On bipartite cages of excess 4

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Abstract

The Moore bound $M(k,g)$ is a lower bound on the order of k-regular graphs of girth g (denoted (k,g)-graphs). The excess e of a (k,g)-graph of order n is the difference $n - M(k,g)$. In this paper we consider the existence of (k,g)-bipartite graphs of excess 4 by studying spectral properties of their adjacency matrices. For a given graph G and for the integers i with $0 \leqslant i \leqslant diam(G)$, the *i-distance matrix* A_i of G is an $n \times n$ matrix such that the entry in position (u,v) is 1 if the distance between the vertices u and v is i, and zero otherwise. We prove that the (k,g)-bipartite graphs of excess 4 satisfy the equation $kJ = (A + kI)(H_{d-1}(A) + E)$, where $A = A_1$ denotes the adjacency matrix of the graph in question, J the $n \times n$ all-ones matrix, $E = A_{d+1}$ the adjacency matrix of a union of vertex-disjoint cycles, and $H_{d-1}(x)$ is the Dickson polynomial of the second kind with parameter $k-1$ and degree $d-1$. We observe that the eigenvalues other than $\pm k$ of these graphs are roots of the polynomials $H_{d-1}(x) + \lambda$, where λ is an eigenvalue of E. Based on the irreducibility of $H_{d-1}(x) \pm 2$, we give necessary conditions for the existence of these graphs. If E is the adjacency matrix of a cycle of order n, we call the corresponding graphs *graphs with cyclic excess*; if E is the adjacency matrix of a disjoint union of two cycles, we call the corresponding graphs *graphs with bicyclic excess*. In this paper we prove the non-existence of (k,g)-graphs with cyclic excess 4 if $k \geqslant 6$ and $k \equiv 1 (\text{mod } 3)$, $g = 8, 12, 16$ or $k \equiv 2 (\text{mod } 3)$, $g = 8$; and the non-existence of (k,g)-graphs with bicyclic excess 4 if $k \geqslant 7$ is an odd number and $g = 2d$ such that $d \geqslant 4$ is even.

Keywords: Cage problem, bipartite graphs, cyclic excess, bicyclic excess

1 Introduction

A k-regular graph of girth g is called a (k,g)-*graph.* A (k,g)-*cage* is a (k,g)-graph with the fewest possible number of vertices, among all (k,g)-graphs. The order of a (k,g)-cage is denoted by $n(k,g)$. The *Cage Problem* or *Degree/Girth Problem* calls for finding cages, and it was considered for the first time by Tutte [17]. It is known that a (k,g)-graph exists for any combination of $k \geqslant 2$ and $g \geqslant 3$, see Erdős and Sachs [10] and Sachs [15]. However, the orders $n(k,g)$ of (k,g)-cages have only been determined for very limited sets of parameters, see Balbuena, González-Moreno and Montellano [1], Exoo and Jajcay [11] and Combinatorics Wiki [5]. A natural lower bound on the order of a (k,g)-graph is called the *Moore bound*, and the form of the bound depends on the parity of g, that is,

$$n(k,g) \geqslant M(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2}, & g \text{ odd}, \\ 2\left(1 + (k-1) + \cdots + (k-1)^{(g-2)/2}\right), & g \text{ even}. \end{cases} \tag{1}$$

The graphs whose orders are equal to the Moore bound are called *Moore graphs*. They are known to exist if $k = 2$ and $g \geqslant 3$, $g = 3$ and $k \geqslant 2$, $g = 4$ and $k \geqslant 2$, $g = 5$ and $k = 2, 3, 7$, or $g = 6, 8, 12$ and a generalized n-gon of order $k-1$ exists, see Bannai and Ito [2], Damerell [7] and Exoo and Jajcay [11]. The existence of a $(57,5)$-Moore graph is an open question.
The *excess* e of a (k,g)-graph is the difference between its order n and the Moore bound $M(k,g)$, that is, $e = n - M(k,g)$. Regarding graphs of even girth we use the following three results:

Theorem 1 (Biggs and Ito [4]). *Let G be a (k,g)-cage of girth $g = 2d \geqslant 6$ and excess e. If $e \leqslant k-2$, then e is even and G is bipartite of diameter $d+1$.*

It is known that these graphs are partially distance-regular. For more information on almost-distance-regular graphs, see Dalfó, van Dam, Fiol, Garriga and Gorissen [6]. For the next theorem, let $D(k,2)$ denote the incidence graph of a symmetric $(v,k,2)$-design.

