Minimal orbits of promotion

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Abstract

We give a bijection between the symmetric group S_n, and the set of standard Young tableaux of rectangular shape m^n, $m \geqslant n$, that have order n under *jeu de taquin* promotion.

1 Introduction

Fix positive integers $m \geqslant n$, and let $\square$ be either the $m \times n$ rectangle, or the $n \times m$ rectangle. The **promotion** map $\partial : \mathsf{SYT}(\square) \to \mathsf{SYT}(\square)$ defines an action of $(\mathbb{Z}, +)$ on the set of standard Young tableaux of shape $\square$. For $T \in \mathsf{SYT}(\square)$, ∂T is computed by deleting the entry 1 from T, decrementing each entry by 1, rectifying, and finally adding an entry mn in the lower-right corner. For example,

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline & 1 & 4 \\ \hline 2 & 3 & 5 \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} = \partial T\,.$$

Interest in this action stems from its connections to geometry and representation theory, and its striking combinatorial properties (see [2, 9, 11, 15, 16]).

Let $\mathcal{O}_r := \{T \in \mathsf{SYT}(\square) \mid \partial^r T = T\}$ denote the set of tableaux whose order under promotion divides r. By a theorem of Haiman [5], $\partial^{mn} T = T$ for all $T \in \mathsf{SYT}(\square)$; hence $\mathcal{O}_r$ is empty if r is coprime to mn. It is also not hard to show that $\mathcal{O}_r$ is empty for $r < n$; this is implicit in the proof of Theorem 3. The minimal orbits of promotion, therefore, have order n.

The action of promotion on $\mathsf{SYT}(\square)$ exhibits a cyclic sieving phenomenon, as defined in [14]: we have $|\mathcal{O}_r| = F(\zeta^r)$, where $F(q)$ is a q-analogue of the hook length formula

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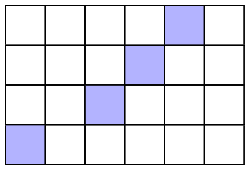

Figure 1: An example of a diagonal of $\square$: here $n = 4$, $m = 6$, $\lambda_+ = 5431$ and $\lambda_- = 432$.

for $|\mathsf{SYT}(\square)|$, and ζ is a primitive $(mn)^{\text{th}}$ root of unity. The quantity $F(\zeta^r)$ appears in a number of other places in representation theory and combinatorics, which proffers a variety of avenues of proof for this cyclic sieving theorem. It was first proved by Rhoades using Kazhdan-Lusztig theory [11]. Subsequently other proofs were found using representation theory of SL_n [16], and the geometry of the Grassmannian [9] and the affine Grassmannian [2]. Simpler, more combinatorial proofs are known in special cases when $n = 2, 3$ [8]. The survey [12] discusses of a number of related results. However, at present there is no known combinatorial proof in general, nor any proof that gives an *effective* description of the sets $\mathcal{O}_r$.

The purpose of this paper is to give an explicit combinatorial construction of the orbits in $\mathcal{O}_n$, i.e. the minimal orbits of promotion. We will assume that the reader is familiar with basic definitions from the combinatorial theory of Young tableau, such as Schensted insertion, *jeu de taquin* operations (sliding, reverse-sliding, rectification), Knuth equivalence and dual equivalence; all of these may be found in [3].

Using Rhoades' cyclic sieving theorem, one can compute that $|\mathcal{O}_n| = n!$. Our main result gives a bijection between the symmetric group S_n and $\mathcal{O}_n$. Under this bijection promotion corresponds to right-multiplication by the n-cycle $(1\ n\ n{-}1\ \ldots\ 2)$. There are a number of arbitrary choices involved in constructing the bijection, and much of the proof is concerned with showing that the construction is in fact well-defined.

To begin, choose a skew shape $\lambda_+/\lambda_- \subset \square$, consisting of n boxes $\square_1, \square_2, \ldots, \square_n$, such that $\square_{i+1}$ is strictly above and strictly right of $\square_i$, for $i = 1, \ldots, n-1$. We call λ_+/λ_- a **diagonal** of $\square$, (see Figure 1). For each permutation $w \in S_n$, we define a tableau $T_w^{\lambda_+}$ of shape λ_+, using a procedure similar to rectification. In the following algorithm, T is a tableau under construction. If $\square$ is a box of λ_+, we write $\square \in \lambda_+$, and $T[\square]$ denotes the entry of T in box $\square$.

Algorithm A. *INPUT:* A permutation $w \in S_n$.

Begin with $T[\square_i] := w(i)$, for $i = 1, \ldots, n$, leaving all boxes of λ_- unfilled;
while shape$(T) \neq \lambda_+$ **do**
  Let $\mu \subset \lambda_+$ be the unfilled boxes to the left of T;
  Choose any corner box $\square \in \mu$;
  Let T' be the tableau obtained by sliding $\square$ through T;
  If the final position of the sliding path is $\square_i$, then set $T'[\square_i] := T[\square_i] + n$;
  Set $T := T'$;
end while
return the resulting tableau, $T_w^{\lambda_+} := T$.

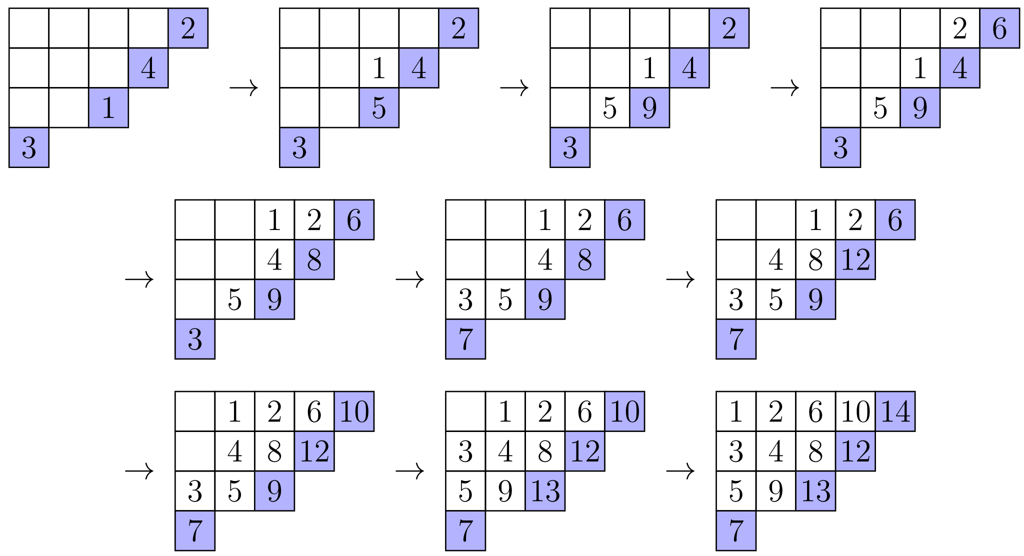

Figure 2: Construction of $T_w^{\lambda_+}$, with $w = 3142$, and $\lambda_+ = 5431$.

An example of Algorithm A is given in Figure 2. Note that in this example, $T_w^{\lambda_+}$ is not a standard Young tableau, since the entries are not $\{1, \dots, |\lambda_+|\}$.

The key difference is between Algorithm A and rectification is that as we slide empty boxes of λ_- through T, we are also refilling the boxes of λ_+/λ_-. If we were not refilling these boxes, we would be performing ordinary rectification on a tableau with reading word w, which produces the insertion tableau of w. Since this difference involves only entries greater that n, the insertion tableau of w is the subtableau $T_w^{\lambda_+}$ formed by entries $1, \dots, n$.

