A Lower Bound on the Diameter of the Flip Graph

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Submitted: Aug 14, 2015; Accepted: Feb 20, 2017; Published: Mar 3, 2017
Mathematics Subject Classifications: 05C62, 68R10

Abstract

The flip graph is the graph whose vertices correspond to non-isomorphic combinatorial triangulations and whose edges connect pairs of triangulations that can be obtained from one another by flipping a single edge. In this note we show that the diameter of the flip graph is at least $\frac{7n}{3} + \Theta(1)$, improving upon the previous $2n + \Theta(1)$ lower bound.

1 Introduction

A *combinatorial triangulation* is a planar graph to which no edge can be added without destroying planarity. The term *triangulation* emphasizes that in any planar drawing of a combinatorial triangulation all the faces are delimited by cycles with three vertices, while the term *combinatorial* stresses the fact that a combinatorial triangulation is not associated with a particular geometric realization. We are interested in *simple* combinatorial triangulations, which have no self-loops or multiple edges. In the following, when we say *triangulation* we always mean *simple combinatorial triangulation*.

Consider a planar drawing Γ, say on the sphere, of a triangulation G and consider an edge (a, b) in G. If (a, b) were removed from Γ, there would exist a unique region of the sphere delimited by a cycle with four vertices; in fact the cycle delimiting such region would be (a, a', b, b'), for some vertices a' and b'. The operation of *flipping* (a, b) consists of removing (a, b) from G and inserting the edge (a', b') inside the region delimited by the cycle (a, a', b, b'). The resulting triangulation G' might not be simple though. In the following, we only refer to flips that maintain the triangulations simple.

*Supported by MIUR Project "MODE" under PRIN 20157EFM5C and by H2020-MSCA-RISE project 73499 – "CONNECT".

The *flip graph* $\mathcal{G}_n$ describes the possibility of transforming n-vertex triangulations using flips. The vertex set of $\mathcal{G}_n$ is the set of non-isomorphic n-vertex triangulations; two n-vertex triangulations G and H are connected by an edge in $\mathcal{G}_n$ if there exists an edge e of G such that flipping e in G results in H.

Various properties of the flip graph have been studied. A particular attention has been devoted to the *diameter* of $\mathcal{G}_n$, which is the length of the longest (among all pairs of vertices) shortest path; refer to the surveys [3, 5]. A first proof that the diameter of $\mathcal{G}_n$ is finite goes back to almost a century ago [11]. A sequence of deep improvements [4, 7, 8, 9, 10] have led to the current best upper bound of $5n + \Theta(1)$, which was proved recently by Cardinal et al. [7]. Significantly less results and techniques have been presented for the lower bound. We are only aware of a $2n + \Theta(1)$ lower bound on the diameter of $\mathcal{G}_n$, which was proved by Komuro [8] by exploiting the existence of triangulations with "very different" vertex degrees. The main contribution of this note is the following theorem.

Theorem 1. *For every $n \geqslant 3$, the diameter of the flip graph is at least $\frac{7n}{3} - 34$.*

2 Proof of the Main Result

In this section we prove Theorem 1. Let $n \geqslant 3$. For a triangulation G, we denote by $V(G)$ and $E(G)$ its vertex and edge set, respectively.

Consider any n-vertex triangulation G_1. A path incident to G_1 in $\mathcal{G}_n$ is a sequence of n-vertex triangulations such that the first triangulation in the sequence is G_1 and any two triangulations which are consecutive in the sequence can be obtained from one another by flipping a single edge. Thus, a path incident to G_1 in $\mathcal{G}_n$ corresponds to a *valid sequence* $\sigma = (u_1, v_1), \dots, (u_k, v_k)$ of flips, where $u_1, \dots, u_k, v_1, \dots, v_k$ are vertices in $V(G_1)$ and (u_i, v_i) is an edge of the triangulation obtained starting from G_1 by performing flips $(u_1, v_1), \dots, (u_{i-1}, v_{i-1})$ in this order. For a valid sequence σ of flips, denote by G_1^σ the n-vertex triangulation obtained starting from G_1 by performing the flips in σ. Observe that $V(G_1) = V(G_1^\sigma)$, given that a flip only modifies the edge set of a triangulation, and not its vertex set.

Now consider any two n-vertex triangulations G_1 and G_2 and consider a simple path in $\mathcal{G}_n$ between them. This path corresponds to a valid sequence σ of flips transforming G_1 into G_2. By the definition of $\mathcal{G}_n$, the n-vertex triangulations G_1^σ and G_2 are isomorphic; that is, there exists a bijective mapping $\gamma : V(G_1^\sigma) \to V(G_2)$ such that $(u, v) \in E(G_1^\sigma)$ if and only if $(\gamma(u), \gamma(v)) \in E(G_2)$.

