Strongly connected multivariate digraphs

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Abstract

Generalizing the idea of viewing a digraph as a model of a linear map, we suggest a multi-variable analogue of a digraph, called a hydra, as a model of a multi-linear map. Walks in digraphs correspond to usual matrix multiplication while walks in hydras correspond to the tensor multiplication introduced by Robert Grone in 1987. By viewing matrix multiplication as a special case of this tensor multiplication, many concepts on strongly connected digraphs are generalized to corresponding ones for hydras, including strong connectedness, periods and primitiveness, etc. We explore the structure of all possible periods of strongly connected hydras, which turns out to be related to the existence of certain kind of combinatorial designs. We also provide estimates of largest primitive exponents and largest diameters of relevant hydras. Much existing research on tensors are based on some other definitions of multiplications of tensors and so our work here supplies new perspectives for understanding irreducible and primitive nonnegative tensors.

Keywords: De Bruijn form, cyclic decomposition, diameter, Markov operator, period, phase space, primitive exponent, hydra, tensor multiplication.

Contents

1 Hydras and weighted hydras

We reserve the notation $\mathbb{N}$ for the set of positive integers. For every $n \in \mathbb{N}$, we write $[n]$ for the set of the smallest n positive integers. For a set K and a positive integer t, we understand K^t as $K^{[t]}$, namely the set of all maps from $[t]$ to K. Thus, an element $x \in K^t$ can be specified by its values $x(i)$ for $i \in [t]$. We will also often directly write x_i for $x(i)$, as K^t can be understood as the set of all length-t words on K. The *binary semifield*, denoted by T, is the ring consisting of two elements, 0 and 1, in which the arithmetic operation is the same with the binary field, excepting that $1+1$ now equals to 1, not to 0 as in the binary field. Note that each element $f \in \mathrm{T}^K$ can be identified with its support, i.e., $\{k \in K : \ f(k) = 1\}$, and then, T^K is often naturally identified with the power set of K, which is more commonly denoted by 2^K. It is clear that the addition/multiplication operation in T^K corresponds to the taking union/intersection operation in 2^K.

Let K be a set. A *digraph* Γ on K consists of a pair (K, E) where $E \subseteq K \times K = K^2$. Usually, we call K the *vertex set* of Γ and E the *arc set* of Γ, and denote them by $\mathrm{V}(\Gamma)$ and $\mathrm{A}(\Gamma)$. More generally, given a ring R, we can consider a weight function w from $\mathrm{A}(\Gamma)$ to R and get an R-weighted digraph. For every $(k, \ell) \in (K \times K) \setminus \mathrm{A}(\Gamma)$, we can think that w assigns weight $0 \in R$ to (k, ℓ) and so, in this way, a general *weighted digraph* on K with weights/variables from R is simply a map w_Γ from $K \times K$ to R. A digraph Γ given as a pair $(\mathrm{V}(\Gamma), \mathrm{A}(\Gamma))$ can be viewed as a T-weighted digraph, where the associated weight function w_Γ defined on $\mathrm{V}(\Gamma) \times \mathrm{V}(\Gamma)$ sends (k, ℓ) to $1 \in \mathrm{T}$ if and only if $(k, \ell) \in \mathrm{A}(\Gamma)$.

An R-weighted digraph Γ on K can be represented as a map f from K to R^K where, for every $k \in K$, $f(k) \in R^K$ sends each $\ell \in K$ to $\mathrm{w}_\Gamma((k, \ell)) \in R$. This map f from K to R^K and the weight function w_Γ from $K \times K$ to R surely determines each other and so we will write Γ_f for this digraph Γ corresponding to f. In the case that $R = \mathrm{T}$,

$$f(k) = \{\ell \in K : \ (k, \ell) \in \mathrm{A}(\Gamma_f)\} \in 2^K$$

for each $k \in K$, and

$$\mathrm{A}(\Gamma_f) = \{(k, \ell) \in K^2 : \ \ell \in f(k)\}.$$

This observation allows us to identify an R-weighted digraph on K with a single variable map from K to R^K, and especially, to identify a digraph with a single variable map from K to 2^K. Then, is there a multivariate counterpart for weighted digraphs? Yes, for any positive integer t, we could simply call any map from K^t to R^K a t-variable digraph on K with variables from a ring R. But, does it lead to any interesting mathematics?

A Markov chain is a sequence of possible events in which the probability of every event relies linearly on the probability of the states attained in the previous event. More generally, for any positive integer t, an order-t Markov chain is a sequence of possible events in which the probability of each event depends multi-linearly on the probability of the states attained in the previous t events. It is widely believed that everything about higher-order Markov chains can be encoded as something about Markov chains and so it does not make sense to study higher-order Markov chains separately [LM95, Example 1.5.10, Proposition 1.5.12]. We suggested multivariate graph theory in [WXZ16] as a framework to understand multi-linear phenomena, including higher-order Markov chains.

When modeling higher-order Markov chains with multivariate graph theory, we will be looking at quantitative/probabilistic properties if we take variables from the ring of non-negative reals, and we will be examining the qualitative/topological properties if we take variables from the binary semifield. This paper will focus on the case that the variables are from the binary semifield and develop relevant basic properties of multivariate graph theory. We hope to convince the readers that there is something new from multivariate graph theory and the encoding of results from Markov chains to their higher-order version may not be so trivial. Dobzhansky [Dob73] said that "Nothing in biology makes sense except in the light of evolution". In general, to understand the global pictures of various evolution models is a fundamental question in sciences [Fur14, Kra15, LM95]. Though multivariate graph theory can be studied from many different approaches, this paper will pay attention to some basic concepts from the dynamical system point of view.

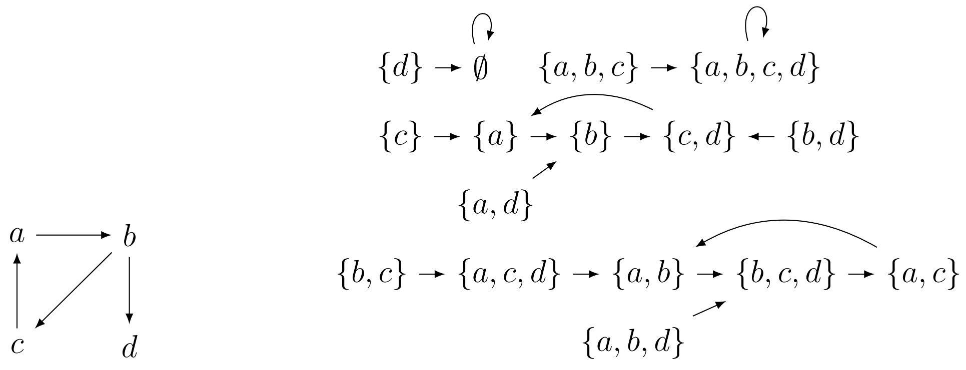
After its invention by Eilenberg and Mac Lane [EML45], category theory has extended its tentacles into most parts of mathematics [Lei14, p. 9] and even sciences [Coe06, DDH83]. To justify our formalism of the concept of digraphs as maps from K to 2^K, we try to see if this leads to a system of related objects among which we have maps and compositions of maps. At first sight, we cannot compose two digraphs as maps from K to 2^K and so we can locate neither dynamics nor category structures around this formalism. However, as suggested by [Lei14, Example 0.4], each map f from K to 2^K induces a Boolean linear map M_f from 2^K to 2^K, called the *Markov operator associated to* f, such that $\mathrm{M}_f(A) = \cup_{k\in A} f(k)$ for each $A \in 2^K$. It is useful to think of $A \in 2^K$ as its characteristic column vector and think of M_f as the K by K matrix whose kth column is $f(k)$; in this way, you can understand $\mathrm{M}_f(A)$ as the product of the column vector A left multiplied by the matrix M_f, namely $\mathrm{M}_f(A) = \mathrm{M}_f\, A$. For two maps f and g from $(2^K)^K$, we can naturally compose M_f and M_g to get $\mathrm{M}_f\,\mathrm{M}_g$, and we thus view $\mathrm{M}_f\,\mathrm{M}_g$ as the product/composition of f and g, namely we set fg to be the element of $(2^K)^K$ such that $\mathrm{M}_{fg} = \mathrm{M}_f\,\mathrm{M}_g$. This viewpoint not only allows us define compositions of digraphs but also corresponds to important mathematical models in applications. Indeed, in a Markov chain modelled by a digraph $f \in (2^K)^K$, an arc (k, ℓ) in Γ_f stands for a possible transition from state k to state ℓ; if A is the set of states with positive probability at present, $\mathrm{M}_f(A) = \mathrm{M}_f\, A$ will be the set of states with positive probability at next time slot. This connection to Markov chains explains why we call M the Markov operator. Besides the local transition as represented by an arc, we need to understand the global dynamics, i.e., the asymptotic behavior of the trajectories, or, in even more concrete words, a long sequence of consecutive state transitions as signified by a path. We often use $k \to \ell$ or $k\ell$ for an arc (k, ℓ), as this allows us, for example, to write

$$k_1 \to k_2 \to k_3 \to k_4$$

or

$$k_1k_2k_3k_4$$

for the three consecutive arcs (k_1, k_2), (k_2, k_3), (k_3, k_4) in the digraph Γ_f in a succinct way, and so on. The *phase space of* f, denoted by $\mathcal{PS}_f$, has vertex set $\mathrm{V}(\mathcal{PS}_f) = 2^K$ and



f $\qquad\qquad$ $\mathcal{PS}_f$

Figure 1: A digraph f on $\{a, b, c, d\}$ and its phase space $\mathcal{PS}_f$.

arc set $\mathrm{A}(\mathcal{PS}_f) = \{A \to \mathrm{M}_f(A) :\ A \in 2^K\}$; see Figure 1 for an example. The structure of $\mathcal{PS}_f$ tells us the dynamical behaviour of M_f. For any positive integer n, let $\mathbb{Z}_n$ denote the cyclic group of integers modulo n and let $\mathscr{C}_n$ denote the *n-cycle*, which is the digraph with $\mathrm{V}(\mathscr{C}_n) = \mathbb{Z}_n$ and $\mathrm{A}(\mathscr{C}_n) = \{i \to i+1 :\ i \in \mathbb{Z}_n\}$. When K is finite, a traveller driven by M_f in $\mathcal{PS}_f$ will eventually run around a limit cycle repeatedly after staying for a while, so-called transient time, in the transient part, which is an in-tree attached to the limit cycle.

Let K be a set and t a positive integer. As advised earlier in this note, let us define a *t-variable digraph* f on K to be a map from K^t to 2^K. The map f induces a Boolean t-linear map from $(2^K)^t$ to $(2^K)^t$, called the *Markov operator associated to* f and denoted by M_f, such that

$$\mathrm{M}_f(A_1, \ldots, A_t) = \big(A_2, \ldots, A_t, \cup_{(k_1,\ldots,k_t)\in A_1\times\cdots\times A_t} f(k_1, \ldots k_t)\big)$$

for all $(A_1, \ldots, A_t) \in (2^K)^t$. The *phase space* of f, denoted by $\mathcal{PS}_f$, is the digraph which has vertex set $\mathrm{V}(\mathcal{PS}_f) = (2^K)^t$ and arc set $\mathrm{A}(\mathcal{PS}_f) = \{A \to \mathrm{M}_f(A) :\ A \in (2^K)^t\}$. For an order-$t$ Markov chain, we can establish its nonparametric model with a t-variable digraph in an obvious way [WXZ16, Eq. (3.3)] and the phase space of this t-variable digraph displays the evolution of the supports of the random events. Surely, when K is finite, to understand $\mathcal{PS}_f$ is also to understand those limit cycles, transient time and transient parts, as in the case of $t = 1$.

For $A = (A_1, \ldots, A_t) \in (2^K)^t$ and $B = (B_1, \ldots, B_t) \in (2^K)^t$, we write $A \preceq B$ provided $A_i \subseteq B_i$ for $i \in [t]$. We identify the set K with $\binom{K}{1}$ and so view K^t as a subset of $(2^K)^t$. Therefore, for each $A \in (2^K)^t$ and $a \in K^t$, $a \in A$ is equivalent to $a \preceq A$. For any two t-variable digraphs f and g on K, we say that f is *bigger* than g provided $f \neq g$ and $f(a) \supseteq g(a)$ for all $a \in K^t$. Under this partial order, the biggest t-variable digraph on K is the digraph $\mathrm{f}_{t,K}$ satisfying $\mathrm{f}_{t,K}(a) = K$ for all $a \in K^t$.

For any t-variable digraph f on K, its *De Bruijn form* Γ_f is the digraph with vertex set K^t and arc set

$$\{(k_1, \ldots, k_t) \to (k_2, \ldots, k_{t+1}) : k_{t+1} \in f(k_1, \ldots, k_t), k_1, \ldots, k_t \in K\}.$$

We often directly call the set $K^t = \mathrm{V}(\Gamma_f)$ the *vertex set* of f, denoted by $\mathrm{V}(f)$, and call $\mathrm{A}(\Gamma_f)$ the *arc set* of f, denoted by $\mathrm{A}(f)$. The De Bruijn form of $\mathrm{f}_{t,K}$ is just the famous *dimension-t De Bruijn digraph on the alphabet K* [dB46, DW05, Goo46], which we denote by $\mathrm{B}(t, K)$, and the De Bruijn form of any digraph of several variables must be a spanning subgraph of a De Bruijn digraph. Note that $\mathrm{B}(t, K)$ has vertex set $\mathrm{V}(\mathrm{f}_{t,K}) = K^t$ and arc set

$$\mathrm{A}(\mathrm{f}_{t,K}) = \{(k_1, \ldots, k_t) \to (k_2, \ldots, k_{t+1}) : \ k_1, \ldots, k_{t+1} \in K\},$$

the latter corresponding to K^{t+1} in a natural way. By mapping a multivariate digraph to its De Bruijn form and mapping a De Bruijn form to its arc set, we get a bijection from the set of t-variable digraphs on K to the set of spanning subgraphs of $\mathrm{B}(t, K)$ and then to the set of subsets of K^{t+1}. This suggests that multivariate digraphs are generalizations of both **hy**pergraphs [Ber89, Bol86, Wan08] and **d**ig**ra**phs [BJG09, BM08, BR91]. For this reason, we propose to call a t-variable digraph a *t-head hydra*, or simply a *t-hydra*. When we mention a digraph below, we mean a 1-hydra; we speak of a hydra for any t-hydra for some $t \in \mathbb{N}$. We should mention that a natural generalization of both digraphs and hypergraphs, called directed hypergraphs, have been well studied in the literature [All14, GLPN93].

For a t-hydra f, one can view Γ_f as the local dynamical mechanism and $\mathcal{PS}_f$ as the global evolving picture. The question is to see how to link the local with the global. In light of the higher-order Markov chain model, one can also think of Γ_f as the particle version of f and $\mathcal{PS}_f$ as the wave version of f. Both versions encode full information about f in some way but the transformation between different representations may involve nontrivial mathematics. Note that the study of $\mathcal{PS}_f$ is nothing but the study of the t-linear map M_f. The aim of this paper is to develop some hopefully new perspective for getting a better appreciation of multilinear phenomena.

Let K be a set, t a positive integer and R a ring. A *t-fold tensor over R^K* is a map f from K^t to R. Following the convention in graph algebra [Rae05], we define the *adjacency tensor* of a t-hydra f on K, denoted by $\mathcal{A}(f)$, to be the $(t+1)$-fold tensor over T^K, also called a Boolean $\overbrace{K \times \cdots \times K}^{t+1}$-array, whose $(i_1, \ldots, i_{t+1})$-entry, where $(i_1, \ldots, i_{t+1}) \in K^{t+1}$, is given by

$$\mathcal{A}(f)_{i_1,\ldots,i_{t+1}} = \begin{cases} 1, & \text{if } (i_{t+1}, \ldots, i_2) \to (i_t, \ldots, t_1) \in \mathrm{A}(\Gamma_f); \\ 0, & \text{otherwise.} \end{cases}$$

When $t = 1$, $\mathcal{A}(f)$ coincides with M_f and it is more commonly known as the adjacency matrix of f. The set of all Boolean $\overbrace{K \times \cdots \times K}^{t+1}$-arrays are often named as order-$(t+1)$ dimension-K *Boolean tensors* (in coordinate forms). Note that every order-$(t+1)$ tensor is the adjacency tensor of a corresponding t-hydra. Therefore, hydras and tensors are different representations of the the same objects and we often do not distinguish between a hydra f and its associated tensor $\mathcal{A}(f)$ in the rest of the paper. For any two t-hydras f and g on K, we can surely do the composition of M_f and M_g. But, unlike the digraph case, we should warn the reader that there may not exist another t-hydra h such that $\mathrm{M}_h = \mathrm{M}_f \, \mathrm{M}_g$. In an obvious way, we can introduce R-weighted hydras for any ring R. All the concepts discussed above for hydras, which are simply T-weighted hydras or Boolean hydras, can be extended to general weighted cases. Especially, t-fold tensors over R^K can be viewed the same as R-weighted $(t+1)$-hydras via the general concept of adjacency tensors. Note that tensors and its practical applications [Lan12, Stu16] have been a very active field of research in last decade. Especially, some definitions of tensor multiplication/composition are available from [Gro87, Sha13, Wil73, Yam65]. By resorting to the higher-order Markov chain background, the following definition of tensor multiplication looks to be natural and it can be checked to be isomorphic with the one [Gro87, Eq. (2)] posed by Grone in 1987. For any two $(t+1)$-fold tensors $\mathcal{B}$ and $\mathcal{C}$ over R^K, we define the product of $\mathcal{B}$ and $\mathcal{C}$ to be the $(t+1)$-fold tensor over R^K, which is denoted by $\mathcal{B}\,\mathcal{C}$ and sends $(k_1, \ldots, k_{t+1}) \in K^{t+1}$ to

$$\mathcal{B}\,\mathcal{C}(k_1, \ldots, k_{t+1}) = \sum_{k \in K} \mathcal{B}(k_1, k, k_2, \ldots, k_t)\, \mathcal{C}(k, k_2, \ldots, k_{t+1}). \tag{1}$$

Surely, when K is an infinite set, to make Eq. (1) well-defined, we need to impose some local finiteness assumption or some convergence assumption. For $t = 1$, Eq. (1) becomes the rule for the usual matrix multiplication:

$$\mathcal{B}\,\mathcal{C}(k_1, k_2) = \sum_{k \in K} \mathcal{B}(k_1, k)\, \mathcal{C}(k, k_2).$$

The idea of Eq. (1) is that the weight for a transition from $\ell_1 \cdots \ell_t$ at initial t slots to to ℓ_{t+2} at time $t+2$ via $\mathcal{B}\,\mathcal{C}$ should be the sum of all weights as given by the following transition

$$\ell_1 \cdots \ell_t \overset{\mathcal{C}(\ell_{t+1}\ell_t \cdots \ell_1)}{\longrightarrow} \ell_2 \cdots \ell_{t+1} \overset{\mathcal{B}(\ell_{t+2}\ell_{t+1} \cdots \ell_2)}{\longrightarrow} \ell_3 \cdots \ell_{t+2},$$

where ℓ_{t+1} runs through all possible states of the system at time $t+1$. That is, matrix multiplication corresponds to walks in weighted complete digraphs while tensor multiplication corresponds to walks in general weighted De Bruijn digraphs. With the multiplication given in Eq. (1), the set of $(t+1)$-fold tensors over R^K form the tensor algebra $\mathbb{A}(t+1, R, K)$. The map f that sends $(k_1, \ldots, k_{t+1}) \in K^{t+1}$ to δ_{k_1,k_2} is the left identity for the tensor multiplication in $\mathbb{A}(t+1, R, K)$. Like the Lie bracket product in Lie algebras, the tensor product defined in Eq. (1) may not be associative and so $\mathbb{A}(t+1, R, K)$ is generally not an associative algebra. We should not forget that dynamics is the study of change and change takes place within time [Fur14, Preface]. Since our multiplication is abstracted from the evolution of a dynamical system, it is natural that we should do the composition of maps according to the flow of time, that is, multiplication should be done from right to left, and so it does not really make sense to require the associativity. To understand the structure of $\mathbb{A}(t+1, R, K)$ as an nonassociative algebra [Sch95] may be an interesting direction.

$$\begin{tikzcd} aa \arrow[d] & ba \arrow[l] \\ ab \arrow[ur] & bb \arrow[u] \end{tikzcd} \qquad\qquad \begin{tikzcd} aa \arrow[r] & ab \arrow[dl] \arrow[d] \\ ba \arrow[u] & bb \end{tikzcd}$$

$$\Gamma_{\overleftarrow{f}} \qquad\qquad\qquad \Gamma_f$$

Figure 2: A 2-hydra and its reversal.

