A q-Analog of Foulkes' Conjecture

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Abstract

We propose a q-analog of classical plethystic conjectures due to Foulkes. In our conjectures, a divided difference of plethysms of Hall-Littlewood polynomials $H_n(\boldsymbol{x};q)$ replaces the analogous difference of plethysms of complete homogeneous symmetric functions $h_n(\boldsymbol{x})$ in Foulkes' conjecture. At $q = 0$, we get back the original statement of Foulkes, and we show that our version holds at $q = 1$. We discuss further supporting evidence, as well as various generalizations, including a (q,t)-version.

Keywords: Foulkes' conjecture; Macdonald polynomials; q-analog

Contents

1 Introduction

The Foulkes' conjecture, which dates back to 1950 (see [10]), has a long and interesting history. Some headway has been made on it, but it remains open in general. A survey

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of the current state of affairs can be found in [6, 14], and related papers include [3, 5, 7, 17, 18, 19, 25]. In its original form, the conjecture states that, for any positive integers a and b, with $a < b$, the difference of plethysms[1] of complete homogeneous symmetric polynomials

$$h_b \circ h_a - h_a \circ h_b, \tag{1}$$

expands with positive (integer) coefficients in the Schur basis $\{s_\mu\}_{\mu \vdash n}$ of symmetric polynomials (here μ runs over the set of partitions of $n = a\,b$). Instances of this positivity are

$$h_4 \circ h_2 - h_2 \circ h_4 = s_{422} + s_{2222}, \qquad \text{and} \qquad h_4 \circ h_3 - h_3 \circ h_4 = s_{732} + s_{5421} + s_{6222}.$$

From the point of view of representation theory, one may interpret Foulkes' conjecture as saying that there is a $\mathrm{GL}(V)$-module inclusion of the composite of symmetric powers $S^a(S^b(V))$ inside $S^b(S^a(V))$. Therefore each $\mathrm{GL}(V)$-irreducible occurs with smaller multiplicity in $S^a(S^b(V))$ than it does in $S^b(S^a(V))$, and the conjecture reflects this at the level of the corresponding characters (with Schur polynomials appearing as characters of irreducible representations). Many interesting point of view may be considered, and some of these are nicely discussed in [14] both with an historical perspective[2], and explanations of ties with Geometric Complexity Theory.

Here, we consider symmetric "polynomials" (we will often say function) in a denumerable set of variables $\boldsymbol{x} = x_1, x_2, x_3, \ldots$, which are typically not mentioned explicitly. This makes it so that our statements hold irrespective of the number of variables occurring in the symmetric functions considered (*ergo* the dimension of the vector space V). Notice that the case of Foulke's conjecture when this dimension is 2 is a theorem. Indeed, it corresponds to Hermite's *Law of reciprocity* (see [11]), which says that the modules $S^a(S^b(V))$ and $S^b(S^a(V))$ coincide when V has dimension 2 (over $\mathbb{C}$). Another fact that is worth recalling is that Brion [4] has shown that (1) holds if b is large enough with respect to a.

Our q-analog replaces the relevant homogeneous symmetric function $h_n = h_n(\boldsymbol{x})$ by the Macdonald (Hall-Littlewood) polynomial

$$H_n(\boldsymbol{x}; q) := \sum_{\mu \vdash n} K_\mu(q) s_\mu(\boldsymbol{x}), \qquad \text{where} \qquad K_\lambda(q) = \sum_\tau q^{c(\tau)},$$

with τ running through the set of standard tableaux of shape λ, and $c(\tau)$ standing for the charge statistic. As we will recall further below, one has $H_n(\boldsymbol{x}; 0) = h_n(\boldsymbol{x})$. Hence we are considering a slightly different notion of q-analog, in which it is the specialization at $q = 0$ (rather than $q = 1$) that gives back the original statement.

Conjecture 1 (***q***-Foulkes)**.** For any integers $0 < a \leqslant b$, the Schur function expansion of the divided difference

$$F_{a,b}(\boldsymbol{x}; q) := \frac{H_b \circ H_a - H_a \circ H_b}{1 - q} \tag{2}$$

[1]This is an important operation on symmetric functions which was introduced by Littlewood [15]. Its explicit definition is recalled further below.

[2]Where it is underlined that the question raised by Foulkes apparently goes back to Hadamard [9].

has coefficients in $\mathbb{N}[q]$.

For short, one says that $F_{a,b}(\boldsymbol{x};q)$ is *Schur positive*. This has been checked to hold whenever $ab \leqslant 25$, and we will prove in the sequel that the conjecture is true at $q=1$. We have the specialization $F_{a,b}(\boldsymbol{x};0) = f_{a,b} := h_b \circ h_a - h_a \circ h_b$. We will show later that it makes sense to divide by $1-q$. For instance, with $a=2$ and $b=3$, we find after calculation that expression (2) expands in the Schur basis as as

$$\begin{aligned} F_{2,3}(\boldsymbol{x};q) \;=\; & \left(q^2+2\,q^3+q^4\right) s_{33} + \left(1+q+q^2+q^3\right) s_{222} \\ & + \left(q+2\,q^2+2\,q^3+2\,q^4+q^5\right) s_{321} \\ & + \left(q+2\,q^2+3\,q^3+3\,q^4+2\,q^5+q^6\right) s_{2211} \\ & + \left(q^2+2\,q^3+q^4\right) s_{3111} + \left(q^2+2\,q^3+3\,q^4+2\,q^5\right) s_{21111} \\ & + \left(q^3+q^4+q^5+q^6\right) s_{111111}. \end{aligned}$$

This does specialize, at $q=0$, to the corresponding case of Foulkes' conjecture:

$$f_{2,3} = h_3 \circ h_2 - h_2 \circ h_3 = s_{222}$$

A second part of Foulkes' conjecture, shown to be true by Brion [4], concerns the stability of coefficients as b grows while a remains fixed. To simplify its statement, we consider the linear operator which sends a Schur function $s_\mu(\boldsymbol{x})$ to $s_{\overline{\mu}}(\boldsymbol{x})$, where $\overline{\mu}$ is the partition obtained by removing the largest part in μ. Let us write $\overline{f}$ for the effect of this operator on a symmetric function f. For example, we get $\overline{s_{622}+s_{442}+s_{4222}+s_{22222}} = s_{22}+s_{42}+s_{222}+s_{2222}$, which is clearly not homogeneous. Using this notation convention, the second part of Foulkes' conjecture states that, for all $a \leqslant b$, the Schur expansion of

$$\overline{f_{a,b+1}} - \overline{f_{a,b}} = \overline{(h_{b+1}\circ h_a - h_a \circ h_{b+1})} - \overline{(h_b \circ h_a - h_a \circ h_b)} \tag{3}$$

also affords positive integers polynomials as coefficients. Observe that the "Bar" operator allows the comparison of homogeneous functions of different degrees, namely $\overline{f_{a,b+1}}$ of degree $a(b+1)$ with $f_{a,b}$ of degree ab. For instances of, one calculates $\overline{f_{2,4}} - \overline{f_{2,3}} = s_{222}$, since the left-hand side is equal to $\overline{(s_{422}+s_{2222})} - \overline{s_{222}}$, in which $\overline{s_{422}}$ cancels out with $\overline{s_{222}}$.