Theorem 2 (Biggs and Ito [4]). *Let G be a (k,g)-cage of girth $g = 2d \geqslant 6$ and excess 2. Then $g = 6$, G is a double-cover of $D(k,2)$, and $k \not\equiv 5, 7 \pmod 8$.*

Theorem 3 (Jajcayová, Filipovski and Jajcay [13]). *Let $k \geqslant 6$ and $g = 2d > 6$. No (k,g)-graphs of excess 4 exist for parameters k, g satisfying at least one of the following conditions:*

1) *$g = 2p$, with $p \geqslant 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod p$;*

2) *$g = 4 \cdot 3^s$ such that $s \geqslant 4$, and k is divisible by 9 but not by 3^{s-1};*

3) *$g = 2p^2$, with $p \geqslant 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod p$ and k even;*

4) *$g = 4p$, with $p \geqslant 5$ a prime number, and $k \not\equiv 0, 1, 2, 3, p-2 \pmod p$;*

5) *$g \equiv 0 \pmod{16}$, and $k \equiv 3 \pmod g$.*

Motivated by the result in Theorem 3, which was obtained through counting cycles in a hypothetical graph with given parameters and excess 4, in this paper we address the question of the existence of (k,g)-graphs of excess 4 using spectral properties of

their adjacency matrices. The question of the existence of (k, g)-graphs of excess 4 is wide open, and prior to the publication of Jajcayová, Filipovski and Jajcay [13], no such results were known. The results contained in our paper extend further our understanding of the structure of the potential graphs of excess 4. Throughout, we assume that $k \geqslant 6$, $g = 2d \geqslant 6$ and G is a (k, g)-graph of excess 4 and order n. Due to Biggs's result stated in Theorem 1, the restriction of the parameters k, g given above allows us to conclude that G is a bipartite graph with diameter $d + 1$.
For each integer i in the range $0 \leqslant i \leqslant d + 1$, we define the $n \times n$ matrix $A_i = A_i(G)$ as follows. The rows and columns of A_i correspond to the vertices of G, and the entry in position (u, v) is 1 if the distance $d(u, v)$ between the vertices u and v is i, and zero otherwise. Clearly, $A_0 = I, A_1 = A$, the usual adjacency matrix of G. The last non-zero matrix is the matrix A_{d+1}, which we denote E and refer to it as the *excess matrix*, that is, E is the adjacency matrix of the graph with the same vertex set V as G such that two vertices of V are adjacent if and only if they are at distance $d + 1$. We call this graph the *excess graph* of G and we denote it $G(E)$. If J is the all-ones matrix, the sum of the *i-distance matrices* A_i, for $0 \leqslant i \leqslant d$, and the matrix E yields $\sum_{i=0}^{d} A_i + E = J$. To apply the last identity, we use Lemma 4 from Jajcayová, Filipovski and Jajcay [13]. Employing the methodology used by Bannai and Ito in [2] and [3], later by Biggs and Ito in [4], Delorme, Jørgensen, Miller and Villavicencio in [8] and Garbe in [12], we show that the eigenvalues of G other than $\pm k$ are the roots of the polynomials $H_{d-1}(x) + \lambda$. Here, $H_{d-1}(x)$ is the Dickson polynomial of the second kind with parameter $k - 1$ and degree $d - 1$, and λ is an eigenvalue of the excess matrix E. Furthermore, for odd $k \geqslant 7$ and $d \geqslant 4$, we prove that the polynomial $H_{d-1}(x) \pm 2$ is irreducible over $\mathbb{Q}[x]$, which leads to necessary conditions for the existence of (k, g)-graphs of excess 4, see Theorem 10.

We say that a graph G has a *cyclic excess* if the excess graph $G(E)$ is a cycle of length n, and a graph G has a *bicyclic excess* if $G(E)$ is a disjoint union of two cycles. In [9] Delorme and Villavicencio considered graphs with cyclic defect and excess 2, proving the non-existence of infinitely many such graphs. The paper describes the cycle structure of the excess graphs of the known non-trivial graphs of excess 2:

1) The excess graph of the only $(3, 5)$-graph of excess 2 is a disjoint union of a 9-cycle and a 3-cycle or a disjoint union of an 8-cycle and 4-cycle.

2) The excess graph of the unique $(4, 5)$-graph of excess 2 (the Robertson graph) is a disjoint union of a 3-cycle, a 12-cycle and a 4-cycle.

3) The excess graph of the unique $(3, 7)$-graph of excess 2 (the McGee graph) is a disjoint union of six 4-cycles.