The key idea for the proof of Theorem 1 is to consider the bijective mapping γ *before* the flips in σ are applied to G_1 and to derive a lower bound on the number of flips in σ based on properties of γ. In fact, the property we employ is the number of common edges of G_1 and G_2 according to γ.

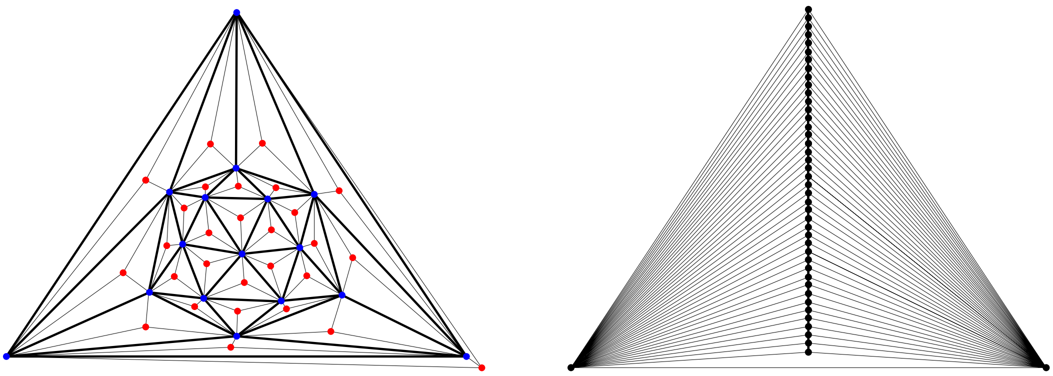
More precisely, for a bijective mapping $\gamma : V(G_1) \to V(G_2)$ between the vertex sets of two triangulations G_1 and G_2, we define the *number c_γ of common edges with respect to γ* as the number of distinct edges $(u, v) \in E(G_1)$ such that $(\gamma(u), \gamma(v)) \in E(G_2)$. We have the following.

Lemma 2. *For any two n-vertex triangulations G_1 and G_2, the number of flips needed to transform G_1 into G_2 is at least $3n - 6 - \max_\gamma c_\gamma$, where the maximum is over all the bijective mappings $\gamma : V(G_1) \rightarrow V(G_2)$.*

Proof. The statement descends from the following two observations. First, two isomorphic n-vertex triangulations have $3n - 6$ common edges according to the bijective mapping γ realizing the isomorphism. Second, for any two n-vertex triangulations H and L that have ℓ common edges with respect to any bijective mapping γ, flipping any edge in H results in a combinatorial triangulation H' such that H' and L have at most $\ell + 1$ common edges with respect to γ. $\square$

It remains to define two n-vertex triangulations G_1 and G_2 such that *any* bijective mapping γ between their vertex sets has a small number c_γ of common edges.

- Triangulation G_1 is defined as follows (see Fig. 1a). Let H be any triangulation of maximum degree six with $\lfloor \frac{n}{3} \rfloor + 2$ vertices. Note that the number of faces of H is $2(\lfloor \frac{n}{3} \rfloor + 2) - 4 = 2\lfloor \frac{n}{3} \rfloor$. If $n \equiv 2$ modulo 3, if $n \equiv 1$ modulo 3, or if $n \equiv 0$ modulo 3, then insert a vertex inside each face of H, insert a vertex inside each face of H except for one face, or insert a vertex inside each face of H except for two faces, respectively. When a vertex is inserted inside a face of H, it is connected to the three vertices of H incident to the face. Denote by G_1 the resulting n-vertex triangulation. We say that the vertices of G_1 in H are *blue*, while the other vertices of G_1 are *red*.



(a) (b)

Figure 1: Triangulations G_1 (a) and G_2 (b).

- Triangulation G_2 is defined as follows (see Fig. 1b). Starting from a path P with $n - 2$ vertices, connect all the vertices of P to two further vertices a and b, and connect a with b. Interestingly, the triangulation G_2, which is sometimes called the "canonical triangulation", has been previously used in order to prove upper bounds for the diameter of the flip graph [4, 7, 8, 9, 11], whereas in this paper we use G_2 in order to prove a lower bound for the same parameter.

We have the following.