If the adjacency matrix $\mathcal{A}(f)$ of a digraph f is symmetric, the digraph f is called a *symmetric digraph* or simply a *graph*. A large part of graph theory is about graphs and so, for the purpose of extending that part to multivariable case, let us define symmetric hydras. Let f be a t-hydra on a set K. The *reversal* of f, denoted by $\overleftarrow{f}$, is the t-hydra on K such that

$$k_1 \cdots k_t \to k_2 \dots k_{t+1}$$

is an arc in Γ_f if and only if

$$k_t \dots k_1 \leftarrow k_{t+1} \cdots k_2$$

is an arc in $\Gamma_{\overleftarrow{f}}$. In Figure 2, we depict a 2-hydra on $\{a, b\}$ and its reversal. We call a t-hydra *symmetric* provided $f = \overleftarrow{f}$. Though the local correspondence between a hydra and its reversal is quite straightforward, the relationship between their phase spaces seems not so trivial to tell in case the hydra has more than one variables.

Recall that a digraph f on a set K is strongly connected if for all $a, b \in K$ there exists a nonnegative integer N such that $b \in \mathrm{M}_f^N(a)$. Here is an easy way to generalize this concept for hydras. Let f be a t-hydra on a set K. For $a, b \in K^t$, we define $\mathcal{RI}_f(a, b)$ to be the set

$$\{N \geqslant 0 : \ b \preceq \mathrm{M}_f^N(a)\} = \{N \geqslant 0 : \ b \in \mathrm{M}_f^N(a)\}$$

and call it the set of *reachable indices* of f from a to b. An element from $(2^K)^t$ which has $\emptyset$ as one of its t components is called a *vacant element*. It is clear that for any vacant element a in $(2^K)^t$, $\mathbb{N} \setminus \mathcal{RI}_f(a, \emptyset^t)$ is a finite subset, that is, $\mathrm{M}_f^N(a) = \emptyset^t$ when N is large enough. We say that f is *strongly connected* if $\mathcal{RI}_f(a, b) \neq \emptyset$ for all $a, b \in K^t$. When f is strongly connected and $|K| \geqslant 2$, both $\emptyset^t$ and K^t give rise to length-1 limit cycles in $\mathcal{PS}_f$, i.e., they are fixed points of M_f. We call f *primitive* if there exists an integer $N > 0$ such that for all $a \in K^t$, $\mathrm{M}_f^N(a) = K^t$. Equivalently, f is primitive if and only if we can find $N \in \mathbb{N}$ so that

$$\mathrm{M}_f^N(a) = \begin{cases} \emptyset^t, & \text{if } a \in (2^K)^t \text{ is vacant;} \\ K^t, & \text{if } a \in (2^K)^t \text{ is not vacant.} \end{cases}$$

For a primitive t-hydra f on K, the *primitive exponent* of f, which we denote by $\mathrm{g}(f)$, is the minimum positive integer N such that $\mathrm{M}_f^N(a) = K^t$ for all $a \in K^t$. If g is a primitive t-hydra on K and if f is bigger than g, we can derive that f is primitive and $\mathrm{g}(f) \leqslant \mathrm{g}(g)$.

Note that $\mathrm{f}_{t,K}$ is primitive with

$$\mathrm{g}(\mathrm{f}_{t,K}) = \begin{cases} 1, & \text{if } |K| = 1; \\ t, & \text{if } |K| > 1. \end{cases}$$

In terms of the concept of reachable indices, many more basic concepts for digraphs can be naturally extended to hydras, say distance, diameter, radius, girth, etc. For example, let us define the *distance from a to b in f* to be

$$\mathrm{Dist}_f(a,b) = \min \mathcal{RI}_f(a,b)$$

for every $a, b \in K^t$, and the *diameter of f* to be

$$\mathrm{Dia}(f) = \max_{a,b \in K^t} \mathrm{Dist}_f(a,b).$$

So far, we have introduced some basic concepts in multivariate graph theory via the phase space, namely by examining the action of the Markov operator. It is surely possible to give these definitions via the concept of tensor multiplication as presented in Eq. (1). The current approach focuses on the combinatorial core of many situations. But, to start from Eq. (1) will allow us go beyond binary semifield to max-times semiring [Ser09] or to real/complex numbers [CPZ08, Lim05], and even go from homogeneous to nonhomogeneous [WZ15].

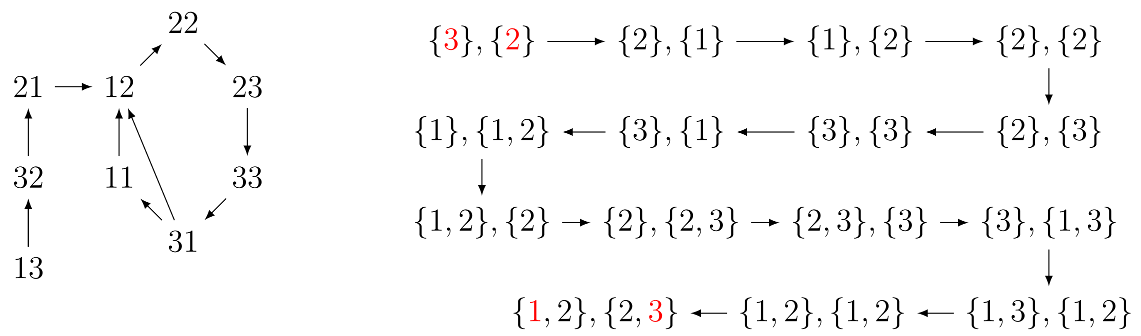
In graph theory, more precisely, in one-variable graph theory, there are very rich results about strongly connected digraphs, say the cyclicity theorem [BCOQ92, Theorem 3.112][BR91, Lemma 3.4.1], Wielandt's theorem on primitive exponent [BR91, Theorem 3.5.6], Hoffman's theorem on Perron pairs [BR91, Theorem 5.1.3][WD06, Theorem 2.8], Harary-Moser's strong tournament theorem [BJG09, Theorem 11.7.2], Robbins' theorem on strong orientation [BD15, CT78, Rob39], Bessy-Thomassé Theorem (Gallai's Conjecture) [BT04][BM08, Theorem 19.11], just to name a few. Then, should we expect some new mathematics in multivariate graph theory? We will first display in § 2 some possibly counter-intuitive examples to acquaint the reader with some concepts in multivariate graph theory and to show the difference caused by several variables. Our main contribution in this paper will be an analysis of the structure of strongly connected hydras, including some results related to the aforementioned cyclicity theorem and Wielandt's theorem. We summarize our main observations on strongly connected hydras in § 3 and then, in § 4, § 5 and § 6 we develop some technical apparatus to verify our claims in § 3.

2 Surprises from several variables

Let f be a hydra on a finite set K. Clearly, $f = \Gamma_f$ is strongly connected if and only if $\mathrm{Dia}(f) < \infty$. If f is a strongly connected digraph, it is trivial that

$$\mathrm{Dia}(f) \leqslant |\,\mathrm{V}(f)| - 1. \tag{2}$$

Note that in [WXZ16, Example 3.5] we construct a primitive 3-hydra f on [2] whose De Bruijn form is not strongly connected and $6 = \mathrm{g}(f) = \mathrm{Dia}(f) = \mathrm{Dist}_f(122, 212) = |\,\mathrm{V}(f)| - 2$. Our new example below says that Eq. (2) does not hold for general hydras f.



Γ_{f_3} $\qquad$ $\mathcal{PS}_{f_3}$

Figure 3: The De Bruijn form Γ_{f_3} of a 2-hydra f_3 on $[3]$ and the path in $\mathcal{PS}_{f_3}$ showing $\mathrm{Dist}_{f_3}(32, 13) = 14$.

Example 1. In Figure 3 we depict the De Bruijn form of a primitive 2-hydra f_3 on $[3]$. It holds $\mathrm{g}(f_3) = 23 > \mathrm{Dia}(f_3) = \mathrm{Dist}_{f_3}(32, 13) = 14 > 8 = 3^2 - 1 = |\,\mathrm{V}(f_3)| - 1$. Also observe that Γ_{f_3} is not strongly connected.

Let us recall the classical wheels-within-wheels theorem of Knuth on strongly connected digraphs [Knu74, Lemma 1].

Theorem 2 (Knuth, 1974)**.** *Every strongly connected digraph f (possibly infinite) is either a single vertex with no arcs, or it can be represented as in Figure 4 for some $n \in \mathbb{N}$. Here $\Gamma_1, \ldots, \Gamma_n$ are strongly connected digraphs; x_i and y_i are (possibly equal) vertices of Γ_i; and e_i is an arc from y_i to x_{i+1}. The original digraph f consists of the vertices and arcs of $\Gamma_1, \ldots, \Gamma_n$ plus the arcs $e_1, \ldots, e_n$.*

In fact, if σ is any given cycle of f, there exists such a representation in which each of the e_i is contained in σ.

The next example is indeed reporting our failure in extending the wheels-within-wheels theorem to hydras.

Example 3. Let f be the 2-hydra on $[4]$ whose De Bruijn form Γ_f is shown in Figure 5. The hydra f is strongly connected but its De Bruijn form Γ_f has two weakly connected components. There is no way to partition $\mathrm{V}(f) = [4]^2$ into $V_1, \ldots, V_n$ for some $n \in \mathbb{N}$ such that the following hold.

(a) For every $i \in \mathbb{Z}_n$ and all $x, y \in V_i$, we have $\mathcal{RI}_{h_i}(x, y) \neq \emptyset$, where h_i is the 2-hydra on $[4]$ such that $\mathrm{A}(\Gamma_{h_i}) = \mathrm{A}(\Gamma_f) \cap \big(V_i \times V_i\big)$;

(b) There exist $x_i, y_i \in f_i$ for all $i \in \mathbb{Z}_n$ such that $\mathrm{A}(\Gamma_f) = \cup_{i \in \mathbb{Z}_n}\big(\mathrm{A}(\Gamma_{h_i}) \cup \{x_i \to y_{i+1}\}\big)$.

For a strongly connected digraph f, its reversal must also be strongly connected and we can establish a natural one-to-one length-preserving correspondence between the limit cycles of $\mathcal{PS}_{\overleftarrow{f}}$ and those of $\mathcal{PS}_f$. However, when we enter the world of several variables graph theory, the arrow of time plays some magic, as Example 4 below illustrates.

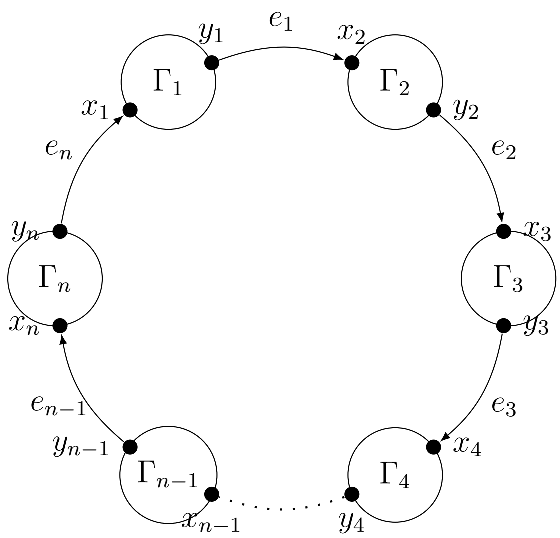


Figure 4: Wheels within wheels.

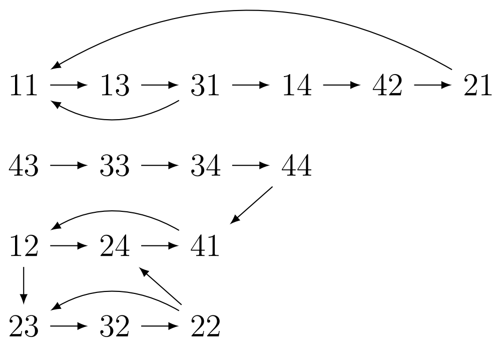


Figure 5: The De Bruijn form Γ_f of a 2-hydra f on [4].

Example 4. In Figure 6, we demonstrate a primitive 2-hydra f on $K = \{a, b\}$ whose reversal is not strongly connected. Note that $\mathcal{PS}_f$ contains two limit cycles while $\mathcal{PS}_{\overleftarrow{f}}$ has three limit cycles.

Question 5. For any hydra f, let $\mathrm{w}(f)$ be the number of weakly connected components of Γ_f. If f is a strongly connected t-hydra, is there any good upper bound estimate for $\mathrm{w}(f)$? Surely, $\mathrm{w}(f)$ equals 1 when $t = 1$.

As with most non-numerical properties, the study of nonnegative tensors is equivalent to the study of corresponding Boolean tensors; see our brief discussion in [WXZ16, p. 404]. Let us then formulate some discussions in the literatures on nonnegative tensors in the language of Boolean tensors below. Following [CPZ08, Lim05], we call an order-t dimension-K Boolean tensor $\mathcal{A}$ *irreducible* if for all nonempty subsets $I \subsetneq K$ we can find $i_1 \in I$ and $i_2, \dots i_t \in K \setminus I$ such that $\mathcal{A}_{i_1, i_2, \dots, i_t} = 1$. A hydra is *irreducible* whenever so is its adjacency tensor. It is well-known that a digraph is strongly connected if and

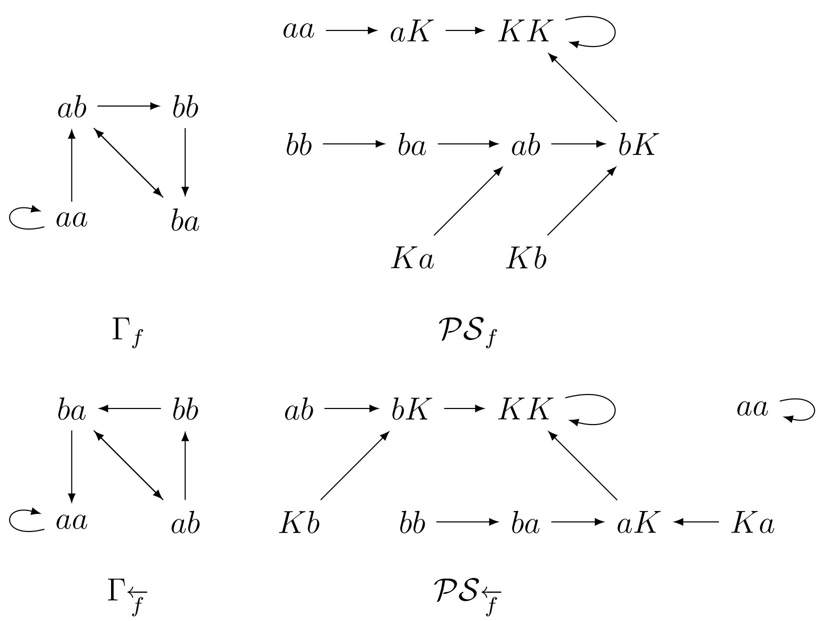


Figure 6: The 2-hydra f on $K = \{a, b\}$ is primitive while its reversal is not strongly connected. We do not include those vacant vertices when displaying the phase spaces. The symbols aa and aK appeared in the two phase spaces should be understood as $\{a\} \times \{a\}$ and $\{a\} \times K$, respectively, and so on.

only if it is irreducible. It is easy to show one direction of it for general hydras. But the equivalence itself cannot be generalized to hydras.

Proposition 6. *Let f be a t-hydra on a set K. If f is strongly connected, then the adjacency tensor $\mathcal{A}(f)$ is irreducible.*

Proof. Assume to the contrary that there is a set I such that $\emptyset \subsetneq I \subsetneq K$ and that $\mathcal{A}(f)_{i_1,\ldots,i_{t+1}} = 0$ for every $i_1 \in I$ and $i_2, \ldots, i_{t+1} \in K \setminus I$. Pick any $a \in (K \setminus I)^t$ and $b \in I^t$. It is clear that $\mathcal{RI}_f(a, b) = \emptyset$ and so f is not strongly connected, yielding a contradiction. $\square$

Example 7. Let f be the 2-hydra on $[2]$ with arc set $\mathrm{A}(f) = \{(1, 2) \to (2, 1), (2, 1) \to (1, 2)\}$. Then, f is not strongly connected but is irreducible.