Theorem 1. *The definition of $T_w^{\lambda_+}$ is independent of the choices in Algorithm A.*

Theorem 1 is analogous to the well known fact that ordinary rectification is well-defined [13]. However, despite the apparent similarity between Algorithm A and rectification, one cannot easily deduce one fact from the other. We discuss some of the difficulties in Section 5.

Similarly, we define a tableau $T_w^{\square/\lambda_-}$ of shape $\square/\lambda_-$, using reverse slides.

Algorithm B. *INPUT:* A permutation $w \in S_n$.

Begin with $T[\square_i] := w(i) + (m-1)n$, for $i = 1, \dots, n$, and all boxes of $\square/\lambda_+$ unfilled;
while shape$(T) \neq \square/\lambda_-$ **do**
  Let $\mu \subset \square/\lambda_-$ be the unfilled boxes to the right of T;
  Choose any corner box $\square$ of μ;
  Let T' be the tableau obtained by reverse-sliding $\square$ through T;
  If final position of the sliding path is $\square_i$, then set $T'[\square_i] := T[\square_i] - n$;
  Set $T := T'$;

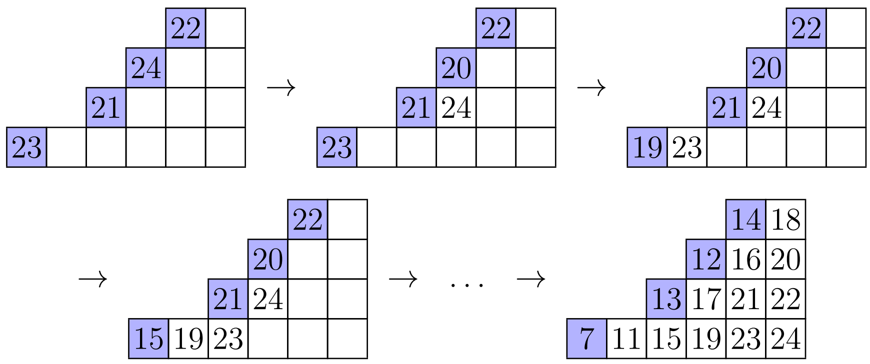

Figure 3: Construction of $T_w^{\Box/\lambda_-}$, with $w = 3142$, $m = 6$, and $\lambda_+ = 5431$.

end while
return the resulting tableau, $T_w^{\Box/\lambda_-} := T$.

An example is given in Figure 3. Since Algorithm B is essentially "Algorithm A turned upside-down", Theorem 1 implies that the definition of $T_w^{\Box/\lambda_-}$ is independent of choices.

We combine these two constructions to produce a tableau T_w of shape $\Box$: for each box $\square \in \Box$, let

$$T_w[\square] := \begin{cases} T_w^{\lambda_+}[\square] & \text{if } \square \in \lambda_+ \\ T_w^{\Box/\lambda_-}[\square] & \text{otherwise.} \end{cases}$$

For example, for $w = 3142$, $m = 6$, we combine the tableaux in Figures 2 and 3 to obtain

$$T_w = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 10 & 14 & 18 \\ \hline 3 & 4 & 8 & 12 & 16 & 20 \\ \hline 5 & 9 & 13 & 17 & 21 & 22 \\ \hline 7 & 11 & 15 & 19 & 23 & 24 \\ \hline \end{array}\,.$$

Since the definition of T_w is piecewise, it is not immediately clear that this is always a sensible construction. We will show that the constructions in Algorithms A and B agree on the diagonal, i.e. $T_w^{\lambda_+}[\square_i] = T_w^{\Box/\lambda_-}[\square_i]$, for $i = 1, \ldots, n$. This is the first step in proving:

Theorem 2. *T_w is a standard Young tableau. Moreover, the definition of T_w is independent of the choice of diagonal λ_+/λ_-.*

Our main result states that this construction gives the minimal orbits of promotion.

Theorem 3. *The map $w \mapsto T_w$ defines a bijection between S_n and $\mathcal{O}_n$. Specifically the following hold:*

(i) *For all $w \in S_n$, $\partial T_w = T_{wc}$, where $c = (1\ n\ n{-}1\ \ldots\ 2)$. In particular $T_w \in \mathcal{O}_n$.*

(ii) *If $w, w' \in S_n$ and $w(i) \neq w'(i)$, then $T_w[\square_i] \neq T_{w'}[\square_i]$. In particular $w \mapsto T_w$ is injective.*

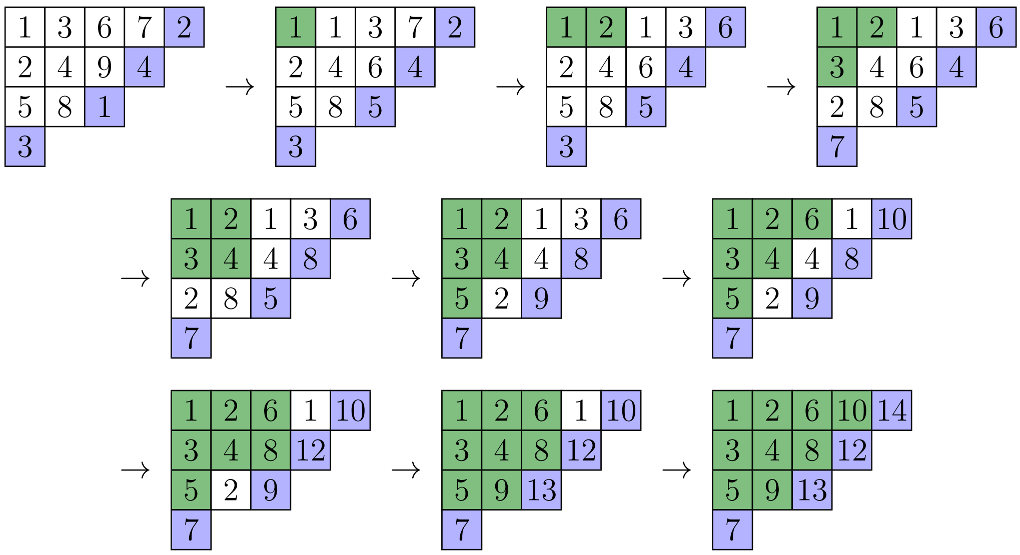

Figure 4: An alternative way to compute $T^{\lambda_+}_{3142}$, using the same order for the boxes of λ_- as the example in Figure 2.

(iii) For each $T \in \mathcal{O}_n$, consider the function $w : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $w(i) \equiv T[\square_i] \pmod{n}$, for $i = 1, \ldots, n$. We have $w \in S_n$, and $T_w = T$. In particular $w \mapsto T_w$ is surjective.

The rest of this paper is organized as follows. In Section 2 we develop a reduction strategy for proving Theorems 1 and 2. This strategy is implemented in Section 3, where we prove two lemmas: the first reducing the problem to one we can solve, and the second solving it. All three theorems are proved in Section 4. Finally, in Section 5 we discuss some additional facts that are true, and some that we would like to be true.

