mathbb{S}_n$-modules such as: the coinvariant space of $\mathbb{S}_n$, the space of $\mathbb{S}_n$-harmonic polynomials, and the cohomology ring of the full-flag variety. More precisely, this makes explicit the graded decomposition into irreducibles of these spaces. Thus, formula (**??**) corresponds to the Hilbert series[3] of their isotropic component of type

[3]Graded dimension.

λ, which lies in $\mathbb{N}[q]$. Using (2.3) to expand H_n, the global Hilbert series of these modules can be simply obtained by computing the scalar product

$$\begin{aligned}
\langle p_1^n, H_n\rangle &= \sum_{\mu\vdash n} \langle p_1^n, s_\mu\rangle\, q^{n(\mu)} [n]_q! \prod_{(i,j)\in\mu} \frac{1-q}{1-q^{h_{ij}}}, &(2.9)\\
&= \sum_{\mu\vdash n} \langle p_1^n, p_\mu\rangle \frac{p_\mu(1/(1-q))}{z_\mu} \prod_{k=1}^{n}(1-q^k) &(2.10)\\
&= \left(\frac{1}{1-q}\right)^n \prod_{k=1}^{n}(1-q^k) &(2.11)\\
&= [n]_q!. &(2.12)
\end{aligned}$$

To see this, recall that $\langle p_\mu, p_\lambda\rangle$ is zero if $\mu \neq \lambda$, and $\langle p_\mu, p_\mu\rangle = z_\mu$. To complete the picture, let us also recall that $\langle p_\mu, s_\lambda\rangle$ is equal to the value, on the conjugacy class μ, of the character of the irreducible representation associated to λ. In particular, it follows that

$$H_n(\mathbf{x}; 1) = p_1^n = \sum_{\mu\vdash n} f_\mu\, s_\mu(\mathbf{x}). \tag{2.13}$$

This is the Frobenius characteristic of the regular representation of $\mathbb{S}_n$, for which the multiplicities f_μ are given by the number of standard Young tableaux of shape μ.

Beside this notion of Frobenius transform that "formally" encodes $\mathbb{S}_n$-irreducibles as Schur function, another more direct interpretation of the above formulas is in terms of characters of polynomial representations of $\mathrm{GL}(V)$, with V an N-dimensional space over $\mathbb{C}$. Recall that the character, of a representation $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$, is a symmetric function of $\chi_\rho(x_1, x_2, \ldots, x_N)$ of the eigenvalues of operators in $\mathrm{GL}(V)$. Through Schur-Weyl duality, out of any $\mathbb{S}_n$-module R and any $\mathrm{GL}(V)$-module U, one may construct a representation of $\mathrm{GL}(V)$:

$$R(U) := R \otimes_{\mathbb{C}\mathbb{S}_n} U^{\otimes n},$$

where $\mathbb{S}_n$ acts on $U^{\otimes n}$ by permutation of components. This construction is functorial:

$$R : GL(V)\text{-Mod} \longrightarrow GL(V)\text{-Mod},$$

and the character of $R(U)$ is the plethysm $f[g(x_1 x_2, \ldots, x_N)]$, whenever f is the Frobenius characteristic of R and g the character of U. Furthermore, under this construction, irreducible polynomial representations of $\mathrm{GL}(V)$ correspond to irreducible $\mathbb{S}_n$-modules R. If such is the case, one writes $S^\lambda(V)$ when R is irreducible of type λ. The corresponding character is the Schur function $s_\lambda(x_1, x_2, \ldots, x_N)$. For the special case $\lambda = (n)$, we get the symmetric power $S^a(V)$ whose character is $h_a(x_1, x_2, \ldots, x_N)$, hence the character of $S^a(S^b(V))$ is the plethysm $h_a[h_b]$.

3. Supporting facts and results

Beside having checked our conjectures by explicit computer calculation for all $1 < a\,b \leq 25$, we now show that they hold true at $q = 1$. Moreover we give a few interesting related results.