Let $\mathcal{A}$ be an order-$(t + 1)$ dimension-K Boolean tensor and let f be the corresponding hydra. Let $\mathfrak{C}_{\mathcal{A}}$ be the map from 2^K to 2^K that sends $A \in 2^K$ to $B \in 2^K$ where $(\overbrace{A, \ldots, A}^{t}, B) = \mathrm{M}_f(\overbrace{A, \ldots, A}^{t+1})$. Following [CPZ11, WXZ16], we call $\mathcal{A}$ a *CPZ-primitive* tensor/digraph if there exists a positive integer N such that $\mathfrak{C}^n_{\mathcal{A}}(A) = K$ for all integers $n \geqslant N$ and for all $A \in 2^K \setminus \{\emptyset\}$. Similar to Proposition 6, a CPZ-primitive tensor is necessarily irreducible. For digraphs, being primitive and being CPZ-primitive are the same. But they are different properties for general hydras. In [WXZ16, Example 3.5],

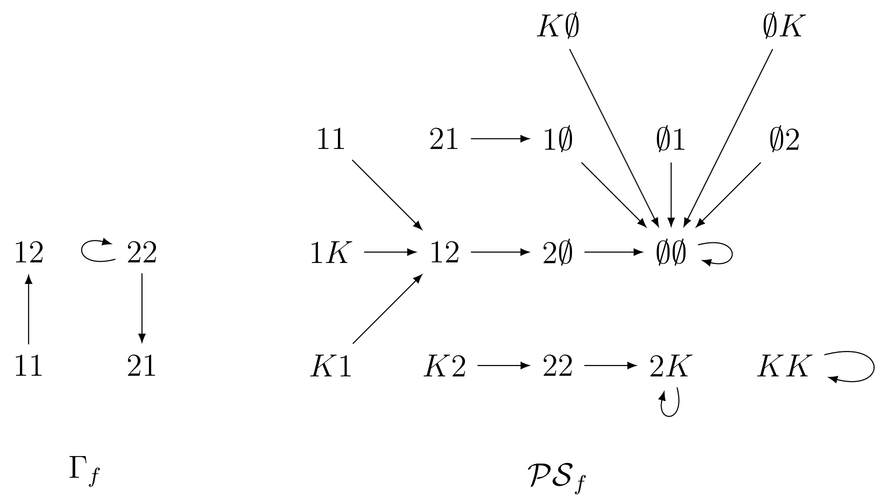


Figure 7: A CPZ-primitive but not strongly connected hydra.

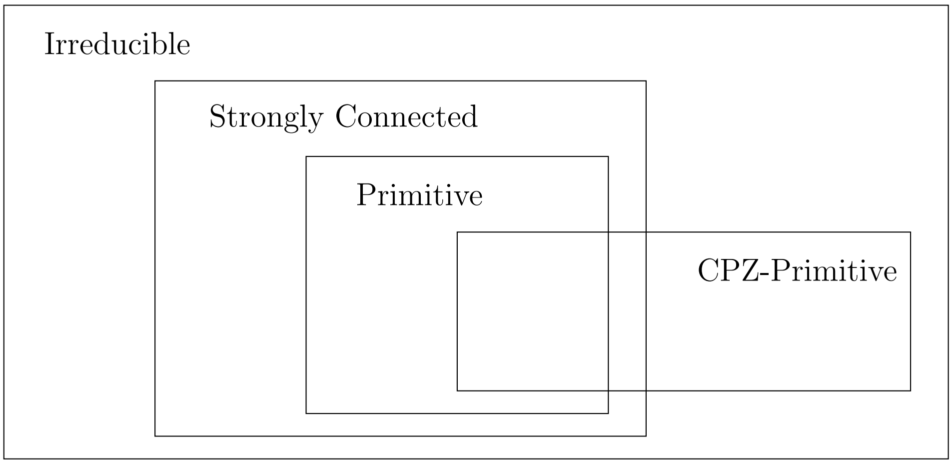


Figure 8: Relationship among several classes of hydras.

we display a primitive hydra which is not CPZ-primitive. The next example, Example 8, shows that being CPZ-primitive may not imply being strongly connected. In Figure 8, we briefly demonstrate the relationship among primitive, CPZ-primitive, strongly connected and irreducible hydras.

Example 8. Let f be the 2-hydra on $[2]$ such that $\mathcal{A}(f)_{i,j,k} = 1$ if and only if

$$ijk \in \{211, 222, 122\}.$$

We depict Γ_f and $\mathcal{PS}_f$ in Figure 7. We can check that $\mathcal{A}(f)$ is CPZ-primitive but not strongly connected.

Let $\mathcal{A}$ be a t-hydra on a set K. Let us define $\mathcal{A}^{\natural 1}$ to be the digraph on K with $\mathcal{A}^{\natural 1}_{i,j} = \mathcal{A}_{i_1,\ldots,i_{t+1}}$ where $i_1 = i, j_1 = \cdots = j_t = j$, and then, for every $n \in \mathbb{N}$, define

$\mathcal{A}^{\natural n+1}$ inductively by setting $\mathcal{A}^{\natural n+1}_{i,j} = \max\{\mathcal{A}_{i,i_1,\ldots,i_t}\mathcal{A}^{\natural n}_{i_1,j}\cdots\mathcal{A}^{\natural n}_{i_t,j} : i_1,\ldots,i_t \in K\}$. For every $j \in K$, $\mathcal{A}$ is said to be *j-primitive* [YHY14, Definition 2.14] provided there exists $n \in \mathbb{N}$ such that $\mathcal{A}^{\natural n}_{i,j} = 1$ holds for all $i \in K$. One can check that $\mathcal{A}^{\natural n}_{i,j} > 0$ if and only if $i \in \mathfrak{C}^n_{\mathcal{A}}(\{j\})$ for every $i, j \in K$. Therefore, $\mathcal{A}$ is CPZ-primitive if and only if $\mathcal{A}$ is j-primitive for all $j \in K$. For $t = 1$, it is well-known that $\mathcal{A}$ is primitive if and only if it is CPZ-primitive and if and only if it is irreducible and j-primitive for any $j \in K$. Yuan et al. posed the conjecture [YHY15, Conjecture 4.5] that this observation for digraphs holds for general hydras, namely a hydra $\mathcal{A}$ on K is CPZ-primitive if and only if $\mathcal{A}$ is irreducible and there exists some $j \in K$ such that $\mathcal{A}$ is j-primitive. The following example shows that the situation is more intricate than expected.

Example 9. Let $\mathcal{A}$ be the 2-hydra on [3] such that $\mathcal{A}_{i,j,k} = 1$ if and only if

$$ijk \in \{111, 211, 311, 132, 123, 322, 233\}.$$

We can check that $\mathcal{A}$ is irreducible. For $n \in \mathbb{N}$, we can also check that

$$\mathfrak{C}^n_{\mathcal{A}}(\{1\}) = [3],$$

$$\mathfrak{C}^n_{\mathcal{A}}(\{2\}) = \begin{cases} \{3\}, & \text{if } n \text{ is odd,} \\ \{2\}, & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad \mathfrak{C}^n_{\mathcal{A}}(\{3\}) = \begin{cases} \{2\}, & \text{if } n \text{ is odd,} \\ \{3\}, & \text{if } n \text{ is even.} \end{cases}$$

This means that $\mathcal{A}$ is neither 2-primitive nor 3-primitive, and hence not CPZ-primitive, but 1-primitive. Note that this refutes [YHY15, Conjecture 4.5]. Also observe that $\mathcal{A}$ is not strongly connected as $\mathcal{RI}_{\mathcal{A}}(13, b) = \emptyset$ for all $b \in [3]^2 \setminus \{13\}$.

3 Main results

Let f be a t-hydra on K. It is natural to consider the *asymptotic equivalence for f*, denoted by $\sim_f$, which is the binary relation on K^t such that, for all $a, b \in K^t$, $a \sim_f b$ if and only if there exists a positive integer m such that $\mathrm{M}^m_f(a) = \mathrm{M}^m_f(b)$. It is clear that asymptotic equivalence is an equivalence relation. Let $\mathcal{C}(f)$ be the set consisting of all equivalence classes of $\sim_f$. We construct the digraph f^* with vertex set $\mathcal{C}(f)$ and arc set $\{C_1 \to C_2 : C_1, C_2 \in \mathcal{C}(f) \text{ and } \mathrm{M}_f(C_1) \cap C_2 \neq \emptyset\}$. Let $\mathrm{per}(f)$ denote the greatest common divisor of $\bigcup_{a\in K^t} \mathcal{RI}_f(a, a)$ and call it the *period* of f. Having in mind the partial order relation of divisibility on the nonnegative integers, we adopt the convention that the greatest common divisor of the set $\{0\}$ is 0. We call a digraph on one vertex without any arc a *0-cycle*. Note that both 0-cycle and 1-cycle are strongly connected, but among them only 1-cycle is primitive. If f is strongly connected and $\mathrm{per}(f) = 0$, we can deduce that $\mathrm{V}(f)$ is a singleton set and $\mathrm{A}(f) = \emptyset$ and hence Γ_f is a 0-cycle. The next result means that the cyclicity theorem for digraphs indeed holds for all hydras, namely every strongly connected hydra with a finite diameter looks like a cycle when viewed from a distance.

Theorem 10. *Let f be a strongly connected hydra with a finite diameter. Then f^* is a* $\operatorname{per}(f)$*-cycle. Moreover, if* $\operatorname{per}(f) > 0$*, then there exists $m \in \mathbb{N}$ such that* $\mathrm{M}_f^n(b)$ *is an equivalence class of $\sim_f$ for every $b \in \mathrm{V}(f)$ and $n \geqslant m$.*

Here is an immediate consequence of Theorem 10. It is noteworthy that a counterpart of it for the CPZ-primitiveness has been established by Cui et al. [CLN15, Theorem 4] in the language of directed hypergraph.

Corollary 11. *Let f be a hydra with a finite diameter. Then, f is primitive if and only if f is strongly connected and* $\operatorname{per}(f) = 1$*.*

Example 12. Let $\mathbb{Z}$ be the set of all integers. Let f be the digraph with $\mathrm{V}(f) = \mathbb{Z}$ and $\mathrm{A}(f) = \{a \to b: \ a \in \mathbb{Z}, b - a = 1\} \cup \{a \to b: \ a \in \mathbb{N}, b = -a - 1\}$. Then f is strongly connected, $\operatorname{per}(f) = 2$, no two different vertices of f can be $\sim_f$ equivalent and so surely f^* is not a 2-cycle.

Let $t \in \mathbb{N}$ and let K be a set. We use $\mathcal{P}(t, K)$ to stand for the set of periods of all those strongly connected t-hydras f on K with $\mathrm{A}(f) \neq \emptyset$ and $\mathrm{Dia}(f) < \infty$. For any $k \in \mathbb{N}$, we write $\mathcal{P}(t, k)$ for $\mathcal{P}(t, [k])$. Let $\mathcal{P}(t)$ stand for $\cup_{k \in \mathbb{N}} \mathcal{P}(t, k)$. Considering all cycles, we see that $\mathcal{P}(1) = \mathbb{N}$.

The periods of strongly connected hydras turn out to be related to a problem on combinatorial design, namely the construction of cyclic decompositions. We elucidate this problem in the sequel. For a map Φ defined on a cyclic group $\mathbb{Z}_p$, we often write Φ_i for $\Phi(i)$ for any $i \in \mathbb{Z}_p$. Let t be a positive integer, K a set, and $X \subseteq K^t$. A *cyclic decomposition* of (X, K, t) with *period* $p \in \mathbb{N}$, also called a *cyclic decomposition of X relative to K with period p*, is a map $\Phi: \ \mathbb{Z}_p \to 2^K \setminus \{\emptyset\}$ such that

$$\bigcup_{i \in \mathbb{Z}_p} \big(\Phi_i \times \cdots \times \Phi_{i+t-1}\big) = X \tag{3}$$

and

$$\big(\Phi_i \times \cdots \times \Phi_{i+t-1}\big) \bigcap \big(\Phi_j \times \cdots \times \Phi_{j+t-1}\big) = \emptyset \tag{4}$$

for all $\{i, j\} \in \binom{\mathbb{Z}_p}{2}$. We use the notation $\operatorname{per}(\Phi)$ for the period p of Φ.

Theorem 13. *Let K be a set and t be a positive integer. There exists a cyclic decomposition Φ of (K^t, K, t) with* $\operatorname{per}(\Phi) = p$ *if and only if $p \in \mathcal{P}(t, K)$.*

Theorem 14. *Let t be a positive integer and let K be any infinite set. Then $\mathcal{P}(t) = \mathcal{P}(t, K) \supseteq \cdots \supsetneq \mathcal{P}(t, 4) \supsetneq \mathcal{P}(t, 3) \supsetneq \mathcal{P}(t, 2) \supsetneq \mathcal{P}(t, 1) = \{1\}$.*

Theorem 15. *For every $t \in \mathbb{N}$, it holds $|\mathbb{N} \setminus \mathcal{P}(t)| < \infty$.*

Proposition 16. *Let k and t be two positive integers. If $([k]^t, [k], t)$ admits a cyclic decomposition of period p, then either $p = 1$ or $p \geqslant 2t$.*

Theorem 17. $\bigcap_{t \in \mathbb{N}} \mathcal{P}(t) = \{1\}$.

For any two integers m and n, we often write $[m,n]$ for $\{\ell :\ m \leqslant \ell \leqslant n\}$. Note that $[1,n] = [n]$. Consider $t \in \mathbb{N}$. By Theorem 15, there exists a minimum integer $\alpha(t)$ such that $[\alpha(t), \infty] \subseteq \mathcal{P}(t)$. By Theorem 13 and Proposition 16, we have $2t - 1 \notin \mathcal{P}(t)$ for all $t \geqslant 2$. Accordingly, we have $\alpha(t) \geqslant 2t$ for all $t \geqslant 2$. We need a bit more effort to deduce the ensuing Proposition 18. In our subsequent work [QWZ17], we will show that $\alpha(t) \geqslant 2^t - t$ for all $t \in \mathbb{N}$.

Proposition 18. *Take $t \in \mathbb{N}$. If $t \geqslant 2$, then $2t+1 \notin \mathcal{P}(t)$; if $t \geqslant 4$, then $2t+2 \notin \mathcal{P}(t)$. In particular, $\alpha(t) \geqslant 2t+2$ when $t \geqslant 2$ and $\alpha(t) \geqslant 2t+3$ when $t \geqslant 4$.*

Conjecture 19. For every $t \in \mathbb{N}$, there exists a minimum integer $\beta(t)$ such that

$$[\alpha(t), k^t - \beta(t)] \subseteq \mathcal{P}(t,k)$$

for all $k \in \mathbb{N}$.

Note that $\alpha(1) = 1$ and $\beta(1) = 0$. If Conjecture 19 is true, then the next proposition means that $\beta(t) \geqslant 2t$ for $t \geqslant 2$.

Proposition 20. *Let t and k be two integers such that $t \geqslant 2$ and $k \geqslant 2t$. Then $k^t - 2t + 1 \notin \mathcal{P}(t,k)$.*

Proposition 21. *For every $k \in \mathbb{N}$, it holds $k^2 - 1 \notin \mathcal{P}(2,k)$.*

Theorem 22. $\mathcal{P}(2) = \mathbb{N} \setminus \{2,3,5,6,7\}$.

Example 23. Theorem 22 asserts that $2 \times 2 + 2 = 6 \notin \mathcal{P}(2)$. But we cannot strengthen the claim in Proposition 18 to $2t+2 \notin \mathcal{P}(t)$ for $t \geqslant 2$. Indeed, we let Φ be the map from $\mathbb{Z}_8$ to $[2]$ that sends $i \in \mathbb{Z}_8$ to

$$\Phi(i) := \begin{cases} \{1\}, & \text{if } i \in \{1,2,3,7\}, \\ \{2\}, & \text{if } i \in \{4,5,6,8\}. \end{cases}$$

We can check that Φ is a cyclic decomposition of $([2]^3, [2], 3)$ with period 8 – Lemma 41 will indicate a more general result about this construction. This establishes that $2 \times 3 + 2 = 8 \in \mathcal{P}(3,2)$.

For any $t \in \mathbb{N}$, we have found that $2t \in \mathcal{P}(t)$ if and only if $t \in \{1,2,4\}$ while $2t+3 \in \mathcal{P}(t)$ if and only if $t = 1$ [QWZ17]. In view of this, as well as Theorem 13, Proposition 16, Proposition 18, Theorem 22 and Example 23, we can obtain the structure of the set $\mathcal{P}(t) \cap [2t+2]$ for every $t \in \mathbb{N}$. Going one step further, we [QWZ17] will find that

$$\mathcal{P}(t) \cap [3t-2] = \{1\}$$

for $t \geqslant 5$ and determine $\mathcal{P}(t) \cap [2t+3]$ for $t \leqslant 4$ as summarized in Table 1:

	$t=1$	$t=2$	$t=3$	$t=4$
$\mathcal{P}(t) \cap [2t+3]$	$[2t+3]$	$\{1, 2t\}$	$\{1, 2t+2\}$	$\{1, 2t\}$

Table 1: Periods of strongly connected t-hydras in $[2t+3]$.

We can conclude that $9 \notin \mathcal{P}(3)$, for which we do not include a proof here due to the length constraint. We mention that this implies $\alpha(3) \geqslant 10$. Note that Theorem 22 shows $\alpha(2) = 8 > 1 = \alpha(1)$.

Conjecture 24. It holds $\alpha(t+1) > \alpha(t)$ for all $t \in \mathbb{N}$.

Theorem 25. *$\mathcal{P}(2,1) = \{1\}$, $\mathcal{P}(2,2) = \{1,4\}, \mathcal{P}(2,3) = \{1,4,9\}$ and $\mathcal{P}(2,4) = \{1, 4, 8, 9, 10, 11, 12, 14, 16\}$.*

Proposition 26. *$\mathcal{P}(3) \subsetneq \mathcal{P}(2)$.*

For $t = 1, 2$, the next conjecture, Conjecture 27, is verified by Theorem 22 and Proposition 26, respectively.