We note that no (k, g)-graph of cyclic excess 2 are known, while examples of graphs with bicyclic excess 2 can be found among the $(3, 5)$-graphs of excess 2. Proving that the excess graphs of bipartite graphs of excess 4 form a disjoint union of cycles, while also inspired by the results in Delorme and Villavicencio [9], in Section 3 we consider the existence of bipartite graphs of excess 4 with cyclic or bicyclic excess 4. Based on the irreducibility of $H_{d-1}(x) \pm 2$ and $H_{d-1}(x) - 1$ over $\mathbb{Q}[x]$, we prove the non-existence of infinitely many such graphs of girth at least 8.

2 Necessary conditions for the existence of graphs of even girth and excess 4

Let $k \geqslant 6$, $g = 2d \geqslant 6$, and let G be a (k, g)-graph of excess 4. Then G is a bipartite graph of diameter $d+1$. Let $N_G(u, i)$ denote the set of vertices of G whose distance from u in G is equal to i, for $1 \leqslant i \leqslant d+1$. The subgraph of G induced by the set of vertices of G whose distance from u is at most $\frac{g-2}{2}$ and whose distance from v is by one larger than their distance from u induces a tree of depth $\frac{g-2}{2}$ rooted at u, which we call it $\mathcal{T}_u$. Also, the subgraph of G induced by the set of vertices of G whose distance from v is at most $\frac{g-2}{2}$ and whose distance from u is by one larger than their distance from v induces a tree of depth $\frac{g-2}{2}$ rooted at v, which we call it $\mathcal{T}_v$. Since G is of girth g, the trees $\mathcal{T}_u$ and $\mathcal{T}_v$ are disjoint and contain no cycles. Since each vertex of G is of degree k, the order of $\mathcal{T}_u \bigcup \mathcal{T}_v$ is equal to $2(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\frac{g-2}{2}})$. We call the union of the trees $\mathcal{T}_u, \mathcal{T}_v$ with the edge f *Moore tree of* G *rooted at* f; it is the subtree of G that is the basis of the Moore bound for even g. The graph G must contain 4 additional vertices w_1, w_2, w_3, w_4, which do not belong to either $\mathcal{T}_u$ or $\mathcal{T}_v$, and whose distance from both u and v is greater than $\frac{g-2}{2}$. We call these vertices *the excess vertices with respect to* f and denote this set $X_f = \{w_1, w_2, w_3, w_4\}$; we call the edges not contained in the Moore tree of G *horizontal edges*.

The following lemma restricts the possible ways in which the four excess vertices are attached to the Moore tree.

Lemma 4 (Jajcayová, Filipovski and Jajcay [13]). *Let $k \geqslant 6$ and $g = 2d \geqslant 6$. Let G be a (k, g)-graph of excess 4, u, v be two adjacent vertices in G, and $X_f = \{w_1, w_2, w_3, w_4\}$ be the four excess vertices with respect to the edge $f = \{u, v\}$. The induced subgraph $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$ (two disjoint copies of K_2) or $\mathcal{P}_3$ (a path of length 3).*

Next, let us define the following polynomials:

$$F_0(x) = 1,\ F_1(x) = x,\ F_2(x) = x^2 - k;$$

$$G_0(x) = 1,\ G_1(x) = x + 1;$$

$$H_{-2}(x) = -\tfrac{1}{k-1},\ H_{-1}(x) = 0,\ H_0(x) = 1,\ H_1(x) = x;$$

$$P_{i+1}(x) = xP_i(x) - (k-1)P_{i-1}(x) \text{ for } \begin{cases} i \geqslant 2, & \text{if } P_i = F_i, \\ i \geqslant 1, & \text{if } P_i = G_i, \\ i \geqslant 1, & \text{if } P_i = H_i. \end{cases} \tag{2}$$

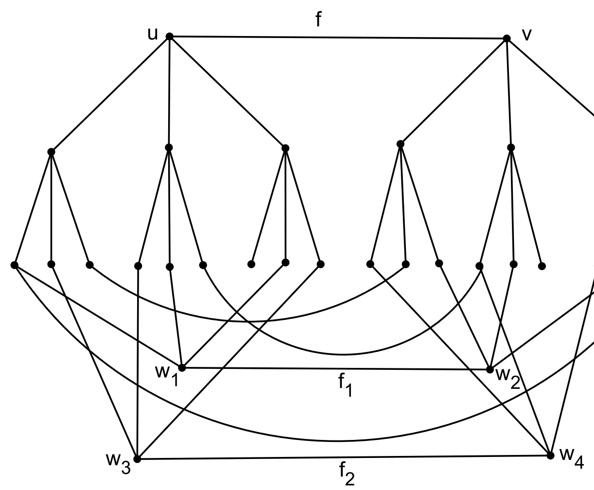

Figure 1: The Moore tree and some of the horizontal edges in a potential (4, 6)-graph of excess 4