First off, we start by showing that division by $1-q^2$ makes sense in the statement of (1.4). Toward a verification of this fact, the first relevant observation is that

Lemma 3.1. *For any a and b, we have*

$$H_a[H_b](\mathbf{x};1) = H_b[H_a](\mathbf{x};1), \qquad \text{and} \qquad H_a[H_b](\mathbf{x};-1) = H_b[H_a](\mathbf{x};-1).$$

Proof. The first equality is easy, since $H_n(\mathbf{x};1) = p_1^n$, so that $H_a[H_b](\mathbf{x};1) = p_1^{ab}$. To get the second, we first show that

$$H_n(\mathbf{x};-1) = \begin{cases} p_2^k, & \text{if } n = 2\,k, \\ p_1\, p_2^k, & \text{if } n = 2\,k+1. \end{cases} \tag{3.1}$$

Indeed, using (2.6) and assuming that $k := \lfloor n/e \rfloor$, we get

$$H_n(\mathbf{x};q) = \sum_{d_1+2d_2+\ldots+nd_n=n} \frac{p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}}{d_1! 2^{d_2} d_2! \cdots n^{d_n} d_n!} (1-q)^{n-\ell}(1+q)^{k-j} r(q), \tag{3.2}$$

with $\ell := d_1 + d_2 + \ldots + d_k \leq n$ and $j = d_2 + d_4 + \ldots + d_{2k} \leq k$, for some polynomial $r(q)$. Clearly this expression vanishes at $q=-1$, except if $k=j$. Thus, the only term that survives is when $d_2 = k$, so that d_1 is either 1 or 0 depending on the parity of n. One easily checks that the coefficient of the only remaining term evaluates to 1 at $q=-1$. Now, a simple calculation shows that the plethysms involved in the second identity of the lemma coincide at this specialization $q=-1$, which commutes with the involved plethysms. □

The above lemma implies that the q-polynomial $F_{a,b}(\mathbf{x};q)$ (with coefficients in Λ) vanishes at $q = \pm 1$. Thus it is divisible by $1-q^2$. On the other hand, even small case exhibit instance of negative signs occurring in $H_b[H_a] - H_a[H_b]$, when $a<b$. The remarkable fact, which led to our conjecture, is that no sign remains after division[4] by $1-q^2$.

Dimension Count. As mentioned previously, when an homogeneous degree n symmetric function f occurs as a (graded) Frobenius transform of the character of an $\mathbb{S}_n$-module, the dimension (Hilbert series) of this module may be readily calculated by taking its scalar product with p_1^n. On the other hand, general principles insure that there exists such a module (albeit not explicitly known) whenever f expands positively (with coefficients in $\mathbb{N}[q]$) in the Schur function basis. Finding an explicit formula for this "dimension" may give a clue on what kind of module one should look for in order to prove the conjectures. With this in mind, let us set the notation

$$\dim(f) := \langle p_1^n, f \rangle.$$

For instance, we may easily calculate that

$$\dim(h_a[h_b]) = \frac{(ab)!}{a!\, b!^a}, \tag{3.3}$$

since p_1^{ab} may only occur in the plethysm $h_a[h_b]$ as

$$\frac{p_1^a}{a!}\left[\frac{p_1^b}{b!}\right] = \frac{p_1^{ab}}{a!\, b!^a}.$$

[4]Division by $1-q$ would suffice, but the result is further divisible by $1+q$, begging to be simplified.

In a classical combinatorial setup, formula (3.3) is easily interpreted as the number of partitions of a set of cardinality ab, into blocks each having size b. We say that this is a b^a-partition. Indeed, using a general framework such as the Theory of Species (see [1]), it is well understood that $h_a[h_b]$ may be interpreted as the Polya cycle index enumerator of such partitions, i.e.:

$$h_a[h_b] = \frac{1}{n!} \sum_{\sigma\in\mathbb{S}_n} \mathrm{fix}_\sigma\, p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n},$$

where $n = ab$, and d_k denotes the number of cycles of size k in σ. Here, we further denote by fix_σ the number of b^a-partitions that are fixed by a permutation σ, of the underlying elements. It follows that

$$\dim(F_{a,b}(\mathbf{x};0)) = (ab)! \left(\frac{1}{b!\,a!^b} - \frac{1}{a!\,b!^a}\right), \tag{3.4}$$

is the difference between the number of a^b-partitions and b^a-partitions. Some authors have attempted to exploit this fact to prove Foulkes conjecture (for positive and negative results along these lines see [15, 16, 17]).