In [4], Brion has shown (3) reducing (see Appendix for notations) it to showing that

$$\langle h_a \circ h_b, s_\lambda \rangle \leqslant \langle h_a \circ h_{b+1}, s_{\lambda+(a)} \rangle,$$

where the **sum** $\lambda+\mu$ of two partitions λ and μ, is the partition whose parts are $\lambda_i + \mu_i$ (with the convention that $\lambda_i = 0$ if i greater than the number of parts of λ). More results along these lines may be found in [12, 19]. A similar phenomenon also seems to hold in our context, leading us to state the following.

Conjecture 2 (***q***-stability)**.** For any integers $0 < a \leqslant b$, and any Schur function s_λ, we have

$$\langle \overline{F_{a,b+1}(\boldsymbol{x};q)} - \overline{F_{a,b}(\boldsymbol{x};q)}, s_\lambda(\boldsymbol{x}) \rangle \in \mathbb{N}[q]. \tag{4}$$

The smallest non-trivial example is:

$$\begin{aligned}\overline{F_{2,4}(\boldsymbol{x};q)}-\overline{F_{2,3}(\boldsymbol{x};q)} \;=\;& (1+q)\Big(\left(q^3+2\,q^4+2\,q^5+q^6\right)s_3+\left(q^2+2\,q^3+3\,q^4+3\,q^5+2\,q^6+q^7\right)s_{21}\\
&+\left(q^3+2\,q^4+2\,q^5+q^6\right)s_{111}+\left(q^2+2\,q^4+q^6\right)s_4\\
&+\left(q+2\,q^2+6\,q^3+7\,q^4+9\,q^5+6\,q^6+4\,q^7+q^8\right)s_{31}\\
&+\left(q+3\,q^2+4\,q^3+7\,q^4+5\,q^5+6\,q^6+2\,q^7+2\,q^8\right)s_{22}\\
&+\left(2\,q^2+6\,q^3+10\,q^4+13\,q^5+11\,q^6+8\,q^7+3\,q^8+q^9\right)s_{211}\\
&+\left(q^3+4\,q^4+6\,q^5+7\,q^6+4\,q^7+2\,q^8\right)s_{1111}\\
&+\left(q+2\,q^2+5\,q^3+6\,q^4+8\,q^5+6\,q^6+5\,q^7+2\,q^8+q^9\right)s_{32}\\
&+\left(3\,q^2+4\,q^3+10\,q^4+9\,q^5+11\,q^6+6\,q^7+4\,q^8+q^9\right)s_{311}\\
&+\left(2\,q+4\,q^2+9\,q^3+12\,q^4+15\,q^5+13\,q^6+11\,q^7+6\,q^8+3\,q^9+q^{10}\right)s_{221}\\
&+\left(2\,q^2+6\,q^3+11\,q^4+16\,q^5+17\,q^6+14\,q^7+9\,q^8+4\,q^9+q^{10}\right)s_{2111}\\
&+\left(q^3+3\,q^4+6\,q^5+7\,q^6+8\,q^7+5\,q^8+3\,q^9+q^{10}\right)s_{11111}\\
&+\left(1+2\,q^2+q^3+4\,q^4+2\,q^5+5\,q^6+q^7+3\,q^8+q^{10}\right)s_{222}\\
&+\left(q+q^2+4\,q^3+5\,q^4+9\,q^5+8\,q^6+9\,q^7+5\,q^8+4\,q^9+q^{10}+q^{11}\right)s_{2211}\\
&+\left(q^2+q^3+5\,q^4+5\,q^5+9\,q^6+7\,q^7+7\,q^8+3\,q^9+2\,q^{10}\right)s_{21111}\\
&+\left(q^3+q^4+3\,q^5+3\,q^6+4\,q^7+3\,q^8+3\,q^9+q^{10}+q^{11}\right)s_{111111}\\
&+\left(q^4+q^6+q^8+q^{10}\right)s_{1111111}\Big),\end{aligned}$$

and we do observe that this specializes to the (much simpler) classical Foulkes case when we set $q=0$. Another stability, in the vein of Manivel [18], that seems to hold in our q-context is that

$$(\overline{F_{a+1,b+1}(\boldsymbol{x};q)}-\overline{F_{a+1,b}(\boldsymbol{x};q)})-(\overline{F_{a,b+1}(\boldsymbol{x};q)}-\overline{F_{a,b}(\boldsymbol{x};q)}),$$

is Schur positive, for all $a<b$.

Extensions

In her thesis, partly presented in [24], Vessenes attributes[3] the following generalization of Foulkes' conjecture to Doran [8]. As before, let $a<b$ and consider two extra integers c and d, both lying between a and b, such that $ab=cd$ (observe that we do not assume that $c<d$). Then Vessenes's extension of Foulkes' conjecture states that

$$h_c\circ h_d-h_a\circ h_b \tag{5}$$

expands with positive coefficients in the Schur basis. Setting $c=b$ and $d=a$ clearly gives back the original statement of Foulkes. For example, one calculates that

$$h_3\circ h_4-h_2\circ h_6=s_{93}+s_{444}+s_{642}+s_{741}+s_{822}.$$

[3]However, it is not clear in [8] where this exact statement can be found, if at all.

Vessenes proves that (5) is indeed Schur positive whenever $a = 2$, and more instances may be shown to hold using formulas of [13] for the plethysms $s_\mu \circ h_k$, for partitions μ of 3, 4 and 5. We have also checked it explicitlly for all cases when the overall degree is less or equal to 36.

Extensive computer algebra experiments[4] suggest that there are two natural (new) extensions to this conjecture, namely

Conjecture 3. Assume that $a \leqslant c, d \leqslant b$, with $ab = cd$, and k is any positive integer, then we have Schur positivity of both the differences

$$h_c \circ (h_d^k) - h_a \circ (h_b^k), \qquad \text{and} \qquad h_c \circ s_\delta - h_a \circ s_\beta, \tag{6}$$

where we consider the rectangular shape partitions

$$\delta = \underbrace{dd\cdots d}_{k}, \qquad \text{and} \qquad \beta = \underbrace{bb\cdots b}_{k}.$$

More experiments (for all cases when the overall degree is less than 18) suggest that all the above conjectures are all encompassed in the following one, which involves the (combinatorial) two parameters Macdonald polynomials $H_\mu(\boldsymbol{x}; q, t)$ indexed by rectangular shape partitions.