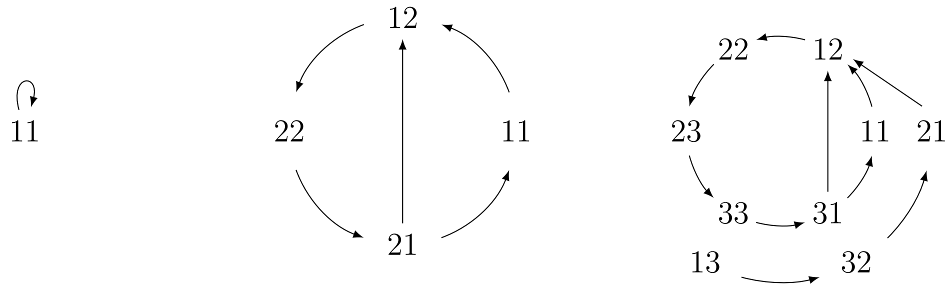
Conjecture 27. For every $t \in \mathbb{N}$, $\mathcal{P}(t+1) \subsetneq \mathcal{P}(t)$.

The minimum possible m that satisfies the requirement in Theorem 10 is called the *transient time of f*. When f is primitive, the transient time is just the primitive exponent $\mathrm{g}(f)$. A good estimate of transient time for a strongly connected hydra may not be easy. We only have a look at the primitive exponent in this paper. For any $t, k \in \mathbb{N}$, define $\gamma(t,k)$ to be the maximum possible value of the primitive exponent of a primitive t-hydra on $[k]$.

Theorem 28 (Wielandt [Sch02])**.** *$\gamma(1,k) = (k-1)^2 + 1$ for all $k \in \mathbb{N}$.*

Let f and g be two t-hydras on a set K. We say that f is *weakly isomorphic* to g if there exists a bijection ϕ from $\mathrm{V}(f)$ to $\mathrm{V}(g)$, called a *weak isomorphism*, such that for all $x, y \in \mathrm{V}(f)$, $xy \in \mathrm{A}(f)$ if and only if $\phi(x)\phi(y) \in \mathrm{A}(g)$. We say that f is *isomorphic* to g if there exists a permutation τ on K such that $(\underbrace{\tau, \ldots, \tau}_{t})$ gives rise to a weak isomorphism from f to g. It is not obvious if there will be any relationship between $\mathcal{PS}_f$ and $\mathcal{PS}_g$ provided f and g are weakly isomorphic. But if f and g are isomorphic, their dynamical behaviours can be said to be really of no difference.

Example 29. We use computer programming to enumerate all 2-hydras on sets with at most 3 elements. It is found that $\gamma(2,1) = 1, \gamma(2,2) = 7$ and $\gamma(2,3) = 23$. Up to isomorphism, the hydras f_1, f_2 and f_3 given in Figure 9 are the only primitive 3-hydras on one, two and three elements whose primitive exponents equal $\gamma(2,1)$, $\gamma(2,2)$ and $\gamma(2,3)$, respectively.



Γ_{f_1} Γ_{f_2} Γ_{f_3}

Figure 9: $\gamma(2,1) = \mathrm{g}(f_1) = 1, \gamma(2,2) = \mathrm{g}(f_2) = 7, \gamma(2,3) = \mathrm{g}(f_3) = 23$.

Let r_k be the minimum number of multiplications to multiply two k by k complex matrices. By coincidence, $r_1 = 1 = \gamma(2,1), r_2 = 7 = \gamma(2,2), r_3 \leqslant 23 = \gamma(2,3)$ [Lan12, p. 5, p. 283]. Can we expect

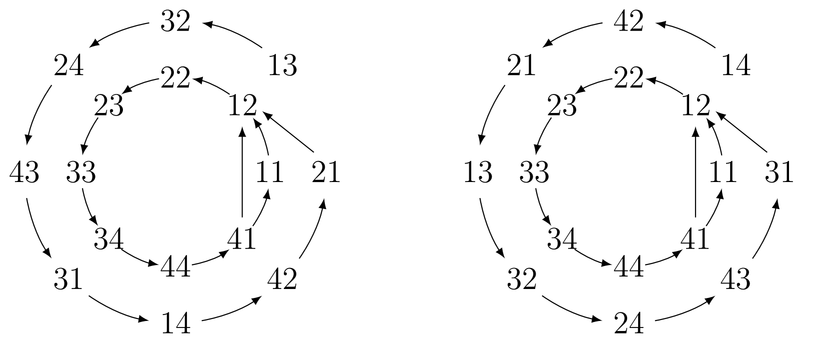
$$r_k \leqslant \gamma(2,k) \tag{5}$$

for all $k \in \mathbb{N}$? Note that we [WXZ16, Conjecture 3.6] have made the conjecture that

$$\gamma(2,k) = O(k^2) \tag{6}$$

when k approaches the infinity. If both Eq. (5) and Eq. (6) are correct, we will have the astounding result that, asymptotically, it is as easy to multiply matrices as it is to add them.

Example 30. In Figure 10 we display two primitive 2-hydras f_4 and g_4 on $[4]$. It holds $\mathrm{g}(f_4) = \mathrm{g}(g_4) = 50$. Note that f_4 and g_4 are weakly isomorphic but not isomorphic. We mention that $\mathrm{Dia}(g_4) = 28 > 23 = \mathrm{Dia}(f_4)$.



Γ_{f_4} Γ_{g_4}

Figure 10: $\gamma(2,4) \geqslant \mathrm{g}(f_4) = \mathrm{g}(g_4) = 50 > 28 = \mathrm{Dia}(g_4) = \mathrm{Dist}_{g_4}(42,14) > 23 = \mathrm{Dia}(f_4) = \mathrm{Dist}_{f_4}(32,13)$.

The parameter 50 appeared in Example 30 can be expressed as $50 = (2 \times 4 - 1)^2 + 1$. Our Theorem 31 explains why this expression has special meaning for us.

Theorem 31. *It holds* $\gamma(2, k) \geqslant (2k - 1)^2 + 1$ *for all* $k \geqslant 4$.

Theorem 32. *It holds* $\gamma(t, k) \geqslant k^t$ *for all* $k, t \in \mathbb{N}$.

In [CW17], we will show that $\gamma(t, k) \leqslant t(k - 1)(k^t - 1) + 1$ for all $k, t \in \mathbb{N}$.

Unlike the case of $t = 1$, to determine $\gamma(t, k)$ for general t and k looks to be quite challenging. In contrast, we mention that for the concept of CPZ-primitiveness, it turns out that the maximum primitive exponents of Boolean tensors are not related to the order of the tensors [YHY14, Theorem 1.2], namely the number of variables does not make any difference then.

For any $t, k \in \mathbb{N}$, let $\mathrm{D}_{t,k}$ be the maximum possible diameter of a strongly connected t-hydra f on a set of size k.

Example 33. Making use of computer search, we can determine that $\mathrm{D}_{2,2} = 4$ and $\mathrm{D}_{2,3} = 15$. Up to isomorphism, there are four 2-hydras on [2] whose diameters achieve $\mathrm{D}_{2,2} = 4$ and twenty-six 2-hydras on [3] whose diameters achieve $\mathrm{D}_{2,3} = 15$. It is surprising that all these 2-hydras are primitive while the reversal of none of them is strongly connected. Indeed, those four 2-hydras on [2] all have primitive exponent 4 and the primitive exponents of those twenty-six 2-hydras on [3] range among $15, 16, 17$ and 18. We display three such extremal hydras in Figure 11. The hydra g_4 in Figure 10 shows the fact that $\mathrm{D}_{2,4} \geqslant 28$.

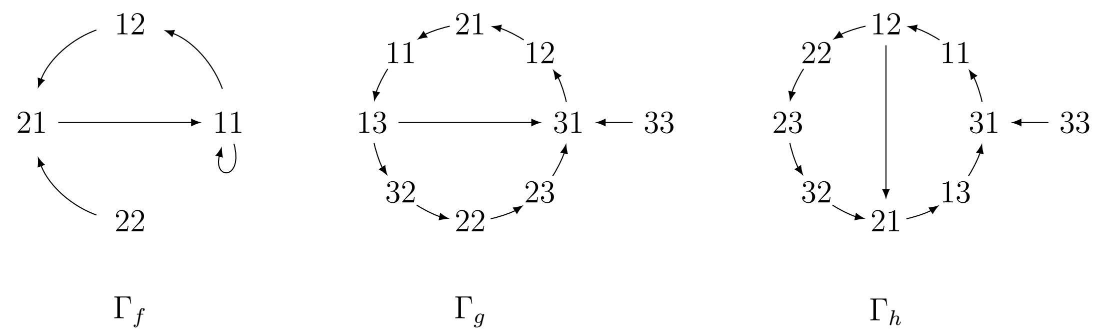


Figure 11: $\mathrm{D}_{2,2} = \mathrm{Dist}_f(12, 22) = \mathrm{g}(f) = 4$, $\mathrm{D}_{2,3} = \mathrm{Dist}_g(32, 33) = \mathrm{g}(g) = \mathrm{Dist}_h(22, 33) = 15$ and $\mathrm{g}(h) = 18$.

Example 33 is about our knowledge of $\mathrm{D}_{2,k}$ for $k \leqslant 4$. We now provide an estimate of $\mathrm{D}_{2,k}$ for $k \geqslant 5$.

Theorem 34. *For any integer k greater than 4, it holds*

$$\mathrm{D}_{2,k} \geqslant \begin{cases} 2k^2, & \textit{if } k \textit{ is odd}, \\ 2k^2 - k + 1, & \textit{if } k \textit{ is even}. \end{cases}$$

Our proof of Theorem 31 and Theorem 34 relies on the construction of a family of 2-hydras introduced at the beginning of § 6. The experience of going through that proof leads us to the following conjecture.

Conjecture 35. $\gamma(2,k) = (2k-1)^2 + 1$ for $k \geqslant 4$ and $\lim_{k\to\infty} \frac{\mathrm{D}_{2,k}}{k^2} = 2$.

In order to unfold deep secrets hidden in seemingly messy high dimensional data, it is often crucial to get a "correct" definition of a cycle in a hypergraph [Wan08, Wan11]. For hydras, we also face the problem of deciding what is a real cycle or a real hole of a multilinear map. A cycle in the usual digraph setting could be characterized in many ways. For example, the n-cycle $\mathscr{C}_n$ could be characterized as the strongly connected digraph on n vertices with maximum possible diameter and then with minimum number of arcs. We leave it as an open problem to find a good definition of a cycle hydra, which hopefully can bring forth many new concepts for hydra theory, say zeta functions [Ter11] or homology theory [Big97], besides those followed from the definition of reachable indices.

4 Generalized cycles

This section is mainly devoted to a proof of Theorem 10, which closely follows the usual proof of the corresponding result for digraphs. With the help of Theorem 10, we will deduce Theorem 13, a result which becomes trivial for digraphs.

Lemma 36. *Take a set K, an integer $t \in \mathbb{N}$, and $a, b \in K^t$. Let f be a t-hydra on K. If $i \in \mathcal{RI}_f(a,b)$ and $j \in \mathcal{RI}_f(b,c)$, then $i+j \in \mathcal{RI}_f(a,c)$.*

Proof. From $b \in \mathrm{M}_f^i(a)$ and $c \in \mathrm{M}_f^j(b)$, we surely get $c \in \mathrm{M}_f^{i+j}(a)$, that is, $i+j \in \mathcal{RI}_f(a,c)$. $\square$

Lemma 37. *Let f be a strongly connected t-hydra on a set K.*

(a) For every $a \in K^t$, the greatest common divisor of $\mathcal{RI}_f(a,a)$ equals to $\mathrm{per}(f)$.

(b) For any $a, b \in K^t$, the elements of $\mathcal{RI}_f(a,b)$ are congruent modulo $\mathrm{per}(f)$.

Proof. (a) Let a and b be two vertices of f. Let p_a and p_b denote the greatest common divisors of $\mathcal{RI}_f(a,a)$ and $\mathcal{RI}_f(b,b)$, respectively. It suffices to show $p_a = p_b$. Let $s \in \mathcal{RI}_f(a,a)$. Due to the symmetry between a and b, we need only verify that p_b is a divisor of s. Since f is strongly connected, we can take $t_1 \in \mathcal{RI}_f(b,a)$ and $t_2 \in \mathcal{RI}_f(a,b)$. By Lemma 36, $t_1 + t_2$ and $s + t_1 + t_2$ are contained in $\mathcal{RI}_f(b,b)$ and therefore p_b divides s, as required.

(b) Take $t_1, t_2 \in \mathcal{RI}_f(a,b)$. Pick $s \in \mathcal{RI}_f(b,a)$ and hence, by Lemma 36, $s+t_1, s+t_2 \in \mathcal{RI}_f(a,a)$. According to (a), $\mathrm{per}(f)$ is a divisor of both $s+t_1$ and $s+t_2$. This shows that $t_1 - t_2 = (s+t_1) - (s+t_2)$ is a multiple of $\mathrm{per}(f)$, finishing the proof. $\square$

For any hydra f with period p, any $a \in \mathrm{V}(f)$ and $i \in \mathbb{N}$, use $\mathrm{C}_{i,a}(f)$ as a shorthand for $\bigcup_{\substack{m\in\mathbb{N}\cup\{0\}\\ m\equiv i \pmod p}} \mathrm{M}_f^m(a)$. Note that $\mathrm{C}_{i,a}(f) = \mathrm{C}_{i+p,a}(f)$ and so it is often natural to view the parameter i of $\mathrm{C}_{i,a}(f)$ as an element of $\mathbb{Z}_p$.

Lemma 38. *Let f be a strongly connected t-hydra on a set K with period $p > 0$.*

(a) For each $a \in \mathrm{V}(f)$, $\{\mathrm{C}_{i,a}(f) : i \in [p]\}$ form a partition of $\mathrm{V}(f)$.

(b) For all $a \in \mathrm{V}(f)$ and $i \in [p]$, $\mathrm{M}_f\left(\mathrm{C}_{i,a}(f)\right) = \mathrm{C}_{i+1,a}(f)$.

(c) Let a, b and c be three vertices of f. If $b \sim_f c$, then there exists $i \in [p]$ such that $b, c \in \mathrm{C}_{i,a}(f)$.

Proof. (a) Since f is a strongly connected, $\bigcup_{i\in[p]} \mathrm{C}_{i,a}(f) = \mathrm{V}(f)$. By Lemma 37(b), $\mathrm{C}_{1,a}(f), \ldots, \mathrm{C}_{p,a}(f)$ are mutually disjoint.

(b) It is obvious from the definition that $\mathrm{M}_f\left(\mathrm{C}_{i,a}(f)\right) \subseteq \mathrm{C}_{i+1,a}(f)$. By Lemma 37(a), $\mathcal{RI}_f(a, a)$ contains a positive multiple of p, say rp for some $r \in \mathbb{N}$. Since $p > 0$, $rp-1 \geqslant 0$ holds. Take any $b \in \mathrm{C}_{i+1,a}(f)$. By definition, there exists a nonnegative integer j such that $b \in \mathrm{M}_f^j(a)$ and $j \equiv i + 1 \pmod p$. Consequently,

$$b \in \mathrm{M}_f^j(a) \subseteq \mathrm{M}_f^{rp+j}(a) = \mathrm{M}_f\left(\mathrm{M}_f^{rp-1+j}(a)\right) \subseteq \mathrm{M}_f\left(\mathrm{C}_{i,a}(f)\right).$$

This gives $\mathrm{M}_f\left(\mathrm{C}_{i,a}(f)\right) = \mathrm{C}_{i+1,a}(f)$.

(c) This is a consequence of (a) and (b). □

The next result, Lemma 39, is in the folklore [KS76, Theorem 1.4.1] [RA05, Theorem 1.0.1]. The number $N(X)$ appeared in it is the so-called Frobenius number in the Postage Stamp Problem. Its estimate may be useful in bounding the transient time of a strongly connected hydra.

Lemma 39. *Let X be a nonempty subset of positive integers closed under integer addition and let $\gcd(X) = p$. Then there exists a smallest nonnegative integer $N(X)$ such that, for all integers $x > N(X)$, $x \in X$ holds if and only if $p \mid x$.*

Lemma 40. *Let f be a strongly connected t-hydra on set K with $\mathrm{Dia}(f) < \infty$ and $\mathrm{per}(f) = p \in \mathbb{N}$. Let $a \in K^t$. There exists a number $m \in \mathbb{N}$ such that $\mathrm{M}_f^{\ell}(b) = \mathrm{C}_{j,a}(f)$ whenever $b \in \mathrm{C}_{i,a}(f)$, $\ell > m$ and $\ell + i \equiv j \pmod p$.*

Proof. Applying Lemma 39 for $X = \mathcal{RI}_f(a, a)$, we know the existence of a positive integer R such that $pr \in \mathcal{RI}_f(a, a)$ whenever $r \geqslant R$. Lemma 36 then implies that $x + pr \in \mathcal{RI}_f(a, b)$ for any $b \in \mathrm{V}(f), x \in \mathcal{RI}_f(a, b)$ and $r \geqslant R$. By Lemma 38(b), we can take $m = \mathrm{Dia}(f) + Rp$, completing the proof. □

Proof of Theorem 10. The result is trivial when $\mathrm{per}(f) = 0$. When $\mathrm{per}(f) \in \mathbb{N}$, it follows from Lemma 38 and Lemma 40. □

Proof of Theorem 13. Let Φ be a cyclic decomposition of (K^t, K, t) with $\mathrm{per}(\Phi) = p \in \mathbb{N}$. For each $i \in \mathbb{Z}_p$, we let C_i be the nonempty set $\Phi_i \times \cdots \times \Phi_{i+t-1}$. Consider the t-hydra f on K with

$$\mathrm{A}(\Gamma_f) = \{(a_1, \ldots, a_t) \to (a_2, \ldots, a_{t+1}) : \exists i \in \mathbb{Z}_p, (a_1, \ldots, a_t) \in C_i, (a_2, \ldots, a_{t+1}) \in C_{i+1}\}.$$

It is clear that f is a strongly connected t-hydra with $\mathrm{Dia}(f) \leqslant p+t$ and with C_i, $i \in \mathbb{Z}_p$, being all the $\sim_f$ equivalence classes. It then follows from Theorem 10 that $p = \mathrm{per}(f) \in \mathcal{P}(t, K)$.