It is interesting that we have the following very nice q-analog (at 0) of (3.4).

Proposition 3.1. *For all $a < b$, we have*

$$\dim(F_{a,b}(\mathbf{x};q)) = \frac{(ab)!}{1-q^2} \left(\frac{[b]_q!}{b!} \frac{([a]_q!)^b}{a!^b} - \frac{[a]_q!}{a!} \frac{([b]_q!)^a}{b!^a}\right), \tag{3.5}$$

and, letting $q \mapsto 1$, we find that

$$\dim(F_{a,b}(\mathbf{x};1)) = \frac{(ab)!(a-1)(b-1)(b-a)}{8}. \tag{3.6}$$

Proof. We first calculate $\dim(H_a[H_b])$ directly as follows

Now, exploiting classical properties of the logarithmic derivative $\mathrm{D}_{\log} f := f'/f$ (with respect to q), we easily calculate that

$$\mathrm{D}_{\log}\, [b]_q!\, ([a]_q!)^b\Big|_{q=1} = \frac{1}{2}\binom{b-1}{2} + \frac{b}{2}\binom{a-1}{2}.$$

From this we may readily obtain that $\lim_{q\to 1} \dim(F_{a,b}(\mathbf{x};q))$ gives (3.6). □

Observe that $\dim(F_{a,b}(\mathbf{x};q))$ is divisible by $[j]_q$, for any $1 \le j \le b$. It follows that

$$\dim(F_{a,b}(\mathbf{x};q))\big|_{\mathbf{mod}\ (q^j-1)} = \frac{[j]_q}{j} \dim(F_{a,b}(\mathbf{x};1)). \tag{3.7}$$

For instance, we have $\dim(F_{2,4}(\mathbf{x};1)) = 30240$ and

$$\begin{aligned}
\dim(F_{2,4}(\mathbf{x};q))\big|_{\mathbf{mod}\ (q^2-1)} &= 15120\,q + 15120,\\
\dim(F_{2,4}(\mathbf{x};q))\big|_{\mathbf{mod}\ (q^3-1)} &= 10080\,q^2 + 10080\,q + 10080,\\
\dim(F_{2,4}(\mathbf{x};q))\big|_{\mathbf{mod}\ (q^4-1)} &= 7560\,q^3 + 7560\,q^2 + 7560\,q + 7560.
\end{aligned}$$

Equation (3.7) also implies that the Schur expansion of $F_{a,b}(\mathbf{x};q)$ affords a similar distribution regularity, with respect to the coefficients of powers of q modulo j, for $j \leq b$. Indeed, we first observe that

$$F_{a,b}(\mathbf{x};q)\big|_{\mathbf{mod}\ (q^j-1)} = \frac{[j]_q}{j} F_{a,b}(\mathbf{x};1) + (q-1)G_{a,b}(\mathbf{x};q),$$

where $\dim(G_{a,b}(\mathbf{x};q)) = 0$. Denoting by $f^k_{a,b}(\mathbf{x})$ the Schur positive[5] coefficient of q^k in $F_{a,b}(\mathbf{x};q)$, the above statement is equivalent to saying that

$$\sum_{(k\ \mathbf{mod}\, j)=i} \dim(f^k_{a,b}(\mathbf{x})), \qquad 0 \leq i \leq j,$$

is independent of i. Thus, the announced regularity (of dimension) concerns the decomposition

$$F_{a,b}(\mathbf{x};q)\big|_{\mathbf{mod}\ (q^j-1)} = \sum_{i=0}^{j-1} q^i \sum_{(k\ \mathbf{mod}\, j)=i} f^k_{a,b}(\mathbf{x}).$$

