The Polytope of k-star Densities

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Abstract

This paper describes the polytope $\mathbf{P}_{k;N}$ of i-star counts, for all $i \leqslant k$, for graphs on N nodes. The vertices correspond to graphs that are regular or as regular as possible. For even N the polytope is a cyclic polytope, and for odd N the polytope is well-approximated by a cyclic polytope. As N goes to infinity, $\mathbf{P}_{k;N}$ approaches the convex hull of the moment curve. The affine symmetry group of $\mathbf{P}_{k;N}$ contains just a single non-trivial element, which corresponds to forming the complement of a graph.

The results generalize to the polytope $\mathbf{P}_{I;N}$ of i-star counts, for i in some set I of non-consecutive integers. In this case, $\mathbf{P}_{I;N}$ can still be approximated by a cyclic polytope, but it is usually not a cyclic polytope itself.

Polytopes of subgraph statistics characterize corresponding exponential random graph models. The elongated shape of the k-star polytope gives a qualitative explanation of some of the degeneracies found in such random graph models.

Keywords: polytope, k-star model, exponential random graph model, vertex degrees, convex support

1 Introduction

In this work all graphs are assumed to be undirected and simple, i.e. there are no multiple edges between the same nodes, and there are no loops. Let G be a graph with N nodes[1]. For each node x of G denote by d_x the degree of x, i.e. the number of edges of G containing x. A *k-star* is a graph with one node of degree k and k nodes of degree one.

[1] The nodes of a graph are often called *vertices*. In this manuscript, the term *vertex* will be reserved for the vertices of polytopes.

The *number of k-stars* in G is defined as

$$N_k(G) := \sum_x \binom{d_x}{k} = \frac{1}{k!}\sum_x [d_x]_k,$$

where $[d_x]_k = d_x(d_x-1)\cdots(d_x-k+1)$. It is the number of subgraphs of G that are isomorphic to a k-star. Alternatively, it is the number of injective graph homomorphisms from a k-star to G. The *k-star density* is

$$n_k(G) = \frac{1}{N\binom{N-1}{k}} N_k(G) = \frac{1}{N[N-1]_k}\sum_x [d_x]_k.$$

$n_k(G)$ is normalized such that $0 \leqslant n_k(G) \leqslant 1$, with equality for the empty graph and the full graph, respectively. When the graph G is understood, n_k and N_k will be written instead of $n_k(G)$ and $N_k(G)$.

The object of study of this work is the convex hull of the set of possible values of the vector $\vec{n}_k := (n_1, \ldots, n_k)$ of i-star statistics:

$$\mathbf{P}_{k;N} := \operatorname{conv}\{\vec{n}_k(G) \,:\, G \text{ a graph on } N \text{ nodes}\}.$$

See Figures 1, 2 and 3 for examples with $k = 2$ and $k = 3$. As shown in [6], $\mathbf{P}_{k;N+1} \subset \mathbf{P}_{k;N}$. Hence there is a well-defined limit object $\mathbf{P}_{k;\infty} := \bigcap_N \mathbf{P}_{k;N}$.

The polytope $\mathbf{P}_{k;N}$ arises in the following statistical context: If $\hat{G}$ is a random graph on N nodes, then the expected value of $\vec{n}_k(\hat{G})$ lies in $\mathbf{P}_{k;N}$, and any element of $\mathbf{P}_{k;N}$ arises in this way. In general, different distributions may have the same expected value. Choosing for each expected value the distribution that maximizes the entropy, one obtains the mean value parametrization of (the closure of) an exponential family. In this way, one can associate an exponential family to any polytope, and in this context, the polytope is called the convex support of the exponential family. See [1] for an introduction to exponential families and [14, 2] for overviews on exponential random graph models. The polytope $\mathbf{P}_{k;N}$ is the convex support of the exponential random graph model known as the k-star model. The main results in this paper are independent of this interpretation, and hence this connection will not be explained here. Some remarks in this direction will be made in Section 6.

The numbers $n_1, \ldots, n_{N-1}$ are related to the degree distribution. Denote by $\tilde{N}_d$ the number of nodes of degree d in G, and let $p_d = \frac{1}{N}\tilde{N}_d$. The numbers $(p_0, \ldots, p_{N-1})$ form a probability distribution on $\{0, \ldots, N-1\}$, called *degree distribution*. Then

$$n_k = \frac{1}{[N-1]_k}\sum_{d=0}^{N-1} p_d [d]_k. \tag{1}$$

Therefore, $\mathbf{P}_{k;N}$ is a projection of the polytope of degree distributions $\mathbf{D}_N$, defined as the convex hull of all probability distributions on $\{0, \ldots, N-1\}$ that arise as degree distributions of graphs with N nodes. For even N and arbitrary $d \in \{0, \ldots, N-1\}$ there exists a d-regular graph with N nodes (recall that G is *d-regular*, if $d_x = d$ for all nodes x of G). Hence $\mathbf{D}_N$ is a simplex if N is even. For odd N the polytope $\mathbf{D}_N$ is described in Section 3.

Expression (1) can be further transformed using the expansion

$$[d]_k = \sum_{i=0}^{k} s_{k,i} d^i.$$

The coefficients $s_{k,i}$ are called *Stirling numbers of the first kind.* Denote by $\mu_i = \sum_x \frac{d_x^i}{N} = \sum_{d=0}^{N-1} p_d d^i$ the i-th moment of the degree distribution. As observed in [11], the Stirling numbers relate the k-star densities with the moments of the degree distribution:

$$n_k = \frac{1}{[N-1]_k} \sum_{i=0}^{k} s_{k,i} \sum_x \frac{d_x^i}{N} = \frac{1}{[N-1]_k} \sum_{i=0}^{k} s_{k,i} \mu_i. \tag{2}$$

Hence the polytope $\mathbf{P}_{k;N}$ is affinely equivalent to the polytope of the first k moments of the degree distribution.

This paper is organized as follows: The limit polytope is described in Section 2. Moreover, a cyclic polytope $\tilde{\mathbf{P}}_{k;N}$ is found that can serve as an approximation of $\mathbf{P}_{k;N}$, with error $O(1/N)$. For even N, the two polytopes $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ agree. For odd N, the vertices of $\mathbf{P}_{k;N}$ are described in Section 3, and a coarse classification of the facets is found. Section 4 discusses the related polytope $\mathbf{P}_{I;N}$ obtained when dropping some coordinates of the vector $\vec{n}_k$. The resulting polytope is usually not a cyclic polytope, but can be approximated by cyclic polytopes with error $O(1/N)$. The affine symmetry group of $\mathbf{P}_{k;N}$ is computed in Section 5. For $N > k+1$ the only non-trivial symmetry is the involution induced from the permutation that replaces each graph by its complement graph. Section 6 discusses implications for the corresponding exponential families.

2 The limit polytope

By (1), the vector $\vec{n} := (n_1, \cdots, n_{N-1})$ equals the expected value of the vector

$$\left(\frac{d}{N-1}, \frac{[d]_2}{[N-1]_2}, \dots, \frac{[d]_{N-1}}{(N-1)!}\right)$$

under the degree distribution. Denote by D the $k \times N$-matrix with entries $D_{i,d} = \frac{[d]_i}{[N-1]_i}$ for $i = 1, \dots, k$ and $d = 0, \dots, N-1$, and let $\tilde{\mathbf{P}}_{k;N}$ be the convex hull of the columns of D. When multiplied by the degree distribution $p(G)$ of a graph G (considered as a column vector), the matrix D computes the vector $\vec{n}(G)$. Therefore, $\vec{n}(G) \in \tilde{\mathbf{P}}_{k;N}$ for any graph G on N nodes, and so $\mathbf{P}_{k;N} \subseteq \tilde{\mathbf{P}}_{k;N}$. If N is even, then $\tilde{\mathbf{P}}_{k;N} = \mathbf{P}_{k;N}$. This follows from the fact that if N is even and $0 \leqslant d \leqslant N-1$, then there exists a d-regular graph on N nodes. On the other hand, if N is odd and if d is odd, then there is no d-regular graph on N nodes. Nevertheless, $\tilde{\mathbf{P}}_{k;N}$ is a valuable relaxation of the polytope $\mathbf{P}_{k;N}$, and many