In [16], Singleton gives many relationships between these polynomials. We use two of them. Given any $i \geqslant 0$,

$$G_i(x) = \sum_{j=0}^{i} F_j(x), \tag{3}$$

$$G_{i+1}(x) + (k-1)G_i(x) = (x+k)H_i(x). \tag{4}$$

The above defined polynomials have a close connection to the properties of a graph G. Namely, for $t < g$, the element $(F_t(A))_{x,y}$ counts the number of paths of length t joining vertices x and y of G. It follows from (3) that $G_t(A)$ counts the number of paths of length at most t joining pairs of vertices in G. All of the preceding claims can be found in Delorme, Jørgensen, Miller and Villavicencio [8].

The next lemma is based on the structure of G described in Lemma 4.

Lemma 5. *Let $k \geqslant 6$ and $g = 2d \geqslant 6$, and let G be a (k,g)-graph of excess 4. If A is the adjacency matrix of G and E is the excess matrix of G, then*

$$F_d(A) = kA_d - AE.$$

Proof. Let $f = \{u, v\}$ be a base edge of the Moore tree and let $f_1 = \{w_1, w_2\}, f_2 = \{w_3, w_4\}$ be the edges of the subgraph induced by X_f. Also, let us assume that $d(u, w_1) = d(u, w_3) = d$ and $d(u, w_2) = d(u, w_4) = d+1$. We consider the case when $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$, in which case the excess vertices do not share a common neighbour. The other cases when $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$ and the excess vertices share a common neighbour or the subgraph induced by the excess vertices contains $\mathcal{P}_3$ are analogous. Since there are $k-1$ paths of length d from u to w_1 and w_3, by the definition of $F_i(x)$, we have $(F_d(A))_{u,w_1} = (F_d(A))_{u,w_3} = k-1$. Considering the vertices at distance d from u, there are also the $(k-1)^{d-1}$ leaves of the subtree rooted at v. For $2(k-1)$ of these vertices, there exist $k-1$ paths of length d from u to them. Namely, they are the vertices adjacent to w_2 or w_4. For all the other leaves, there are k paths between them and u. Thus, $(F_d(A))_{u,s} = 0$ if $d(u,s) \neq d$, $(F_d(A))_{u,s} = k$ if s is a leaf of a branch rooted at v and not adjacent to w_2 and w_4, and $(F_d(A))_{u,s} = k-1$ if s is w_1, w_3 or a leaf of a branch rooted at v and adjacent to w_2 or w_4. This yields the matrix kA_d, such that $(kA_d)_{u,s} = k$ if $d(u,s) = d$ and $(kA_d)_{u,s} = 0$ if $d(u,s) \neq d$. Now, let s be a vertex of G such that $d(u,s) = d$ and s is adjacent to w_2 or w_4. If $s = w_1$ or $s = w_3$, then it is easy to see that $(AE)_{u,s} = 1$. On the other hand, since s is adjacent to the subtree rooted at u through $k-2$ different horizontal edges, it follows that, between the $k-1$ branches of the subtree rooted at u, there exists one sub-branch that is not adjacent to s through a horizontal edge. Let s_1 be the root of that sub-branch. Then, $d(s,s_1) = d+1$ and $d(u,s_1) = 1$, which implies $(A)_{u,s_1} = 1$ and $(E)_{s_1,s} = 1$. Let s_2 be the other vertex at distance $d+1$ from s. Because all neighbours of u, except s_1, are at distance smaller than $d+1$ from s, we have $(A)_{u,s_2} = 0$ and $(E)_{s_2,s} = 1$. Thus $(AE)_{u,s} = 1$. If s is a vertex of G such that $d(u,s) = d$ and s is not adjacent to w_2 or w_4, then the distance between s and the neighbours of u is $d-1$. In this case, $(AE)_{u,s} = 0$. If $d(u,s) \neq d$, then the distance between s and the neighbours of u is different from $d+1$, and therefore $(AE)_{u,s} = 0$. The required identity follows from summing up the above conclusions. □