2 Strategy

To prove Theorem 1, we need to formulate it in a different way. As with ordinary rectification, each possible sequence of choices of boxes in Algorithm A can be encoded by a standard Young tableau $U \in \mathsf{SYT}(\lambda_-)$, by putting $U[\square] := |\lambda_-| + 1 - k$ if $\square$ is the box chosen in the k^{th} iteration of the loop. Let T^U_w denote the result of applying Algorithm A with order of boxes encoded by U. For example, Figure 2 computes T^U_{3142}, with

$$U = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 7 \\ \hline 2 & 4 & 9 & \\ \cline{1-3} 5 & 8 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}\ .$$

Theorem 1 states that T^U_w is independent of $U \in \mathsf{SYT}(\lambda_-)$.

The steps of a rectification-type algorithm can be performed in a variety of different but equivalent orders. In particular, instead of sliding the entries of U through T, from largest to smallest, one can reverse-slide the entries of T though U, from smallest to largest, (see [1] for full details). Figure 4 illustrates this in the context of Algorithm A, using the example from Figure 2.

This perspective not only gives a reformulation of Algorithm A, but allows us generalize it to inputs that are not permutations. Let $\sigma = \sigma_1\sigma_2\sigma_3\dots$ be an infinite sequence, with $\sigma_k \in \{1,\dots,n\}$ for $k = 1,2,3,\dots$. We construct a sequence $\boldsymbol{\square}^U_\sigma = \square_1\,\square_2\,\square_3\,\dots$ of boxes of λ_+, as follows.

Algorithm C. *INPUT:* The pair (σ, U).

Let $U_0 := U$;
for $k = 1,2,3,\dots$ **do**
  Let U_k be the tableau obtained by reverse-sliding box $\blacksquare_{\sigma_k}$ through U_{k-1};
  Define $\square_k$ to be the final position of the sliding path;
  Delete the entry in $\blacksquare_{\sigma_k}$ from U_k, if one exists;
end for
return $\boldsymbol{\square}^U_\sigma := \square_1\,\square_2\,\square_3\,\dots$.

We use this sequence to define a function $\delta^U_\sigma : \{1,\dots,n\} \to \mathbb{Z}_{\geqslant 0}$,

$$\delta^U_\sigma(i) := \#\{k \mid \sigma_k = i \text{ and } \square_k \neq \blacksquare_i\}\,,$$

where $\boldsymbol{\square}^U_\sigma = \square_1\,\square_2\,\square_3\,\dots$. This function will be key in proving Theorem 2. If $\boldsymbol{\square}^U_\sigma$ is independent of $U \in \mathsf{SYT}(\lambda_-)$, we write $\boldsymbol{\square}^{\lambda_+}_\sigma := \boldsymbol{\square}^U_\sigma$, and $\delta^{\lambda_+}_\sigma := \delta^U_\sigma$.

Strictly speaking, Algorithm C is not a proper algorithm, in that it does not terminate; however, we are really only interested in a finite part of $\boldsymbol{\square}^U_\sigma$. Once k is sufficiently large, we have $\square_k = \blacksquare_{\sigma_k}$. For $N \geqslant 0$, denote the truncation of a sequence at its N^{th} term by $\mathrm{Trunc}_N(a_1a_2a_3\dots) := (a_1a_2\dots a_N)$.

For $w \in S_n$, define w^* to be the repeating sequence

$$w^* := a_1a_2\dots a_n\,a_1a_2\dots a_n\,a_1a_2\dots,$$

where $a_1a_2\dots a_n$ is the word representing w^{-1} in one line notation (i.e. $a_i = w^{-1}(i)$ for $i = 1,\dots,n$). The following proposition precisely states the relationship between Algorithms A and C.

Proposition 4. *Write* $\boldsymbol{\square}^U_{w^*} = \square_1\,\square_2\,\square_3\,\dots$. *For each box* $\square \in \lambda_+$, $T^U_w[\square]$ *is the smallest* k *such that* $\square_k = \square$. *Hence, for some* $N \geqslant 0$, $\mathrm{Trunc}_N\big(\boldsymbol{\square}^U_{w^*}\big)$ *determines* T^U_w. *In addition, we have* $T^U_w[\blacksquare_i] = w(i) + n \cdot \delta^U_{w^*}(i)$.

Proof. The first two statements are simply a more precise formulation of the remarks at the beginning of this section. For the third, note that the k^{th} term of w^* is equal to i, for $k = w(i), w(i)+n, w(i)+2n, \dots$. By the definition of δ^U_σ, the $(\delta^U_{w^*}+1)^{\text{th}}$ term in this sequence is the smallest k such that $\square_k = \blacksquare_i$. The former is $w(i) + n\cdot\delta^U_{w^*}(i)$, and the latter is $T^U_w[\blacksquare_i]$. $\square$

Partially order the boxes of λ_+: let $\square \leqslant \square'$ if $\square'$ is both weakly right of $\square$ and weakly above $\square$.

Proposition 5. *Write* $\boldsymbol{\square}^U_\sigma = \square_1 \square_2 \square_3 \dots$ *. If* $\sigma_k < \sigma_{k+1}$ *then* $\square_k < \square_{k+1}$*; if* $\sigma_k > \sigma_{k+1}$ *then* $\square_k > \square_{k+1}$.

Proof. This follows from the fact that *jeu de taquin* preserves horizontal (and vertical) strips. $\square$

For a sequence $a_1 a_2 a_3 \dots$ with terms from a partially ordered set (e.g. the numbers $\{1, \dots, n\}$ or the boxes of λ_+), define the **strict Knuth transformations** to be the operations

$$\kappa_k(a_1 a_2 a_3 \dots) = a_1 a_2 \dots a_{k-1} xyz a_{k+3} a_{k+4} \dots,$$

where

$$xyz = \begin{cases} a_k a_{k+2} a_{k+1} & \text{if } \ a_{k+1} < a_k < a_{k+2} \quad \text{or} \quad a_{k+2} < a_k < a_{k+1} \\ a_{k+1} a_k a_{k+2} & \text{if } \ a_k < a_{k+2} < a_{k+1} \quad \text{or} \quad a_{k+1} < a_{k+2} < a_k \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In the third case $\kappa_k(a_1 a_2 a_3 \dots)$ is also undefined. These are similar to elementary Knuth transformations on sequences, except that the inequalities are required to be strict. We define two sequences $a_1 a_2 a_3 \dots$ and $b_1 b_2 b_3 \dots$ to be **equivalent** if for every $N \geqslant 0$, there exists a finite sequence $\kappa_{k_1}, \kappa_{k_2}, \dots, \kappa_{k_m}$ of strict Knuth transformations such that

$$\mathrm{Trunc}_N\big(\kappa_{k_1} \circ \kappa_{k_2} \circ \dots \circ \kappa_{k_m}(a_1 a_2 a_3 \dots)\big) = \mathrm{Trunc}_N\big(b_1 b_2 b_3 \dots\big)\,.$$

When $a_1 a_2 a_3 \dots$ is a sequence of boxes, this generalizes of the notion of dual equivalence on tableaux [5].