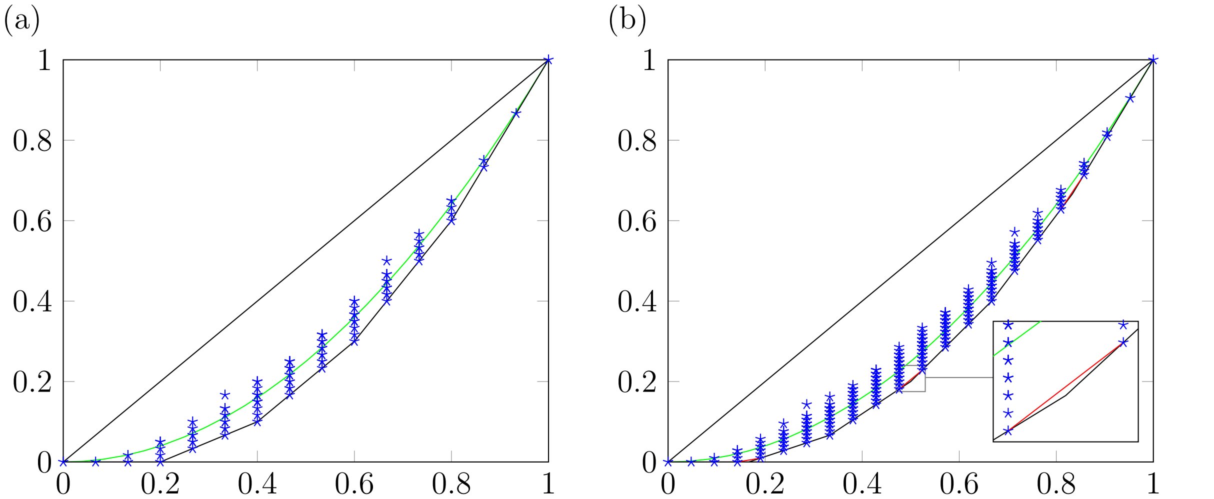

Figure 1: (a) The polytope $\mathbf{P}_{2;6}$. (b) The polytopes $\tilde{\mathbf{P}}_{2;7}$ and $\mathbf{P}_{2;7}$. In both figures, each star is a possible value of $\vec{n}_2$ for some graph with six or seven nodes, and the moment curve is marked in green. In (b), the polytope $\tilde{\mathbf{P}}_{2;7}$ is bounded by the black edges, while $\mathbf{P}_{2;7}$ is bounded by the black and red edges. The red edges (which are barely visible, since they lie very close to neighbouring black edges) bound the special facets defined in Section 3.

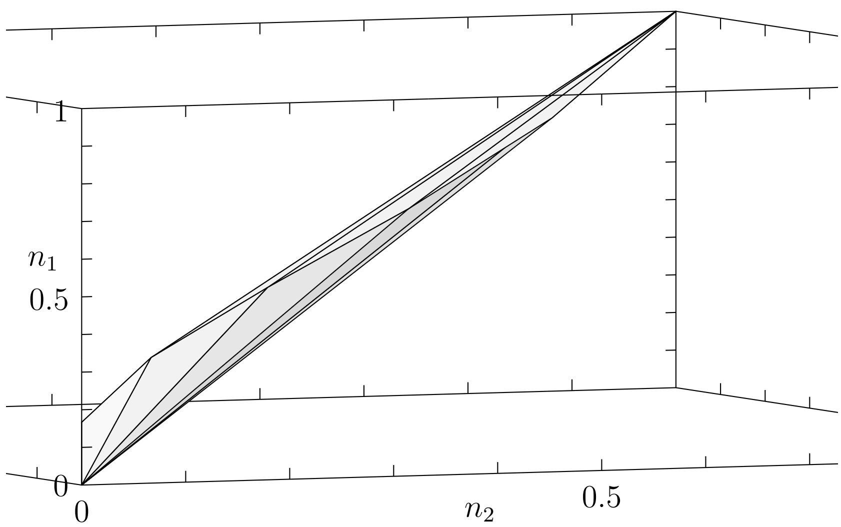

Figure 2: The polytope $\tilde{\mathbf{P}}_{3,7}$, plotted using Sketch [12]. The volume of $\tilde{\mathbf{P}}_{3,7}$, according to Polymake [7], is 7/675, while the diameter is $\sqrt{3}$, hence the polytope is stretched along the diagonal of the unit cube. This makes it difficult to obtain a good perspective of $\tilde{\mathbf{P}}_{3,7}$.

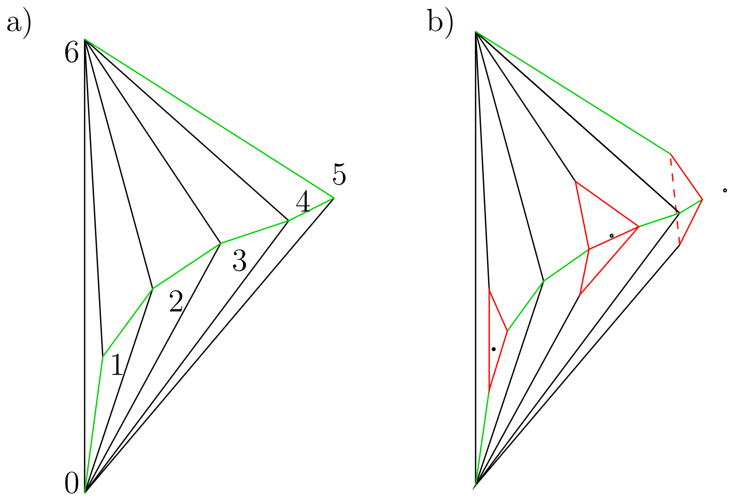

Figure 3: Sketches of the polytopes $\tilde{\mathbf{P}}_{3,7}$ (a) and $\mathbf{P}_{3,7}$ (b), motivated by the pictures in [9]. The number d labels the vertex n^d of $\tilde{\mathbf{P}}_{3,7}$. The green edges in $\tilde{\mathbf{P}}_{3,7}$ are the edges from n^d to n^{d+1}. Hence, the green line corresponds to the moment curve. The red edges in $\mathbf{P}_{3,7}$ mark the difference between $\mathbf{P}_{3,7}$ and $\tilde{\mathbf{P}}_{3,7}$.

properties of $\mathbf{P}_{k;N}$ can be deduced from $\tilde{\mathbf{P}}_{k;N}$. The difference between $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ is characterized in Section 3; see also Figures 1 and 3.

To describe $\mathbf{P}_{k;\infty}$ the following definition is needed: The *moment curve* is the curve

$$s^k : [0,1] \to \mathbb{R}^k, p \mapsto (p^1, p^2, \ldots, p^k).$$

The moment curve corresponds to the *spine* in [6]. It contains the expected values of $\vec{n}_k$ for the *Erdős-Rényi random graphs* $G(N,p)$: Under $G(N,p)$, for each $x, y \in \{1, \ldots, N\}$, $x < y$, an independent Bernoulli variable $\theta_{x,y}$ with parameter p is drawn, and (x,y) is an edge in $G(N,p)$ if and only if $\theta_{x,y} = 1$. Since $s^k(p)$ equals the expectated value of $\vec{n}_k$ under $G(N,p)$, the moment curve is a subset of $\mathbf{P}_{k;N}$ for all N (this corresponds to the fact that the spine always belongs to the convex support polytope, see [6, Proposition 2.1]).

The convex hull of the moment curve has a nice interpretation: It consists of all vectors $(\mu_1, \ldots, \mu_k) \in \mathbb{R}^k$ such that there exists a probability distribution p on the unit interval $[0,1]$ such that $\mu_1, \ldots, \mu_k$ are the first k moments of p. In [10] the convex hull of s^k is studied from this point of view.

Theorem 1. $\mathbf{P}_{k;\infty}$ *equals the convex hull of the moment curve.*

Proof. From $\mathbf{P}_{k;2l} = \tilde{\mathbf{P}}_{k;2l} \subset \mathbf{P}_{k;2l-1} \subset \tilde{\mathbf{P}}_{k;2l-1}$ follows $\mathbf{P}_{k;\infty} = \bigcap_N \tilde{\mathbf{P}}_{k;N}$. Now,

$$\begin{aligned}
D_{j,d} = \frac{[d]_j}{[N-1]_j} &= \sum_{i=0}^{j} s_{j,i} \frac{d^i}{[N-1]_j} \\
&= \left(\frac{d}{N-1}\right)^j + \frac{d^j}{N-1}\left(\frac{1}{[N-2]_{j-1}} - \frac{1}{(N-1)^{j-1}}\right) + \sum_{i=0}^{j-1} s_{j,i}\frac{d^i}{[N-1]_j} \\
&= \left(\frac{d}{N-1}\right)^j + \frac{d^j}{N-1}\left(\frac{g_j(N)}{[N-2]_{j-1}(N-1)^{j-1}}\right) + \sum_{i=0}^{j-1} s_{j,i}\frac{d^i}{[N-1]_j},
\end{aligned}$$

where $g_j(N)$ is a polynomial in N of degree at most $j-2$. Therefore, since $d/(N-1) \leqslant 1$,

$$D_{j,d} - \left(\frac{d}{N-1}\right)^j = O(\frac{1}{N}).$$

Therefore, the columns of D lie close to the point $s^k(\frac{d}{N-1})$. Since the vertices lie in an $O(1/N)$-neighbourhood of the moment curve, any point in $\mathbf{P}_{k;N}$ lies in an $O(1/N)$-neighbourhood of the convex hull of the moment curve; hence $\mathbf{P}_{k;\infty}$ is contained in the convex hull of the moment curve. Conversely, since $\mathbf{P}_{k;N}$ contains the moment curve for all N, it is clear that $\mathbf{P}_{k;\infty}$ contains the convex hull of the moment curve. □

Remark 2. Theorem 1 disproves the following Conjecture 4.3 from [6]: *For any finite collection of subgraphs there exists an integer m such that for any $\epsilon > 0$ the limit object of the polytopes of subgraph statistics as $N \to \infty$ can be approximated by a polytope by intersecting the limit object with at most m closed half-spaces that cut away at most a volume of ϵ.* However, it is not possible to approximate the convex hull of the moment curve arbitrarily close in this way with a fixed number of hyperplanes. For example, when the collection of subgraphs consists of a one-star and a two-star, the corresponding limit object is $P_{2;\infty}$, the convex hull of a piece of a parabola.