This is somewhat surprising when one considers each individual term, which are quite different as illustrated by the following example:

$$\begin{aligned} F_{2,3}(\mathbf{x};q)\big|_{\mathbf{mod}\ (q^2-1)} &= (s_{33} + 2\,s_{222} + 2\,s_{321} + 2\,s_{2211} + s_{3111} + 3\,s_{21111}) \\ &\quad + (s_{33} + 2\,s_{321} + 4\,s_{2211} + s_{3111} + s_{21111} + 2\,s_{111111})\,q. \end{aligned}$$

It would be nice to have a direct explanation why all terms have the same "dimension", even though they break up differently into irreducibles.

The conjecture holds at $q = 1$. We start with an explicit formula that will be helpful in the sequel, setting the simplifying notation

$$\mathcal{E}_b := ((h_2 + e_2)^b + (h_2 - e_2)^b)/2, \qquad \text{and} \qquad \mathcal{O}_b := \frac{1}{2}\left((h_2+e_2)^b - (h_2-e_2)^b\right), \tag{3.8}$$

for the odd-part of $(h_2 + e_2)^b$ in the "variable" e_2.

Lemma 3.2. *For all a and b, we have the divided difference evaluation*

$$\lim_{q\to 1} \frac{h_1^{ab} - H_a[H_b]}{1-q} = a\binom{b}{2} h_1^{ab-2}\, e_2 + \binom{a}{2} h_1^{(a-2)b}\, \mathcal{O}_b. \tag{3.9}$$

Proof. This is a straigthforward calculation using rules (2.4) and (2.6). □

It immediately follows that we have the following formula.

Proposition 3.2. *For any $a > 1$ and $c > 0$, we have*

$$F_{a,b}(\mathbf{x};1) = \frac{1}{4}\left(ab\,(b-a)\,h_1^{ab-2}\,e_2 + a\,(a-1)\,h_1^{(a-2)b}\,\mathcal{O}_b - b\,(b-1)\,h_1^{a(b-2)}\,\mathcal{O}_a\right). \tag{3.10}$$

Moreover, this a positive integer coefficient polynomial in h_1, h_2 and e_2; hence, it expands positively in the Schur basis.

[5]Depending on whether Conjecture (1.4) is true or not.

We also have the following recursive approach to the calculation of $F_{a,b}$, as a polynomial in h_1, h_2 and e_2.

Proposition 3.3.

$$F_{a,b+1}(\mathbf{x};1) = h_1^a\, F_{a,b}(\mathbf{x};1) + h_1^{(a-2)b}\, \theta_a(b), \tag{3.11}$$

with $\Theta_a(b)$ satisfying the recurrence

$$\Theta_a(b) = (3\,h_2 + e_2)\,\Theta_a(b-1) - h_1^2\,(3\,h_2 - e_2)\,\Theta_a(b-2) + h_1^4\,(h_2 - e_2)\,\Theta_a(b-3),$$

with initial conditions: $\Theta_a(a) = \Theta_{a-1}(a)$, $\Theta_a(a+1) = (a^2 e_2/2)\,\mathcal{E}_a + (a\,h_2/2)\,\mathcal{O}_a$, and

$$\Theta_a(a+2) = \frac{((a+1)^2-2)\,e_2}{2}\,\mathcal{E}_{a+1} - \frac{(a+1)\,h_2}{2}\,\mathcal{O}_{a+1} + e_2\,(a\,e_2 + h_2)\,(h_2 - e_2)^a.$$