Conjecture 4. Assume that $a \leqslant c, d \leqslant b$, with $ab = cd$, and k is any positive integer. Then, we have $\mathbb{N}[q]$-Schur positivity of

$$\frac{H_c \circ H_\delta(\boldsymbol{x}; q, t) - H_a \circ H_\beta(\boldsymbol{x}; q, t)}{1-q}, \qquad \text{with} \qquad \delta = \underbrace{dd\cdots d}_{k}, \quad \text{and} \quad \beta = \underbrace{bb\cdots b}_{k}. \tag{7}$$

We will show why division by $1 - q$ makes sense and how it implies previous conjectures in the following section. It is interesting to observe that, at $t = 1$, formula (7) specializes to the nice q-analog

$$F^{(k)}_{a,b;c,d}(\boldsymbol{x}; q) := \frac{H_c \circ H_d^k(\boldsymbol{x}; q) - H_a \circ H_b^k(\boldsymbol{x}; q)}{1-q}, \tag{8}$$

which thus have to be Schur positive if Conjecture 4 is to hold.

2 Supporting facts and implications

To further discuss and exploit the several implications of conjectures 1 and 4, and their ties to conjecture 3, we rapidly recall some properties and specializations of the symmetric polynomials $H_\mu(\boldsymbol{x}; q, t)$.

[4]For all cases when the overall degree if less or equal to 32.

Combinatorial Macdonald polynomials

Recall that the polynomial $H_n(\boldsymbol{x};q)$ is a special instance of the "combinatorial" Macdonald polynomials $H_\mu = H_\mu(\boldsymbol{x};q,t)$. Since the parameter t only appears in the H_μ's that are indexed by partitions having at least 2 parts, it makes sense to avoid its mention in H_n. For instance, we have

$$\begin{aligned} H_3(\boldsymbol{x};q,t) &= s_3(\boldsymbol{x}) + \left(q^2+q\right) s_{21}(\boldsymbol{x}) + q^3 s_{111}(\boldsymbol{x}), \\ H_{21}(\boldsymbol{x};q,t) &= s_3(\boldsymbol{x}) + (q+t)\, s_{21}(\boldsymbol{x}) + q\,t\, s_{111}(\boldsymbol{x}), \\ H_{111}(\boldsymbol{x};q,t) &= s_3(\boldsymbol{x}) + \left(t^2+t\right) s_{21}(\boldsymbol{x}) + t^3 s_{111}(\boldsymbol{x}). \end{aligned}$$

The H_μ are orthogonal with respect to the scalar product defined on the power-sum basis by the formula

$$\langle p_\mu(\boldsymbol{x}), p_\mu(\boldsymbol{x})\rangle_{q,t} := (-1)^{n-\ell(\mu)}\, z_\mu \prod_{k\in\mu}(1-q^k)(1-t^k),$$

with $\langle p_\mu(\boldsymbol{x}), p_\lambda(\boldsymbol{x})\rangle_{q,t} = 0$ whenever $\mu \neq \lambda$. Here, $\ell(\mu)$ stands for the number of parts of μ, and k runs over the parts of μ. They afford the following specializations:

$$\text{a)}\quad H_\mu(\boldsymbol{x};0,0) = h_n(\boldsymbol{x}), \tag{9}$$

$$\text{b)}\quad H_\mu(\boldsymbol{x};1,1) = h_1^n(\boldsymbol{x}), \tag{10}$$

$$\text{c)}\quad H_\mu(\boldsymbol{x};0,1) = h_\mu(\boldsymbol{x}), \tag{11}$$

$$\text{d)}\quad H_\mu(\boldsymbol{x};0,t)\big|_{t^{\text{maxdeg}}} = s_\mu(\boldsymbol{x}). \tag{12}$$

Furthermore we have

$$H_\mu(\boldsymbol{x};q,1) = \prod_{k\in\mu} H_k(\boldsymbol{x};q), \qquad \text{with} \qquad H_k(\boldsymbol{x};q) = \prod_{i=1}^{n}(1-q^i)\, h_k\left[\frac{\boldsymbol{x}}{1-q}\right], \tag{13}$$

which may be written in the following form using plethystic rules of calculation, and formula (32):

$$H_n(\boldsymbol{x};q) = \sum_{\mu\vdash n} \frac{[n]_q!}{z_\mu\, [\mu_1]_q[\mu_2]_q\cdots[\mu_\ell]_q}\,(1-q)^{n-\ell(\mu)}\; p_\mu(\boldsymbol{x}), \tag{14}$$

with $\ell = \ell(\mu)$ standing for the number of parts of μ. We also have the symmetry

$$H_{\mu'}(\boldsymbol{x};q,t) = H_\mu(\boldsymbol{x};q,t). \tag{15}$$

Hence it follows that

$$H_\mu(\boldsymbol{x};1,t) = \prod_{k\in\mu'} H_k(\boldsymbol{x};t). \tag{16}$$

Now, we may show that division by $1-q$ makes sense in formulas (2) and (7). Indeed, for any a and b, formula (10) implies that we have $(H_a \circ H_b)(\boldsymbol{x};1) = h_1(\boldsymbol{x})^{ab}$, since the evaluation at $q = 1$ is compatible[5] with plethysm. Hence, the numerator of the right-hand side of (2) vanishes at $q = 1$, and is thus divisible by $1 - q$. More generally, using (13) with $\mu = bb \cdots b$ and hence

$$\mu' = \underbrace{kk \cdots k}_{b},$$

one sees that

$$H_a \circ H_{bb\cdots b}(\boldsymbol{x}; 1, t) = h_1^a \circ (H_k(\boldsymbol{x}; t)^b) = H_k(\boldsymbol{x}; t)^{ab},$$

it follows that the coefficients of each Schur function in the expression of (7) are polynomials in q.

Several specializations of (7) are of interest. In particular, at $q = 0$, we get

$$h_a \circ H_{bb\cdots b}(\boldsymbol{x}; 0, t) \preceq_s h_c \circ H_{dd\cdots d}(\boldsymbol{x}; 0, t),$$

where we write $f \preceq_s g$ if $g - f$ is Schur positive. Comparing coefficients of the highest power of t on both sides of this gives

$$h_a \circ s_{bb\cdots b} \preceq_s h_c \circ s_{dd\cdots d},$$

which is precisely the second statement in (6). Likewise, setting $t = 1$ we get

$$h_a \circ (h_b^k) \preceq_s h_c \circ (h_d^k),$$

Specializing at $q = 0$ and $t = 0$ also gives the following instance of (5)

$$h_a \circ h_{(kb)} \preceq_s h_c \circ h_{(kd)}.$$

We will also consider in the sequel the following[6] q-analog of Schur functions:

$$S_\mu(\boldsymbol{x}; q) := \omega\, q^{n(\mu')}\, H_\mu(\boldsymbol{x}; 1/q, 0),$$

defined in terms of specialization at $t = 0$ of the combinatorial Macdonald polynomials[7] $H_\mu(\boldsymbol{x}; q, t)$, with ω standing for the "usual" linear involution that sends $s_\mu(\boldsymbol{x})$ to $s_{\mu'}(\boldsymbol{x})$. In particular, one may check that $S_\mu(\boldsymbol{x}; 0) = s_\mu(\boldsymbol{x})$ and $S_\mu(\boldsymbol{x}; 1) = e_{\mu'}(\boldsymbol{x})$. For instance, we have

$$S_{32}(\boldsymbol{x}; q) = s_{32}(\boldsymbol{x}) + q\, s_{311} + q\,(q + 1)\, s_{221}(\boldsymbol{x}) + q^2\,(q + 1)\, s_{2111}(\boldsymbol{x}) + q^4 s_{11111}(\boldsymbol{x}),$$

and $S_{32}(\boldsymbol{x}; 1) = e_1(\boldsymbol{x})\, e_2(\boldsymbol{x})^2$. Observe that all terms in the Schur expansion of $S_\mu(\boldsymbol{x}; q)$: are indexed by partitions that are dominated by μ. Moreover we get back our previous context for $\mu = (a)$, since $S_{(a)}(\boldsymbol{x}; q) = H_a(\boldsymbol{x}, q)$.