The moment curve is a part of the *algebraic moment curve* $\gamma^k : t \mapsto (t^1, t^2, \ldots, t^k)$. The convex hull of N distinct points on the algebraic moment curve is a *cyclic polytope*. In a sense, $\mathbf{P}_{k;\infty}$ equals a cyclic polytope with an infinite number of vertices.

The face lattice of a cyclic polytope is independent of the choice of the N points. A polytope is $C(k;N)$ if it is combinatorially equivalent to a cyclic polytope with N vertices on γ^k, i.e. if it has the same face lattice as such a cyclic polytope.

Cyclic polytopes appear not only in the limit $N \to \infty$:

Theorem 3. *$\tilde{\mathbf{P}}_{k;N}$ is a $C(k;N)$ cyclic polytope. Therefore, if N is even, then $\mathbf{P}_{k;N}$ is a $C(k;N)$ cyclic polytope.*

Proof. The affine map $\varphi : \mathbb{R}^k \to \mathbb{R}^k$, defined via

$$\varphi(m_1, \ldots, m_k)_j = \frac{1}{[N-1]_j}\left(\sum_{i=1}^{j} s_{j,i} m_i + s_{j,0}\right)$$

maps the point $(d, d^2, \ldots, d^k)$ on the algebraic moment curve to a vertex of $\tilde{\mathbf{P}}_{k;N}$. □

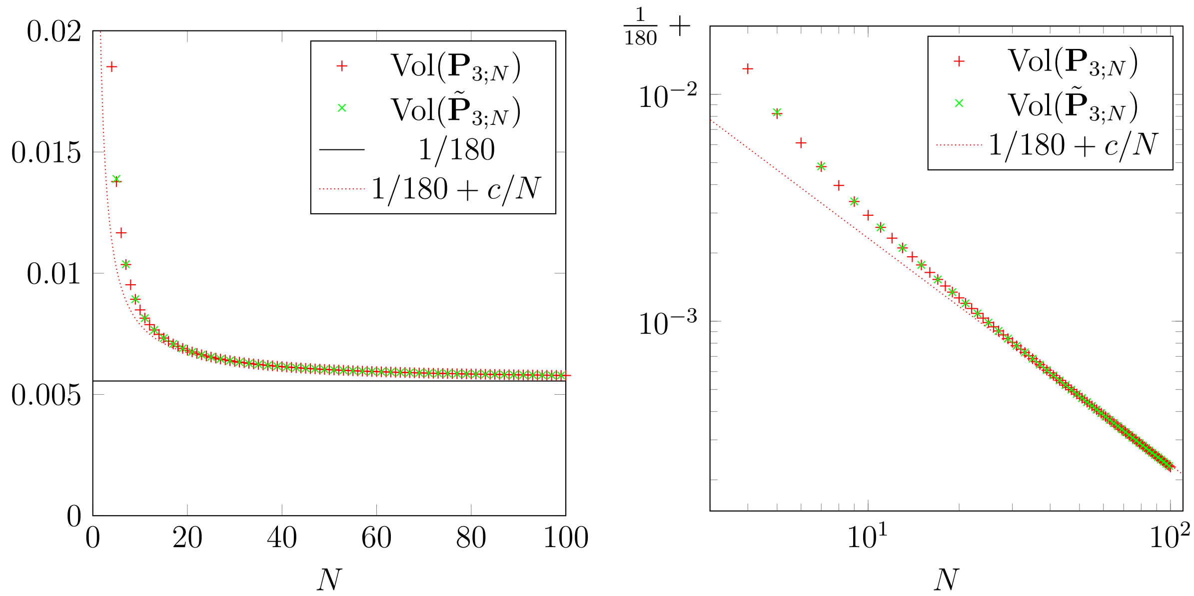

Figure 4: The volumes of $\mathbf{P}_{3;N}$, $\tilde{\mathbf{P}}_{3;N}$ and $\mathbf{P}_{3;\infty} = 1/180$ (solid line). The dashed line is the function $\mathrm{Vol}(\mathbf{P}_{3;\infty}) + c/N$, with $c = 0.0232249$ fit by gnuplot [15]. The right hand side is a double-logarithmic plot after substracting $1/180$.

The facet description of the cyclic polytopes is known, and the facet description of $\tilde{\mathbf{P}}_{k;N}$ can be deduced from this:

Gale's evenness condition. *Let $t_0 < t_1 < \cdots < t_{N-1}$, and let $\mathbf{C}(t_0, \ldots, t_{N-1})$ be the convex hull of the points $\{\gamma^k(t_i) : i = 0, \ldots, N-1\}$ on the algebraic moment curve. Let $0 \leqslant d_1 < d_2 < \cdots < d_k \leqslant N-1$ be integers. Then the vertices $\gamma^k(t_{d_1}), \ldots, \gamma^k(t_{d_k})$ define a facet of $\mathbf{C}(t_0, \ldots, t_{N-1})$ if and only if for any two integers $d, d' \in \{0, \ldots, N-1\} \setminus \{d_1, \ldots, d_k\}$ the set $\{i : t_d < t_{d_i} < t_{d'}\}$ has even cardinality.*

Figure 4 shows a plot of the volumes of $\mathbf{P}_{3;N}$ and $\tilde{\mathbf{P}}_{3;N}$. It can be seen that the volume of $\tilde{\mathbf{P}}_{3;N} \setminus \mathbf{P}_{3;N}$ is negligible, and the volume of $\mathbf{P}_{3;N} \setminus \mathbf{P}_{3;\infty}$ decreases as c/N, with $c \approx 0.0232249$. According to [10], the volume of the convex hull of the moment curve is

$$\mathrm{Vol}(\mathbf{P}_{k;\infty}) = \prod_{i=0}^{k-1} \frac{(i!)^2}{(2i+1)!}.$$

In particular, $\mathrm{Vol}(\mathbf{P}_{3;\infty}) = 1/180$. Note that the diameter of $\mathbf{P}_{k;\infty}$ equals the distance of the two points $(0, \ldots, 0)$ and $(1, \ldots, 1)$ corresponding to the empty and the full graph, respectively; and so $\mathrm{diam}(\mathbf{P}_{k;\infty}) = \sqrt{k}$.

3 The vertices and facets of $\mathbf{P}_{k;N}$

In this section the polytope $\mathbf{P}_{k;N}$ for odd N is described. First a description of the polytope $\mathbf{D}_N$ of degree distributions of graphs with N nodes is needed.

A sequence $(d_i)_{i=1}^N$ of integers is *graphical* if there exists a graph with node set $\{1,\dots,N\}$ and degree sequence $(d_i)_{i=1}^N$. Similarly, a distribution p on $\{0,\dots,N-1\}$ is *graphical*, if there exists a graph with N nodes and degree distribution p. The following theorem, due to Erdős and Gallai, characterizes graphical sequences.

Erdős-Gallai theorem. *Assume that $d_1 \geqslant d_2 \geqslant \dots \geqslant d_N$. The sequence is graphical if and only if $\sum_{i=1}^N d_x$ is even and, for all $k = 1,\dots,N$,*

$$\sum_{i=1}^k d_i \leqslant k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}. \tag{3}$$

In particular, the constant degree sequence $d_i = d$ is graphical, provided that N is even or d is even. Moreover, if N and d are odd but d' is even, then the degree sequence with $d_1 = \dots = d_{N-1} = d$ and $d_N = d'$ is graphical.

Let δ^d be the distribution on $\{0,\dots,N-1\}$ concentrated on the degree d, and let $\delta^{d,d'}$ be the distribution on $\{0,\dots,N-1\}$ with $\delta^{d,d'}_d = \frac{N-1}{N}$ and $\delta^{d,d'}_{d'} = \frac{1}{N}$. Assume that N is odd. By what was said above, δ^d is graphical if d is even, and $\delta^{d,d'}$ is graphical if d is odd and d' is even. Any graphical degree distribution is a convex combination of degree distributions of this form. Denote by $\mathbf{D}_N$ the convex hull of the set of graphical degree distributions of graphs on N nodes. It is easy to see that δ^d for even d and $\delta^{d,d'}$ for odd d and even d' are not convex combinations of other graphical distributions; and hence they are the vertices of $\mathbf{D}_N$.

This description of the vertices also gives a representation of $\mathbf{D}_N$ as a Minkowski sum[2]: Denote by Δ_{N-1} the simplex of all probability distributions on $\{0,\dots,N-1\}$ (this is the convex hull of all δ^d with odd or even d), and let $\Delta_{N-1,\text{even}}$ be the subsimplex with vertices δ^d with even d. Then

$$\mathbf{D}_N = \frac{1}{N}\Delta_{N-1} + \frac{1}{N}\Delta_{N-1,\text{even}}. \tag{4}$$

Since $\Delta_{N-1,\text{even}}$ is a face of Δ_{N-1}, it is easy to obtain the following combinatorial description of $\mathbf{D}_N$ from this Minkowski sum representation[3]:

Proposition 4. *If N is odd, then $\mathbf{D}_N$ is combinatorially and projectively equivalent to a product of two $(N-1)/2$-simplices. (This means that there is a rational bijection $\mathbf{D}_N \cong \Delta_{(N-1)/2} \times \Delta_{(N-1)/2}$ that induces an isomorphism of the corresponding face lattices.)*

Proof. The statement follows from the fact that $\mathbf{D}_N$ is the Minkowski sum of a simplex with one of its faces. The projective map can also be constructed explicitly and has a probabilistic interpretation: Namely, given a graphical degree distribution $p = (p_0,\dots,p_{N-1})$, we have $p_{\text{even}} := \sum_{d \text{ even}} p_d > 0$. Thus, the numbers $p_{d|\text{even}} := p_d/p_{\text{even}}$ are well-defined. For even d, the number $p_{d|\text{even}}$ equals the probability of obtaining a node of degree d when

[2] I thank the anonymous reviewer for this hint, which helped elucidate a mistake in an early version of this manuscript.