Conversely, assume there exists a strongly connected t-hydra f on K with period $p \in \mathbb{N}$ and finite diameter. Observe that, for any $b \in K^t$ and $n \in \mathbb{N}$, $\mathrm{M}_f^n(b)$ must be of the form $L_1 \times \cdots \times L_t$, where L_i are nonempty subsets of K for $i \in [t]$. Therefore, according to Theorem 10, there exists a cyclic decomposition Φ of (K^t, K, t) with period $p \in \mathbb{N}$, for which $\Phi_i \times \cdots \times \Phi_{i+t-1}$ is an equivalence class of $\sim_f$ for every $i \in \mathbb{Z}_p$. $\square$

5 Periods

For any set K, $t \in \mathbb{N}$ and $X \subseteq K^t$, we call a cyclic decomposition Φ of (X, K, t) a *discrete cyclic decomposition* if $\mathrm{per}(\Phi) = |X|$.

Lemma 41. *Let K be a set and and t be a positive integer. Then (K^t, K, t) admits a discrete cyclic decomposition if and only if $|K| \in \mathbb{N}$.*

Proof. The existence of a discrete decomposition Φ forces $|K^t| = \mathrm{per}(\Phi) \in \mathbb{N}$ and hence $|K| \in \mathbb{N}$. The other direction is simply the well-known fact that "Every De Bruijn digraph has a Hamiltonian cycle;" see [BJG09, Theorem 6.9.8] [BM08, Exercise 3.4.10] [dB46, Goo46]. $\square$

Take $k, t \in \mathbb{N}$. If a cyclic decomposition Φ of $X \subseteq [k]^t$ relative to $[k]$ satisfies $\Phi_i = \{1\}$ for all $i \in [t-1]$, we call Φ a *strong cyclic decomposition of* $(X, [k], t)$. For any $X \subseteq [k]^t$, we set

$$\mathcal{P}^*(X, [k], t) := \{\mathrm{per}(\Phi) : \ \Phi \text{ is a strong cyclic decomposition of } (X, [k], t)\}.$$

Let $\mathcal{P}^*(t, k) := \mathcal{P}^*([k]^t, [k], t)$ and $\mathcal{P}^*(t) := \bigcup_{\ell \in \mathbb{N}} \mathcal{P}^*(t, \ell)$.

Lemma 42. *Let t be a positive integer.*

(a) If K_1 and K_2 are two nonempty sets satisfying $K_1 \subseteq K_2$, then $\mathcal{P}(t, K_1) \subseteq \mathcal{P}(t, K_2)$.

(b) If k_1 and k_2 are two integers such that $1 < k_1 \leqslant k_2$, then $\mathcal{P}^*(t, k_1) \subseteq \mathcal{P}^*(t, k_2)$.

Proof. For any map Φ from $\mathbb{Z}_p$ to $2^K \setminus \{\emptyset\}$, any $k \in K$ and any set X disjoint from K, we put $\Phi^{k,X}$ to be the map from $\mathbb{Z}_p$ to $2^{K \cup X} \setminus \{\emptyset\}$ satisfying

$$\Phi_i^{k,X} = \begin{cases} \Phi_i, & \text{if } k \notin \Phi_i, \\ \Phi_i \cup X, & \text{if } k \in \Phi_i, \end{cases}$$

for all $i \in \mathbb{Z}_p$. If Φ is a cyclic decomposition of (K_1^t, K_1, t) with period $p \in \mathbb{N}$, we can check that, for any $k \in K_1$, $\Phi^{k, K_2 \setminus K_1}$ is a cyclic decomposition of (K_2^t, K_2, t) with period p. Claim (a) now follows from Theorem 13. If Φ is a strong cyclic decomposition of $([k_1]^t, [k_1], t)$ and $k_2 \geqslant k_1 > 1$, $\Phi^{k_1, [k_1+1, k_2]}$ is a strong cyclic decomposition of $([k_2]^t, [k_2], t)$ with period p, yielding (b). $\square$

Lemma 43. *Let $\ell, k, t \in \mathbb{N}$ and assume $\ell \leqslant k$. Then $\ell^t \in \mathcal{P}(t,k)$. If $1 < \ell$, then we also have $\ell^t \in \mathcal{P}^*(t,k)$.*

Proof. This follows immediately from Lemma 41 and Lemma 42. □

Proof of Theorem 14. Let $k \in \mathbb{N}$. By Lemma 41, $(k+1)^t \in \mathcal{P}(t,k+1)$. Since $(k+1)^t > |[k]^t|$, Theorem 13 says that $(k+1)^t \notin \mathcal{P}(t,k)$. Therefore, thanks to Lemma 42(a), it remains to show $\mathcal{P}(t,K) \subseteq \mathcal{P}(t)$.

Take $p \in \mathcal{P}(t,K)$. By Theorem 13, there exists a cyclic decomposition Φ of (K^t, K, t) with period p. For any $x \in K^t$, let $\xi(x)$ denote the unique element $i \in \mathbb{Z}_p$ such that $x \in \Phi_i \times \cdots \times \Phi_{i+t-1}$. Let $\widehat{\Phi}$ be the map from K to $\{0,1\}^{\mathbb{Z}_p}$ such that

$$\widehat{\Phi}(x)_i = \begin{cases} 0, & \text{if } x \notin \Phi_i, \\ 1, & \text{if } x \in \Phi_i, \end{cases}$$

for $i \in \mathbb{Z}_p$. Let $C \subseteq \{0,1\}^{\mathbb{Z}_p} \setminus \{\emptyset\}$ be the image of the map $\widehat{\Phi}$. Note that $|C| < 2^p < \infty$. Let Ψ be the map from $\mathbb{Z}_p$ to 2^C such that $\Psi_i = \{\widehat{\Phi}(x) : x \in \Phi_i\}$ for all $i \in \mathbb{Z}_p$. For every $(c_1, \ldots, c_t) \in C^t$ and any $x, y \in \widehat{\Phi}^{-1}(c_1) \times \cdots \times \widehat{\Phi}^{-1}(c_t)$, the definition of $\widehat{\Phi}$ guarantees that $y \in \Phi_{\xi(x)} \times \cdots \times \Phi_{\xi(x)+t-1}$, and so, the definition of a cyclic decomposition ensures $\xi(x) = \xi(y)$, which means that $\xi(x)$ is the only element i in $\mathbb{Z}_p$ such that $(c_1, \ldots, c_t) \in \Psi_i \times \cdots \times \Psi_{i+t-1}$. We now conclude that Ψ is a cyclic decomposition of (C^t, C, t) with period p, and so, by Theorem 13, $p \in \mathcal{P}(t, |C|) \subseteq \mathcal{P}(t)$, as was to be shown. □

For a map Φ defined on $\mathbb{Z}_p$, we let $\sigma_p(\Phi)$ be the map on $\mathbb{Z}_p$ such that

$$\big(\sigma_p(\Phi)\big)_i = \Phi_{i+1} \tag{7}$$

for all $i \in \mathbb{Z}_p$. If Φ is a map defined on $[p]$, we naturally identify it as a map on $\mathbb{Z}_p$ and in this way we can also talk about the map $\sigma_p(\Phi)$.

Lemma 44. *Let K be a set and t be a positive integer. Let $X \subseteq K^t$. If Φ is a cyclic decomposition of (X, K, t) with period p, then so is $\sigma_p(\Phi)$.*

Proof. Write $\Psi = \sigma_p(\Phi)$. Firstly, $\Psi_i = \Phi_{i+1} \neq \emptyset$ for $i \in \mathbb{Z}_p$. Secondly,

$$\bigcup_{i \in \mathbb{Z}_p} \big(\Psi_i \times \cdots \times \Psi_{i-t+1}\big) = \bigcup_{i \in \mathbb{Z}_p} \big(\Phi_{i+1} \times \cdots \times \Phi_{i-t+2}\big) = X.$$

Lastly, for any $\{i,j\} \in \binom{\mathbb{Z}_p}{2}$,

$$\begin{aligned} &\big(\Psi_i \times \cdots \times \Psi_{i+t-1}\big) \bigcap \big(\Psi_j \times \cdots \times \Psi_{j+t-1}\big) \\ =&\big(\Phi_{i+1} \times \cdots \times \Phi_{i+t}\big) \bigcap \big(\Phi_{j+1} \times \cdots \times \Phi_{j+t}\big) = \emptyset. \end{aligned}$$

This finishes the proof. □

For any $p \in \mathbb{N}$, when we are considering the cyclic group $\mathbb{Z}_p$ of p elements, we often think of it as the set of all residue classes modulo p. Thus, we may use $[m,n]_p$ for the residue classes to modulus p as represented by $m, m+1, \ldots, n$, and view it as a subset of $\mathbb{Z}_p$; we sometimes drop the subscript p from the notation $[m,n]_p$ if it is clear from the context. Let Φ be a map on $\mathbb{Z}_p$ and Ψ a map on $\mathbb{Z}_q$. We view them as words indexed by $\mathbb{Z}_p$ and $\mathbb{Z}_q$ and so it is natural to use $\Phi\Psi$ to denote the map defined on $\mathbb{Z}_{p+q}$, called the *concatenation* of Φ and Ψ, such that, for $i \in \mathbb{Z}_{p+q}$, it holds

$$(\Phi\Psi)_i = \begin{cases} \Phi_i, & \text{if } i \in [p]_{p+q}, \\ \Psi_{i-p}, & \text{if } i \in [p+1, p+q]_{p+q}. \end{cases}$$

Lemma 45. *Take a set K, $t \in \mathbb{N}$ and $X, Y \subseteq K^t$. Assume that $X \cap Y = \emptyset$. Let Φ be a cyclic decomposition of (X, K, t) with period p and let Ψ be a cyclic decomposition of (Y, K, t) with period q. If $\Phi_i = \Psi_i$ for $i \in [t-1]$, then $\Phi\Psi$ is a cyclic decomposition of $(X \cup Y, K, t)$ with* $\mathrm{per}(\Phi\Psi) = p + q$.

Proof. Put $\Delta = \Phi\Psi$. Note that

$$\begin{aligned}
&\bigcup_{i\in[p]_{p+q}} \big(\Delta_i \times \cdots \times \Delta_{i+t-1}\big) \\
=&\Big(\bigcup_{i\in[p-t+1]} \Delta_i \times \cdots \times \Delta_{i+t-1}\Big) \cup \Big(\bigcup_{i\in[p-t+2,p]} \Delta_i \times \cdots \times \Delta_{i+t-1}\Big) \\
=&\Big(\bigcup_{i\in[p-t+1]} \Phi_i \times \cdots \times \Phi_{i+t-1}\Big) \cup \Big(\bigcup_{i\in[p-t+2,p]} \Phi_i \times \cdots \times \Phi_p \times \Psi_1 \times \cdots \times \Psi_{i+t-1-p}\Big) \\
=&\Big(\bigcup_{i\in[p-t+1]} \Phi_i \times \cdots \times \Phi_{i+t-1}\Big) \cup \Big(\bigcup_{i\in[p-t+2,p]} \Phi_i \times \cdots \times \Phi_{i+t-1}\Big) \\
=&\bigcup_{i\in\mathbb{Z}_p} \big(\Phi_i \times \cdots \times \Phi_{i+t-1}\big) = X.
\end{aligned}$$

Similarly, we have $\bigcup_{i\in[p+1,p+q]_{p+q}} \big(\Delta_i \times \cdots \times \Delta_{i+t-1}\big) = Y$. By checking some other obvious conditions, we conclude that $\Phi\Psi$ is a cyclic decomposition of $(X \cup Y, K, t)$. ☐

For any $t \in \mathbb{N}$ and any set K, we define $\mathrm{d}_{K,t}$ to be the map on $K^t \times K^t$ which sends $(x, y) \in K^t \times K^t$ to

$$\mathrm{d}_{K,t}(x,y) := \begin{cases} 0, & \text{if } x = y, \\ t + 1 - \min\{i \in [t] :\ x_i \neq y_i\}, & \text{if } x \neq y. \end{cases}$$

For any two subsets X and Y of K^t, we write $\mathrm{d}_{K,t}(X,Y)$ for $\min\{\mathrm{d}_{K,t}(x,y) :\ x \in X, y \in Y\}$. If two nonempty subsets X and Y of K^t fulfils $\mathrm{d}_{K,t}(X,Y) > 1$, it is apparent that $(X \cup Y, K, t)$ has no discrete cyclic decomposition.

Lemma 46. *Let K be a set and t a positive integer. Let $X, Y \subseteq K^t$ and let Φ and Ψ be discrete cyclic decompositions of (X, K, t) and (Y, K, t), respectively. If $\mathrm{d}_{K,t}(X,Y) = 1$, then there is a discrete cyclic decomposition of $X \cup Y$ relative to K.*

Proof. Suppose $\mathrm{d}_{K,t}(X,Y)=1$. Then, in light of Lemma 44, we can assume $\Phi_i=\Psi_i$ for $i\in[t-1]$. Applying Lemma 45, we see that $\Phi\Psi$ is a cyclic decomposition of $(X\cup Y,K,t)$, as desired. $\square$

Lemma 47. *Let c and t be two positive integers. Put $X=[c+1]^t\setminus[c]^t$. If $t>1$, then there exists a discrete cyclic decomposition Δ of $(X,[c+1],t)$ such that*

$$\Delta_i=\begin{cases}\{c+1\}, & \text{if } i\in[2t-1]\setminus\{t\},\\ \{c\}, & \text{if } i=t.\end{cases}\tag{8}$$

Proof. If $t=2$, we let

$$\Delta_i=\begin{cases}\{c+1\}, & \text{if } i\in\{1,3,\ldots,2c+1\},\\ \{c+1-\frac{i}{2}\}, & \text{if } i\in\{2,4,\ldots,2c\}.\end{cases}\tag{9}$$

and the resulting Δ as a map on $\mathbb{Z}_{2c+1}$ is what we want.

We proceed to deal with the case of $t>2$.

We view the elements of $[c+1]^t$ as maps on $\mathbb{Z}_t$ and this allows us consider the action of σ_t on the elements of X – recall the definition of σ as given in Eq. (7). For each $x\in[c+1]^t$, let $\overline{x}$ refer to the orbit of x under the action of σ_t. It is clear that σ_t gives rise to a permutation of X, namely X is the disjoint union of several orbits of σ_t.

For any $x\in X$, let

$$\begin{cases}\mathrm{n}_x:=|\{x_i:\ x_i=c+1\}|\in\mathbb{N},\\ \mathrm{o}_x:=|\overline{x}|\in\mathbb{N},\end{cases}$$

and define Λ^x to be the map from $\mathbb{Z}_{\mathrm{o}_x}$ to $2^{[c+1]}$ that sends $i\in\mathbb{Z}_{\mathrm{o}_x}$ to $\{x_i\}$. For our subsequent proof, the important thing is that

$$\Lambda^x \text{ is a discrete cyclic decomposition of } (\overline{x},[c+1],t).\tag{10}$$

For any $i\in[t]$, let $\Omega_i:=\{x\in X:\ \mathrm{n}_x=i\}$ and $\Pi_i:=\{x\in X:\ \mathrm{n}_x\leqslant i\}$, which are both closed under the action of σ_t. Let $\overline{\Omega_i}=\{\overline{x}:\ \mathrm{n}_x=i,x\in X\}$ and $\overline{\Pi_i}=\{\overline{x}:\ \mathrm{n}_x\leqslant i,x\in X\}$ for $i\in[t]$. Let G be the digraph with vertex set $\mathrm{V}(G)=\overline{\Pi_{t-2}}$ and arc set $\mathrm{A}_G=\{(\overline{x},\overline{y}):\ \mathrm{d}_{[c+1],t}(\overline{x},\overline{y})=1,\overline{x},\overline{y}\in\mathrm{V}(G)\}$. Note that $G[\overline{\Pi_1}]$ is a connected graph with diameter at most $t-1$ and, for $i\in[t-3]$, in $G[\overline{\Pi_{i+1}}]$ every vertex from $\overline{\Omega_{i+1}}$ is adjacent to some vertex in $\overline{\Omega_i}\subseteq\overline{\Pi_i}$. By successive applications of Lemma 46 and fact (10), we derive the existence of a discrete cyclic decomposition $\mathcal{Q}$ of $(\Pi_{t-2},[c+1],t)$.

Let

$$y=(\underbrace{c+1,\ldots,c+1}_{t-1},c)\in X\subseteq[c+1]^t.$$

Let $Y:=\Omega_{t-1}\setminus\overline{y}$. Note that $\mathrm{d}_{[c+1],t}(Y,\Pi_{t-2})=1$ and $\mathrm{d}_{[c+1],t}(\Omega_t,Y\cup\Pi_{i-2})=1$. Combining Lemma 46 and fact (10), we can start from $\mathcal{Q}$ to get a discrete cyclic decomposition Φ of $(X\setminus\overline{y},[c+1],t)$.

Since $(\overbrace{c+1,\ldots,c+1}^{t}) \in X \setminus \overline{y}$, we can assume, in view of Lemma 44, that Φ begins with

$$(\Phi_1,\ldots,\Phi_{t-1}) = (\overbrace{\{c+1\},\ldots,\{c+1\}}^{t-1}).$$

Putting $\Delta = \Lambda^y\Phi$, then from Lemma 45 and (10) we obtain that Δ is a discrete cyclic decomposition of $(X,[c+1],t)$ with

$$(\Delta_1,\ldots,\Delta_{2t-1}) = (\overbrace{\{c+1\},\ldots,\{c+1\}}^{t-1},\{c\},\overbrace{\{c+1\},\ldots,\{c+1\}}^{t-1}),$$

as wanted. $\square$

Lemma 48. *Let k and t be two integers greater than 1 and let $X = [k+1]^t \setminus [k]^t$. Then $\{3^t - 2^t + jt : \ j \in [0,k-2]\} \subseteq \mathcal{P}^*(X,[k+1],t)$.*

Proof. Let $r = 3^t - 2^t$. Applying Lemma 47 for $c = 2$, we can get a discrete cyclic decomposition Δ of $([3]^t \setminus [2]^t)$ relative to $[3]$ such that

$$(\Delta_1,\ldots,\Delta_{2t-1}) = (\overbrace{\{3\},\ldots,\{3\}}^{t-1},\{2\},\overbrace{\{3\},\ldots,\{3\}}^{t-1}). \tag{11}$$

Now we construct $\Phi^{(0)} : \ \mathbb{Z}_r \to 2^{[k+1]}$ by putting

$$\Phi^{(0)}_i = \begin{cases} \{1\}, & \text{if } \Delta_i = \{1\},\\ [2,k], & \text{if } \Delta_i = \{2\},\\ \{k+1\}, & \text{if } \Delta_i = \{3\}. \end{cases}$$

For $j \in [k-2]$, we then construct $\Phi^{(j)} : \ \mathbb{Z}_{r+jt} \to 2^{[k+1]}$ by putting

$$\begin{cases} \Phi^{(j)}_{[t]} & = (\overbrace{\{k+1\},\ldots,\{k+1\}}^{t-1},\{2\}),\\ \Phi^{(j)}_{[t+1,2t]} & = (\overbrace{\{k+1\},\ldots,\{k+1\}}^{t-1},\{3\}),\\ \quad\vdots & \qquad\qquad\vdots\\ \Phi^{(j)}_{[(j-1)t+1,jt]} & = (\overbrace{\{k+1\},\ldots,\{k+1\}}^{t-1},\{j+1\}),\\ \Phi^{(j)}_{[jt+1,(j+1)t]} & = (\overbrace{\{k+1\},\ldots,\{k+1\}}^{t-1},[j+2,k]), \end{cases}$$

and, for $i \in [r-t]$,

$$\Phi^{(j)}_{(j+1)t+i} = \begin{cases} \{1\}, & \text{if } \Delta_{i+t} = \{1\},\\ [2,k], & \text{if } \Delta_{i+t} = \{2\},\\ \{k+1\}, & \text{if } \Delta_{i+t} = \{3\}. \end{cases}$$

For $j \in [0, k-2]$, it follows from Eq. (11) that

$$\Phi^{(j)}_{[(j+1)t+1,(j+2)t-1]} = (\overbrace{\{k+1\}, \ldots, \{k+1\}}^{t-1}).$$

This enables us to check that $\Phi^{(j)}$ is a cyclic decomposition of $(X, [k+1], t)$ with period $r + jt$ for $j \in [0, k-2]$.