[5]Meaning that they commute as operators. Observe that this is not so with the evaluation at $q = -1$.
[6]This is well-known in the theory of Macdonald polynomials, and all properties also mentioned.
[7]See Appendix A for various notations used here.

Dimension Count

As mentioned previously, when a homogeneous degree n symmetric function f occurs as a (graded) Frobenius transform of the character of a $\mathbb{S}_n$-module, the dimension (Hilbert series) of this module may be readily calculated by taking its scalar product with h_1^n. On the other hand, general principles insure that there exists such a module (albeit not explicitly known) whenever f expands positively (with coefficients in $\mathbb{N}[q]$) in the Schur function basis. Finding an explicit formula for this "dimension" may give a clue on what kind of module one should look for in order to prove the conjectures. With this in mind, let us set the notation

$$\dim(f) := \langle h_1^n, f\rangle.$$

For instance, we may easily calculate that

$$\dim(h_a \circ h_b) = \frac{(ab)!}{a!\, b!^a}, \tag{17}$$

since h_1^{ab} may only occur in the plethysm $h_a \circ h_b$ as

$$\frac{h_1^a}{a!}\left[\frac{h_1^b}{b!}\right] = \frac{h_1^{ab}}{a!\, b!^a}.$$

In a classical combinatorial setup, formula (17) is easily interpreted as the number of partitions of a set of cardinality ab, into blocks each having size b. We say that this is a b^a-partition. Indeed, using a general framework such as the Theory of Species (see [2]), it is well understood that $h_a \circ h_b$ may be interpreted as the Polya cycle index enumerator of such partitions, i.e.:

$$h_a \circ h_b = \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n} \mathrm{fix}_\sigma\, p_1^{d_1} p_2^{d_2}\cdots p_n^{d_n},$$

where $n = ab$, and d_k denotes the number of cycles of size k in σ. Here, we further denote by fix_σ the number of b^a-partitions that are fixed by a permutation σ, of the underlying elements. It follows that

$$\dim(F_{a,b}(\boldsymbol{x};0)) = (ab)!\left(\frac{1}{b!\, a!^b} - \frac{1}{a!\, b!^a}\right), \tag{18}$$

is the difference between the number of a^b-partitions and b^a-partitions. Some authors have attempted to exploit this fact to prove Foulkes' conjecture (for positive and negative results along these lines see [20, 21, 22, 23]).

It is interesting that we have the following very nice q-analog (at 0) of (18).

Proposition 5. *For all $a < b$, we have*

$$\dim(F_{a,b}(\boldsymbol{x};q)) = \frac{(ab)!}{1-q}\left(\frac{[b]_q!}{b!}\frac{([a]_q!)^b}{a!^b} - \frac{[a]_q!}{a!}\frac{([b]_q!)^a}{b!^a}\right), \tag{19}$$

and, letting $q \mapsto 1$, we find that

$$\dim(F_{a,b}(\boldsymbol{x};1)) = \frac{(ab)!(a-1)(b-1)(b-a)}{4}. \tag{20}$$

Proof. We first calculate $\dim(H_a \circ H_b)$ directly as follows

Now, exploiting classical properties of the logarithmic derivative $\mathrm{D}_{\log} f := f'/f$ (with respect to q), we easily calculate that

$$\mathrm{D}_{\log}\, [b]_q!\, ([a]_q!)^b \Big|_{q=1} = \frac{1}{2}\binom{b-1}{2} + \frac{b}{2}\binom{a-1}{2}.$$

From this we may readily obtain that $\lim_{q\to 1} \dim(F_{a,b}(\boldsymbol{x};q))$ gives (20). □

3 The q-conjecture holds at $q = 1$

We start with an explicit formula that will be helpful in the sequel, setting the simplifying notation

$$\mathcal{E}_b := ((h_2+e_2)^b + (h_2-e_2)^b)/2, \qquad \text{and} \qquad \mathcal{O}_b := \frac{1}{2}\left((h_2+e_2)^b - (h_2-e_2)^b\right), \quad (21)$$

respectively for the even-part and odd-part of $(h_2+e_2)^b$ in the "variables" h_2 and e_2. We thing of these as the homogenous and elementary symmetric functions, for which we have the power-sum expansions $p_1^2 = h_2 + e_2$ and $p_2 = h_2 - e_2$.

Lemma 6. *For all a, b and k, we have the divided difference evaluation*

$$\lim_{q\to 1} \frac{h_1^{abk} - H_a \circ H_b^k}{1-q} = ak\binom{b}{2} h_1^{abk-2}\, e_2 + \binom{a}{2} h_1^{(a-2)bk}\, \mathcal{O}_{bk}. \quad (22)$$

Proof. The limit on the left hand-side is the evaluation at 1 of the derivative of $H_a \circ H_b^k$ with respect to q. We use formula (14) to calculate this, exploiting the fact that the evaluation at 1 of the q-derivative of $g(q) = (1-q)^m f(q)$ is

$$g'(1) = \begin{cases} f'(1) & \text{if m=0,} \\ -f(1) & \text{if m=1,} \\ 0 & \text{otherwise.} \end{cases}$$

Using (14) and the rules of plethysm to expand $H_a \circ H_b^k$, and observing that the only partitions μ of a such that $a - \ell(\mu) \leqslant 1$ are either $\mu = 1^a$ or $\mu = 21^{a-2}$, we find that we have the expansion

$$H_a(\boldsymbol{x};q) = \frac{[a]!_q}{a!}\, p_1(\boldsymbol{x})^a + (1-q)\frac{[a]!_q}{2(a-2)!(1+q)}\, p_1(\boldsymbol{x})^{a-2}\, p_2(\boldsymbol{x}) + (1-q)^2\, G(\boldsymbol{x};q),$$

with $G(\boldsymbol{x};1) \neq 0$, and where $[a]!_q$ stands for the q-analog of $a!$. Thus, we get that the left hand-side of (22) evaluates as