[3] This result was also suggested by the anonymous reviewer.

sampling uniformly from the nodes of even degree. Observe that $p_{\text{even}} + \sum_{d \text{ odd}} p_d = 1$. Now it is easy to check that

$$(p_0, \ldots, p_{N-1}) \mapsto (p_{0|\text{even}}, p_{2|\text{even}}, \ldots, p_{N-1|\text{even}}; p_{\text{even}}, p_1, p_3, \ldots, p_{N-2})$$

is indeed a well-defined rational bijection $\mathbf{D}_N \cong \Delta_{(N-1)/2} \times \Delta_{(N-1)/2}$. $\square$

The next result describes the inequalities of the facets of $\mathbf{D}_N$.

Theorem 5. *Let N be odd. The polytope $\mathbf{D}_N$ has $N+1$ facets:*

- *The inequality $\sum_{i \text{ odd}} p_i \leqslant 1 - \frac{1}{N}$ defines a facet H of $\mathbf{D}_N$ containing the vertices $\delta^{d,d'}$ with odd d and even d'. The facet H is a Cartesian product of two simplices of dimension $\frac{N-3}{2}$ and $\frac{N-1}{2}$.*
- *All other facets are of the form $F \cap \mathbf{D}_N$, where F is a facet of Δ_{N-1}. They correspond to the inequalities $p_i \geqslant 0$ for $i = 0, \ldots, N-1$.*

Proof. $\mathbf{D}_N$ arises from the simplex Δ_{N-1} by replacing the points δ^d with odd d by the vertices $\delta^{d,d'}$ with even d'. All vertices of $\mathbf{D}_N$ satisfy $\sum_{i \text{ odd}} p_i \leqslant 1 - \frac{1}{N}$. Conversely, all probability distributions on $\{0, \ldots, N-1\}$ satisfying this inequality can be written as a convex combination of the vertices of $\mathbf{D}_N$. Therefore, the inequalities in the statement of the theorem are the defining inequalities of $\mathbf{D}_N$.

The simplex Δ_{N-1} can be identified with the join of the simplex $\Delta_{N-1,\text{even}}$ over $\{0, 2, \ldots, N-1\}$ and the simplex $\Delta_{N-1,\text{odd}}$ over $\{1, 3, \ldots, N-2\}$. The facet H corresponds to a slice of this join, and so it is affinely equivalent to the Cartesian product $\Delta_{N-1,\text{even}} \times \Delta_{N-1,\text{odd}}$. $\square$

The polytope $\mathbf{P}_{k;N}$ agrees with the convex hull of the image of the set of all degree distributions under the matrix D, and the vertices of $\mathbf{P}_{k;N}$ are among the images of the vertices of $\mathbf{D}_N$. It remains to decide, which vertices of $\mathbf{D}_N$ yield vertices of $\mathbf{P}_{k;N}$. Note that $n^d := D\delta^d$ equals the dth column of D. Let $n^{d,d'} = D\delta^{d,d'}$.

Theorem 6. *Let N be odd. The set of vertices of $\mathbf{P}_{k;N}$ consists of the points n^d for even d and the points $n^{d,d'}$ for odd d and all d' satisfying the following conditions:*

- *If $k = 2$, then $d' \in \{d-1, d+1\}$.*
- *If $k = 3$, then $d' \in \{0, d-1, d+1, N-1\}$.*
- *If $k \geqslant 4$, then d' is any even number satisfying $0 \leqslant d' \leqslant N-1$.*

For an illustration see Figure 3.

Proof. If d is even, then n^d is a vertex of $\mathbf{P}_{k;N}$, because it is a vertex of $\tilde{\mathbf{P}}_{k;N}$. To see when $n^{d,d'}$ is a vertex, one needs to know the edges of $\tilde{\mathbf{P}}_{k;N}$ containing the vertex n^d for odd d. Gale's evenness condition implies:

- If $k = 2$, then all edges are of the form $(d, d+1)$ or $(0, N-1)$ (here, the integer d is identified with the vertex n^d).
- If $k = 3$, then the facets are of the form $(0, d, d+1)$ or $(d, d+1, N-1)$. Hence, the edges are of the form $(d, d+1)$, $(0, d)$ and $(d, N-1)$.
- If $k \geqslant 4$, then any pair (d, d') is an edge, since any such pair is contained in some facet.

Hence it suffices to show that, if d is odd and d' is even, then $n^{d,d'}$ is a vertex of $\mathbf{P}_{k;N}$ if and only if (d, d') is an edge of $\tilde{\mathbf{P}}_{k;N}$.

If (d, d') is an edge, then $n^{d,d'}$ lies on an edge of $\tilde{\mathbf{P}}_{k;N}$. Therefore, there is one unique way of expressing $n^{d,d'}$ as a convex combination of the vertices of $\tilde{\mathbf{P}}_{k;N}$, namely $n^{d,d'} = \frac{N-1}{N} n^d + \frac{1}{N} n^{d'}$. Since only $n^{d'}$ belongs to $\mathbf{P}_{k;N}$, but not n^d, it follows that $n^{d,d'}$ is an extreme point, or vertex, of $\mathbf{P}_{k;N}$. In particular, if $k \geqslant 4$, all points $n^{d,d'}$ are vertices.

To show that $n^{d,d'}$ is not a vertex of $\mathbf{P}_{k;N}$ if (d, d') is not an edge of $\tilde{\mathbf{P}}_{k;N}$, it suffices to write $n^{d,d'}$ as a convex combination of other points of $\mathbf{P}_{k;N}$. The calculations are deferred to the appendix, see Lemmas 19 and 20. □

Assume that N is odd. The two-dimensional polytope $\mathbf{P}_{2;N}$ is a polygon with $\frac{3N-1}{2}$ facets: $\mathbf{P}_{2;N}$ is obtained from $\tilde{\mathbf{P}}_{2;N}$ by replacing every vertex n^d of $\tilde{\mathbf{P}}_{2;N}$ with odd d by the two vertices $n^{d,d-1}$ and $n^{d,d+1}$, and hence $\mathbf{P}_{2;N}$ has $N + \frac{N-1}{2} = \frac{3N-1}{2}$ vertices and the same number of facets. For $k \geqslant 3$, the facet description is more complicated. Equation (4) shows that $\mathbf{P}_{k;N}$ is a Minkowski sum

$$\mathbf{P}_{k;N} = \frac{N-1}{N} \tilde{\mathbf{P}}_{k;N} + \frac{1}{N} \tilde{\mathbf{P}}_{k;N;\mathrm{even}},$$

where $\tilde{\mathbf{P}}_{k;N;\mathrm{even}}$ is the convex hull of the vectors $\vec{n}_k(G)$ for all d-regular graphs G for all even d. Thus, $\mathbf{P}_{k;N}$ is a Minkowski sum of two cyclic polytopes, one of which arises from the other by dropping almost half of the vertices. This implies that any face of $\mathbf{P}_{k;N}$ is also a Minkowski sum of two corresponding faces.

To distinguish faces of $\mathbf{P}_{k;N}$ we need the following finer notion: The Minkowsky sum of two polytopes $\mathbf{P}$ and $\mathbf{P}'$ is called *vertex-preserving*, if for any pair of vertices $v \in \mathbf{P}$, $v' \in \mathbf{P}'$ the sum $v + v'$ is a vertex of $\mathbf{P} + \mathbf{P}'$. This condition is weaker than $\mathbf{P} + \mathbf{P}' \cong \mathbf{P} \times \mathbf{P}'$, since it may happen $\dim(\mathbf{P} + \mathbf{P}') < \dim(\mathbf{P} \times \mathbf{P}') = \dim(\mathbf{P}) + \dim(\mathbf{P}')$ for a vertex-preserving Minkowski sum (an example is given below). We need the following property:

- For any edge e in a non-trivial vertex-preserving Minkowski sum there exists a pair of vertices $v, v' \in (\mathbf{P} + \mathbf{P}') \setminus e$ such that the line segment (v, v') is parallel to e and has the same length.

To see this, note that any edge of $\mathbf{P} + \mathbf{P}'$ is of the form $(v_1 + v_1', v_1 + v_2')$ or $(v_1 + v_1', v_2 + v_1')$, with vertices $v_1, v_2 \in \mathbf{P}$ and $v_1', v_2' \in \mathbf{P}'$. It suffices to consider an edge of the first form. Let $v_2 \neq v_1$ be another vertex of $\mathbf{P}$. Then we can take $v = v_2 + v_1'$ and $v' = v_2 + v_2'$ in the claim.