Proposition 6. *Let* σ *be a sequence with terms from* $\{1, \dots, n\}$*. If* $\kappa_k(\sigma)$ *is defined, then* $\kappa_k(\boldsymbol{\square}^U_\sigma)$ *is defined, and*

(i) $\boldsymbol{\square}^U_{\kappa_k(\sigma)} = \kappa_k(\boldsymbol{\square}^U_\sigma)$*;*

(ii) $\delta^U_{\kappa_k(\sigma)}(i) = \delta^U_\sigma(i)$*, for* $i = 1, \dots, n$*.*

Proof. Write $\widehat{\sigma} := \kappa_k(\sigma)$. Let $\boldsymbol{\square}^U_\sigma = \square_1 \square_2 \square_3 \dots$, and let $U_0, U_1, U_2, \dots$ be the sequence of tableaux produced in Algorithm C. Let $\boldsymbol{\square}^U_{\widehat{\sigma}} = \widehat{\square}_1 \widehat{\square}_2 \widehat{\square}_3 \dots$, and $\widehat{U}_0, \widehat{U}_1, \widehat{U}_2, \dots$ be the corresponding objects for $\widehat{\sigma}$. Since $\sigma_j = \widehat{\sigma}_j$ for $j < k$, we have $\square_j = \widehat{\square}_j$ and $U_j = \widehat{U}_j$, for $j < k$. In particular $U_{k-1} = \widehat{U}_{k-1}$. Given 3 boxes $\square, \square', \square''$ let $T(\square, \square', \square'')$ denote the standard Young tableau with entries 1, 2, and 3, in boxes $\square$, $\square'$ and $\square''$ respectively. The pair of tableaux $T(\square_{\sigma_k}, \square_{\sigma_{k+1}}, \square_{\sigma_{k+2}})$ and $T(\square_{\widehat{\sigma}_k}, \square_{\widehat{\sigma}_{k+1}}, \square_{\widehat{\sigma}_{k+2}})$ form a dual equivalence class. It follows from [5, Corollary 2.8] that $U_{k+2} = \widehat{U}_{k+2}$, and since $\sigma_j = \widehat{\sigma}_j$ for $j > k+2$ we have $\square_j = \widehat{\square}_j$ for $j > k+2$. By [5, Lemma 2.3], $T(\square_k, \square_{k+1}, \square_{k+2})$ and $T(\widehat{\square}_k, \widehat{\square}_{k+1}, \widehat{\square}_{k+2})$ also form a dual equivalence class, which implies that the three-term sequences $(\square_k, \square_{k+1}, \square_{k+2})$ and $(\widehat{\square}_k, \widehat{\square}_{k+1}, \widehat{\square}_{k+2})$ are related by a strict Knuth transformation. This proves (i), and (ii) is straightforward. $\square$

This leads to our strategy for proving Theorems 1 and 2, which is outlined in the next proposition. For the second statement in Theorem 2 we will need to consider how the constructions in Algorithms A and B are related for different choices of diagonal. This is facilitated by the following definition. Let λ'_+/λ'_- be another diagonal of $\square$. Let $\square_1 \square_2 \square_3 \ldots$ be a sequence of boxes of λ_+, and let $\square'_1 \square'_2 \square'_3 \ldots$ be a sequence of boxes of λ'_+. We say that these sequences are **compatible** if $\square'_k = \square_k$ whenever $\square'_k \in \lambda_+$ and $\square_k \in \lambda'_+$.

Proposition 7. *Suppose that σ is equivalent to w^*, and $\blacksquare^{\lambda_+}_\sigma$ is well-defined (i.e. $\blacksquare^U_\sigma$ is independent $U \in \mathsf{SYT}(\lambda_-)$). Then the following are true.*

(i) $T^{\lambda_+}_w$ is well-defined (i.e. T^U_w is independent of $U \in \mathsf{SYT}(\lambda_-)$).

(ii) $T^{\lambda_+}_w[\square_i] = w(i) + n \cdot \delta^{\lambda_+}_\sigma(i)$.

(iii) Suppose λ'_+ is obtained from λ_+ by adding one box. If $\blacksquare^{\lambda'_+}_\sigma$ is well-defined and compatible with $\blacksquare^{\lambda_+}_\sigma$, then the tableaux $T^{\lambda_+}_w$ and $T^{\lambda_+{}'}_w$ coincide on λ_+.

Proof. Let $U, \widehat{U} \in \mathsf{SYT}(\square)$. To prove (i), we must show that $T^U_w = T^{\widehat{U}}_w$. By Proposition 4, there exists $N \geqslant 0$ so that T^U_w and $T^{\widehat{U}}_w$ are determined by $\mathrm{Trunc}_N(\blacksquare^U_{w^*})$ and $\mathrm{Trunc}_N(\blacksquare^{\widehat{U}}_{w^*})$; therefore it is enough to show that the latter two are equal. Since w^* is equivalent to σ there is a sequence $\kappa_{k_1}, \kappa_{k_2}, \ldots, \kappa_{k_M}$ of strict Knuth transformations such that

$$\mathrm{Trunc}_N\big(\kappa_{k_1} \circ \kappa_{k_2} \circ \cdots \circ \kappa_{k_M}(\sigma)\big) = \mathrm{Trunc}_N\big(w^*\big)$$

Since $\blacksquare^{\lambda_+}_\sigma = \blacksquare^U_\sigma = \blacksquare^{\widehat{U}}_\sigma$, by Proposition 6(i) we have

$$\mathrm{Trunc}_N\big(\blacksquare^U_{w^*}\big) = \mathrm{Trunc}_N\big(\kappa_{k_1} \circ \kappa_{k_2} \circ \cdots \circ \kappa_{k_M}(\blacksquare^{\lambda_+}_\sigma)\big) = \mathrm{Trunc}_N\big(\blacksquare^{\widehat{U}}_{w^*}\big)\,,$$

as required. Similarly, (ii) follows from Proposition 4 and Proposition 6(ii). For (iii), it is easy to see that $\blacksquare^{\lambda_+}_\sigma$ is compatible with $\blacksquare^{\lambda'_+}_\sigma$ if and only if $\kappa_k(\blacksquare^{\lambda_+}_\sigma)$ is compatible with $\kappa_k(\blacksquare^{\lambda'_+}_\sigma)$. By Proposition 6(i), $\blacksquare^{\lambda_+}_{w^*}$ is compatible with $\blacksquare^{\lambda'_+}_{w^*}$; the result follows by Proposition 4. $\square$

In the next section we will construct a suitable σ for each permutation w, enabling us to prove Theorems 1 and 2. The construction of σ is based on the *cyclage* operation of Lascoux and Schützenberger [7]. The following two facts will be used to establish equivalence:

Proposition 8. *Let $w, \widehat{w} \in S_n$ be two permutations. If w^{-1} and $\widehat{w}^{-1}$ have the same insertion tableau, then w^* is equivalent to $\widehat{w}^*$.*

Proof. Fix $N \geqslant 0$. Let $a_1 a_2 \ldots a_n$ and $b_1 b_2 \ldots b_n$ be the words representing w^{-1} and $\widehat{w}^{-1}$ respectively. Since these words have the same insertion tableau, they are related by a finite sequence of elementary Knuth transformations, and since $a_i \neq a_j$ for $i \neq j$, each of these is a strict Knuth transformation. It follows that w^* can be transformed into any sequence of the form

$$b_1 b_2 \ldots b_n \, b_1 b_2 \ldots b_n \;\cdots\; b_1 b_2 \ldots b_n \, a_1 a_2 \ldots a_n \, a_1 a_2 \ldots a_n \, a_1 a_2 \;\ldots$$

using a finite sequence of strict Knuth transformations. If there are at least N/n copies of $b_1 b_2 \ldots b_n$, then truncating at the N^{th} term gives $\mathrm{Trunc}_N(\widehat{w}^*)$, as required. $\square$