$$\begin{aligned}
(H_a \circ H_b^k)'(1) &= \frac{d}{dq}\left(\frac{[a]!_q}{a!}\, p_1^a \circ H_b^k + (1-q)\frac{[a]!_q}{2(a-2)!(1+q)}(p_1^{a-2}\, p_2) \circ H_b^k\right)\Big|_{q=1},\\
&= \frac{d}{dq}\left(\frac{[a]!_q}{a!}\, H_b^{ak}\right)\Big|_{q=1} + \frac{d}{dq}\left((1-q)\frac{[a]!_q}{2(a-2)!(1+q)}\, H_b^{(a-2)k}\cdot (p_2 \circ H_b^k)\right)\Big|_{q=1},\\
&= \frac{d}{dq}\left(\frac{[a]!_q}{a!}\, H_b^{ak}\right)\Big|_{q=1} - \left(\frac{[a]!_q}{2(a-2)!(1+q)} H_b^{(a-2)k}\cdot (p_2 \circ H_b^k)\right)\Big|_{q=1},\\
&= \frac{1}{2}\binom{a}{2} p_1^{abk} + ak\, p_1^{(ak-1)b}\frac{d}{dq}(H_b)\Big|_{q=1} - \frac{1}{2}\binom{a}{2} p_1^{(a-2)bk} p_2^{bk},\\
&= \frac{1}{2}\binom{a}{2} p_1^{abk} + ak\, p_1^{(ak-1)b}\binom{b}{2}\frac{(p_1^b - p_1^{(b-2)}p_2)}{2} - \frac{1}{2}\binom{a}{2} p_1^{(a-2)bk} p_2^{bk},
\end{aligned}$$

recalling that $H_n(\boldsymbol{x};1) = p_1^n$ and that $p_1^n \circ F = F^n$. Since $p_1 = h_1$, $p_1^2 = h_2 + e_2$ and $p_2 = h_2 - e_2$, this last expression is clearly equal to the right-hand side of equation (22). □

It immediately follows that we have the following formula as a difference of two expressions obtained from the lemma.

Proposition 7. *For any $1 < a \leqslant c, d \leqslant b$ and $k \geqslant 1$, with $n := abk = cdk$, we have*

$$F^{(k)}_{a,b;c,d}(\boldsymbol{x};1) = \left(ak\binom{b}{2} - ck\binom{d}{2}\right) h_1^{n-2}\, e_2 + \binom{a}{2} h_1^{(a-2)bk}\, \mathcal{O}_{bk} - \binom{c}{2} h_1^{(c-2)dk}\, \mathcal{O}_{dk}. \quad (23)$$

Moreover, this a positive integer coefficient polynomial in h_1, h_2 and e_2; hence, it expands positively in the Schur basis.

For instance, we get

$$\begin{aligned}
F^{(2)}_{2,6;3,4} &= 24\, {e_2}^{12} + 252\, {e_2}^{11} h_2 + 1224\, {e_2}^{10} {h_2}^2 + 3868\, {e_2}^{9} {h_2}^3 + 7152\, {e_2}^{8} {h_2}^4\\
&\quad + 10680\, {e_2}^{7} {h_2}^5 + 9744\, {e_2}^{6} {h_2}^6 + 7512\, {e_2}^{5} {h_2}^7 + 3192\, {e_2}^{4} {h_2}^8\\
&\quad + 1228\, {e_2}^{3} {h_2}^9 + 168\, {e_2}^{2} {h_2}^{10} + 12\, e_2 {h_2}^{11}.
\end{aligned}$$

To make the positivity in the previous proposition more apparent, we exploit the following recursive approach to the calculation of $F_{a,b} = F^{(1)}_{a,b;b,a}$, as a polynomial in h_1, h_2 and e_2, together with Appendix 4.

Proposition 8.

$$F_{a,b+1}(\boldsymbol{x};1) = h_1^a\, F_{a,b}(\boldsymbol{x};1) + 2\, h_1^{(a-2)b}\, \Theta_a(b), \quad (24)$$

with $\Theta_a(b)$ defined recursively as

$$\Theta_a(b) = (3\, h_2 + e_2)\, \Theta_a(b-1) - h_1^2\, (3\, h_2 - e_2)\, \Theta_a(b-2) + h_1^4\, (h_2 - e_2)\, \Theta_a(b-3),$$

with initial conditions: $\Theta_a(a) = \Theta_{a-1}(a)$, $\Theta_a(a+1) = (a^2 e_2/2)\, \mathcal{E}_a + (a\, h_2/2)\, \mathcal{O}_a$, and

$$\Theta_a(a+2) = \frac{((a+1)^2 - 2)\, e_2}{2}\, \mathcal{E}_{a+1} - \frac{(a+1)\, h_2}{2}\, \mathcal{O}_{a+1} + e_2\, (a\, e_2 + h_2)\, (h_2 - e_2)^a.$$

Moreover, $\Theta_a(b)$ lies in $\mathbb{N}[h_1, h_2, e_2]$.

Using these calculation techniques, we find that

$$\begin{aligned}
F_{2,3} &= 4\,e_2^3, & F_{2,4} &= 8\,e_2^3\,(e_2+2\,h_2)\\
& & F_{2,5} &= 8\,e_2^3\,(2\,e_2^2+5\,h_2\,e_2+5\,h_2^2),\\
F_{3,4} &= 24\,h_1^4\,e_2^3\,h_2, & F_{3,5} &= 8\,h_1^5\,e_2^3\,(e_2^2+5\,h_2\,e_2+10\,h_2^2),\\
& & F_{3,6} &= 12\,h_1^6\,e_2^3\,(e_2^3+9\,e_2^2\,h_2+15\,e_2\,h_2^2+15\,h_2^3),\\
F_{4,5} &= 16\,h_1^{10}\,e_2^3\,(e_2^2+5\,h_2^2), & F_{4,6} &= 24\,h_1^{12}\,e_2^3\,(e_2^3+4\,e_2^2\,h_2+5\,e_2\,h_2^2+10\,h_2^3),\\
& & F_{4,7} &= 24\,h_1^{14}\,e_2^3\,(2\,e_2^4+7\,e_2^3\,h_2+21\,e_2^2\,h_2^2+21\,e_2\,h_2^3+21\,h_2^4).
\end{aligned}$$

In particular, for all $b > a > 1$, we have.

$$F_{a,b+1}(\boldsymbol{x};1) - h_1^a F_{a,b}(\boldsymbol{x};1) \in \mathbb{N}[h_1, h_2, e_2],$$

which implies the analog at $q = 1$ of the stability portion of Foulkes' conjecture, namely

Proposition 9. *For all $a < b$, and all partition λ, we have*

$$\langle \overline{F_{a,b+1}(\boldsymbol{x};1)} - \overline{F_{a,b}(\boldsymbol{x};1)}, s_\lambda(\boldsymbol{x})\rangle \in \mathbb{N}.$$