Lemma 6. *Let $k \geqslant 6$ and $g = 2d \geqslant 6$, and let G be a (k,g)-graph of excess 4. If A is the adjacency matrix of G, E is the excess matrix of G and J is the all-ones matrix, then*

$$kJ = (A+kI)(H_{d-1}(A) + E).$$

Proof. By the definition of the polynomials $G_i(x)$ and using the fact that G has diameter $d+1$, we conclude $J = G_{d-1}(A) + A_d + E$. The relation (3), setting $i = d$, asserts $G_d(A) = G_{d-1}(A) + F_d(A)$. Substituting this identity in (4), where we fix $i = d-1$, we get $kG_{d-1}(A) + F_d(A) = (A+kI)H_{d-1}(A)$. Due to Lemma 5 the last identity is equivalent to $kG_{d-1}(A) + kA_d + kE = (A+kI)(H_{d-1}(A) + E)$. From $kJ = kG_{d-1}(A) + kA_d + kE$ follows $kJ = (A+kI)(H_{d-1}(A) + E)$. □

The next theorem gives a relationship between the eigenvalues of the matrices A and E (this result is an analogue of Theorem 3.1 in Delorme, Jørgensen, Miller and Villavicencio [8]).

Theorem 7. *If $\mu (\neq \pm k)$ is an eigenvalue of A, then*

$$H_{d-1}(\mu) = -\lambda,$$

where λ is an eigenvalue of E.

Proof. Let us suppose that μ is an eigenvalue of A. Since G is a k-regular graph, the all-ones matrix J is a polynomial in A. This implies that any eigenvector of A is also an eigenvector of J. From $kJ = (A+kI)(H_{d-1}(A)+E)$ and since $H_{d-1}(A)$ is also a polynomial in A, we have that E is a polynomial in A, and consequently, every eigenvector of A is an eigenvector of E. Therefore, the eigenvalues of kJ are of the form $(\mu+k)(H_{d-1}(\mu)+\lambda)$. As is well known, the eigenvalues of kJ are kn (with multiplicity 1) and 0 (with multiplicity $n-1$). The eigenvalue kn corresponds to $\mu = k$, and so all the remaining eigenvalues, except for $-k$, satisfy the above equation. □

Since the eigenvalues of a disjoint union of cycles are known, we are now in a position to determine the spectrum of A.

Lemma 8. *Let $k \geqslant 6$ and $g = 2d \geqslant 6$, and let G be a (k, g)-graph of excess 4. If A and E are, respectively the adjacency matrix and the excess matrix of G, then:*

(1) The matrix E is the adjacency matrix of a graph $G(E)$, consisting of a disjoint union of c cycles C_i of length l_i with $1 \leqslant i \leqslant c$. Moreover, if d is odd and V_1 and V_2 are the two partition sets of the bipartite graph G, then every cycle in $G(E)$ is completely contained either in V_1 or V_2.

(2) The spectrum of A consists of:

(2.1) $\pm k$, $c-2$ solutions of $H_{d-1}(x) = -2$, and one solution of each equation $H_{d-1}(x) = -2\cos(\frac{2\pi j}{l_i})$, for $j = 1, \ldots, l_i - 1, 1 \leqslant i \leqslant c$ and d odd.

(2.2) $\pm k$, $c-1$ solutions of $H_{d-1}(x) = -2$, and one solution of each equation (except one) $H_{d-1}(x) = -2\cos(\frac{2\pi j}{l_i})$, for $j = 1, \ldots, l_i - 1, 1 \leqslant i \leqslant c$ and d even.

Proof. (1) Our proof is analogous to that of Kovács [14] for girth 5, and Garbe's proof [12] for odd girth $g = 2k+1 > 5$. Let $f = \{u, v\}$ be a base edge of a bipartite Moore tree of G. Lemma 4 asserts that there exist exactly two vertices of G at distance $d+1$ from u. Namely, they are the excess vertices adjacent to the leaves of the subtree rooted at v. The excess matrix E is the adjacency matrix for the graph $G(E)$ with same vertex set V as G such that two vertices of $G(E)$ are adjacent if and only if they are at distance $d+1$. Because, for each vertex $u \in V(G)$, there are exactly two vertices at distance $d+1$ from u, every component of $G(E)$ is a cycle. Let c be the number of these cycles and let l_i, for $i = 1, \ldots, c$, be the lengths of these cycles ordered in an arbitrary manner. Moreover, if d is an odd number, any two vertices of G at distance $d+1$ lie in the same partite set. Therefore, any connected component of $G(E)$ is entirely contained either in V_1 or V_2.
(2) The eigenvalues of an n-cycle are known and are equal to $2\cos(\frac{2\pi j}{n})$, for $j = 0, \ldots, n-1$. Therefore the eigenvalues of $G(E)$ are $2\cos(\frac{2\pi j}{l_i})$, for $j = 0, 1, \ldots, l_i - 1$ and $1 \leqslant i \leqslant c$, (see Garbe [12]). Since G is a k-regular bipartite graph, it has (among others) the eigenvalues k and $-k$. Let V_1 and V_2 be the partition sets of G. Hence, the eigenvector of A corresponding to k consists of the all-ones vector j, and the eigenvector corresponding to $-k$ is the vector j' with values 1 on V_1 and values -1 on V_2. If d is an odd number, then two vertices of $G(E)$ are adjacent if and only if they are in the same partite set.