Using one of these calculation techniques, we find that

$$\begin{aligned}
&F_{2,3} = 2\,e_2^3, && F_{2,4} = 4\,e_2^3\,(e_2 + 2\,h_2)\\
& && F_{2,5} = 4\,e_2^3\,(2\,e_2^2 + 5\,h_2\,e_2 + 5\,h_2^2),\\
&F_{3,4} = 12\,h_1^4\,e_2^3\,h_2, && F_{3,5} = 4\,h_1^5\,e_2^3\,(e_2^2 + 5\,h_2\,e_2 + 10\,h_2^2),\\
& && F_{3,6} = 6\,h_1^6\,e_2^3\,(e_2^3 + 9\,e_2^2\,h_2 + 15\,e_2\,h_2^2 + 15\,h_2^3),\\
&F_{4,5} = 8\,h_1^{10}\,e_2^3\,(e_2^2 + 5\,h_2^2), && F_{4,6} = 12\,h_1^{12}\,e_2^3\,(e_2^3 + 4\,e_2^2\,h_2 + 5\,e_2\,h_2^2 + 10\,h_2^3),\\
& && F_{4,7} = 12\,h_1^{14}\,e_2^3\,(2\,e_2^4 + 7\,e_2^3\,h_2 + 21\,e_2^2\,h_2^2 + 21\,e_2\,h_2^3 + 21\,h_2^4).
\end{aligned}$$

Moreover, we have the following stability result.

Lemma 3.3. *For all $b > a > 1$,*

$$F_{a,b+1}(\mathbf{x};1) - h_1^a F_{a,b}(\mathbf{x};1) \in \mathbb{N}[h_1, h_2, e_2].$$

This implies the analog at $q = 1$ of the stability portion of Foulkes conjecture, namely

Proposition 3.4. *For all $a < b$, and all partition λ, we have*

$$\langle \overline{F_{a,b+1}(\mathbf{x};1)} - \overline{F_{a,b}(\mathbf{x};1)}, s_\lambda(\mathbf{x})\rangle \in \mathbb{N}.$$

Proof. Indeed, using the classical Pieri rule for the calculation of $h_1\, s_\lambda$, it is easy to see that

$$\overline{h_1^a F_{a,b}(\mathbf{x};1))} - \overline{F_{a,b}(\mathbf{x};1)}$$

is Schur positive, since one of the terms in $h_1^a s_\lambda$ is the Schur function indexed by the partition obtained from λ by adding a boxes to its firts line. Hence the lemma directly implies that

$$\overline{F_{a,b+1}(\mathbf{x};1)} - \overline{F_{a,b}(\mathbf{x};1)} = (\overline{F_{a,b+1}(\mathbf{x};1)} - \overline{h_1^a F_{a,b}(\mathbf{x};1))}) + (\overline{h_1^a F_{a,b}(\mathbf{x};1))} - \overline{F_{a,b}(\mathbf{x};1)})$$

is Schur positive. $\square$

4. Extensions

In her thesis, partly presented in [18], Vessenes attributes the following generalization of Foulkes conjecture to Doran [5]. However, it is not clear in the paper cited where this exact statement can be found, even if Vessenes does state in the following form. For a and b, let c be a divisor of $n := ab$, lying between a and b, and set $d = n/c$. Then the generalized conjecture of Doran states that

$$h_c[h_d] - h_a[h_b] \in \mathbb{N}[s_\mu \mid \mu \vdash n]. \tag{4.1}$$

This is to say that this difference is Schur positive. For example, one calculates that

$$h_3[h_4] - h_2[h_6] = s_{93} + s_{444} + s_{642} + s_{741} + s_{822}.$$

Our experiments suggest that this extends to our q-context, in a manner that is compatible with our previous discussion.

Conjecture 3. *Let c be a divisor of $n := ab$, with $a \leq c \leq b$, and set $d = n/c$. Then the divided difference*

$$\frac{H_c[H_d] - H_a[H_b]}{1-q^2},$$

is Schur positive.