Proof. Indeed, using the classical Pieri rule for the calculation of $h_1\,s_\lambda$, it is easy to see that

$$\overline{h_1^a F_{a,b}(\boldsymbol{x};1))} - \overline{F_{a,b}(\boldsymbol{x};1)}$$

is Schur positive, since one of the terms in $h_1^a s_\lambda$ is the Schur function indexed by the partition obtained from λ by adding a boxes to its first line. Hence the lemma directly implies that

$$\overline{F_{a,b+1}(\boldsymbol{x};1)} - \overline{F_{a,b}(\boldsymbol{x};1)} = (\overline{F_{a,b+1}(\boldsymbol{x};1)} - \overline{h_1^a F_{a,b}(\boldsymbol{x};1))}) + (\overline{h_1^a F_{a,b}(\boldsymbol{x};1))} - \overline{F_{a,b}(\boldsymbol{x};1)})$$

is Schur positive. □

It is interesting to calculate how $F_{a,b}(\boldsymbol{x};q)$ expands explicitly as a polynomial in q. Indeed, by a direct calculation, one gets

$$F_{a,b}(\boldsymbol{x};q) = (h_b\circ h_a - h_a\circ h_b) + ((h_{b-1}\circ h_a)\,h_{a-1}\,h_1 - (h_{a-1}\circ h_b)\,h_{b-1}\,h_1)\,q + \dots$$

with similar (but more intricate expressions as illustrated below) for higher degree terms. Hence, the conjectured Schur-positivity of $F_{a,b}(\boldsymbol{x};q)$ implies that we have Schur positivity of

$$((h_{b-1}\circ h_a)\cdot h_{a-1} - (h_{a-1}\circ h_b)\cdot h_{b-1})\cdot h_1,$$

but we may show that in fact

$$(h_{b-1}\circ h_a)\cdot h_{a-1} - (h_{a-1}\circ h_b)\cdot h_{b-1} = h_1^\perp(h_b\circ h_a - h_a\circ h_b). \tag{25}$$

Indeed, it follows readily from the definitions that

$$h_1^\perp(h_n \circ h_k) = h_{k-1} \cdot (h_{n-1} \circ h_k). \tag{26}$$

In a sense this is because $h_1^\perp$ acts as a derivation, sending h_n to h_{n-1}, and this is a form of chain-rule. Hence the positivity of the coefficient of q in $F_{a,b}(\boldsymbol{x};q)$ is a consequence of the classical version Foulkes' conjecture, since Schur positivity is preserved by both operations of multiplication by h_1 and its adjoint[8] $h_1^\perp$.

However, for higher degree, it does not seem that we can calculate coefficients as easily. To illustrate, we have calculated that the coefficient of q^2 in $F_{a,b}(\boldsymbol{x};q)$ is equal to

$$\begin{aligned}(h_{b-2} \circ h_a) \cdot (h_2 \circ h_{a-1}\, h_1) + (h_{b-1} \circ h_a) \cdot (h_2\, h_{a-2} + h_1\, h_{a-1} - h_a) - h_b \circ h_a \\ -(h_{a-2} \circ h_b) \cdot (h_2 \circ h_{b-1}\, h_1) - (h_{a-1} \circ h_b) \cdot (h_2\, h_{b-2} + h_1\, h_{b-1} - h_b) + h_a \circ h_b\end{aligned}$$

Property (25) extends to the wider context of (5), so that the coefficient of q in the right-hand side of (7) (for $k = 1$) is indeed Schur positive (assuming that (5) holds), since it is equal to $h_1 \cdot h_1^\perp(h_c \circ h_d - h_a \circ h_b)$.

4 Expanding Foulkes' conjecture to more general diagrams

For partitions α, β, γ, and δ, none of which equal to (1) and such that $|\alpha|\cdot|\beta| = |\gamma|\cdot|\delta| = n$, let us say that $\langle\alpha, \beta, \gamma, \delta\rangle$ is a **Foulkes configuration** for n, if and only if

$$s_\alpha \circ s_\beta \;\preceq_s\; s_\gamma \circ s_\delta. \tag{27}$$

Clearly, for $a < b$, Foulkes' conjecture says that $\langle a, b, b, a\rangle$ is a Foulkes configuration. Likewise statement (5), under the conditions there specified, is equivalent to saying that $\langle a, b, c, d\rangle$ is a Foulkes configuration. Other cases are possible. Indeed, by direct explicit calculation we find the following:

$\langle 2,3,3,2\rangle$,	$\langle 11,111,3,11\rangle$,	$\langle 111,2,11,21\rangle$,	$\langle 111,11,2,21\rangle$,
$\langle 2,4,4,2\rangle$,	$\langle 2,1111,4,11\rangle$,	$\langle 11,4,31,2\rangle$,	$\langle 11,22,31,2\rangle$,
$\langle 11,22,31,11\rangle$,	$\langle 11,31,211,2\rangle$,	$\langle 11,211,211,11\rangle$,	$\langle 11,1111,31,11\rangle$,
$\langle 22,2,2,31\rangle$,	$\langle 22,11,2,211\rangle$,	$\langle 211,2,11,31\rangle$,	$\langle 211,11,11,211\rangle$,
$\langle 1111,2,2,31\rangle$,	$\langle 1111,11,2,211\rangle$,	$\langle 2,5,5,2\rangle$,	$\langle 2,221,311,11\rangle$,
$\langle 2,2111,311,11\rangle$,	$\langle 11,32,311,2\rangle$,	$\langle 11,41,311,2\rangle$,	$\langle 11,11111,5,11\rangle$,
$\langle 11111,2,2,311\rangle$,	$\langle 11111,11,11,311\rangle$.		

Under the same assumptions as in (27) for the partitions involved, we say the we have a ***q*-Foulkes configuration** denoted $\langle\alpha, \beta, \gamma, \delta\rangle_q$, if and only if

$$0 \preceq_s \frac{S_\gamma \circ S_\delta - S_\alpha \circ S_\beta}{1-q}, \tag{28}$$

[8]For the usual scalar product on symmetric functions

with the right-hand side having polynomial coefficients in q. In particular, this last condition requires that, at $q = 1$ we have the equality

$$(S_\alpha \circ S_\beta)\Big|_{q=1} = (S_\gamma \circ S_\delta)\Big|_{q=1},$$

which is equivalent to

$$e_{\alpha'} \circ e_{\beta'} = e_{\gamma'} \circ e_{\delta'}. \tag{29}$$

For instance, it is easy to check that this last equality holds when

$$\alpha = a, \qquad \beta = \underbrace{bb\cdots b}_{k}, \qquad \gamma = c, \qquad \text{and} \qquad \delta = \underbrace{dd\cdots d}_{k}, \tag{30}$$

for any a, b, c, d, k in $\mathbb{N}$, such that $ab = cd$, since both sides of (29) evaluate to e_k^{a+b}. Evidently, all q-Foulkes configurations are also Foulkes configurations, but most Foulkes configurations do not satisfy the extra requirement that (30) holds. Explicit calculations reveal that this condition significantly reduces the number of possibilities.

An intriguing development, explicitly checked out for all cases[9] with n up to 30, is that having both the necessary conditions (27) and (29) holding seems to be equivalent to having the full q-Schur positivity (28) holding too. In other words, we have the following general statement, which would reduce all q-versions to the $q = 0$ case.