Proposition 7. *Let $k \geqslant 3$, and let N be odd.*

1. *For any face F of $\tilde{\mathbf{P}}_{k;N}$, the intersection $\hat{F} := F \cap \mathbf{P}_{k;N}$ is a face of $\mathbf{P}_{k;N}$. Any face that contains a vertex n^d with even d is of this form. If $\hat{F} \neq \emptyset$, then $\dim(\hat{F}) = \dim(F)$. Moreover, if F is a facet, then $\hat{F} \neq \emptyset$, and so $\hat{F}$ is a facet of $\mathbf{P}_{k;N}$. In this case, $\hat{F}$ is neither a simplex nor a non-trivial vertex-preserving Minkowski sum.*

2. *Let F' be a face that is not of the form $\hat{F}$. The vertex set of F' is of the form $\{n^{d,d'} : d \in I, d' \in I'\}$, where I and I' are subsets of $\{0, \dots, n-1\}$ such that I only contains odd numbers and I' only contains even numbers. F' is either a simplex or a non-trivial vertex-preserving Minkowski sum of two cyclic polytopes.*

Proof. If F is a face of $\tilde{\mathbf{P}}_{k;N}$, then $\hat{F} = F \cap \mathbf{P}_{k;N}$ is a face of $\mathbf{P}_{k;N}$ (possibly empty). Any face of $\mathbf{P}_{k;N}$ that contains a vertex n^d must be of the form $\hat{F}$, because the edges of $\mathbf{P}_{k;N}$ containing n^d are in one-to-one correspondence with the edges of $\tilde{\mathbf{P}}_{k;N}$ containing n^d, and so the vertex figures of n^d in $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ agree.

If F is a facet, then, by Gale's evenness condition, F contains a vertex of the form n^d with even d, and n^d is also a vertex of $\hat{F}$. When F is an arbitrary face, then $\hat{F} \neq \emptyset$ if and only if F contains a vertex n^d with even d. In both cases, the intersection of any edge of F containing n^d with $\mathbf{P}_{k;N}$ is an edge of $\mathbf{P}_{k;N}$ (possibly shorter), and therefore $\hat{F}$ has the same dimension as F and is a facet of $\mathbf{P}_{k;N}$ if and only if F is a facet of $\tilde{\mathbf{P}}_{k;N}$.

By Gale's evenness condition, any facet F of $\tilde{\mathbf{P}}_{k;N}$ contains a vertex of the form $n^{d'}$ with odd d'. Since $k \geqslant 3$, it follows that $\hat{F}$ has more vertices than the simplex F (namely, $n^{d'}$ is replaced by more than one vertices of the form $n^{d',d''}$), and thus it is not a simplex. Moreover, F contains an edge with vertices n^{d_1} and n^{d_2}, where both d_1 and d_2 are even. The line segment (n^{d_1}, n^{d_2}) is an edge of $\mathbf{P}_{k;N}$. To prove that $\hat{F}$ is not a non-trivial vertex-preserving Minkowski sum, it suffices to check that no other pair of vertices of $\hat{F}$ that is parallel to (n^{d_1}, n^{d_2}) has the same length. This follows since F is a simplex with edge (n^{d_1}, n^{d_2}), and any other line segment parallel to (n^{d_1}, n^{d_2}) that has the same length contains a point that lies outside of F. This finishes the proof of 1.

Let F' be a face of $\mathbf{P}_{k;N}$ with vertices $n^{d_1,d'_1}, \dots, n^{d_l,d'_l}$. Since preimages of faces are faces, the distributions $\delta^{d_1,d'_1}, \dots, \delta^{d_l,d'_l}$ define a face H' of $\mathbf{D}_N$, and H' must be a face of the facet H from Theorem 5. Hence H' can be identified with the Cartesian product $H'_1 \times H'_2$, where H'_1 is the subsimplex generated by $\delta^{d_1}, \dots, \delta^{d_l}$, and H'_2 is the subsimplex generated by $\delta^{d'_1}, \dots, \delta^{d'_l}$. Thus, the vertex set of F' is $\{n^{d,d'} : d = d_1, \dots, d_l,\, d' = d'_1, \dots, d'_l\}$. This also shows that F' is a (possiby trivial) vertex-preserving Minkowski sum. Both summands are cyclic polytopes, since the polytopes

$$\mathbf{C}_d := \mathrm{conv}\{n^{d,d'} : d' = 0, 2, \dots, N-1\} \text{ for odd } d, \text{ and}$$
$$\mathbf{C}_{d'} := \mathrm{conv}\{n^{d,d'} : d = 1, 3, \dots, N-2\} \text{ for even } d'$$

are cyclic polytopes.

Let F' be a face of $\mathbf{P}_{k;N}$ with vertex set $\{n^{d,d'} : d \in I, d' \in I'\}$. When either of I or I' is a singleton, the vertex-injective Minkowski sum representation is trivial. In this case, it remains to show that F' is a simplex. If $I = \{d\}$ is a singleton, then F' is also a face of the cyclic polytope $\mathbf{C}_d$. Either $\mathbf{C}_d$ is itself a simplex, or $\mathbf{C}_d$ is full-dimensional, and hence F' is a strict face of $\mathbf{C}_d$. In either case, F' is a simplex. The same conclusion holds true if I' is a singleton. □

The facets of $\mathbf{P}_{k;N}$ that do not arise from facets of $\tilde{\mathbf{P}}_{k;N}$ are called *special facets* in the following. As mentioned above, being a non-trivial vertex-preserving Minkowski sum is weaker than being a non-trivial Cartesian product. For example, $\mathbf{P}_{4;11}$ has a facet with vertices

$$n^{1,0}, n^{1,2}, n^{1,4}, n^{1,6},\ n^{5,0}, n^{5,2}, n^{5,4}, n^{5,6} = \{n^{d,d'} : d = 1, 5, d' = 1, 2, 4, 6\}.$$

This threedimensional facet is a Minkowski sum of the threedimensional simplex over $n^{1,0}, n^{1,2}, n^{1,4}, n^{1,6}$ and an edge.

For small values of k, the special facets can be described explicitly. The following lemma describes the case $k = 3$; see also Figures 2 and 3.

Lemma 8. *If $k = 3$, then any special facet is a simplex with vertices $n^{d,d'_1}, n^{d,d'_2}, n^{d,d'_3}$, with $d'_1, d'_2, d'_3 \in \{0, d-1, d+1, N-1\}$.*

Proof. Let $d_1 < d_2$ be two odd integers and d' be even such that $n^{d_1,d'}$ and $n^{d_2,d'}$ are two vertices. Then either $d' \in \{0, N-1\}$, or $d_2 - 1 = d' = d_1 + 1$. In any case, the line segment L between n^{d_1} and n^{d_2} intersects the convex hull $\mathbf{C}$ of n^0, n^{d_1+1} and n^{N-1}. The line segment L' between $n^{d_1,d'}$ and $n^{d_2,d'}$ is parallel to L, with both end points shifted in the direction of the same vertex of $\mathbf{C}$; hence L' also intersects $\mathbf{C}$. Therefore, L' is not an edge. This shows that all vertices $n^{d,d'}$ of F' have the same d. The statement now follows from the fact that the points $n^{d,0}, n^{d,d-1}, n^{d,d+1}, n^{d,N-1}$ define a cyclic polytope, and F' must be a facet of this cyclic polytope, hence F' is a simplex. □

Remark 9. By (2), the polytope $\mathbf{P}_{k;N}$ is affinely equivalent to the polytope of the first k moments of the degree distribution. For $k \geqslant N-1$ the map that maps a probability distribution on $\{0, \dots, N-1\}$ to its first k moments is an injective linear map (it is described by a Vandermonde matrix). This shows that, for $k \geqslant N-1$, the polytope $\mathbf{P}_{k;N}$ is affinely equivalent to the polytope $\mathbf{D}_N$ of all graphical degree distributions. For $k < N-1$, $\mathbf{P}_{k;N}$ is a projection of $\mathbf{D}_N$, and Theorem 6 shows that this projection preserves the number of vertices if $k > 3$.

4 Subsets of *k*-stars

Some of the above results generalize to the situation where one is interested not in the complete vector $\vec{n}_k$, but in some subvector $(n_i)_{i\in I}$ for some set $I \subseteq \{1, \dots, k\}$. Denote the corresponding polytopes by $\mathbf{P}_{I;N}$, and let s^I be the curve $[0,1] \to \mathbb{R}^I$ defined via $s^I_i(t) = t^i$ for all $i \in I$. Then $\mathbf{P}_{I;N}$ and s^I are orthogonal projections of $\mathbf{P}_{k;N}$ and s^k into a coordinate hyperplane. Hence the limit $\mathbf{P}_{I;\infty} = \bigcap_N \mathbf{P}_{I;N}$ equals the convex hull of s^I.

See Figure 5 a and b) for illustrations. The convex hull of N points on the curve s^I is also a cyclic polytope. This can be seen as follows:

Let $\mathbb{I} \subseteq \mathbb{R}$ be an interval. A curve $\gamma : \mathbb{I} \mapsto \mathbb{R}^k$ is *of order k* if for any hyperplane H in $\mathbb{R}^k$ the cardinality of $\gamma(\mathbb{I}) \cap H$ is at most k. It is known that the convex hull of N points on a curve of order k in $\mathbb{R}^k$ is a $C(k; N)$ cyclic polytope, see [4].