Lemma 3. *For any bijective mapping $\gamma : V(G_1) \to V(G_2)$, we have $c_\gamma \leqslant 2\lfloor \frac{n}{3} \rfloor + 28$.*

Proof. Consider any bijective mapping $\gamma : V(G_1) \to V(G_2)$. First, note that each vertex $v \in V(G_1)$ has degree at most twelve. Namely, v has at most six blue neighbors; further, v has at most six incident faces in H, hence it has at most six red neighbors. It follows that, whichever vertex in $V(G_1)$ is mapped to a according to γ, at most twelve out of the $n-1$ edges incident to a are shared by G_1 and G_2 with respect to γ. Analogously, at most twelve out of the $n-1$ edges incident to b are shared by G_1 and G_2 with respect to γ. It remains to bound the number of edges of P that are shared by G_1 and G_2 with respect to γ. This proof uses a pretty standard technique (see, e.g., [6, 7]). Since G_1 has no edge connecting two red vertices, the number of edges of P that are shared by G_1 and G_2 with respect to γ is at most the number of edges of P that have at least one of their end-vertices mapped to a blue vertex; since $\lfloor \frac{n}{3} \rfloor + 2$ vertices of G_1 are blue, there are at most $2\lfloor \frac{n}{3} \rfloor + 4$ such edges of P. It follows that the number of edges shared by G_1 and G_2 with respect to γ is at most $2\lfloor \frac{n}{3} \rfloor + 28$. □

By Lemma 3, we have that G_1 and G_2 are two n-vertex triangulations such that, for any bijective mapping $\gamma : V(G_1) \to V(G_2)$, we have $c_\gamma \leqslant 2\lfloor \frac{n}{3} \rfloor + 28$. By Lemma 2, the number of flips needed to transform G_1 into G_2 is at least $3n - 6 - 2\lfloor \frac{n}{3} \rfloor - 28 \geqslant \frac{7n}{3} - 34$. This concludes the proof of Theorem 1.

3 Conclusions

In this note we have presented a lower bound of $\frac{7n}{3} + \Theta(1)$ on the diameter of the flip graph for n-vertex triangulations. One of the main ingredients for this lower bound is a lemma stating that there exist two n-vertex triangulations such that any bijective mapping γ between their vertex sets creates at most $c_\gamma \leqslant \frac{2n}{3} + \Theta(1)$ common edges.

It not clear to us whether the bound resulting from this approach can be improved further. That is, is it true that, for every two n-vertex triangulations, there exists a bijective mapping γ between their vertex sets creating $c_\gamma \geqslant \frac{2n}{3} + \Theta(1)$ common edges? The only lower bound on the value of c_γ we are aware of comes as a corollary of the fact that every n-vertex triangulation has a matching of size at least $\frac{n+4}{3}$, as proved in [2], hence $c_\gamma \geqslant \frac{n+4}{3}$.

It is an interesting fact that, for every n-vertex triangulation H, a bijective mapping $\gamma : V(H) \to V(G_2)$ exists creating $c_\gamma = \frac{2n}{3} + \Theta(1)$ common edges, where G_2 is the canonical triangulation from the proof of Theorem 1. In fact, every n-vertex triangulation H has a set of $\frac{n}{3} + \Theta(1)$ vertex-disjoint simple paths covering its vertex set $V(H)$, as proved by Barnette [1] (this bound is the smallest possible [6]). Mapping these paths to sub-paths of the path P in G_2 provides the desired bijective mapping γ.

We conclude with a remark on the definition of the problem we addressed in this paper. Consider a planar drawing Γ of a triangulation G and denote by $\sigma_G(v)$ the clockwise order of the edges incident to each vertex $v \in V(G)$ in Γ. By Whitney's theorem [12], in *any* planar drawing of G either the clockwise order incident to each vertex $v \in V(G)$ is $\sigma_G(v)$, or the clockwise order incident to each vertex $v \in V(G)$ is the reverse of $\sigma_G(v)$. Some papers in fact define a combinatorial triangulation so that each vertex v is associated with a clockwise order $\sigma_G(v)$ of its incident edges; then two triangulations G and G' are isomorphic if two conditions are satisfied: (i) there exists a bijective mapping $\gamma : V(G) \to V(G')$ such that $(u, v) \in E(G)$ if and only if $(\gamma(u), \gamma(v)) \in E(G')$; and (ii) $\sigma_G(v)$ coincides with $\sigma_{G'}(\gamma(v))$ for every vertex $v \in V(G)$. This somehow changes the structure of the flip graph; for example, with this definition the flip graph has twice as many vertices as with the definition that was given in our paper. All the bounds known in the literature for the diameter of the flip graph hold true in both settings; in particular our lower bound immediately extends to the setting in which the vertices of the triangulations are associated with a clockwise order of their incident edges. However we ask: Is the diameter of the flip graph different with the two different definitions? How many flips does it take to transform a triangulation with a certain clockwise order of the edges incident to each vertex into the triangulation in which such orders are reversed?

Acknowledgments

Thanks to Michael Hoffmann for an inspiring seminar he gave when visiting the author.

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