Conjecture 10. For partitions α, β, γ, and δ, such that $e_{\alpha'} \circ e_{\beta'} = e_{\gamma'} \circ e_{\delta'}$, we have

$$s_\alpha \circ s_\beta \preceq_s s_\gamma \circ s_\delta, \qquad \text{if and only if} \qquad 0 \preceq_s \frac{S_\gamma \circ S_\delta - S_\alpha \circ S_\beta}{1-q}. \tag{31}$$

Clearly, when both α and γ are one part partitions, respectively equal to a and c, the second condition in (31) is simply that $e^a_{\beta'} = e^c_{\delta'}$. Observe that this implies that $\beta' = \mu^i$ and $\delta' = \mu^j$ for some partition μ, and $ai = cj$. Only this simpler version is needed in all cases explicitly calculated. In other words, all configurations that we have found to satisfy

$$e_{\alpha'} \circ e_{\beta'} = e_{\gamma'} \circ e_{\delta'}$$

are such that α and γ are reduced to one part, and thus of the simple form stated. It seems that this should be easy to prove.

Appendix A: Background on symmetric functions and plethysm

Trying to make this text self-contained, we now rapidly recall most of the necessary background on symmetric functions. For more background, see [1, 16]. As is usual, we often write symmetric functions without explicit mention of the variables. Thus, we denote by p_k (as in [16]) the power-sum symmetric functions

$$p_k = p_k(x_1, x_2, x_3, \ldots) := x_1^k + x_2^k + x_3^k + \ldots,$$

[9] Involving 67 configurations in total.

using which, we can expand the complete homogeneous symmetric functions as

$$h_n = \sum_{\mu \vdash n} \frac{p_\mu}{z_\mu}, \qquad \text{with} \qquad p_\mu := p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}, \tag{32}$$

where $d_k = d_k(\mu)$ is the multiplicity of the part k in the partition μ of n, and z_μ stands for the integer

$$z_\mu := \prod j^{d_j}\, d_j!.$$

For instance, we have the very classical expansions

$$h_2 = \frac{p_1^2}{2} + \frac{p_2}{2}, \qquad h_3 = \frac{p_1^3}{6} + \frac{p_1 p_2}{2} + \frac{p_3}{3}.$$

As is also very well known, the homogeneous degree n component λ_n of the graded ring Λ of symmetric functions, affords as a linear basis the set of Schur functions $\{s_\mu\}_{\mu \vdash n}$, indexed by partitions of n. Among the manifold interesting formulas regarding these, we will need the Cauchy-kernel identity.

$$h_n(\boldsymbol{xy}) = \sum_{\mu \vdash n} s_\mu(\boldsymbol{x}) s_\mu(\boldsymbol{y}) \tag{33}$$

$$= \sum_{\mu \vdash n} \frac{p_\mu(\boldsymbol{x}) p_\mu(\boldsymbol{y})}{z_\mu}, \tag{34}$$

with $h_n(\boldsymbol{xy}) = h_n(\ldots, x_i y_j, \ldots)$ corresponding to the evaluation of h_n in the "variables" $x_i y_j$. Otherwise stated, we may express this by the generating function identity

$$\sum_{n \geqslant 0} h_n(\boldsymbol{xy})\, z^n = \prod_{i,j} \frac{1}{1 - x_i y_j\, z}.$$

Rules of plethysm

Plethysm is an associative operation on symmetric functions, characterized by the following properties. Let f_1, f_2, g_1 and g_2 be any symmetric functions, and α and β be in $\mathbb{Q}$, then

a) $(\alpha\, f_1 + \beta\, f_2) \circ g = \alpha\,(f_1 \circ g) + \beta\,(f_2 \circ g)$, (left linearity)

b) $(f_1 \cdot f_2) \circ g = (f_1 \circ g) \cdot (f_2 \circ g)$, (left multiplicativity)

c) $p_k \circ (\alpha\, g_1 + \beta\, g_2) = \alpha\,(p_k \circ g_1) + \beta\,(p_k \circ g_2)$,

d) $p_k \circ (g_1 \cdot g_2) = (p_k \circ g_1) \cdot (p_k \circ g_2)$,

e) $p_k \circ p_j = p_{kj}$.

The first four properties reduce any calculation of plethysm to instances of the fifth one. For a given symmetric function f, one may consider the plethysm $f \circ (-)$ as an operator on symmetric functions. In fact, this operator may naturally be extended to any rational fraction in the underlying variables. It is sometimes more convenient to use the alternate notation $f[-]$ for this operator and "add" the further rules

a) $p_k[g_1/g_2] = p_k[g_1]/p_k[g_2]$,

b) $p_k[x] = x^k$, if x is a variable,

c) $p_k[c] = c$, if c is a constant.

Then, considering variable sets as sums $\boldsymbol{x} = x_1 + x_2 + x_3 + \ldots$, one observes that $f[\boldsymbol{x}]$ corresponds to the evaluation of the symmetric function f in the variables $\boldsymbol{x}$. Moreover, Cauchy's formula gives an explicit expression for the expansion of

$$h_n[\boldsymbol{x}\boldsymbol{y}] = h_n[(x_1 + x_2 + x_3 + \ldots)(y_1 + y_2 + y_3 + \ldots)].$$

Likewise $f[1/(1-q)] = f[1 + q + q^2 + \ldots]$, corresponds to the evaluation $f(1, q, q^2, \ldots)$.

Another interesting classical property of Schur functions may be expressed as

$$s_\mu[A + B] = \sum_{\nu \subseteq \mu} s_{\mu/\nu}[A]\, s_\nu[B], \tag{35}$$

where $s_{\mu/\nu}$ stand for the **skew Schur** function characterized by

$$\langle s_{\mu/\nu}, s_\lambda \rangle = \langle s_\mu, s_\nu^\perp s_\lambda \rangle,$$

writing $\nu \subseteq \mu$ if $\nu_i \leqslant \mu_i$ for all i, and $f^\perp$ standing for the dual operator of multiplication by f for the usual scalar product $\langle -, - \rangle$ on symmetric functions (for which the Schur functions form an orthonormal basis). It is well known that $s_{\mu/\nu}$ is Schur positive, and $s_{\mu/0} = s_\mu$.

Macdonald polynomials

With all this at hand, the polynomial $H_n(\boldsymbol{x}; q)$ can be explicitly defined as

$$H_n(\boldsymbol{x}; q) := [n]_q!\, (1-q)^n\, h_n \left[\frac{\boldsymbol{x}}{1-q} \right] \tag{36}$$

as before $[n]_q!$ stands for classical the q-analog of $n!$:

$$[n]_q! := [1]_q\, [2]_q \cdots [n]_q, \qquad \text{with} \qquad [k]_q = 1 + q + \ldots + q^{k-1}.$$

To get a Schur expansion for $H_n(\boldsymbol{x}; q)$, we recall the hook length expression

$$\begin{aligned} s_\mu[1/(1-q)] &= s_\mu(1, q, q^2, q^3, \ldots) \\ &= q^{n(\mu)} \prod_{1 \leqslant i \leqslant \mu_j} \frac{1}{1 - q^{h_{ij}}}, \end{aligned} \tag{37}$$

where $h_{ij} = h_{ij}(\mu)$ is the hook length of a cell (i,j) of the Ferrers diagram of μ, and $n(\mu) := \sum_{(i,j)} j$. Now, using Cauchy's formula (33), with $\boldsymbol{y} = 1 + q + q^2 + \ldots$, we find that

$$H_n(\boldsymbol{x};q) = \sum_{\mu \vdash n} \frac{q^{n(\mu)}[n]_q!}{\prod_{1 \leqslant i \leqslant \mu_j} [h_{ij}]_q} \; s_\mu(\boldsymbol{x}). \tag{38}$$