Lemma 10. *Let $I = \{i_1, \ldots, i_l\}$ be a set of positive integers. The curve $\gamma^I : [0, \infty) \to \mathbb{R}^l$ defined by $\gamma^I(t)_j = t^{i_j}$ is of order l.*

Proof. Any hyperplane $H \subseteq \mathbb{R}^l$ is defined as the vanishing set of some affine function $f(x) = \sum_{j=1}^{l} a_j x_j - b$. The intersection points of γ^I and H correspond to the zeros of the polynomial

$$f \circ \gamma^I(t) = \sum_{j=1}^{l} a_j t^{i_j} - b.$$

According to Décartes' rule of signs, there are at most l such zeros. □

Corollary 11. *The convex hull of N points on the curve s^I is a $C(|I|; N)$ cyclic polytope.*

The corollary shows that, if N is large, then $\mathbf{P}_{I;N}$ is almost a cyclic polytope. To be precise, the proof of Theorem 1 shows the following: There is a $C(|I|; N)$ cyclic polytope $\mathbf{C}$ such that for any vertex v of $\mathbf{P}_{I;N}$ there is a vertex v' of $\mathbf{C}$ with $\|v - v'\| = O(1/N)$. However, in general $\mathbf{P}_{I;N}$ is not a $C(|I|; N)$ cyclic polytope, even if N is even, as the following lemma shows:

Lemma 12.

1. *If $|I| > 2$ and $2 \notin I$ and $N > 3$ is even, then $\mathbf{P}_{I;N}$ has less than $|N|$ vertices.*

2. *Suppose that I is not a sequence of consecutive integers and suppose that $|I| > 2$. Let j be the smallest positive integer not in I. If $N > j > 3$, then $\mathbf{P}_{I;N}$ and $\tilde{\mathbf{P}}_{I;N}$ are not cyclic polytopes.*

Proof. Let $\tilde{\mathbf{P}}_{I;N}$ be the convex hull of the N columns of the $(|I| \times N)$-matrix D^I with entries $D^I_{i,d} = \frac{[d]_i}{[N-1]_i}$ for $i \in I$ and $d = 0, \ldots, N-1$. If N is even, then $\mathbf{P}_{I;N} = \tilde{\mathbf{P}}_{I;N}$.

1. If $1 \notin I$, then the first two columns of D^I are zero vectors. If $1 \in I$, then the first three columns of D^I are

$$\begin{pmatrix} 0 & \frac{1}{N-1} & \frac{2}{N-1} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

and so the second column lies in the convex hull of the first and the third column.

2. If $i \in I$, then $D^I_{i,d} = 0$ for $d \leqslant i$ and $D^I_{i,d} > 0$ for $d \geqslant i$. The equations $n_i = 0$ for $i \in I$, $i > j$, define a proper face F of $\tilde{\mathbf{P}}_{I;N}$. The face F is affinely equivalent to the convex hull of the columns of the first k columns of the matrix $D^{\{1,\ldots,j-1\}}$, where k is the

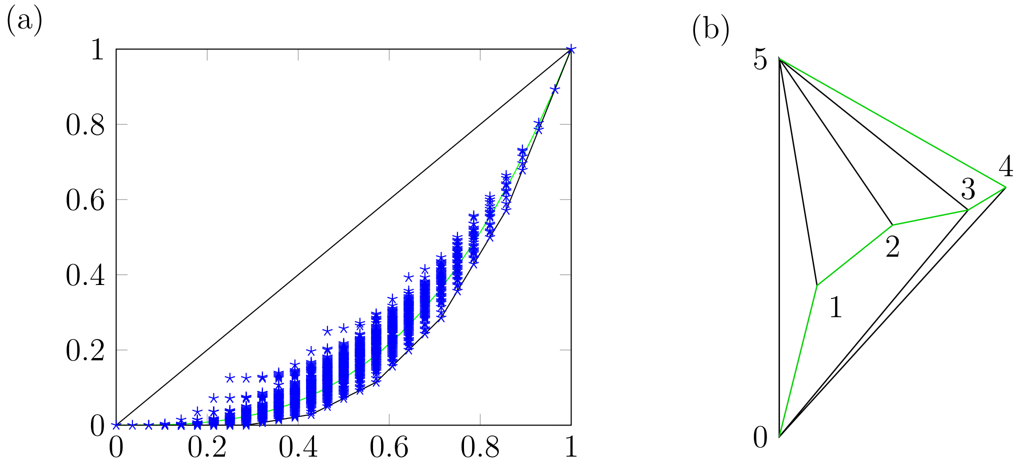

Figure 5: (a) The polytope $\mathbf{P}_{\{1,3\};8}$. Each cross is a possible value of $\vec{n}_2$ for some graph with eight nodes, and the moment curve is marked in green. (b) A sketch of $\mathbf{P}_{\{1,2,4\},6}$, similar to Figure 3. The number d labels the vertex n^d of $\tilde{\mathbf{P}}_{3,7}$. The green edges in $\tilde{\mathbf{P}}_{3,7}$ are the edges from n^d to n^{d+1}. Hence, the green line corresponds to the moment curve.

smallest integer in I larger than j. Hence F is a $C(j-1;k)$ cyclic polytope. In particular, F is not a simplex. However, all proper faces of cyclic polytopes are simplices. Hence $\tilde{\mathbf{P}}_{I;N}$ is not a cyclic polytope.

If N is odd, then $\mathbf{P}_{I;N}$ is a strict subset of $\tilde{\mathbf{P}}_{I;N}$, obtained by cutting off some of the vertices. Since $\tilde{\mathbf{P}}_{I;N}$ contains a face which is not a simplex, the same is true for $\mathbf{P}_{I;N}$. $\square$

Remark 13. Lemma 12 shows that the main argument in the proof of Proposition 4.1 in [6] is false[4]. In fact, the idea behind the proof of Lemma 12 can be used to construct a counter-example to [6, Proposition 4.1].

The proof of Lemma 12b) that $\tilde{\mathbf{P}}_{I;N}$ is in general not a cyclic polytope relies on the fact that the first few columns of D induce a face which is itself a cyclic polytope. It turns out that $\tilde{\mathbf{P}}_{I;N}$ becomes a cyclic polytope when enough columns of D are dropped from the beginning:

Lemma 14. *Assume that $k := \max(I) > N$. Then the convex hull of the last $N-k$ columns of D, $\operatorname{conv}\{n^k, n^{k+1}, \ldots, n^{N-1}\} \subset \tilde{\mathbf{P}}_{I;N}$, is a cyclic polytope.*

Proof. It suffices to show that the curve $t \in [k;\infty) \mapsto ([t]_i)_{i\in I}$ is of order k (note the domain of definition). As in the proof of Lemma 10, this follows from the fact that the falling factorial powers $[t]_i$ for $i \leqslant k$ satisfy Déscartes' rule of sign on the interval $[k;\infty)$, a result that is due to Runge [13] (see also [5] for newer results in this direction). $\square$

By Lemma 12 and Corollary 11, in general, $\tilde{\mathbf{P}}_{I;N}$ is not a cyclic polytope, but it is $O(1/N)$-close to a cyclic polytope. For $I = \{1,2,4\}$, even more is true, see Figure 5 b):

[4]Note that only the preprint version contains the proofs. In the preprint, the Proposition is numbered 6.1.

An arbitrarily small deformation of the polytopes' vertices suffices to turn $\tilde{\mathbf{P}}_{\{1,2,4\};N}$ into a cyclic polytope. It is the author's conjecture that the same holds true for any I and N.

5 The complement symmetry

The objects of study of this section are the affine symmetries of the polytopes $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$, i.e. affine maps $\mathbb{R}^k \to \mathbb{R}^k$ that restrict to bijections of the corresponding polytopes. It turns out that, if $N > k+1$, then there is just a single non-trivial affine symmetry,

The automorphism group of the face lattice of $C(k;N)$ cyclic polytopes is known, see Theorem 8.3 in [9]. The automorphism group of the "standard cyclic polytope," corresponding to N points on the moment curve s^k (or on the algebraic moment curve) with equidistant values of t is probably also known, but difficult to find in the literature. Theorem 16 below shows that these polytopes have only one non-trivial symmetry for $N > k+1$.

The complement $\tilde{G} = (V, \tilde{E})$ of a graph $G = (V, E)$ is a graph with the same node set V as G and with edge set $\tilde{E} = \{(x,y) \in V^2 : x \neq y, (x,y) \notin E\}$. The complement $\tilde{G}$ satisfies $\tilde{d}_i = N - 1 - d_i$. The operation $\tilde{G} \mapsto G$ is an involution of the set of graphs with N nodes. It induces an involution on the polytopes $\mathbf{D}_N$, $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$:

Theorem 15. *The map $\phi : \mathbb{R}^k \to \mathbb{R}^k$ defined by $\phi(x)_j = \sum_{i=0}^{j} (-1)^i \binom{j}{i} x_i$ is an involution that satisfies $\phi(\vec{n}(G)) = \vec{n}(\tilde{G})$, whenever $\tilde{G}$ is the complement of the graph G. Both polytopes $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ are invariant under ϕ.*

Proof. The following formula

$$\binom{a}{b} = \sum_{i=0}^{b} (-1)^i \binom{c}{i} \binom{a+c-i}{b-i}$$

is needed, which follows from induction and $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$. For any $0 \leqslant d \leqslant N-1$,

$$\frac{[N-1-d]_k}{[N-1]_k} = \frac{\binom{N-1-d}{k}}{\binom{N-1}{k}} = \frac{1}{\binom{N-1}{k}} \sum_{i=0}^{k} (-1)^i \binom{d}{i} \binom{N-1-i}{k-i}$$
$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{[d]_i}{[N-1]_i}. \quad (5)$$

Therefore, ϕ permutes the vertices of $\tilde{\mathbf{P}}_{k;N}$. The equality $\phi(\vec{n}(G)) = \vec{n}(\tilde{G})$ and the statement about $\mathbf{P}_{k;N}$ follow by taking expected values of (5) under the degree distribution of G. Since ϕ is a linear map that restricts to an involution on the full-dimensional polytope $\tilde{\mathbf{P}}_{k;N}$, it follows that ϕ is an involution on all of $\mathbb{R}^k$. □

Theorem 16.