Therefore $E \cdot j' = 2j'$, which implies that from the set of c solutions on $H_{d-1}(x) = -2$, we need to subtract two multiplicities for the eigenvalues k and $-k$. If d is an even number, then two vertices of $G(E)$ are adjacent if and only if they are in different partite sets. Thus $E \cdot j' = -2j'$. In this case, from the set of c solutions on $H_{d-1}(x) = -2$, we need to subtract one multiplicity for the eigenvalue k and from the set of all solutions on $H_{d-1}(x) = 2$, we need to subtract one multiplicity for the eigenvalue $-k$. □

Lemma 9. *Let $k \geqslant 6$ and $g = 2d \geqslant 6$ and let G be a (k, g)-graph of excess 4. Let c be the number of cycles of $G(E)$ and c_2 be the number of cycles of even length. Then:*

(1) If $H_{d-1}(x) - 2$ is irreducible over $\mathbb{Q}[x]$, then $d - 1$ divides $c - 1$ or $c - 2$.

(2) If $H_{d-1}(x) + 2$ is irreducible over $\mathbb{Q}[x]$, then $d - 1$ divides $c_2 - 1$ or c_2.

Proof. (1) Combining Theorem 7 and Lemma 8 (2), we obtain that $H_{d-1}(x) - 2$ is an irreducible factor of the characteristic polynomial of A. Realizing that all the roots of an irreducible factor of a characteristic polynomial of a given rational symmetric matrix have the same multiplicities, (see Kovács [14]), from Lemma 8 (2) we have the following: If d is an even number, then the $d - 1$ roots of $H_{d-1}(x) - 2$ have multiplicity $\frac{c-1}{d-1}$, which has to be a positive integer. If d is odd, then the $d - 1$ roots have multiplicity $\frac{c-2}{d-1}$.
(2) This proof follows the same reasoning as (1). □

We can base the testing of irreducibility of $H_{d-1}(x) \pm 2$ on the well known Eisenstein's criterion that asserts for a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and a prime p that divides a_i for all $0 \leqslant i < n$, does not divide a_n and p^2 does not divide a_0. Now we are ready for the main result of this section.

Theorem 10. *Let $k(\geqslant 7)$ be an odd number and let $g = 2d \geqslant 8$. Let c be the number of cycles of $G(E)$ and c_2 be the number of cycles with even length. If there exists a (k, g)-graph of excess 4, then:*

(1) If d is an odd number, then $d - 1$ divides $c - 2$ and c_2.

(2) If d is an even number, then $d - 1$ divides $c - 1$ and $c_2 - 1$.

Proof. According to Lemma 9, it is enough to prove that the polynomials $H_{d-1}(x) - 2$ and $H_{d-1}(x) + 2$ are irreducible. We prove, using induction on $d \geqslant 4$, that $H_{d-1}(x) = x^{d-1} + (k-1)P_{d-3}(x)$, where $P_{d-3}(x)$ is an integer polynomial of degree $d-3$. We calculate $H_3(x) = x^3 - 2(k-1)x$. Let us suppose that the above formula holds for $H_{d-2}(x)$ and $H_{d-3}(x)$. That yields

$$\begin{aligned} H_{d-1}(x) = x(x^{d-2} + (k-1)P_{d-4}(x)) - (k-1)(x^{d-3} + (k-1)P_{d-5}(x)) = \\ = x^{d-1} + (k-1)P_{d-3}(x). \end{aligned}$$