Let $j \in [0, k-2]$. Since $\Phi^{(j)}$ is a cyclic decomposition of $(X, [k+1], t)$, there exists a unique $i^{(j)} \in \mathbb{Z}_{r+jt}$ such that

$$(\overbrace{1, 1, \ldots, 1}^{t-1}, k+1) \in \ \Phi^{(j)}_{i^{(j)}} \times \Phi^{(j)}_{i^{(j)}+1} \times \cdots \times \Phi^{(j)}_{i^{(j)}+t-1}.$$

Checking our construction of $\Phi^{(j)}$, we see that, for any $s \in \mathbb{Z}_{r+jt}$, $1 \in \Phi^{(j)}_s$ is possible only if $\{1\} = \Phi^{(j)}_s$. Thus, by virtue of Lemma 44, $\sigma^{r+jt+1-i^{(j)}}_{r+jt}(\Phi^{(j)})$ is a strong cyclic decomposition of $(X, [k+1], t)$ with period $r + jt$. $\square$

Lemma 49. *For every $t \in \mathbb{N}$, the set $\mathcal{P}^*(t)$ contains a complete residue system modulo t.*

Proof. For $i = 0, \ldots, t-1$, we shall show inductively the existence of $p_i \in \mathcal{P}^*(t)$ such that $p_i \equiv i \pmod t$.

When $i = 0$, by Lemma 41 there exists a discrete cyclic decomposition of $[t]^t$ relative to $[t]$. Since $t \mid t^t$, we can set $p_0 = t^t \in \mathcal{P}^*(t)$.

Assume that $i \leqslant t-1$ is a positive integer and we have known the existence of the required p_{i-1}. We are going to prove the existence of p_i. Let us assume that $p_{i-1} \in \mathcal{P}^*(t, d)$ for some $d \in \mathbb{N}$, namely there exists a strong cyclic decomposition Φ of $([d]^t, [d], t)$ with period p_{i-1}. Let c be a positive multiple of t satisfying $c \geqslant d$. By Lemma 42(b), we can find a strong cyclic decomposition Ψ of $([c]^t, [c], t)$ with period p_{i-1}. By Lemma 44 and Lemma 47, we can find a strong cyclic decomposition Δ of $[c+1]^t \setminus [c]^t$ relative to $[c+1]$ with period $(c+1)^t - c^t$. We then employ Lemma 45 to deduce that $\Psi\Delta$ is a strong cyclic decomposition of $([c+1]^t, [c+1], t)$ with period $p_{i-1} + (c+1)^t - c^t$. Noting that $(c+1)^t - c^t \equiv 1 \pmod t$, we can take $p_i = p_{i-1} + (c+1)^t - c^t$, finishing the proof. $\square$

Lemma 50. *Let t be a positive integer. Then $|\mathbb{N} \setminus \mathcal{P}^*(t)| < \infty$.*

Proof. Let $r = 3^t - 2^t$. By Lemma 49, $\mathcal{P}^*(t)$ contains a complete residue system modulo t, say $p_1, \ldots, p_t$.

Take $i \in [t]$. According to Lemma 42(b), there is $s_i \in \mathbb{N}$ such that

$$p_i \in \cap_{k_i \in [s_i, \infty]} \mathcal{P}^*(t, k_i). \tag{12}$$

For any $k_i \in [s_i, \infty]$, choose a strong cyclic decomposition Φ_{p_i,k_i} of $([k_i]^t, [k_i], t)$ with period p_i. By Lemma 48, for any

$$p \in \{r, r+t, r+2t, \ldots, r+(k_i-2)t\}, \tag{13}$$

there exists a strong cyclic decomposition Ψ_{p,k_i} of $([k_i+1]^t \setminus [k_i]^t, [k_i+1], t)$ with period p. From Lemma 45 we derive that $\Phi_{p_i,k_i}\Psi_{p,k_i}$ is a cyclic decomposition of $([k_i+1]^t, [k_i+1], t)$ with period $p+p_i$. Putting together Eq. (12) and Eq. (13), we see that p can take all values of the form $r+\ell t, \ell \in \mathbb{N}$. As $p_1, \ldots, p_t$ form a complete residue system modulo t, the freedom in choosing i from $[t]$ says that every large enough integer can be expressed as $p+p_i$, the period of $\Phi_{p_i,k_i}\Psi_{p,k_i}$ for suitable i, k_i and p. This demonstrates that $|\mathbb{N} \setminus \mathcal{P}^*(t)| < \infty$. $\square$

Proof of Theorem 15. It follows from Theorem 13 that $\mathcal{P}^*(t) \subseteq \mathcal{P}(t)$. Thus, the claim is direct from Lemma 50. $\square$

Let k, t and p be three positive integers and let Φ be a map from $\mathbb{Z}_p$ to $2^{[k]}$. We write $\rho_{\Phi,t}$ for the map from $\mathbb{Z}_p$ to $2^{[k]}$ such that $\rho_{\Phi,t}(i) = \bigcap_{j\in[t]} \Phi_{i-1+j}$ for $i \in \mathbb{Z}_p$.

Lemma 51. *Let k, t and p be three positive integers and let Φ be a cyclic decomposition of $([k]^t, [k], t)$ with period p. We adopt the shorthand ρ for the map $\rho_{\Phi,t}$.*

(a) If $\rho(i) \neq \emptyset$, $\rho(j) \neq \emptyset$ and $i \neq j$, then $\rho(i) \cap \rho(j) = \emptyset$.

(b) $\bigcup_{i\in\mathbb{Z}_p} \rho(i) = [k]$.

(c) If $\rho(i) \neq \emptyset$ and $p \geqslant 2$, then $\rho(i+1) = \rho(i+2) = \cdots = \rho(i+t-1) = \emptyset$.

(d) If $p \geqslant 2$, then $X = \{i \in \mathbb{Z}_p : \ \rho(i) \neq \emptyset\}$ contains at least two elements.

(e) If $\rho(i) \neq \emptyset$ and $p \geqslant 2$, then $\Phi_{i+t} \cap \rho(i) = \emptyset = \Phi_{i-1} \cap \rho(i)$.

Proof. (a) Assume, for the sake of contradiction, that $a \in \rho(i) \cap \rho(j)$. Then $\Phi_i \times \cdots \times \Phi_{i+t-1}$ and $\Phi_j \times \cdots \times \Phi_{j+t-1}$ contain $(\overbrace{a, \ldots, a}^{t})$ as a common element. This violation with Eq. (4) says that Φ cannot be a cyclic decomposition.

(b) By Eq. (3),

$$\bigcup_{i\in\mathbb{Z}_p} \big(\Phi_i \times \cdots \times \Phi_{i+t-1}\big) \supseteq \{(\overbrace{a, \ldots, a}^{t}) : \ a \in K\}, \tag{14}$$

and thus the claim follows from the definition of $\rho_{\Phi,t}$.

(c) Assume that $a \in \rho(i)$ and $b \in \rho(i+j)$ for some $j \in [t-1]$. Then

$$(\overbrace{a, \ldots, a}^{j}, \overbrace{b, \ldots, b}^{t-j}) \in \big(\Phi_i \times \cdots \times \Phi_{i+t-1}\big) \bigcap \big(\Phi_{i+1} \times \cdots \times \Phi_{i+t}\big).$$

This is impossible, as Φ satisfies Eq. (4).

(d) Claim (b) asserts that $X \neq \emptyset$. If X contains only one element, say i, then Claim (b) shows that $\Phi_i \times \cdots \times \Phi_{i+t-1} = K^t$, and hence Eq. (4) forces $p = 1$.

(e) If $a \in \rho(i) \cap \Phi_{i+t}$, then

$$(\overbrace{a, \ldots, a}^{t}) \in \big(\Phi_i \times \cdots \times \Phi_{i+t-1}\big) \bigcap \big(\Phi_{i+1} \times \cdots \times \Phi_{i+t}\big);$$

if $a \in \rho(i) \cap \Phi_{i-1}$, then

$$(\overbrace{a, \ldots, a}^{t}) \in \big(\Phi_i \times \cdots \times \Phi_{i+t-1}\big) \bigcap \big(\Phi_{i-1} \times \cdots \times \Phi_{i+t-2}\big).$$

In both cases, Eq. (4) becomes impossible to hold, which is a desired contradiction. □

Proof of Proposition 16. It is immediate from Lemma 51 (c) and Lemma 51(d). □

Proof of Theorem 17. This is a result of Theorem 13 and Proposition 16. □

Proof of Proposition 18. First we prove that $2t+1 \notin \mathcal{P}(t)$ when $t \geqslant 2$. If it were not true, then Theorem 13 says that there exists a cyclic decomposition Φ of $([k]^t, [k], t)$ for some $k \in \mathbb{N}$ with $\mathrm{per}(\Phi) = 2t+1$. Let $\rho = \rho_{\Phi,t}$ and let $X = \{i \in \mathbb{Z}_{2t+1} : \ \rho(i) \neq \emptyset\}$. By Lemma 44 and Lemma 51(a)(b)(c)(d), there is no loss of generality in assuming that $X = \{1, t+1\}$, and $\rho(1)$ and $\rho(t+1)$ form a partition of $[k]$. Using Lemma 51 (e) for $i \in \{1, t+1\}$, we obtain $\Phi_{2t+1} \cap \rho(1) = \Phi_{2t+1} \cap \rho(t+1) = \emptyset$. This ensures $\Phi_{2t+1} = \emptyset$, which is absurd.

Next we show $2t+2 \notin \mathcal{P}(t)$ when $t \geqslant 4$. By Theorem 13, our job is to demonstrate the nonexistence of any cyclic decomposition Φ of $([k]^t, [k], t)$ with $\mathrm{per}(\Phi) = 2t+2$ for every $k \in \mathbb{N}$. By way of contradiction, assume that such a Φ exists. Let $\rho = \rho_{\Phi,t}$ and let $X = \{i \in \mathbb{Z}_{2t+2} : \ \rho(i) \neq \emptyset\}$. By Lemma 44 and Lemma 51(a)(b)(c)(d), we need only address the following two cases.

Case 1. $X = \{1, t+2\}$, and $\rho(1)$ and $\rho(t+2)$ form a partition of $[k]$.

Using Lemma 51 (e) for $i \in \{1, t+2\}$, we obtain $\Phi_{t+1} \cap \rho(1) = \emptyset$ and $\Phi_{t+1} \cap \rho(t+2) = \emptyset$. This gives $\Phi_{t+1} = \emptyset$, a contradiction.

Case 2. $X = \{1, t+1\}$, and $\rho(1)$ and $\rho(t+1)$ form a partition of $[k]$.

Using Lemma 51 (e) for $i \in \{1, t+1\}$, we obtain

$$\Phi_t = \rho(1), \Phi_{2t+1} \subseteq \rho(1), \tag{15}$$

and

$$\Phi_{2t+2} \subseteq \rho(t+1). \tag{16}$$

Take $a \in \Phi_{2t+1}$ and $b \in \Phi_{2t+2}$. From $b \in \Phi_1$ we would obtain $(b, \overbrace{a, \ldots, a}^{t-1}) \in \big(\Phi_{2t+2} \times \cdots \times \Phi_{t-1}\big) \cap \big(\Phi_1 \times \cdots \times \Phi_t\big)$ while from $b \in \Phi_2$ we will derive $(a, b, \overbrace{a, \ldots, a}^{t-2}) \in \big(\Phi_{2t+1} \times \cdots \times \Phi_{t-2}\big) \cap \big(\Phi_1 \times \cdots \times \Phi_t\big)$, both of which surely violate (4). Consequently, we get

$$b \notin \Phi_1 \cup \Phi_2. \tag{17}$$

We also claim that

$$a \notin \Phi_{2t}. \tag{18}$$

If this did not hold, we would have $(\overbrace{b,\dots,b}^{t-1},a) \in \big(\Phi_t\times\cdots\times\Phi_{2t}\big)\cap\big(\Phi_{t+1}\times\cdots\times\Phi_{2t+1}\big)$, which is impossible. By Eq. (3), there exists $i \in \mathbb{Z}_{2t+2}$ such that $(\overbrace{b,\dots,b}^{t-2},a,a) \in \Phi_i\times\cdots\times\Phi_{i+t-1}$. By Eqs. (15) and (17), we see that $b \notin \Phi_1\cup\Phi_2\cup\Phi_t\cup\Phi_{2t+1}$ and so $\{1,2,t,2t+1\}\cap[i,i+t-3]_{2t+2}=\emptyset$. As $t \geqslant 4$, this can happen only if $i \in \{t+1,t+2,t+3\}$. On the other hand, Eq. (16) along with Eq. (18) says that $\{2t,2t+2\}\cap[i+t-2,i+t-1]_{2t+2}=\emptyset$, and therefore $i \notin \{t+1,t+2,t+3\}$. This contradiction completes the proof. □

Proof of Proposition 20. If the assertion would not hold, then, according to Theorem 13, there is a cyclic decomposition Φ of $([k]^t,[k],t)$ with $\mathrm{per}(\Phi)=k^t-2t+1$. We write p for $\mathrm{per}(\Phi)$ and note that $p \geqslant 2t$. It is also obvious that Φ is not discrete and hence $\max_{j\in\mathbb{Z}_p}|\Phi_j|>1$.

Case 1. There exists $j \in \mathbb{Z}_p$ such that $|\Phi_j| \geqslant 3$.

This implies

$$\begin{aligned} k^t &= \sum_{i\in\mathbb{Z}_p}|\Phi_i\times\cdots\times\Phi_{i+t-1}| \\ &\geqslant \Big(\sum_{i\in[j+1,j-t+1+p]_p}+\sum_{i\in[j-t+1,j]_p}\Big)|\Phi_i\times\cdots\times\Phi_{i+t-1}| \\ &\geqslant (k^t-2t+1-t)+3t=k^t+1, \end{aligned}$$

giving a desired contradiction.

Case 2. There is a unique $j\in\mathbb{Z}_p$ such that $|\Phi_j|=2$ and $|\Phi_{j'}|=1$ for all $j'\in\mathbb{Z}_p\setminus\{j\}$.

We now have

$$\begin{aligned} k^t &= \sum_{i\in\mathbb{Z}_p}|\Phi_i\times\cdots\times\Phi_{i+t-1}| \\ &= \Big(\sum_{i\in\mathbb{Z}_p\setminus[j-t+1,j]_p}+\sum_{i\in[j-t+1,j]_p}\Big)|\Phi_i\times\cdots\times\Phi_{i+t-1}| \\ &= (k^t-2t+1-t)+2t=k^t-t+1<k^t, \end{aligned}$$

again a contradiction.

Case 3. There exists $\{j_1,j_2\}\in\binom{\mathbb{Z}_p}{2}$ such that $|\Phi_{j_1}|=|\Phi_{j_2}|=2$.

We first consider the case of $j_1\in[j_2-t+1,j_2+t-1]_p$. Without loss of generality, we assume that $j_1=j_2-a$ where $a\in[t-1]$. It follows that

$$\begin{aligned} k^t &= \sum_{i\in\mathbb{Z}_p}|\Phi_i\times\cdots\times\Phi_{i+t-1}| \\ &= \Big(\sum_{i\in[j_2+1-p,j_1-t]_p}+\sum_{i\in[j_1-t+1,j_2-t]_p}+\sum_{i\in[j_2-t+1,j_1]_p}+\sum_{i\in[j_1+1,j_2]_p}\Big)|\Phi_i\times\cdots\times\Phi_{i+t-1}| \\ &\geqslant (k^t-2t+1-a-t)+2a+4(t-a)+2a=k^t+1+(t-a)>k^t+1, \end{aligned}$$

which is absurd.

It remains to consider the case that $j_1 \notin [j_2 - t + 1, j_2 + t - 1]_p$. We can derive that

$$\begin{aligned}
k^t &= \sum_{i \in \mathbb{Z}_p} |\Phi_i \times \cdots \times \Phi_{i+t-1}| \\
&= \Big(\sum_{i \in \mathbb{Z}_p \setminus ([j_1-t+1, j_1]_p \cup [j_2-t+1, j_2]_p)} + \sum_{i \in [j_1-t+1, j_1]_p} + \sum_{i \in [j_2-t+1, j_2]_p} \Big) |\Phi_i \times \cdots \times \Phi_{i+t-1}| \\
&\geqslant \big((k^t - 2t + 1) - 2t\big) + 2t + 2t = k^t + 1,
\end{aligned}$$

which is a contradiction, and so the proposition follows. □

Proof of Proposition 21. If $k = 1$, the result is trivial. Assume then $k \geqslant 2$. By Theorem 13, our task is to show the nonexistence of a cyclic decomposition Φ of $([k]^2, [k], 2)$ with period $k^2 - 1$. If such a Φ exists, then there exists $j \in \mathbb{Z}_{k^2-1}$ such that $|\Phi_j| \geqslant 2$. Hence

$$\begin{aligned}
k^2 = \sum_{i \in \mathbb{Z}_{k^2-1}} |\Phi_i \times \Phi_{i+1}| &\geqslant \sum_{i \in \mathbb{Z}_{k^2-1} \setminus \{j-1, j\}} |\Phi_i \times \Phi_{i+1}| + |\Phi_{j-1} \times \Phi_j| + |\Phi_j \times \Phi_{j+1}| \\
&\geqslant (k^2 - 3) + 2 + 2 = k^2 + 1.
\end{aligned}$$

This is impossible and hence we are done. □

Let k be a positive integer and let Φ be a cyclic decomposition of $([k]^2, [k], 2)$. For any $x \in [k]$, define $\delta(x, \Phi) := \{i : x \in \Phi_i\}$, $\delta^+(x, \Phi) := \{i + 1 : i \in \delta(x, \Phi)\}$ and $\delta^-(x, \Phi) := \{i - 1 : i \in \delta(x, \Phi)\}$.