- *If $N \leqslant k+1$, then $\tilde{\mathbf{P}}_{k;N}$ is a simplex. The set of affine automorphisms of $\tilde{\mathbf{P}}_{k;N}$ is the symmetric group $\mathbb{S}_N$.*
- *If $N > k+1$, then ϕ is the only non-trivial affine symmetry of $\tilde{\mathbf{P}}_{k;N}$.*

Proof. The case $N \leqslant k+1$ is trivial, so assume $N > k+1$. The polytope $\tilde{\mathbf{P}}_{k;N}$ is a $C(k;N)$ cyclic polytope.

Suppose $\phi' : x \mapsto Ax + b$ is an affine symmetry of $\tilde{\mathbf{P}}_{k;N}$, where $A \in \mathbb{R}^{k\times k}$ and $b \in \mathbb{R}^k$. Since ϕ' preserves the volume of the full-dimensional polytope $\tilde{\mathbf{P}}_{k;N}$, it follows that $|\det(A)| = 1$. For any subset $S \subset \{0,\dots,N-1\}$ of cardinality $k+1$ let D_S be the submatrix of D consisting of those columns of D indexed by S, and let $\tilde{D}_S$ be the same matrix with an additional row of ones added on top. Then $\tilde{D}_S$ is a Vandermonde matrix with determinant $\det(\tilde{D}_S) = \prod_{i\in S}\prod_{i<j\in S}(j-i)$. In particular, $|\det(\tilde{D}_S)|$ is minimal if and only if S consists of consecutive integers.

The map ϕ' corresponds to a permutation σ of the column indices $0,\dots,N-1$ of D in such a way that $\phi'(n^i) = n^{\sigma(i)}$. Then $\tilde{D}_{\sigma(S)} = \left(\begin{smallmatrix}1 & 0\\ 0 & A\end{smallmatrix}\right)\tilde{D}_S$, and so $|\det(\tilde{D}_{\sigma(S)}))| = |\det(\tilde{D}_S)|$. In particular, σ maps consecutive integers to consecutive integers, and therefore, either σ equals the identity, in which case ϕ' equals the identity of $\mathbb{R}^k$, or σ inverts the order on $\{0,\dots,N-1\}$, in which case $\phi' = \phi$. □

Theorem 17. *Let N be odd.*

- *If $N \leqslant k+1$, then $\mathbf{P}_{k;N}$ is affinely equivalent to $\mathbf{D}_N$, and the group of affine symmetries of $\mathbf{P}_{k;N}$ is the subgroup $\mathbb{S}_{\lceil \frac{n}{2}\rceil}\times\mathbb{S}_{\lfloor \frac{n}{2}\rfloor} \subseteq \mathbb{S}_n$ of the symmetry group of $\tilde{\mathbf{P}}_{k;N}$, where the first factor permutes the vertices n^d with even d, and the second factor permutes the points n^d with odd d.*
- *If $N > k+1$, then ϕ is the only non-trivial symmetry of $\mathbf{P}_{k;N}$.*

The proof makes use of the following result:

Lemma 18. *Any affine symmetry of $\mathbf{P}_{k;N}$ extends to an affine symmetry of $\tilde{\mathbf{P}}_{k;N}$.*

Proof. The statement is trivial if N is even and if $N = 1$, so assume that $N \geqslant 3$ is odd. Let $k = 2$. Then $\mathbf{P}_{k;N}$ is a $(\frac{3N-1}{2})$-gon The symmetry group of the face lattice of this polygon is the dihedral group $\mathbb{D}_{\frac{3N-1}{2}}$. The edge (n^0, n^{N-1}) is characterized by the following property: Consider the lines defined by all edges of $\mathbf{P}_{k;N}$. All intersection points of these lines lie in one closed hyperplane supporting (n^0, n^{N-1}). This property is invariant under affine transformations. Hence, (n^0, n^{N-1}) is mapped to itself under any affine symmetry. There are only two elements of the dihedral group with this property, the identity and the permutation corresponding to the complement symmetry ϕ.

Now assume that $k \geqslant 3$, and let d be odd. By Gale's evenness condition, there is a face F of $\tilde{\mathbf{P}}_{k;N}$ with vertices n^0, n^d and n^{d+1}. Then n^0, $n^{d,0}$, $n^{d,d+1}$ and n^{d+1} are vertices of $\hat{F} = F \cap \mathbf{P}_{k;N}$.

Let ϕ' be an affine symmetry of $\mathbf{P}_{k;N}$. Then ϕ' permutes the facets. Under this action, simplices are mapped to simplices, and non-trivial vertex-preserving Minkowski sums are mapped to non-trivial vertex-preserving Minkowski sums. By Proposition 7, ϕ' maps $\hat{F}$ to another face $\hat{F}' = F' \cap \mathbf{P}_{k;N}$, where F' is a face of $\tilde{\mathbf{P}}_{k;N}$. The vertices of the form $n^{d'}$ for even d' are distinguished by the fact that they are not contained in any special face. Hence ϕ' permutes these vertices among themselves. Moreover, the line connecting $\phi'(n^0)$ and $\phi'(n^{d,0})$ intersects the line connecting $\phi'(n^{d,d+1})$ and $\phi'(n^{d+1})$ in a unique point, which must be of the form $n^{\hat{d}}$ for some odd $\hat{d}$. This shows that ϕ' maps $\tilde{\mathbf{P}}_{k;N}$ into $\tilde{\mathbf{P}}_{k;N}$, and hence ϕ' is a symmetry of $\tilde{\mathbf{P}}_{k;N}$. □

Proof of Theorem 17. The statement for $N > k + 1$ follows from Lemma 18 and Theorem 16. Assume $N \leqslant k + 1$. By Lemma 18, the group of affine symmetries of $\mathbf{P}_{k;N}$ is a subgroup of the group of affine symmetries of $\tilde{\mathbf{P}}_{k;N}$. Moreover, any affine symmetry of $\mathbf{P}_{k;N}$ has to preserve the sets $\{n^d : d \text{ odd}\}$ and $\{n^d : d \text{ even}\}$. Conversely, any affine symmetry of $\tilde{\mathbf{P}}_{k;N}$ that preserves these sets also permutes the vertices $n^{d,d'}$ of $\mathbf{P}_{k;N}$ and thus restricts to a symmetry of $\mathbf{P}_{k;N}$. □

When I is a set of non-consecutive integers, then the polytope $\mathbf{P}_{I;N}$ in general has no non-trivial symmetry; see, for example, Figure 5b).

6 Exponential random graphs

Let $\mathcal{G}_N$ be the set of graphs with node set $\{1, \dots, N\}$, and fix an integer $k \leqslant N - 1$. For all real numbers $\beta_1, \dots, \beta_k$ let $\psi_{\beta_1,\dots,\beta_k}(G) = \exp(\sum_{i=1}^{k} \beta_i n_i(G))$ and $Z_{\beta_1,\dots,\beta_k} = \sum_{G \in \mathcal{G}_N} \psi_{\beta_1,\dots,\beta_k}(G)$. Then $P_{\beta_1,\dots,\beta_k}(G) = \psi_{\beta_1,\dots,\beta_k}(G)/Z_{\beta_1,\dots,\beta_k}$ defines a probability distribution on $\mathcal{G}_N$. The family $\mathcal{E}_{k;N} := (P_{\beta_1,\dots,\beta_k})_{\beta_1,\dots,\beta_k}$ is called the *k-star model*. It is an example of an exponential random graph model, and hence a particular example of an exponential family.

It follows from the general theory of exponential families that the map $(\beta_1, \dots, \beta_k) \mapsto \mathbb{E}_{P_{\beta_1,\dots,\beta_k}}(\vec{n})$ is a homeomorphism of $\mathbb{R}^k$ to the interior of $\mathbf{P}_{k;N}$. It induces a homeomorphism from $\mathcal{E}_{k;N}$ to the interior of $\mathbf{P}_{k;N}$ that extends to a homeomorphism $\overline{\mathcal{E}_{k;N}} \cong \mathbf{P}_{k;N}$, where $\overline{\mathcal{E}_{k;N}}$ denotes the closure of $\mathcal{E}_{k;N}$ with respect to the induced topology when considering a probability distribution P on $\mathcal{G}_N$ as a real vector with $|\mathcal{G}_N|$ components.