It is well known that the coefficient of $s_\mu(\boldsymbol{x})$ occurring here is a positive integer polynomial that q-enumerates standard tableaux with respect to the charge statistic. This is the q-hook formula. Thus, we find the two expansions.

$$\begin{aligned} H_3(\boldsymbol{x};q) &= \frac{[2]_q[3]_q}{6}\, p_1(\boldsymbol{x})^3 + \frac{[3]_q}{2}(1-q)\, p_1(\boldsymbol{x})p_2(\boldsymbol{x}) + \frac{[2]_q}{3}(1-q)^2\, p_3(\boldsymbol{x}), \\ &= s_3(\boldsymbol{x}) + (q+q^2)s_{21}(\boldsymbol{x}) + q^3 s_{111}(\boldsymbol{x}). \end{aligned}$$

It is clear that $H_n(\boldsymbol{x};0) = h_n$. The $H_n(\boldsymbol{x};q)$ function encodes, as a Frobenius transform, the character of several interesting isomorphic graded $\mathbb{S}_n$-modules such as: the coinvariant space of $\mathbb{S}_n$, the space of $\mathbb{S}_n$-harmonic polynomials, and the cohomology ring of the full-flag variety. More precisely, this makes explicit the graded decomposition into irreducibles of these spaces. Thus, the coefficient of $s_\mu(\boldsymbol{x})$ in formula (38) corresponds to the Hilbert series[10] of the isotropic component of type μ of this space. Using (34) to expand H_n, the global Hilbert series of these modules can be simply obtained by computing the scalar product

$$\begin{aligned} \langle p_1^n, H_n \rangle &= \sum_{\mu \vdash n} \langle p_1^n, s_\mu \rangle\, q^{n(\mu)}[n]_q! \prod_{(i,j)\in\mu} \frac{1-q}{1-q^{h_{ij}}}, & (39)\\ &= \sum_{\mu \vdash n} \langle p_1^n, p_\mu \rangle \frac{p_\mu(1/(1-q))}{z_\mu} \prod_{k=1}^{n} (1-q^k) & (40)\\ &= \left(\frac{1}{1-q}\right)^n \prod_{k=1}^{n}(1-q^k) & (41)\\ &= [n]_q!. & (42) \end{aligned}$$

To see this, recall that $\langle p_\mu, p_\lambda \rangle$ is zero if $\mu \neq \lambda$, and $\langle p_\mu, p_\mu \rangle = z_\mu$. To complete the picture, let us also recall that $\langle p_\mu, s_\lambda \rangle$ is equal to the value, on the conjugacy class μ, of the character of the irreducible representation associated to λ. In particular, it follows that

$$H_n(\boldsymbol{x};1) = h_1^n = \sum_{\mu \vdash n} f_\mu\, s_\mu(\boldsymbol{x}). \tag{43}$$

This is the Frobenius characteristic of the regular representation of $\mathbb{S}_n$, for which the multiplicities f_μ are given by the number of standard Young tableaux of shape μ. Beside this notion of Frobenius transform that "formally" encodes $\mathbb{S}_n$-irreducibles as Schur function, another more direct interpretation of the above formulas is in terms of characters

[10]Graded dimension.

of polynomial representations of $\mathrm{GL}(V)$, with V an N-dimensional space over $\mathbb{C}$. Recall that the character, of a representation $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$, is a symmetric function $\chi_\rho(x_1, x_2, \ldots, x_N)$ of the eigenvalues of operators in $\mathrm{GL}(V)$. Through Schur-Weyl duality, out of any $\mathbb{S}_n$-module R and any $\mathrm{GL}(V)$-module U, one may construct a representation of $\mathrm{GL}(V)$:

$$R(U) := R \otimes_{\mathbb{C}\mathbb{S}_n} U^{\otimes n},$$

where $\mathbb{S}_n$ acts on $U^{\otimes n}$ by permutation of components. This construction is functorial:

$$R : GL(V)\text{-Mod} \longrightarrow GL(V)\text{-Mod},$$

and the character of $R(U)$ is the plethysm $(f \circ g)(x_1 x_2, \ldots, x_N)$, where f is the Frobenius characteristic of R and g the character of U. Furthermore, under this construction, irreducible polynomial representations of $\mathrm{GL}(V)$ correspond to irreducible $\mathbb{S}_n$-modules R. If such is the case, one writes $S^\lambda(V)$ when R is irreducible of type λ. The corresponding character is the Schur function $s_\lambda(x_1, x_2, \ldots, x_N)$. For the special case $\lambda = (n)$, we get the symmetric power $S^a(V)$ whose character is $h_a(x_1, x_2, \ldots, x_N)$, hence the character of $S^a(S^b(V))$ is the plethysm $h_a \circ h_b$.

Appendix B: $\mathbb{N}$-positivity of $\Theta_a(b)$

The proof of $\mathbb{N}$-positivity of the solution of the recurrence occurring in Proposition 8 may be directly obtained as follows. Let us consider the positive integer coefficient series[11] defined as:

$$\rho(z; a) := \sum_{k=1}^{\infty} k\, a \binom{a+1}{2k+1} z^{2\,k+1}, \tag{44}$$

and then define recursively $\theta_n(z) = \theta_n(z; a)$ as follows

$$\theta_n(z) = (3+z)\, \theta_{n-1}(z) + (1+z)\,(z-3)\, \theta_{n-2}(z) + (1+z)^2\,(1-z)\, \theta_{n-3}(z),$$

with initial conditions $\theta_0(z) := \rho(z; a-1)$, $\theta_1(z) := \rho(z; a)$, and

$$\theta_2(z) = \sum_{k=1}^{\infty} \left(k(a-1) \binom{a+2}{2k+1} z^{2k+1} + 2k \binom{a+1}{2k+1} (1+z)\, z^{2\,k+1} \right).$$

For any $a > 2$ (in $\mathbb{N}$), $\theta_n(z; a)$ is clearly a degree $a+n$ polynomials in the variable z, with positive integer coefficients. This makes it obvious that the expression $\Theta_a(b)$ occuring in Proposition 8 is $\mathbb{N}$-positive, since

$$\Theta_a(b) = h_2^b\, \theta_{b-a}(e_2/h_2; a).$$

[11] Which is obviously terminating when a is an integer.

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Thanks to Mark Wildon for making publicly available his extensive table of calculations on Foulkes' conjecture:

http://www.ma.rhul.ac.uk/∼uvah099/other.html#FoulkesData

These were very helpful for our calculations which also made use of John Stembridge's SF package. Similar tools are available in Sage. Thanks to the anonymous referee for his/her suggestions, and for drawing our attention to reference [13]. Using the results therein, it seems that much larger experiments could be calculated for some of our conjectures.

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