Lemma 52. *Take $k \in \mathbb{N}$. Let Φ be a cyclic decomposition of $([k]^2, [k], 2)$ and let $x \in [k]$.*

(a) Let S be a subset of $\delta^+(x, \Phi)$ and let A_i be a subset of Φ_i for all $i \in S$. Then, $\bigcup_{i \in S} A_i = [k]$ if and only if $S = \delta^+(x, \Phi)$ and $\Phi_i = A_i$ for all $i \in S$.

(b) Let S be a subset of $\delta^-(x, \Phi)$ and let A_i be a subset of Φ_i for all $i \in S$. Then, $\bigcup_{i \in S} A_i = [k]$ if and only if $S = \delta^-(x, \Phi)$ and $\Phi_i = A_i$ for all $i \in S$.

Proof. (a) By Eq. (3), each element of $\{x\} \times K$ appears in a set of the form $\Phi_{i-1} \times \Phi_i$ for exactly one $i \in S$. The result thus follows.

(b) By Eq. (3), each element of $K \times \{x\}$ appears in a set of the form $\Phi_i \times \Phi_{i+1}$ for exactly one $i \in S$. This implies the claim, as desired. □

Lemma 53. *$\{5, 6\} \cap \mathcal{P}(2) = \emptyset$.*

Proof. By Proposition 18, we only need to show $6 \notin \mathcal{P}(2)$. We assume that there exists a cyclic decomposition Φ of $([k]^2, [k], 2)$ with $\mathrm{per}(\Phi) = 6$ and, in light of Theorem 13, aim to find a contradiction. Let $\rho = \rho_{\Phi,2}$ and let $X = \{i \in \mathbb{Z}_6 : \rho(i) \neq \emptyset\}$. By Lemma 44 as well as Lemma 51(a)(b)(c)(d), it suffices to consider the following cases.

Case 1. $\rho(1)$ and $\rho(3)$ form a partition of $[k]$ and $X = \{1, 3\}$.

Using Lemma 51 (e) for $i = 1$ and $i = 3$, we find $\Phi_6 \cap \rho(1) = \emptyset$ and $\Phi_5 \cap \rho(3) = \emptyset$, respectively. This implies that $\Phi_5 \times \Phi_6 \subseteq \rho(1) \times \rho(3) \subseteq \Phi_2 \times \Phi_3$, a contradiction with Eq. (4).

Case 2. $\rho(1)$ and $\rho(4)$ form a partition of $[k]$ and $X = \{1, 4\}$.

We employ Lemma 51 (e) for $i \in \{1, 4\}$ to get $\Phi_3 \cap \rho(1) = \emptyset$ and $\Phi_3 \cap \rho(4) = \emptyset$. This is impossible as we have assumed that $\Phi_3 \cap \big(\rho(1) \cup \rho(4)\big) = \Phi_3 \cap [k] = \Phi_3 \neq \emptyset$.

Case 3. $\rho(1), \rho(3)$ and $\rho(5)$ form a partition of $[k]$ and $X = \{1, 3, 5\}$.

By Lemma 51 (e), it holds

$$\Phi_6 \cap \rho(1) = \Phi_3 \cap \rho(1) = \emptyset. \tag{19}$$

Case 3.1. $\Phi_4 \cap \rho(1) = \emptyset$ and $\Phi_5 \cap \rho(1) = \emptyset$.

Take $x \in \rho(1)$. By Eq. (19) and the standing assumption for Case 3.1, we have $\delta(x, \Phi) = \{1, 2\}$, $\delta^+(x, \Phi) = \{2, 3\}$ and $\delta^-(x, \Phi) = \{6, 1\}$. It follows from Lemma 52 that

$$\Phi_2 \cup \Phi_3 = [k] \tag{20}$$

and

$$\Phi_6 \cup \Phi_1 = [k]. \tag{21}$$

Take any element $z \in \rho(5)$, which must fall into $\Phi_2 \cup \Phi_3$ by Eq. (20).

We first deal with the case that $z \in \Phi_2$. We have $\{z\} \times \rho(3) \subseteq \Phi_2 \times \Phi_3$. Eq. (4) says $(\Phi_2 \times \Phi_3) \cap (\Phi_5 \times \Phi_6) = \emptyset$. Thus, considering that $z \in \rho(5) \subseteq \Phi_5$, we obtain

$$\Phi_6 \cap \rho(3) = \emptyset. \tag{22}$$

Eq. (4) also says $(\Phi_2 \times \Phi_3) \cap (\Phi_6 \times \Phi_1) = \emptyset$. Thus, considering that $z \in \rho(5) \subseteq \Phi_6$, we obtain

$$\Phi_1 \cap \rho(3) = \emptyset. \tag{23}$$

Note that Eq. (21) says that Eq. (22) and Eq. (23) cannot hold simultaneously, a contradiction.

We now consider the case that $z \in \Phi_3$. We have $\{z\} \times \rho(3) \subseteq \Phi_3 \times \Phi_4$. In the same manner as in the previous case, we get $\Phi_6 \cap \rho(3) = \Phi_1 \cap \rho(3) = \emptyset$, which again contradicts Eq. (21).

Case 3.2. $\Phi_4 \cap \rho(1) = \emptyset$ and $\Phi_5 \cap \rho(1) \neq \emptyset$.

Take $x \in \Phi_5 \cap \rho(1)$. Due to Eq. (19), we have $\delta(x, \Phi) = \{1, 2, 5\}$, $\delta^+(x, \Phi) = \{2, 3, 6\}$ and $\delta^-(x, \Phi) = \{1, 4, 6\}$.

Recall that $\rho(1) \cup \rho(3) \cup \rho(5) = [k]$. Applying Lemma 52(a) for $S = \{2, 3, 6\}$, $A_2 = \rho(1)$, $A_3 = \rho(3)$ and $A_6 = \rho(5)$, we obtain

$$\Phi_2 = \rho(1), \Phi_3 = \rho(3), \Phi_6 = \rho(5); \tag{24}$$

applying Lemma 52(b) for $S = \{1, 4, 6\}$, $A_1 = \rho(1)$, $A_4 = \rho(3)$ and $A_6 = \rho(5)$, we obtain

$$\Phi_1 = \rho(1), \Phi_4 = \rho(3), \Phi_6 = \rho(5). \tag{25}$$

By Lemma 51 (e) for $i = 3$, it holds

$$\Phi_5 \cap \rho(3) = \emptyset. \tag{26}$$

It follows from Eqs. (24), (25) and (26) that $\big(\rho(5) \times \rho(3)\big) \cap \big(\cup_{i\in\mathbb{Z}_6} \Phi_i \times \Phi_{i+1}\big) = \emptyset$, arriving at a contradiction with Eq. (3).

Case 3.3. $\Phi_4 \cap \rho(1) \neq \emptyset$ and $\Phi_5 \cap \rho(1) = \emptyset$.

Analogous to the analysis of Case 3.2, we derive in this case that $\Phi_1 = \Phi_2 = \rho(1), \Phi_3 = \rho(3), \Phi_5 = \Phi_6 = \rho(5)$ and $\Phi_4 \cap \rho(5) = \emptyset$. This leads to $\big(\rho(5)\times\rho(3)\big)\cap\big(\cup_{i\in\mathbb{Z}_6}(\Phi_i\times\Phi_{i+1})\big) = \emptyset$, a contradiction with Eq. (3) as well.

Case 3.4. $\Phi_4 \cap \rho(1) \neq \emptyset$ and $\Phi_5 \cap \rho(1) \neq \emptyset$.

Take $x_1 \in \Phi_4 \cap \rho(1)$ and $x_2 \in \Phi_5 \cap \rho(1)$. Then $(x_1, x_2) \in (\Phi_4 \times \Phi_5) \cap (\Phi_1 \times \Phi_2)$, which violates Eq. (4). □

Lemma 54. $7 \notin \mathcal{P}(2)$.

Proof. By Theorem 13, we need to show that there is no cyclic decomposition Φ of $([k]^2, [k], t)$ with $\mathrm{per}(\Phi) = 7$ for all $k \in \mathbb{N}$. Suppose for a contradiction that such a Φ exists. Let $\rho = \rho_{\Phi,2}$ and let $X = \{i \in \mathbb{Z}_7 : \ \rho(i) \neq \emptyset\}$. By Lemma 44 and Lemma 51(a)(b)(c)(d), there are three cases to consider.

Case 1. $X = \{1, 3\}$, and $\rho(1)$ and $\rho(3)$ form a partition of $[k]$.

Using Lemma 51 (e) for $i = 1$, we have $\Phi_7 \cap \rho(1) = \emptyset$ and hence

$$\emptyset \neq \Phi_7 \subseteq \rho(3); \tag{27}$$

using Lemma 51 (e) for $i = 3$, we have $\Phi_5 \cap \rho(3) = \emptyset$ and hence

$$\emptyset \neq \Phi_5 \subseteq \rho(1). \tag{28}$$

Take $x \in \Phi_6$. It follows from Eq. (4) that $\big(\Phi_2 \times \Phi_3\big) \cap \big(\Phi_6 \times \Phi_7\big) = \emptyset$ and $\big(\Phi_2 \times \Phi_3\big) \cap \big(\Phi_5 \times \Phi_6\big) = \emptyset$. The former means that $x \notin \rho(1)$, as otherwise, by virtue of Eq. (27), (x, x') will appear in both $\Phi_2 \times \Phi_3$ and $\Phi_6 \times \Phi_7$, where x' is any element of Φ_7; the latter shows that $x \notin \rho(3)$, as otherwise, in light of Eq. (28), (x', x) will appear in both $\Phi_2 \times \Phi_3$ and $\Phi_5 \times \Phi_6$, where x' is any element of Φ_5. Since $\rho(1) \cup \rho(3) = [k]$, we have now obtained a required contradiction.

Case 2. $X = \{1, 4\}$, and $\rho(1)$ and $\rho(4)$ form a partition of $[k]$.

Using Lemma 51 (e) for $i \in \{1, 4\}$, we have $\Phi_3 \cap \rho(1) = \emptyset$ and $\Phi_3 \cap \rho(4) = \emptyset$. This forces $\Phi_3 = \emptyset$, which is a contradiction.

Case 3. $X = \{1, 3, 5\}$, and $\rho(1), \rho(3)$ and $\rho(5)$ form a partition of $[k]$.

Utilizing Lemma 51 (e) for $i \in \{1, 5\}$ gives $\Phi_7 \cap \rho(1) = \emptyset$ and $\Phi_7 \cap \rho(5) = \emptyset$, and so

$$\Phi_7 \subseteq \rho(3) \tag{29}$$

follows. Pick $x \in \Phi_7$. Then $\delta(x, \Phi) \supseteq \{3, 4, 7\}$, $\delta^+(x, \Phi) \supseteq \{1, 4, 5\}$ and $\delta^-(x, \Phi) \supseteq \{2, 3, 6\}$.

Remember that $\rho(1) \cup \rho(3) \cup \rho(5) = [k]$. Accordingly, by applying Lemma 52 (a) for $S = \{1, 4, 5\}$, $A_1 = \rho(1), A_4 = \rho(3)$ and $A_5 = \rho(5)$, we can deduce

$$\Phi_1 = \rho(1), \Phi_4 = \rho(3), \Phi_5 = \rho(5). \tag{30}$$

Moreover, making use of Lemma 52(b) for $S = \{2, 3, 6\}$, $A_2 = \rho(1), A_3 = \rho(3)$ and $A_6 = \rho(5)$, we can find that

$$\Phi_2 = \rho(1), \Phi_3 = \rho(3), \Phi_6 = \rho(5). \tag{31}$$

Considering Eqs. (29), (30) and (31), we know that $\big(\rho(5) \times \rho(1)\big) \cap \big(\cup_{i \in \mathbb{Z}_7} (\Phi_i \times \Phi_{i+1})\big) = \emptyset$, arriving at a contradiction with Eq. (3). $\square$

Let Φ be a cyclic decomposition of $(X, [k], t)$ with period p. The *matrix form* of Φ is the $k \times p$ matrix M_Φ whose (i, j) entries, $(i, j) \in [k] \times [p]$, equal to 1 if $i \in \Phi_j$ and equal to 0 otherwise.

Example 55. For each $p \in \{8, 10, 11, 12, 14\}$, we can find a strong cyclic decomposition $\Phi^{(p)}$ of $([4]^2, [4], 2)$ with period p and hence we get

$$\{8, 10, 11, 12, 14\} \subseteq \mathcal{P}^*(2, 4).$$

We display the matrix form of these $\Phi^{(p)}$, $p \in \{8, 10, 11, 12, 14\}$, as below:

$$\mathrm{M}_{\Phi^{(8)}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\mathrm{M}_{\Phi^{(10)}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\mathrm{M}_{\Phi^{(11)}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\mathrm{M}_{\Phi^{(12)}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathrm{M}_{\Phi^{(14)}} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 56. We find a strong cyclic decomposition $\Phi^{(15)}$ of $([5]^2, [5], 2)$ with period 15 and so see that

$$15 \in \mathcal{P}^*(2,5).$$

The matrix form of $\Phi^{(15)}$ is given by

$$\mathrm{M}_{\Phi^{(15)}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 57. *For every integer $k \geqslant 4$, $[8, k^2] \subseteq \mathcal{P}^*(2, k+1)$.*

Proof. We prove the claim by induction on k. First consider the starting case of $k = 4$. Example 56 tells us $15 \in \mathcal{P}^*(2,5)$. By Lemma 42(b), Lemma 43 and Example 55, we have $\{8, 9, 10, 11, 12, 14, 16\} \subseteq \mathcal{P}^*(2,4) \subseteq \mathcal{P}^*(2,5)$. It follows from Lemma 48 that $5 \in \mathcal{P}^*([5]^2 \setminus [4]^2, [5], 2)$. Since we know that $8 \in \mathcal{P}^*(2,4)$, applying Lemma 45 yields $13 = 8 + 5 \in \mathcal{P}^*(2,5)$. In conclusion, we see that $[8, 16] \subseteq \mathcal{P}^*(2,5)$.

We next suppose that $k \geqslant 5$ and $[8, (k-1)^2] \subseteq \mathcal{P}^*(2,k)$. By Lemma 42(b), $[8, (k-1)^2] \subseteq \mathcal{P}^*(2, k+1)$. Owing to Lemmas 45 and 48, we have $\{p+5, p+7, \dots, p+2k+1\} \subseteq \mathcal{P}^*(2, k+1)$ for every $p \in \mathcal{P}^*(2,k)$. Putting together, we get to

$$\mathcal{P}^*(2, k+1) \supseteq \bigcup_{i \in \{5,7,\dots,2k-1\}} [8+i, (k-1)^2 + i] \cup [8, (k-1)^2] = [13, k^2] \cup [8, (k-1)^2].$$

As $k \geqslant 5$, it holds $8 < 13 < (k-1)^2$ and so $[13, k^2] \cup [8, (k-1)^2] = [8, k^2]$. Thus, $\mathcal{P}^*(2, k+1) \supseteq [8, k^2]$ follows. ☐

Proof of Theorem 22. This is a consequence of Theorem 13, Proposition 16, Lemmas 43, 53, 54, 57, and Example 55. ☐

Proof of Theorem 25. A combination of Theorem 13, Proposition 20, Proposition 21, Theorem 22, Lemma 43 and Example 55 completes the proof. ☐

Proof of Proposition 26. By Theorem 22, $\mathcal{P}(2) = \mathbb{N} \setminus \{2, 3, 5, 6, 7\}$. By Theorem 13 and Proposition 16, $\mathcal{P}(3) \subseteq \mathbb{N} \setminus \{2, 3, 4, 5\}$. Therefore, our goal is to show $6, 7 \notin \mathcal{P}(3)$.

Let p be either 6 or 7. In view of Theorem 13, we are going to show the nonexistence of a cyclic decomposition Φ of $([k]^3, [k], 3)$ with period p for any $k \in \mathbb{N}$. For sake of contradiction, assume that such a Φ exists for some $k \in \mathbb{N}$. Let $\rho = \rho_{\Phi,3}$. By Lemma 44 and Lemma 51(a)(b)(c)(d), we may assume that $\rho(1)$ and $\rho(4)$ form a partition of $[k]$.

If $p = 7$, by appealing to Lemma 51 (e) for $i \in \{1, 4\}$, we obtain respectively $\Phi_7 \cap \rho(1) = \emptyset$ and $\Phi_7 \cap \rho(4) = \emptyset$. We now see that $\Phi_7 = \Phi_7 \cap [k] = \big(\Phi_7 \cap \rho(1)\big) \cup \big(\Phi_7 \cap \rho(4)\big) = \emptyset$, which is impossible as Φ is a cyclic decomposition.

It remains to consider $p = 6$. If there exists $x \in \rho(1) \cap \Phi_5$, then $\{x\} \times \rho(4) \times \{x\} \subseteq (\Phi_3 \times \Phi_4 \times \Phi_5) \cap (\Phi_5 \times \Phi_6 \times \Phi_1)$, reaching a contradiction. Applying Lemma 51(e) for

$i = 1$ yields $(\Phi_6 \cup \Phi_4) \cap \rho(1) = \emptyset$ and hence the elements of $\rho(1)$ only appear in Φ_1, Φ_2 and Φ_3. The same reasoning shows that the elements of $\rho(4)$ only appear in Φ_4, Φ_5 and Φ_6. Take $x \in \rho(1)$ and $y \in \rho(4)$. We now see that

$$(x, y, x) \notin \cup_{i \in \mathbb{Z}_6} \big(\Phi_i \times \Phi_{i+1} \times \Phi_{i+2}\big).$$

This contradiction with Eq. (3) finishes the proof. $\square$

The latter part of the above proof of Proposition 26 is devoted to proving $6 \notin \mathcal{P}(3)$. We should mention that it is a special case of the aforementioned result (see Table 1) that $2t \in \mathcal{P}(t)$ if and only if $t \in \{1, 2, 4\}$ [QWZ17].