For $k < l \leqslant N - 1$, $\mathcal{E}_{k;N}$ is a submodel of $\mathcal{E}_{l;N}$, namely the submodel defined by $\beta_{k+1} = \cdots = \beta_l = 0$. The location of $\mathcal{E}_{k;N}$ within $\mathcal{E}_{l;N}$ can be visualized by considering the image of $\mathcal{E}_{k;N}$ under the map $P \mapsto \mathbb{E}_P(\vec{n}_l)$. The closer $\mathbb{E}_P(\vec{n}_l)$ lies to the image of $\mathcal{E}_{k;N}$, the closer P lies to $\mathcal{E}_{k;N}$.

In particular, $\mathcal{E}_{1;N} \subset \mathcal{E}_{k;N}$ for all k. The random graph $P_{\beta_1} \in \mathcal{E}_{1;N}$ is identical to the Erdős-Rényi random graph $G(N,p)$ with parameter $p = \frac{e^{\beta_1}}{1+e^{\beta_1}}$. As β_1 goes from $-\infty$ to $+\infty$, the Erdős-Rényi parameter p goes from zero to one, and hence the expected values of $\vec{n}_k$ retrace the complete moment curve.

The fact that $\mathbf{P}_{k;N}$ converges to the convex hull of the moment curve can be interpreted as follows: The vertices of $\mathbf{P}_{k;N}$ correspond to graphs that are (almost) regular. By the law of large numbers, when N is large, the degree distribution of the Erdős-Rényi graph is concentrated around its mean value. In other words, for large N the Erdős-Rényi graph is almost regular, and the degree can be tuned by varying the parameter p.

When all parameters but one, say β_i, are fixed, then $\mathbb{E}_{P_{\beta_1,\ldots,\beta_k}}(n_i)$ is a monotone function of β_i. More generally, when the parameter vector $\beta = (\beta_1, \ldots, \beta_k)$ moves in a fixed direction, then the relative weights of those graphs whose expected values lie in the same direction increase. In the limit, as $\theta \to +\infty$, the measures $P_{\beta_0+\theta\beta}$ converge to a probability distribution supported on those graphs G that maximize $\sum_{i=1}^{k} \beta_i n_i(G)$. This function $\sum_{i=1}^{k} \beta_i n_i(G)$ is maximized on a face of the polytope $\mathbf{P}_{k;N}$. If the face happens to be a vertex, say, the vertex corresponding to d-regular graphs, then the limit distribution will be the uniform distribution on the d-regular graphs, since every element $P \in \mathcal{E}_{k;N}$ assigns the same probability to all graphs having the same degree distribution.

It has been observed that for large N and certain parameter ranges the k-star models become *degenerate* in the sense that they give a large probability mass to a small number of graphs, see for example [8]. For example, for certain parameter values a significant probability mass is given to almost empty or almost complete graphs. Even worse, the random graph may behave like a mixture of almost empty or almost complete graphs. Clearly, such a random graph is not a good model of real world data.

Degeneracy of exponential random graphs was recently studied by Chatterjee and Diaconis in the paper [2] that studies the limit of exponential random graphs in the language of graph limits. In Theorem 6.4 of that paper, it is shown that in the limit of large N, if the parameters are scaled by $1/N^2$, then the k-star model always behaves like a finite mixture of Erdős-Rényi models.

The fact that for many parameter values the k-star model puts a lot of mass on almost empty or almost comlete graphs can also be understood by looking at $\mathbf{P}_{k;N}$: Note that the two vertices $(0, \ldots, 0)$ and $(1, \ldots, 1)$, corresponding to the empty graph and the complete graph, are the "most exposed" vertices, i.e. there are "many" parameter values $\beta_1, \ldots, \beta_k$ such that the maximum of the function $\sum_{i=1}^{k} \beta_i n_i(G)$ lies either at the empty or the complete graph. This is a consequence of the elongated shape of the polytope. For example, if $\beta_i \geqslant 0$ for all $i = 2, \ldots, k$, then either the complete graph (if $\beta_1 > 0$) or the empty graph (if $\beta_1 \ll 0$) maximize the function $\sum_{i=1}^{k} \beta_i n_i(G)$, and so almost empty or almost complete graphs get the largest weights.

The form of $\mathbf{P}_{k;N}$ also sheds light on the phenomenon that for many parameter values, the k-star model is very close to an Erdős-Rényi random graph by the following heuristic argument: Fix $\beta_1, \ldots, \beta_k$, and consider $P_\theta = P_{\beta_1,\theta\beta_2,\ldots,\theta\beta_k} \in \mathcal{E}_{k;N}$ for $\theta \geqslant 0$. Note that P_0 belongs to the Erdős-Rényi model. Moreover, when $\theta \to \infty$, then P_θ converges to a probability measure P_∞ supported on those graphs that maximize $\sum_{i=1}^{k} \beta_i n_i(G)$. These graphs correspond to a face of the polytope $\mathbf{P}_{k;N}$. For a generic choice of the parameters β_i, this face will be a vertex, and therefore, the random graph will be uniformly distributed on the set of graphs that are d-regular or almost d-regular, where d depends on the parameters β_i.

If N is large, then $\mathbf{P}_{k;N}$ can be approximated by the convex hull of the moment curve. Therefore, P_0 lies on the moment curve, and P_∞ lies close to the moment curve. If the vector $(0, \beta_2, \ldots, \beta_k)$, attached at P_0, points away from the convex hull of the moment curve (for example, if the parameters β_i for $i \geqslant 2$ are negative), then P_0 and P_∞ will be close to each other. In this case, for any non-negative value of θ, the measure P_θ will be well-approximated by the Erdős-Rényi graph P_0. The influence of the parameters $\beta_2, \ldots, \beta_k$ is only small.

While these considerations give a nice geometric picture to think about exponential random graphs, they are not sufficient to give a precise description of what happens for intermediate values of β_i. Quite generally, for exponential families, the corresponding polytope, called convex support, can only tell what happens in asymptotic parameter regimes. For finite values of the parameters, a more fine-grained analysis is needed. The tool used in [2] is a large deviation principle for the Erdős-Rényi graph, due to [3].

A qualitative version of this large deviation principle can be obtained from the description of the polytope as follows: When N is large, the expected value $\vec{n}_k(G(N,p))$ of the vector of i-star statistics for the Erdős-Rényi graph $G(N,p)$ is very close to the boundary of $\mathbf{P}_{k;N}$ (quantitatively: The distance is $O(1/N)$). Note that $\vec{n}_k(G(N,p))$ is a convex combination of points in $\mathbf{P}_{k;N}$. Since this convex combination is very close to the boundary of $\mathbf{P}_{k;N}$, it follows that most of the points that contribute to the convex combination $\vec{n}_k(G(N,p))$ must lie close to $\vec{n}_k(G(N,p))$. This argument shows that the probability mass of $\vec{n}_k(G(N,P))$ is concentrated within a radius of $O(1/N)$ around its mean value. The large deviation principle is much stronger in this respect, since it yields exponential concentration (and also applies to other graph observables that are not a function of $\vec{n}_k$). On the other hand, the polytope $\mathbf{P}_{k;N}$ contains precise informations about all linear inequalities that are valid between the i-star counts. Therefore, while the large deviation results tell where all but a negligeabe fraction of all graphs lie, the polytope $\mathbf{P}_{k;N}$ gives the precise region in which all graphs lie, with no exception. In this sense, the convex support relates to the large deviation result as Chebyshev's inequality relates to the central limit theorem.

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A Proof of Theorem 6

Lemma 19. *Let N be odd. If d is odd, d' is even and $|d'-d| \neq 1$, then $n^{d,d'}$ is an interior point of $\mathbf{P}_{2;N}$.*

Proof. Using the symmetry ϕ it suffices to consider the case $d' < d-1$. By Gale's evenness condition, n^d and $n^{d'}$ do not define an edge in the cyclic polytope with vertices $n^{d'}$, n^{d-1},

n^d and n^{d+1}. Therefore, the line segment $(n^d, n^{d'})$ between n^d and $n^{d'}$ intersects the line segment (n^{d-1}, n^{d+1}). Hence $(n^d, n^{d,d'})$ intersects $(n^{d,d-1}, n^{d,d+1})$ in a point x that lies on $(n^d, n^{d'})$. It follows that $n^{d,d'}$ is a convex combination of x and $n^{d'}$, and so $n^{d,d'}$ is a convex combination of $n^{d,d-1}$, $n^{d,d+1}$ and $n^{d'}$. $\square$

Lemma 20. *Let N be odd. If d is odd, d' is even and $d' \notin \{0, d-1, d+1, N-1\}$, then $n^{d,d'}$ is an interior point of $\mathbf{P}_{3;N}$.*

Proof. Using the symmetry ϕ it suffices to consider the case $d' < d-1$. By Gale's evenness condition, the line segment $(n^d, n^{d'})$ is not an edge of the cyclic polytope with vertices n^0, $n^{d'}$, n^{d-1}, n^d and n^{d+1}. Therefore, $(n^d, n^{d'})$ intersects the convex hull of n^0, n^{d-1} and n^{d+1}. Hence, $(n^d, n^{d,d'})$ also intersects the convex hull C of $n^{d,0}$, $n^{d,d-1}$ and $n^{d,d+1}$ in a unique point x. Since x lies on $(n^d, n^{d'})$, the point $n^{d,d'}$ lies in $(x, n^{d'})$. Therefore, $n^{d,d'}$ is a convex combination of $n^{d,0}$, $n^{d,d-1}$, $n^{d,d+1}$ and $n^{d'}$. $\square$