Proposition 9. *Let $a_1a_2a_3\dots$ and $b_1b_2b_3\dots$ be positive integer sequences. Let A_k be the insertion tableau of the finite word $a_1a_2\dots a_k$, and let B_k be the insertion tableau of $b_1b_2\dots b_k$. Suppose there exists a number M such that the following hold: $A_M = B_M$; $a_k = b_k$ for all $k > M$; and A_k and B_k are row-strict for all $k \leqslant M$. Then $a_1a_2a_3\dots$ is equivalent to $b_1b_2b_3\dots$.*

Proof. Let $A_{k,1}A_{k,2}\dots A_{k,k}$ denote the reading word of A_k. For $k \leqslant M$ there is a finite sequence of elementary Knuth transformations taking

$$A_{k-1,1}A_{k-1,2}\dots A_{k-1,k-1}a_k \;\mapsto\; A_{k,1}A_{k,2}\dots A_{k,k-1}A_{k,k}\,;$$

the precise sequence can be found in many references (e.g. [3, Section 2.1] or [6, Section 6.1]). It is easy to verify that if A_k is row-strict, then all of the transformations in this sequence are *strict* Knuth transformations. This shows that $a_1a_2a_3\dots$ is equivalent to

$$A_{M,1}A_{M,2}\dots A_{M,M}a_{m+1}a_{m+2}\dots = B_{M,1}B_{M,2}\dots B_{M,M}b_{m+1}b_{m+2}\dots$$

which, by the same argument, is equivalent to $b_1b_2b_3\dots$. ☐

3 Descent sequences

Recall that $i \in \{1,\dots,n-1\}$ is a **descent** of w if $w(i) > w(i+1)$; if $w(i) < w(i+1)$, then i is an **ascent** of w. Let $\mathrm{id} \in S_n$ denote the identity element, and let $w_0 \in S_n$ denote the long element, $w_0 = n\,n{-}1\,\dots\,2\,1$. The **major index** of w is defined to be the sum of the descents of w. For example, id is the unique permutation with major index 0, w_0 is the unique permutation with major index $n(n-1)/2$.

Lemma 10. *Let $d_1 > d_2 > \dots > d_t$ be the descents of w in decreasing order, and let $d_i = 0$ for $i > t$. Then w^* is equivalent to the sequence*

$$\sigma_{d_1d_2d_3\dots} := (d_1{+}1, d_1{+}2, \dots, n,\; d_2{+}1, d_2{+}2, \dots, n,\; d_3{+}1, d_3{+}2, \dots, n,\; \dots)\,. \qquad (1)$$

Proof. For a permutation $w \in S_n$ define $\epsilon_w \in \{1,\dots,n\}$ and permutations $\widehat{w}, w' \in S_n$ as follows. Let $b_1b_2\dots b_n$ be the reading word of the insertion tableau of w^{-1}. Let $\epsilon_w := b_1$. Let $\widehat{w}$ and w' be the permutations whose inverses are represented by the words $b_1b_2\dots b_n$ and $b_2b_3\dots b_nb_1$, respectively, in one line notation. Thus

$$\big(\epsilon_w, (w')^*\big) = b_1b_2\dots b_n\,b_1b_2\dots b_n\,b_1b_2\dots = \widehat{w}^*\,.$$

By Proposition 8, w^* is equivalent to $\widehat{w}^* = \big(\epsilon_w, (w')^*\big)$. Using this argument repeatedly, w^* is equivalent to the sequence

$$\big(\epsilon_w, \epsilon_{w'}, \epsilon_{w''}, \dots, \epsilon_{w^{(M-1)}}, (w^{(M)})^*\big)$$

for any $M \geqslant 0$. Since Knuth transformations preserve the descents of the inverse of a permutation, the descents of w are the same as descents of $\widehat{w}$.

Let $M(w)$ be the major index of w_0ww_0, which is equal to $(n-d_1)+\cdots+(n-d_t)$. Suppose $w \neq \mathrm{id}$. Then $\epsilon_w > 1$. Since $\widehat{w}(\epsilon_w) = 1$, and $w'(\epsilon_w) = n$, $\epsilon_w{-}1$ is a descent of w and an ascent of w'. If $\epsilon_w < n$, then ϵ_w is an ascent of w and a descent of w'. For $i \notin \{\epsilon_w{-}1, \epsilon_w\}$, i is a descent of w if and only if i is a descent of w'. It follows from these remarks that $M(w') = M(w) - 1$.

Since $M(w^{(M(w))}) = 0$, $w^{(M(w))} = \mathrm{id}$. Thus we have shown w^* is equivalent $(w^\#, \mathrm{id}^*)$, where

$$w^\# := \left(\epsilon_w, \epsilon_{w'}, \epsilon_{w''}, \ldots, \epsilon_{w^{(M(w)-1)}}\right).$$

We now show, by induction on $M(w)$, that $(w^\#, \mathrm{id}^*)$ is equivalent to $\sigma_{d_1d_2d_3\ldots}$. If $w = \mathrm{id}$, the result is trivial. Suppose $M(w) > 1$ and assume the result is true for w'. Then $(w^\#, \mathrm{id}^*) = (\epsilon_w, (w')^\#, \mathrm{id}^*)$ is equivalent to $(\epsilon_w, \sigma_{d'_1d'_2d'_3\ldots})$, where $d'_1 > \cdots > d'_{t'}$ are the descents of w', and $d'_i = 0$ for $i > t'$. The arguments above show that $\epsilon_w = d_s{+}1$ for some $s \leqslant t$; if $\epsilon_w < n$, then $d'_i = d_i$ for $i \neq s$, and $d'_s = \epsilon_w$; if $\epsilon_w = n$, then $d'_i = d_{i+1}$ for all i. In either case, $\sigma_{d'_1d'_2d'_3\ldots}$ is obtained from $\sigma_{d_1d_2d_3\ldots}$ by deleting the first occurrence of ϵ_w. With this in mind, it follows readily from Proposition 9 that $(\epsilon_w, \sigma_{d'_1d'_2d'_3\ldots})$ is equivalent to $\sigma_{d_1d_2d_3\ldots}$. □

Lemma 11. *Assume that n is the number of columns of $\Box$. Let $\sigma = \sigma_{d_1d_2d_3\ldots}$ be the sequence in* (1).

(i) $\blacksquare^{\lambda_+}_\sigma$ is well-defined: the k^{th} box of this sequence is in column c_k, where

$$c_1c_2c_3\ldots := (1, 2, \ldots, n{-}d_1,\ 1, 2, \ldots, n{-}d_2,\ 1, 2, \ldots, n{-}d_3,\ \ldots).$$

(ii) If C_i is the length of the i^{th} column of λ_+, then

$$\delta^{\lambda_+}_\sigma(i) = C_i - \#\{j \mid d_j \geqslant i\} + \#\{j \mid d_j \geqslant n{+}1{-}i\} - 1 \qquad \text{for } i = 1, \ldots, n.$$

(iii) For any other diagonal λ'_+/λ'_- of $\Box$, $\blacksquare^{\lambda'_+}_\sigma$ is compatible with $\blacksquare^{\lambda_+}_\sigma$.

Proof. We introduce the notation

$$[i,j] := i + \sum_{s=1}^{j-1}(n - d_s), \qquad \text{for } 1 \leqslant i \leqslant n{-}d_j \text{ and } j \geqslant 1.$$

Thus $c_{[i,j]} = i$ and $\sigma_{[i,j]} = d_j{+}i$ for all i, j. Fix $U \in \mathsf{SYT}(\lambda_-)$, and write $\blacksquare^U_\sigma = \Box_1\,\Box_2\,\Box_3\ldots$. Suppose $\Box_k$ is in column e_k. To prove (i), we need to show that $e_{[i,j]} = i$ for all i, j. We will do this by strong induction on j.