Therefore, $H_{d-1}(x) \pm 2 = x^{d-1} + (k-1)P_{d-3}(x) \pm 2$. By the induction hypothesis, it follows that $H_{d-1}(0) = (-1)^{\frac{d-1}{2}}(k-1)^{\frac{d-1}{2}}$ for an odd d, and $H_{d-1}(0) = 0$ for an even d. Hence, for an odd $d(\geqslant 5)$ $|(-1)^{\frac{d-1}{2}}(k-1)^{\frac{d-1}{2}} \pm 2|$ is not divisible by 2^2, and clearly for an even $d(\geqslant 4)$, ± 2 is not divisible by 2^2. Since $k-1$ is even, it follows that every coefficient on $H_{d-1}(x) \pm 2$ except for the coefficient 1 of x^{d-1} is divisible by 2. Thus, the conditions of the Eisenstein's criterion are satisfied, and $H_{d-1}(x) \pm 2$ is irreducible. □

3 The non-existence of bipartite graphs of cyclic or bicyclic excess

In this section we deal with the same family of graphs considered in Section 2. Again, let $k \geqslant 6$ and $g = 2d \geqslant 6$, and let G be a (k, g)-graph of excess 4 and order n. Clearly, n is an even number. We proved that the excess graph $G(E)$ consists of a disjoint union of c cycles C_i, for $1 \leqslant i \leqslant c$. If $c = 1$ and $G(E)$ consists of an n-cycle, then G is of cyclic excess 4, and if $c = 2$ and $G(E)$ consists of a disjoint union of two cycles, then G is of bicyclic excess 4. These are the graphs we study in this section. Note that there is no graph G with cyclic excess 4 if d is an odd number; in this case, we showed that each cycle of $G(E)$ is completely contained either in V_1 or V_2.

Let d be an even number and let L_n be an n-cycle formed by the vertices of $G(E)$. If A' is the adjacency matrix of L_n, its characteristic polynomial $\chi(L_n, x)$ satisfies $\chi(L_n, x) = (x-2)(x+2)(R_n(x))^2$, where R_n is a monic polynomial of degree $\frac{n}{2} - 1$. Consider the factorization $x^n - 1 = \prod_{l|n} \Phi_l(x)$, where $\Phi_l(x)$ denotes the *l-th cyclotomic polynomial.* In the following paragraph, we summarize the properties of cyclotomic polynomials as listed in Delorme and Villavicencio [9].

The cyclotomic polynomial $\Phi_l(x)$ has integral coefficients, it is irreducible over $\mathbb{Q}[x]$, and it is *self-reciprocal* ($x^{\phi(l)}\Phi_l(1/x) = \Phi_l(x)$). From the irreducibility and the self-reciprocity of $\Phi_l(x)$ follows that the degree of $\Phi_l(x)$ is even for $l \geqslant 2$.

Thus, we obtain the following factorization of $R_n(x) : R_n(x) = \prod_{3\leqslant l|n} f_l(x)$, where f_l is an integer polynomial of degree $\frac{\phi(l)}{2}$ satisfying $x^{\phi(l)/2} f_l(x+1/x) = \Phi_l(x)$. Also, f_l is irreducible over $\mathbb{Q}[x]$, $f_3(x) = x+1, f_4(x) = x, f_5(x) = x^2 + x - 1$ and $f_6(x) = x - 1$. Substituting $y = -H_{d-1}(x)$ into $\frac{\chi(L_n,y)}{(y-2)}$, we obtain a polynomial $F(x)$ of degree $(n-1)(d-1)$, which satisfies $F(A)u = 0$ for each eigenvector u of A orthogonal to the all -one vector. Then, $F_{l,k,d-1}(x) = f_l(-H_{d-1}(x))$ yields

$$F(x) = (-H_{d-1}(x) + 2) \prod_{3\leqslant l|n} (F_{l,k,d-1}(x))^2.$$

Lemma 11. *Let $g = 2d > 6$, and $l \geqslant 3$ be a divisor of n. If there is a (k, g)-graph with cyclic excess 4 and order n, then $F_{l,k,d-1}(x)$ must be reducible over $Q[x]$.*

Proof. The degree of $F_{l,k,d-1}(x)$ is equal to $(d-1)\frac{\phi(l)}{2}$. If $F_{l,k,d-1}(x)$ is irreducible over $\mathbb{Q}[x]$, then all its roots must be eigenvalues of A. Employing Observation 3.1. from Delorme and Villavicencio [9], we conclude that there are at most $\phi(l)$ roots of $F_{l,k,d-1}(x)$ that are eigenvalues of A. Thus $(d-1)\frac{\phi(l)}{2}=\phi(l)$, that is, $d = 3$. This contradicts the assumption that $2d > 6$. □

Note that $deg(F_{l,k,d-1}(x)) = d-1$ if and only if $\phi(l) = 2$, that is, if and only if $l \in \{3,4,6\}$.