6 Large primitive exponents and large diameters

Let k be a positive integer. For any subset X of $\mathbb{Z}_k$, we use $\theta_k(X)$ to represent $\{x + 1 : x \in X\} \subseteq \mathbb{Z}_k$. We define two 2-hydras h_k and f_k on $\mathbb{Z}_k$ such that

$$\mathrm{A}(h_k) := \{(i, j) \to (j, i + 1) : \ i, j \in \mathbb{Z}_k\}$$

and

$$\mathrm{A}(f_k) := (\mathrm{A}(h_k) \setminus E_1) \cup E_2,$$

where

$$E_1 := \big\{(q, 1) \to (1, q + 1) : \ q \in [\lceil \frac{k}{2} \rceil]_k\big\}$$

and

$$E_2 := \big\{(q, 1) \to (1, 2) : \ q \in [0, \lceil \frac{k}{2} \rceil]_k\big\}.$$

We refer the readers to Figure 3, Figure 9 and Figure 10 for an illustration of f_1, f_2, f_3 and f_4.

For every $k \in \mathbb{N}$, we write $\mathcal{V}_k$ for the set $\{(q, 1) : \ q \in [0, \lceil \frac{k}{2} \rceil]_k\}$.

Lemma 58. *Let k be a positive integer. Then* $\mathrm{Dist}_{f_k}\big(v, (1, 2)\big) \leqslant 2k$ *for all* $v \in \mathrm{V}(f)$*. If* $k \geqslant 4$*, then* $\mathrm{Dist}_{f_k}\big((1, 3), (1, 2)\big) = 2k$.

Proof. The result is trivial for $k = 1$. We hence assume $k \geqslant 2$. For all $q \in [\lceil \frac{k}{2} \rceil]$, we define

$$C_q := \{(x, y) \in (\mathbb{Z}_k)^2 : \ y - x = q\} \cup \{(x, y) \in (\mathbb{Z}_k)^2 : \ y - x = 1 - q\}.$$

Note that they are all connected components of Γ_{h_k} and they all induce cycles in Γ_{h_k}. Moreover, we have

$$|C_q| = \begin{cases} k, & \text{if } k \text{ is odd and } q = \lceil \frac{k}{2} \rceil, \\ 2k, & \text{otherwise.} \end{cases}$$

For $q \in [2, \lceil \frac{k}{2} \rceil]$, the subgraph of Γ_{f_k} induced by C_q is a path which starts at $(1, q + 1)$ and ends at $(q, 1)$. For $q = 1$, the subgraph of Γ_{f_k} induced by C_q is a $2k$-cycle plus an additional arc $(0,1) \to (1,2)$. Therefore, every element v of $(\mathbb{Z}_k)^2$ can reach a vertex in $\mathcal{V}_k$ in the digraph Γ_{f_k} in at most $2k-1$ steps and then can go one step further there to $(1,2)$. This proves that $\mathrm{Dist}_{f_k}\big(v,(1,2)\big) \leqslant 2k$ for every $v \in (\mathbb{Z}_k)^2$.

If $k \geqslant 4$, then $|C_2| = 2k$, and so the subgraph of Γ_{f_k} induced by C_2 is a path from $(1,3)$ to $(2,1)$ whose length equals to $2k-1$. It tells us that $\mathrm{Dist}_{f_k}\big((1,3),(1,2)\big) = \mathrm{Dist}_{f_k}\big((1,3),(2,1)\big) + \mathrm{Dist}_{f_k}\big((2,1),(1,2)\big) = 2k$, as wanted. $\square$

Let $\mathbb{D} = \{(a,b) \in \mathbb{N} \times \mathbb{N} : a \leqslant b\}$. For any $(a,b) \in \mathbb{D}$, we write

$$\phi_k(a,b) := [a,b]_k \subseteq \mathbb{Z}_k$$

and

$$\mathcal{N}((a,b)) := \{(a,b),(a+1,b),(a,b+1),(a+1,b+1)\} \cap \mathbb{D}.$$

Lemma 59. *Take $x \in \mathbb{D}$ and $x' \in \mathcal{N}(x)$. Let k be an integer greater than 2. If $1 \in \phi_k(x')$, then $\{0,1\} \cap \phi_k(x) \neq \emptyset$.*

Proof. If $\{0,1\} \cap \phi_k(x) = \emptyset$, then $\phi_k(x) \subseteq [2,k-1]_k$ and so, as $x' \in \mathcal{N}(x)$, $\phi_k(x') \subseteq [2,k]_k$, showing that $1 \notin \phi_k(x')$. $\square$

Lemma 60. *Take $k \geqslant 3$, $x \in \mathbb{D}$ and $x' \in \mathcal{N}(x)$. Then exactly one of the following two cases happens.*

(a) It holds

$$(0,1) \notin \phi_k(x) \times \phi_k(x') \text{ or } (1,1) \in \phi_k(x) \times \phi_k(x'), \tag{32}$$

and

$$\mathrm{M}_{f_k}\big(\phi_k(x),\phi_k(x')\big) = \big(\phi_k(x'),\theta_k\phi_k(x)\big). \tag{33}$$

(b) There is $p \in [2,k]$ such that

$$(x,x') = ([p,k]_k,[p,k+1]_k) \text{ or } (x,x') = ([p,k]_k,[p+1,k+1]_k) \tag{34}$$

and

$$\mathrm{M}_{f_k}\big(\phi_k(x),\phi_k(x')\big) = \big(\phi_k(x'),\theta_k\phi_k(x) \cup \{2\}\big). \tag{35}$$

Note that (a) *and* (b) *together tells us that there exists $x'' \in \mathcal{N}(x')$ such that*

$$\mathrm{M}_{f_k}\big(\phi_k(x),\phi_k(x')\big) = \big(\phi_k(x'),\phi_k(x'')\big). \tag{36}$$

Proof. It is obvious that either Eq. (32) or Eq. (34) holds but not both. So we just need to show Eq. (33) follows from Eq. (32) and Eq. (35) follows from Eq. (34). We will only do the former one and leave the routine calculation for the latter to readers.

Since we have

$$\big(\phi_k(x'),\theta_k\phi_k(x)\big) = \mathrm{M}_{h_k}\big(\phi_k(x),\phi_k(x')\big),$$

we shall try to establish

$$\mathrm{M}_{f_k}\big(\phi_k(x),\phi_k(x')\big) = \mathrm{M}_{h_k}\big(\phi_k(x),\phi_k(x')\big) \tag{37}$$

under the assumption of Eq. (32). By the definition of the Markov operator, our task at this moment is to prove

$$\bigcup_{v\in\phi_k(x)\times\phi_k(x')} f_k(v) = \bigcup_{v\in\phi_k(x)\times\phi_k(x')} h_k(v).$$

By the definition of h_k and f_k, we have $h_k(v) = f_k(v)$ for all $v \in \mathbb{Z}_k^2 \setminus \mathcal{V}_k$. Hence we only need to prove the fact that

$$\bigcup_{v\in\left(\phi_k(x)\times\phi_k(x')\right)\cap\mathcal{V}_k} f_k(v) \subseteq \bigcup_{v\in\phi_k(x)\times\phi_k(x')} h_k(v) \tag{38}$$

and

$$\bigcup_{v\in\left(\phi_k(x)\times\phi_k(x')\right)\cap\mathcal{V}_k} h_k(v) \subseteq \bigcup_{v\in\phi_k(x)\times\phi_k(x')} f_k(v). \tag{39}$$

If $\left(\phi_k(x)\times\phi_k(x')\right)\cap\mathcal{V}_k = \emptyset$, Eq. (38) and Eq. (39) will be trivially true. So, we proceed to the case that

$$\left(\phi_k(x)\times\phi_k(x')\right)\cap\mathcal{V}_k \neq \emptyset, \tag{40}$$

which surely implies

$$1 \in \phi_k(x'). \tag{41}$$

By Eq. (41) and Lemma 59,

$$\{(0,1),(1,1)\} \cap \left(\phi_k(x)\times\phi_k(x')\right) \neq \emptyset. \tag{42}$$

According to Eq. (32), $(0,1) \in \phi_k(x)\times\phi_k(x')$ will force $(1,1) \in \phi_k(x)\times\phi_k(x')$. This combined with Eq. (42) gives

$$\bigcup_{v\in\phi_k(x)\times\phi_k(x')} h_k(v) \supseteq \begin{cases} h_k(0,1)\cup h_k(1,1) = \{1,2\}, & \text{if } (0,1)\in\phi_k(x)\times\phi_k(x'), \\ h_k(1,1) = \{2\}, & \text{otherwise.} \end{cases}$$

Note that

$$\bigcup_{v\in(\phi_k(x)\times\phi_k(x'))\cap\mathcal{V}_k} f_k(v) = \begin{cases} \{1,2\}, & \text{if } (0,1)\in\phi_k(x)\times\phi_k(x'), \\ \{2\}, & \text{otherwise.} \end{cases}$$

This proves Eq. (38).

To prove Eq. (39), we distinguish two cases. If $|\phi_k(x')| > 1$, we are able to choose $a \in \phi_k(x') \setminus \{1\}$ and so every element $v \in \left(\phi_k(x)\times\phi_k(x')\right)\cap\mathcal{V}_k$, say $v = (i,1)$ for some $i \in \mathbb{Z}_k$, satisfies

$$h_k(v) = h_k(i,1) = \{i+1\} = f_k(i,a),$$

verifying Eq. (39). It remains to consider the case that $|\phi_k(x')| = 1$. Note that Eq. (41) now implies that $\phi_k(x') = \{1\}$. Since $x' \in \mathcal{N}(x)$, we infer from $\phi_k(x') = \{1\}$ that $\phi_k(x) \in \{\{0\},\{1\},\{0,1\}\}$. By Eq. (32) as well as Eq. (40), we obtain that $\phi_k(x) = \{0,1\}$. This shows that the left hand side of Eq. (39) is $\{1\}$ and the right hand side of Eq. (39) is $[2]$, and so Eq. (39) is valid. $\square$

Lemma 61. *Take an integer $k \geqslant 3$ and let $N = (2k-2)(2k-1) = 4k^2 - 6k + 2$. Then,*

$$\mathrm{M}_{f_k}^{N-1}(\{1\},\{2\}) = (\mathbb{Z}_k \setminus \{1\}, \mathbb{Z}_k)$$

and

$$\mathrm{M}_{f_k}^{N}(\{1\},\{2\}) = (\mathbb{Z}_k, \mathbb{Z}_k).$$

Proof. Pick $p \in [2, k+1]$. By Eq. (36), we can apply Eq. (33) $2k-2$ times to get

$$\mathrm{M}_{f_k}^{2k-2}\left([p,k+1]_k, \theta_k([p,k+1]_k)\right) = \left([p-1,k]_k, [p,k+1]_k)\right) \tag{43}$$

and

$$\mathrm{M}_{f_k}^{2k-2}\left([p,k+1]_k, \theta_k([p-1,k+1]_k)\right) = \left([p-1,k]_k, [p-1,k+1]_k\right). \tag{44}$$

Combining Eq. (35), Eq. (36) and Eq. (43), we find that

$$\mathrm{M}_{f_k}^{2k-1}\left([p,k+1]_k, \theta_k([p,k+1]_k)\right) = \left([p,k+1]_k, \theta_k([p-1,k+1]_k)\right); \tag{45}$$

while combining Eq. (35), Eq. (36) and Eq. (44) we obtain

$$\mathrm{M}_{f_k}^{2k-1}\left([p,k+1]_k, \theta_k([p-1,k+1]_k)\right) = \left([p-1,k+1]_k, \theta_k([p-1,k+1]_k)\right). \tag{46}$$

By repeated usages of Eq. (45) and Eq. (46), we get

$$\mathrm{M}_{f_k}^{N-1}\left([k+1,k+1]_k, \theta_k([k+1,k+1]_k)\right) = \mathrm{M}_{f_k}^{2k-2}\left([3,k+1]_k, \theta_k([2,k+1]_k)\right). \tag{47}$$

This leads to

$$\begin{aligned}
\mathrm{M}_{f_k}^{N-1}(\{1\},\{2\}) &= \mathrm{M}_{f_k}^{N-1}\left([k+1,k+1]_k, \theta_k([k+1,k+1]_k)\right) \\
&= \mathrm{M}_{f_k}^{2k-2}\left([3,k+1]_k, \theta_k([2,k+1]_k)\right) && \text{(By Eq. (47))} \\
&= (\mathbb{Z}_k \setminus \{1\}, \mathbb{Z}_k), && \text{(By Eq. (44))}
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{M}_{f_k}^{N}(\{1\},\{2\}) &= \mathrm{M}_{f_k}\left(\mathrm{M}_{f_k}^{N-1}(\{1\},\{2\})\right) \\
&= \mathrm{M}_{f_k}(\mathbb{Z}_k \setminus \{1\}, \mathbb{Z}_k) \\
&= (\mathbb{Z}_k, \mathbb{Z}_k).
\end{aligned}$$

This is the end of the proof. □

Lemma 62. *Let k be an integer not less than 4. Then the 2-hydra f_k is primitive with primitive exponent $\mathrm{g}(f_k) = 4k^2 - 4k + 2$.*

Proof. Lemma 58 and Lemma 61. □

Proof of Theorem 31. This follows immediately from Lemma 62. □

Proof of Theorem 32. Let f be the t-hydra on $[k]$ such that Γ_f has arc set $\{(k_1,\ldots,k_t) \to (k_2,\ldots,k_{t+1}) :\ k_1,\ldots,k_{t+1} \in [k]\}$. By Lemma 41, Γ_f has Hamiltonian cycles. Let f' be the t-hydra on $[k]$ such that $\Gamma_{f'}$ is a Hamiltonian cycle of Γ_f. Put $a = (\underbrace{1,\ldots,1}_{t})$. Let h be another t-hydra on $[k]$ such that $\mathrm{A}(\Gamma_h) = \mathrm{A}(\Gamma_{f'}) \cup \{a \to a\}$. It is clear that h is strongly connected. Because $\mathrm{per}(h) = \gcd(\mathcal{RI}_h(a,a)) = 1$, we know from Corollary 11 that h is primitive and $\mathrm{g}(h) \geqslant k^t$. Accordingly, $\gamma(t,k) \geqslant \mathrm{g}(h) \geqslant k^t$. $\square$

Lemma 63. *Let k be an integer larger than 4. Then*

$$\mathrm{Dia}(f_k) \geqslant \begin{cases} 2k^2, & \textit{if } k \textit{ is odd,} \\ 2k^2 - k + 1, & \textit{if } k \textit{ is even.} \end{cases} \tag{48}$$

Proof. Let $x = (1,3)$, $y = (1, \lceil \frac{k+3}{2} \rceil)$, and $z = (1,2)$ be vertices of f_k.

First we consider the case that k is odd. Observe that

$$\begin{aligned}
\mathrm{M}_{f_k}^{2k^2-2k}(z) &= \mathrm{M}_{f_k}^{k-1}\left(\mathrm{M}_{f_k}^{(k-1)(2k-1)}([k+1,k+1]_k, \theta_k([k+1,k+1]_k))\right) \\
&= \mathrm{M}_{f_k}^{k-1}\left([\frac{k+3}{2}, k+1]_k, \theta_k([\frac{k+3}{2}, k+1]_k)\right) \\
&= ([1, \frac{k+1}{2}]_k, [2, \frac{k+3}{2}]_k),
\end{aligned}$$

where the last equality is due to Lemma 60 and to obtain the second equality we need to apply Eqs. (45) and (46) $\frac{k-1}{2}$ times. This implies $y = (1, \frac{k+3}{2}) \in \mathrm{M}_{f_k}^{2k^2-2k}(z)$. We can also check that for all nonnegative integers $N < 2k^2 - 2k$,

$$y \notin \mathrm{M}_{f_k}^{N}(z).$$

Therefore $\mathrm{Dist}_{f_k}(z,y) = 2k^2 - 2k$. It then follows $\mathrm{Dia}(f_k) \geqslant \mathrm{Dist}_{f_k}(x,y) = \mathrm{Dist}_{f_k}(z,y) + \mathrm{Dist}_{f_k}(x,z) = \mathrm{Dist}_{f_k}(z,y) + 2k \geqslant 2k^2$.

Next we turn to the case that k is even. By using Eq. (45) and Eq. (46) $\frac{k}{2}$ times and $\frac{k-2}{2}$ times respectively, we have

$$\begin{aligned}
\mathrm{M}_{f_k}^{2k^2-3k+1}(z) &= \mathrm{M}_{f_k}^{(k-1)(2k-1)}\left([k+1,k+1]_k, \theta_k([k+1,k+1]_k)\right) \\
&= \left([\frac{k+4}{2}, k+1]_k, \theta_k([\frac{k+2}{2}, k+1]_k)\right) \\
&= ([\frac{k+4}{2}, k+1]_k, [\frac{k+4}{2}, k+2]_k).
\end{aligned}$$

This says that $y = (1, \frac{k+4}{2}) \in \mathrm{M}_{f_k}^{2k^2-3k+1}(z)$. We can also check that for all nonnegative integers $N < 2k^2 - 3k + 1$,

$$y \notin \mathrm{M}_{f_k}^{N}(z).$$

Therefore $\mathrm{Dist}_{f_k}(z,y) = 2k^2 - 3k + 1$. Finally, we come to $\mathrm{Dia}(f_k) \geqslant \mathrm{Dist}_{f_k}(x,y) = \mathrm{Dist}_{f_k}(z,y) + \mathrm{Dist}_{f_k}(x,z) = \mathrm{Dist}_{f_k}(z,y) + 2k \geqslant 2k^2 - k + 1$, as desired. $\square$

Let us mention that some more complicated computations will show that equality holds in Eq. (48) for all $k \geqslant 5$.

Proof of Theorem 34. Lemma 62 implies that f_k is strongly connected for all $k \in \mathbb{N}$. Thus, the result is direct from Lemma 63. □

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