Fix $j \geqslant 0$. Assume that $e_{[i,s]} = i$ for $1 \leqslant s \leqslant j$, $1 \leqslant i \leqslant n - d_s$. Let $d_0 := n$, $j' := j{+}1$, $p := n - d_j$, $p' := n - d_{j'}$. We will prove the following:

(a) $e_{[i,j']} \geqslant i$ for $1 \leqslant i \leqslant p'$;

(b) $e_{[i,j']} = e_{[i-1,j']} + 1$ for $p < i \leqslant p'$ (where $e_{[0,1]} := 0$);

(c) $e_{[i,j']} \leqslant e_{[i,j]}$ for $1 \leqslant i \leqslant p$.

These imply that $e_{[i,j']} = i$ for $i = 1, \ldots, p'$.

Since

$$\sigma_{[1,j']}\sigma_{[2,j']} \ldots \sigma_{[p',j']} = (d_{j'}{+}1 < d_{j'}{+}2 < \ldots < n)\,,$$

by Proposition 5 we have $\square_{[1,j']} < \square_{[2,j']} < \cdots < \square_{[p',j']}$. Because of the order in which the slides are performed, $\square_{[i+1,j']}$ cannot be above $\square_{[i,j]}$ and in the same column. It follows that $e_{[1,j']} < e_{[2,j']} < \cdots < e_{[p',j']}$, which proves (a). Since all boxes $\square_1, \ldots, \square_{[p,j]}$ are in the first p columns, any box $\square_{[i,j']}$ which is not in the first p columns must be in the first row. In particular this applies when $i > p$, which proves (b).

If $p' = n$, then (c) follows immediately from (a). To complete the proof of (i), suppose that $p' < n$. Then $d_j > d_{j'}$. Let $a_i := \sigma_{[i,j]} = d_j + i$ and let $b_i := \sigma_{[i,j']} = d_{j'} + i$. Consider sequence obtained from σ by changing the subsequence of length $2p$ starting at $\sigma_{[1,j]}$ from

$$(a_1 < a_2 < \ldots < a_p > b_1 < b_2 < \cdots < b_p)$$

to

$$(a_1 > b_1 < a_2 > b_2 < \ldots < a_p > b_p)\,.$$

This transformation can be realized as $K_{p-1} \circ K_{p-2} \circ \cdots \circ K_1(\sigma)$, where

$$K_i := \kappa_{[1,j]+2i-2} \circ \kappa_{[1,j]+2i-1} \circ \kappa_{[1,j]+2i} \cdots \circ \kappa_{[1,j]+i+p-3}$$

is the composition of strict Knuth transformations that moves b_i next to a_i. For example, K_1 performs the following sequence of transformations:

$$\begin{aligned}
&(a_1 < \ldots < a_{p-2} < a_{p-1} < a_p > \boldsymbol{b_1} < b_2 < \ldots < b_p) \\
&\qquad \mapsto\ (a_1 < \ldots < a_{p-2} < a_{p-1} > \boldsymbol{b_1} < a_p > b_2 < \ldots < b_p) \\
&\qquad \mapsto\ (a_1 < \ldots < a_{p-2} > \boldsymbol{b_1} < a_{p-1} < a_p > b_2 < \ldots < b_p) \\
&\qquad \ldots \\
&\qquad \mapsto\ (a_1 > \boldsymbol{b_1} < a_2 < \ldots < a_p > b_2 < \ldots < b_p)\,.
\end{aligned}$$

Here we have recorded only the subsequence of length $2p$ starting at $[1, j]$ — the remaining terms are unaffected by these transformations.

Let $\alpha_i := \square_{[i,j]}$ and let $\beta_j := \square_{[i,j']}$. The corresponding subsequence of $\boldsymbol{\square}^U_\sigma$ is

$$(\alpha_1 < \alpha_2 < \ldots < \alpha_p > \beta_1 < \beta_2 < \ldots < \beta_p)\,.$$

By Proposition 6, $K_{p-1} \circ K_{p-2} \circ \cdots \circ K_1(\boldsymbol{\square}^U_\sigma)$ is defined, and by Proposition 5, each strict Knuth transformation must produce a sequence with the correct descent pattern. Using these two facts, one can deduce (by a straightforward inductive argument) that for all $r = 1, \ldots, p-1$, the corresponding subsequence of $K_r \circ \cdots \circ K_1(\boldsymbol{\square}^U_\sigma)$ must be of the form

$$(\alpha_{q_1} > \gamma_1 < \alpha_{q_2} < \gamma_2 > \ldots < \alpha_{q_r} > \gamma_r < \gamma_{r+1} < \ldots < \gamma_p > \beta_{r+1} < \ldots < \beta_p)\,,$$

where $1 \leqslant q_1 < q_2 < \cdots < q_r \leqslant p$ and $(\gamma_1 < \gamma_2 < \cdots < \gamma_p)$ is obtained from $\alpha_1\alpha_2 \ldots \alpha_p$ by replacing α_{q_i} replaced by β_i for $i = 1, \ldots, r$. In particular, when $r = p-1$, we have $\gamma_p > \beta_p$; thus $\gamma_p = \alpha_p$, and $q_i = i$ for all i. This shows that

$$K_{p-1} \circ \cdots \circ K_1(\boldsymbol{\square}^U_\sigma) = (\ldots\ \alpha_1 > \beta_1 < \alpha_2 > \beta_2 < \ldots < \alpha_p > \beta_p\ \ldots)\,.$$

The descent pattern of this sequence establishes that $\alpha_i > \beta_i$, which proves (c).

For (ii), suppose that $C_i{}^{\text{th}}$ occurrence of i in the sequence $c_1c_2c_3\dots$ occurs at $[s,i]$. Since the subsequence

$$c_{[1,j]}c_{[2,j]}\dots c_{[n-d_j,j]} = (1,2,\dots,n{-}d_j)$$

excludes i if and only if $d_j \geqslant n+1-i$, $s = C_i + \#\{j \mid d_j \geqslant n{+}1{-}i\}$. By (i), $\square_k$ is in column c_k, if and only if $k=[j,i]$ for some j, and $\square_k = \square_i$ when $j \geqslant s$; therefore $\delta^{\lambda_+}_{\sigma}(i)$ is the number of occurrences of i in the sequence $\mathrm{Trunc}_{[n-d_{s-1},s-1]}(\sigma)$. Since the subsequence

$$\sigma_{[1,j]}\dots\sigma_{[n-d_j,j]} = (d_j{+}1, d_j{+}2,\dots,n)$$

excludes i if and only if $d_j \geqslant i$, $\delta^{\lambda_+}_{\sigma}(i) = s-1-\#\{j \mid d_j \geqslant i\}$, as required.

Finally, (iii) follows immediately from (i). □

4 Proofs

We now prove Theorems 1, 2 and 3.