Lemma 12. *Let $k \geqslant 6$ and $g = 2d > 6$, and let n be the order of a (k,g)-graph with cyclic excess 4.*

(1) If $n \equiv 1 \pmod 3$, then $H_{d-1}(x) - 1$ must be reducible over $\mathbb{Q}[x]$.

(2) If $n \equiv 0 \pmod 4$, then $H_{d-1}(x)$ must be reducible over $\mathbb{Q}[x]$.

(3) If $n \equiv 0 \pmod 6$, then $H_{d-1}(x) + 1$ must be reducible over $\mathbb{Q}[x]$.

Proof. It follows directly from Lemma 11, with the additional assumptions $f_3(x) = x + 1, f_4(x) = x$ and $f_6(x) = x - 1$. □

If $n \equiv 0 \pmod 4$, then using the formula for the order of G, $d-1$ must be odd. On the other hand, since $H_1(x) = x, H_3(x) = x^3 - 2(k-1)x$ and $H_{d-1}(x) = xH_{d-2}(x) - (k-1)H_{d-3}(x)$, we see that if $d-1$ is an odd number, then x divides $H_{d-1}(x)$, which implies that $H_{d-1}(x)$ is reducible. Therefore, (2) holds.
The irreducibility of the polynomials $H_{d-1}(x) - 1$ over $\mathbb{Q}[x]$ is examined in Delorme, Jørgensen, Miller and Villavicencio [8], where it is analytically proven that these polynomials are irreducible for $d \in \{4,6,8\}$; and the paper contains a conjecture that if $d \geqslant 10$, then $H_{d-1}(x) - 1$ is irreducible. From the irreducibility of $H_{d-1}(x) - 1$, we obtain the main non-existence result of our paper.

Theorem 13. *If k and g satisfy one of the following conditions, there exists no (k,g)-graph of cyclic excess 4:*

(1) $k \equiv 1,2 \pmod 3$ and $g = 8$.

(2) $k \equiv 1 \pmod 3$ and $g = 12$.

(3) $k \equiv 1 \pmod 3$ and $g = 16$.

Proof. Because the order of the graphs is equal to

$$4 + 2\left(1 + (k-1) + \cdots + (k-1)^{(g-2)/2}\right),$$

we conclude $n \equiv 0 \pmod 3$. Since the polynomial $H_{d-1}(x) - 1$ is known to be irreducible for $d \in \{4,6,8\}$, we get a contradiction to (1) from Lemma 12. □

Remark 14. Since d is an even number, Theorem 10 asserts that $d-1$ divides $c-1$ and $c_2 - 1$. This claim is satisfied because $c = c_2 = 1$.

Next, let us consider graphs of bicyclic excess 4. In this case, we can assume an arbitrary (even or odd) d, as this case does not depend on the parity of d. So, let $G(E)$ be a graph consisting of a disjoint union of two cycles C_1 and C_2. If d is an odd number, then the vertex sets of the cycles C_1 and C_2 correspond to the partite sets V_1 and V_2,

respectively.
If $n \equiv 0 \pmod 4$, d is even, each edge of $C(E)$ has endpoints in V_1 and V_2. Therefore, each of the cycles has even length, that is, $c_2 = 2$. Furthermore, $k - 1$ must be odd. Unfortunately, this will not help us in excluding any family of pairs (k, g) for which G does not exist. In fact, for an odd $d - 1$ and an odd $k - 1$, we cannot conclude the irreducibility of $H_{d-1}(x) + 2$, thus, we cannot employ Lemma 9.
If $n \equiv 2 \pmod 4$ and d is odd, then the lengths of C_1 and C_2 are equal to $\frac{n}{2}$ (clearly, $n = 2s + 1$ is odd). Therefore $c_2 = 0$, and $d - 1$ divides $c - 2$ and c_2.

The main result about the non-existence of graphs G with bicyclic excess 4 is given in the following theorem.

Theorem 15. *If $k(\geqslant 7)$ is odd and $g = 2d \geqslant 8$, where d is an even integer, then there exists no (k, g)-graph with bicyclic excess 4.*

Proof. We have $c = 2$. Theorem 10 implies that $d - 1$ divides $c - 1$, which is a contradiction. □

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