Once again, using Formula (3.9), we can calculate how this specializes at $q = 1$. Setting $n := ab = cd$, we get

$$\lim_{q\to 1} \frac{H_c[H_d] - H_a[H_b]}{1-q^2} = \frac{1}{4}\left(n\,(b-d)\,h_1^{n-2}\,e_2 + a\,(a-1)\,h_1^{n-2b}\,\mathcal{O}_b - c\,(c-1)\,h_1^{n-2d}\,\mathcal{O}_d\right). \tag{4.2}$$

and again the resulting symmetric function lies in $\mathbb{N}[h_1, h_2, e_2]$. For instance, we have

$$\lim_{q\to 1} \frac{H_3[H_4] - H_2[H_6]}{1-q^2} = e_2\left(6\,e_2^5 + 27\,e_2^4 h_2 + 48\,e_2^3 h_2^2 + 58\,e_2^2 h_2^3 + 18\,e_2 h_2^4 + 3\,h_2^5\right).$$

Higher rank generalization. Another intriguing possible extension[6] consists in considering signed-combinations of higher "degree" plethystic compositions, such as

$$h_c[h_b[h_a]]] - h_c[h_a[h_b]]] - h_b[h_c[h_a]]] + h_b[h_a[h_c]]] + h_a[h_c[h_b]]] - h_a[h_b[h_c]]].$$

More generally, we could consider alternating sums, over the symmetric group $\mathbb{S}_n$, of the form

$$\sum_{\sigma\in\mathbb{S}_n} \mathrm{sign}(\sigma)\; h_{a_{\sigma(1)}}[h_{a_{\sigma(2)}}[\cdots[h_{a_{\sigma(n)}}]\cdots]], \tag{4.3}$$

with $a_1 > a_2 > \ldots > a_n > 1$. For $n = 2$, we clearly get back Foulkes conjecture. For $n = 3$, we have checked, although with a rather small number of cases, that (4.3) is indeed Schur positive. We underline that the degree of the symmetric functions involved is $a_1 a_2 \cdots a_n$, which is at least $(n+1)!$ under the hypothesis considered. Hence the verification of the

[6] Which we don't know yet how to extend correctly to the q-context.

Schur positivity of (4.3) rapidly goes beyond our computing capacity[7]. Hence, we may not as safely (beside sheer elegance) as before state the conjecture that (4.3) should always be Schur positive.

Along a related track, C. Reutenauer suggested that we consider immanant analogs of (4.3). The simplest case, besides those already considered or trivial situations, corresponds to

$$2h_c[h_b[h_a]] - h_b[h_a[h_c]] - h_a[h_c[h_b]].$$

Once again, small experiments suggest that we do have Schur positivity for such constructs.

5. Appendix

The proof of $\mathbb{N}$-positivity of the solution of the recurrence occcuring in Proposition 3.3 may be directly translated in terms $\mathbb{N}$-positivity of the following, as a polynomial in z. Let us set

$$\rho(z;a) := \sum_{k=1}^{\infty} k\, a \binom{a+1}{2k+1} z^{2\,k+1}, \tag{5.1}$$

and consider the following recurrence for $\theta_n(z) = \theta_n(z;a)$

$$\theta_n(z) = (3+z)\,\theta_{n-1}(z) + (1+z)\,(z-3)\,\theta_{n-2}(z) + (1+z)^2\,(1-z)\,\theta_{n-3}(z),$$

with initial conditions $\theta_0(z) := \rho(z;a-1)$, $\theta_1(z) := \rho(z;a)$, and

$$\theta_2(z) = \sum_{k=1}^{\infty} k(a-1)\binom{a+2}{2k+1} z^{2k+1} + 2k\binom{a+1}{2k+1}(1+z)\, z^{2\,k+1}.$$

For any $a > 2$ (wth in $\mathbb{N}$), $\theta_n(z;a)$ is clearly a degree $a+n$ polynomials in the variable z, with positives integer coefficients, and the link with our previous setup is simply that

$$\Theta_a(b) = h_2^b\, \theta_{b-a}(e_2/h_2; a).$$

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[7] Especially since we need to consider $n!$ Schur expansions where indexes are partitions of the degree in question. The number of these grows quite fast.

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