Proof of Theorem 1. Since the result is symmetrical with respect to rows and columns, we may assume, without loss of generality, that n is the number of columns of $\square$. By Lemma 10 and Lemma 11(i), w^* is equivalent to a sequence σ such that $\blacksquare^{\lambda_+}_{\sigma}$ is well-defined. The theorem therefore follows from Proposition 7(i). □

Proof of Theorem 2. Again, assume, without loss of generality, that n is the number of columns of $\square$. Using Lemma 10, Lemma 11(ii), Proposition 7(ii), and Proposition 4, we compute that

$$T^{\lambda_+}_w[\square_i] = w(i) + n\cdot\Big(C_i - \#\{j \mid d_j \geqslant i\} + \#\{j \mid d_j \geqslant n{+}1{-}i\} - 1\Big),$$

where $d_1 > d_2 > \dots > d_t$ are the descents of w, and C_i is the length of column i in λ_+. For a partition $\lambda \subset \square$, with row lengths $(\lambda_1,\dots,\lambda_m)$, let $\lambda^\vee$ denote the partition with row lengths $(n-\lambda_m,\dots,n-\lambda_1)$. If $\square$ is the box of $\square$ in column i and row j, let $\square^\vee$ denote the box in column $n+1-i$ and row $m+1-j$. For a skew tableau T of shape $\lambda/\mu \subset \square$, let $T^\vee$ denote the tableau of shape $\mu^\vee/\lambda^\vee$ with entries $T^\vee[\square] := mn+1-T[\square^\vee]$. The relationship between Algorithms A and B is

$$T^{\square/\lambda_-}_w = (T^{\lambda^\vee_-}_{w_0ww_0})^\vee\,. \tag{2}$$

Note that $m{+}1{-}C_i$ is the length of column $n{+}1{-}i$ in $\lambda^\vee_-$, and $n{-}d_1 < n{-}d_2 < \dots < n{-}d_t$ are the descents of w_0ww_0. We compute:

$$\begin{aligned}
T^{\square/\lambda_-}_w[\square_i] &= (T^{\lambda^\vee_-}_{w_0ww_0})^\vee[\square_i]\\
&= mn+1-T^{\lambda^\vee_-}_{w_0ww_0}[\square_i^\vee]\\
&= mn+1-\Big(w_0ww_0(n{+}1{-}i) + n\cdot\delta^{\lambda^\vee_-}_{(w_0ww_0)^*}(n{+}1{-}i)\Big)
\end{aligned}$$

$$= mn + 1 - \left(n{+}1{-}w(i) + n \cdot \left(\begin{array}{r} (m{+}1{-}C_i) + \#\{j \mid n{-}d_j \geqslant n{+}1{-}i\} \\ - \#\{j \mid n{-}d_j \geqslant i\} - 1 \end{array} \right) \right)$$
$$= w(i) + n \cdot \Big(C_i - \#\{j \mid n{-}d_j \geqslant n{+}1{-}i\} + \#\{j \mid n{-}d_j \geqslant i\} - 1 \Big)$$
$$= T_w^{\lambda_+}[\square_i] \,,$$

i.e. $T_w^{\lambda_+}$ and $T_w^{\square/\lambda_-}$ agree on λ_+/λ_-. For each vertically adjacent pair of boxes, either both are in λ_+ or both are in $\square/\lambda_-$. Thus the agreement on λ_+/λ_- shows that T_w is column strict. It also follows now, from Lemma 10, Lemma 11(iii) and Proposition 7(iii), that T_w is independent of the choice of λ_+. Thus we may we assume $\lambda_+ = (n, n{-}1, \ldots, 2, 1)$, which allows us to see that T_w is row-strict.

Let $D := \{T_w[\square_i] \mid i = 1, \ldots, n\}$ be the set of diagonal entries of T_w. Algorithm A ensures that $T_w[\square_i] \equiv w(i) \pmod{n}$, so D contains one number from each congruence class, modulo n. Since the entries of $T_w^{\lambda_+}$ are

$$\Big\{ k \geqslant 1 \;\Big|\; k + nj \in D \text{ for some } j \geqslant 0 \Big\}$$

and the entries of $T_w^{\square/\lambda_-}$ are

$$\Big\{ k \leqslant mn \;\Big|\; k - nj \in D \text{ for some } j \geqslant 0 \Big\} \,,$$

we see that every number in $\{1, 2, \ldots, mn\}$ is an entry of T_w. Therefore T_w is a standard Young tableau. $\square$

Proof of Theorem 3. For (i), we may assume that $\lambda_+ = (n, n{-}1, \ldots, 2, 1)$. This ensures that the sliding path of promotion on any $T \in \mathsf{SYT}(\square)$ passes through exactly one box of λ_+/λ_-. Suppose that $T_w^{\lambda_+}$ is computed using Algorithm A by a sequence of slides whose first step moves the entry 1. After this first step, if we delete the entry 1 and decrement all entries by 1, we are computing $T_{wc}^{\lambda_+}$ instead. This can be done immediately, or at any point during Algorithm A. Compare this with the behaviour of $\partial : T_w \to \partial T_w$ on the entries in λ_+. Suppose the sliding path passes through λ_+/λ_- at $\square_s$. Since the first two steps of promotion delete the entry 1 and decrement all entries by 1, this produces the penultimate step in the construction of $T_{wc}^{\lambda_+}$. Next, we slide the empty box in the upper-left corner of $\square$ through the tableau, which is the almost the same as the final of step in the construction of $T_{wc}^{\lambda_+}$, except that do not yet know what number will appear in $\partial T_w[\square_s]$. This shows that for all boxes of λ_+, with the possible exception of $\square_s$, T_{wc} coincides with ∂T_w. Note that s is the unique number such that $T_{wc}[\square_s] - T_w[\square_s] \neq -1$.

Applying the same argument to $\square/\lambda_-$ and the sliding path of $\partial^{-1} : T_{wc} \to \partial^{-1} T_{wc}$, we see that with the possible exception of one box $\square_{s'}$, T_w coincides with $\partial^{-1} T_{wc}$ on $\square/\lambda_-$. Since s' is the unique number such that $T_{wc}[\square_{s'}] - T_w[\square_{s'}] \neq -1$, we must have $s = s'$. This shows that these two sliding paths are in fact inverse to each other, and hence $\partial T_w = T_{wc}$.

Since $T_w[\square_i] \equiv w(i) \pmod{n}$, (ii) is immediate.

To prove (iii), we use another reformulation of Algorithm A.

Algorithm D. *INPUT:* A permutation $w \in S_n$.

Begin with $T := \varnothing$, the empty tableau, and $\mu := \boxed{\;\;}$;
while μ is not the empty partition **do**
  Choose a corner box $\square \in \mu$;
  if $\square = \blacksquare_i$ for some i **then**
    Set $T[\blacksquare_i] := w(i)$;
  end if
  if $\square \in \lambda_-$ **then**
    Let T' be the tableau obtained by sliding $\square$ through T;
    If the final position of the sliding path is $\blacksquare_i$, then set $T'[\blacksquare_i] := T[\blacksquare_i] + n$;
    Set $T := T'$;
  end if
  Delete the box $\square$ from μ;
end while
return the resulting tableau, $T_w^{\lambda_+} := T$.

It is clear that Algorithm D is equivalent to Algorithm A: when $\square \notin \lambda_+$ nothing happens; when $\square \in \lambda_+/\lambda_-$ we create the initial entries of T; when $\square \in \lambda_-$ we proceed exactly as before.

Suppose $T \in \mathcal{O}_n$. For $i, k = 1, \dots, n$, let $\Delta_{ik} := \partial^k T[\blacksquare_i] - \partial^{k-1} T[\blacksquare_i]$. Thus $\Delta_{ik} \geqslant 0$ if and only if the sliding path of $\partial : \partial^{k-1} T \mapsto \partial^k T$, passes through $\blacksquare_i$, and $\Delta_{ik} = -1$ otherwise. The former can happen for at most one value of i. Since $\partial^n T = T$,

$$\Delta_{i1} + \Delta_{i2} + \dots \Delta_{in} = 0\,.$$

Therefore, for each i, there must be at least one k such that $\Delta_{ik} \geqslant 0$. It follows that for each k there is exactly one i such that $\Delta_{ik} \geqslant 0$, and for each i there is exactly one k such that $\Delta_{ik} \geqslant 0$. From this we see that if $\Delta_{ik} \geqslant 0$ then $\Delta_{ik} = n - 1$, and therefore, for all $k \geqslant 0$,

$$\partial^k T[\blacksquare_i] - (mn - k) \equiv w(i) \pmod{n}\,.$$

For $k = 1, \dots, mn$, construct a tableau T_k by starting with $\partial^k T$, subtracting $mn - k$ from all entries, and deleting any entries for which the result is less than or equal to 0. Let μ_k be the shape formed by the unfilled boxes of T_k. Thus T_0 is empty, $\mu_0 = \boxed{\;\;}$, and T_k is obtained by T_{k-1} as follows: let $\square_k \in \mu_{k-1}$ be the corner of μ_{k-1} on the sliding path of $\partial : \partial^{k-1} T \mapsto \partial^k T$; slide $\square_k$ through T_{k-1}; add entry k in the lower-right corner of $\boxed{\;\;}$.

Let $w : \{1, \dots, n\} \to \{1, \dots, n\}$ be the function defined in the statement of (iii). If $T_k[\blacksquare_i]$ is non-empty, then $T_k[\blacksquare_i] \equiv w(i) \pmod{n}$. Since $\Delta_{ik} < n$ for all k, if $\square_k = \blacksquare_i$ then $T_k[\blacksquare_i] \leqslant n$, i.e. $T_k[\blacksquare_i] = w(i)$; and if $\square_k \in \lambda_-$ and the sliding path of $\square_k$ passes through $\blacksquare_i$, then $T_k[\blacksquare_i] = T_{k-1}[\blacksquare_i] + n$. Thus if we restrict the sequence $T_0, T_1, \dots, T_{mn}$ to λ_+, we obtain precisely a sequence of tableaux produced by Algorithm D. Since $\partial^{mn} T = T$, this shows that $T_w^{\lambda_+}$ is the restriction of T to λ_+. Since T has no repeated entries, $w \in S_n$. By a similar argument $T_w^{\boxed{\;\;}/\lambda_-}$ is the restriction of T to $\boxed{\;\;}/\lambda_-$. Thus $T = T_w$. $\square$

5 Remarks

Here is another way to compute $T_w^{\lambda_+}$. Assume that n is the number of rows of $\square$. Define the **augmented word** of w to be:

$$\begin{aligned} \operatorname{aug}(w) := \; & w(1), w(1)+n, w(1)+2n, \dots w(1)+(m-1)n, \\ & w(2), w(2)+n, w(2)+2n, \dots w(2)+(m-1)n, \\ & \dots \\ & w(n), w(n)+n, w(n)+2n, \dots w(n)+(m-1)n\,. \end{aligned}$$

Theorem 12. *$T_w^{\lambda_+}$ is the insertion tableau of* $\operatorname{aug}(w)$ *restricted to λ_+.*

Proof. Let $\lambda'_+ := (m, m+1, m+2, \dots, m+n-1)$, and compute $T_w^{\lambda'_+}$ using Algorithm A. Choose a sequence of boxes beginning with $m-1$ boxes from row n, followed $m-1$ boxes from row $n-1$, and so on. (After the $m-1$ boxes from row 1, the last $n(n-1)/2$ boxes may be taken in any order.) The first $(m-1)n$ slides produce a tableau whose reading word is $\operatorname{aug}(w)$. Therefore if we restrict $T_w^{\lambda'_+}$ to entries $1, 2, \dots, mn$, we obtain the insertion tableau of $\operatorname{aug}(w)$. By Theorem 2, $T_w^{\lambda_+}$ can be obtained as the restriction of $T_w^{\lambda'_+}$ to λ_+. Since the entries of $T_w^{\lambda_+}$ are a subset of $\{1, 2, \dots, mn\}$, the result follows. $\square$

Theorem 12 provides an alternate definition of $T_w^{\lambda_+}$. It has the advantage of being well-defined, and Theorem 2 can be proved by using Greene's theorem [4] to compute the entries $T[\square_i]$, (see [10, Section 5.2]). Unfortunately, things start to break down at the proof of Theorem 3, which is intimately connected to Algorithm A. The problem is that although the first and last steps of Algorithm A are related to $\operatorname{aug}(w)$, the intermediate steps may not be. For instance, if T is a tableau from one of the intermediate steps it is tempting to define $\operatorname{aug}(T)$ to be the tableau obtained by adding entries $T[\square_i] + n, T[\square_i] + 2n, \dots w(i) + (m-1)n$. to the right of $\square_i$. Unfortunately, it is *not true* the Knuth class of $\operatorname{aug}(T)$ is invariant for all T. There are a number of variations on this idea, and none of them appear to work. We do not know how to construct an invariant of Algorithm A, analogous to the Knuth class of the reading word. In particular, the intermediate tableaux in Algorithm A are not produced by ordinary *jeu de taquin* in any obvious way. This makes it difficult to prove Theorem 3, if one takes Theorem 12 as the definition of T_w.

Another way in which our situation behaves quite differently from ordinary rectification concerns dual equivalence. Consider a generalization of Algorithm C, in which we allow $U \in \mathsf{SYT}(\lambda_-/\mu)$ to be a skew shape, but otherwise the algorithm is performed the same way. This generalization *does not* have the property that $\blacksquare_{w^*}^{U} = \blacksquare_{w^*}^{\widehat{U}}$, when U is dual equivalent to $\widehat{U}$. If this were true, it would provide a more straightforward proof of Theorem 1. We do not know a set of elementary relations that generate the equivalence relation $U \sim \widehat{U} \iff \blacksquare_{w^*}^{U} = \blacksquare_{w^*}^{\widehat{U}}$ for all $w \in S_n$.

Despite the aforementioned difficulties, Theorem 12 can be used as a definition of $T_w^{\lambda_+}$ when λ_+ is an *arbitrary* partition with at most n rows — even in cases where Algorithm A does not make sense. In particular, we can sometimes use this idea to define T_w, when $m < n$. We illustrate this with an example. Take $n = 3$, $m = 2$, $\lambda_+ = 211$, $\lambda_- = 1$, and $w = 132$. The insertion tableau of $\mathrm{aug}(w)$ is

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \Longrightarrow \qquad T_w^{\lambda_+} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \; .$$

Similarly, using (2) as the definition, we compute:

$$T_w^{\square/\lambda_-} = \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \qquad \Longrightarrow \qquad T_w = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array} \; ,$$

which, indeed, has order 3 under promotion. This is very suggestive, but it is unclear what to do with the proof of Theorem 3, when $m < n$.

In the thesis [10], the second author observed that a procedure based on rectification can be used construct the set $\mathcal{O}_{mn/2}$, when one of m, n is even. In this case, other bijections are known, (see [9, Proposition 3.10]); however it is not obvious that they are equivalent. This provides a new perspective, and gives further hints that the methods introduced in this paper may apply beyond the case of minimal orbits.

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