A note on non-$\mathbb{R}$-cospectral graphs

Fenjin Liu*
School of Science
Chang'an University
Xi'an, P.R. China
and
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an, P.R. China
fenjinliu@yahoo.com

Wei Wang
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an, P.R. China
wang_weiw@163.com

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Abstract

Two graphs G and H are called $\mathbb{R}$-cospectral if $A(G)+yJ$ and $A(H)+yJ$ (where $A(G)$, $A(H)$ are the adjacency matrices of G and H, respectively, J is the all-one matrix) have the same spectrum for all $y \in \mathbb{R}$. In this note, we give a necessary condition for having $\mathbb{R}$-cospectral graphs. Further, we provide a sufficient condition ensuring only irrational orthogonal similarity between certain cospectral graphs. Some concrete examples are also supplied to exemplify the main results.

Keywords: $\mathbb{R}$-cospectral graphs; Walk generating function; Irrational orthogonal matrix

1 Introduction

Throughout this paper, we are concerned with undirected simple graphs (loops and multiple edges are not allowed). Let G be a simple graph with $(0, 1)$-adjacency matrix $A(G)$.

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The *spectrum of* G consists of all the eigenvalues (together with their multiplicities) of $A(G)$. Two graphs G and H are called *cospectral* if G and H share the same spectrum. If $A(G)+yJ$ and $A(H)+yJ$ (where J is the all-one matrix) have the same spectrum for all $y \in \mathbb{R}$, then G and H are called $\mathbb{R}$-*cospectral* (also *generalized cospectral*). By taking $y=-1$ we see that $\mathbb{R}$-cospectral graphs have cospectral complements.

A graph G is said to be *determined by its (generalized) spectrum* ((DGS) DS for short) if any graph (generalized) cospectral with G is isomorpic with G. A fundamental problem in spectral graph theory is "which graphs are determined by their spectrum?" Up to now, only a small number of graphs with special structures are known to be DS, simplify because DS is a property difficult to prove. Haemers conjectured that almost all graphs are determined by their spectrum [5], however, it is still open. For excellent surveys on this subject, see [5, 7]. Recently, several new results are published such as some almost complete graphs [2] are shown to be DS as well as the spectral characterization of pineapple graphs [10] are corrected. When it comes to the generalized spectrum, a necessary and sufficient condition for the corona of some graphs to be DGS is established [12], moreover, the DGS property of a large family of graphs $\mathcal{F}_n$ [16] is also obtained [17].

Non-isomorphic cospectral graphs are easily made, early in the last century, Schwenk [14] proved that almost all trees are cospectral. Godsil and McKay [8] invented a powerful and productive method called GM-switching, which can produce numerous pairs of $\mathbb{R}$-cospectral graphs. If G and H are cospectral, then their adjacency matrices $A(G)$ and $A(H)$ are similar. Since both are real symmetric, $A(G)$ and $A(H)$ have a real orthogonal similarity, that is, there exists an orthogonal matrix Q such that $A(G)=Q^TA(H)Q$. An orthogonal matrix Q is *regular* if it has all row and column sums 1. A regular orthogonal matrix Q has *level* ℓ if ℓ is the smallest positive integer such that ℓQ is an integral matrix. Q has level $\ell=\infty$ if it has irrational entries. It easily follows that G and H are $\mathbb{R}$-cospectral if Q is regular. The following result, due to Johnson and Newman [11], gives some equivalent conditions for $\mathbb{R}$-cospectral graphs.

Theorem 1. *[1, 11] Let G and H be two graphs with adjacency matrices $A(G)$ and $A(H)$, respectively, then the following are equivalent:*

(i) G and H are cospectral with cospectral complements.

(ii) G and H are $\mathbb{R}$-cospectral.

(iii) There exists a regular orthogonal matrix Q, such that $A(G)=Q^TA(H)Q$.

It is well known that many simple necessary conditions are achieved for cospectral graphs such as equal number of vertices; edges; triangles [5] etc. However, except for that, such similar necessary conditions for having $\mathbb{R}$-cospectral graphs are seldom considered. In this note, we show that $\mathbb{R}$-cospectral graphs have the same number of cycles C_4; paths P_3; paths P_4 (all the subgraphs C_4, P_3, P_4 above are not necessarily induced).

The rationality property of the entries of Q in Theorem 1 was firstly considered in [15]. If the eigenvalues of graph G are restricted to be all simple (its multiplicity is one) and *main* (it has an associated eigenvector the sum of whose entries is not equal to zero), then Wang and Xu showed that the regular orthogonal matrix Q must be rational.

Theorem 2. *[15, 17] Let G be a graph whose eigenvalues are all main and simple (i.e., controllable graph), $A(G)$ and $A(H)$ be the adjacency matrices of graphs G and H, respectively. Then G and H are $\mathbb{R}$-cospectral if and only if there exists a unique regular rational orthogonal matrix Q such that $A(G) = Q^T A(H)Q$.*

Movitated by this theorem, another two questions arise.

(i) Can we give some families of cospectral graphs such that there only exists real irrational orthogonal similarity between their adjacency matrices?

(ii) Can we find a pair of $\mathbb{R}$-cospectral graphs G and H such that there exists two regular rational orthogonal matrices Q_1, Q_2 and $A(G) = Q_1^T A(H)Q_1$, $A(H) = Q_2^T A(H)Q_2$.

Theorem 2 states that if the walk matrix of a graph G is nonsingular, then there exists a graph H that is cospectral with G w.r.t. the generalized spectrum if and only if there exists a rational regular orthogonal matrix such that the adjacency matrices are similar. So it basically means that the regular orthogonal matrix has a level. Nevertheless, we provide a sufficient condition which ensures only irrational orthogonal similarity between certain cospectral graphs (or equivalently when there is no condition on the walk matrix being nonsingular). Moreover, S. O'rourke and B. Touriz lately [13] shows that the relative number of controllable graphs compared to the total number of simple graphs on n vertices approaches one as n tends to infinity, which gives even more evidence of the importance of the result of item (i) presented in this note. In addition, we supply a pair of $\mathbb{R}$-cospectral graphs, which can also be found in [1], satisfying the above statement (ii).

2 Main results

By a *walk of length k* in a graph we mean any sequence of (not necessarily different) vertices $v_1 v_2 \dots v_k v_{k+1}$ such that for each $i = 1, 2, \dots, k$ there is an edge from v_i to v_{i+1}. The walk is closed if $v_{k+1} = v_1$. We start with the renowned *walk generating function.*

Lemma 3. *[3] Let G be a graph with complement $\bar{G}$, and $H_G(t) = \sum_{k=0}^{\infty} N_k(G)t^k$ be the generating function of the number $N_k(G)$ of walks of length k in G ($k = 0, 1, 2, \dots$). Then $H_G(t) = \frac{1}{t}[(-1)^n P_{\bar{G}}(-\frac{t+1}{t})/P_G(\frac{1}{t}) - 1]$, where $P_G(t)$ and $P_{\bar{G}}(t)$ are characteristic polynomials of G and $\bar{G}$, respectively.*

Since $\mathbb{R}$-cospectral graphs are cospectral with cospectral complements, they have the same characteristic polynomial and complement characteristic polynomial. It easily follows a corollary.

Corollary 4. *Let G and H be $\mathbb{R}$-cospectral graphs. Then G and H have the same walk generating function and the same number of walks of any length k.*

Consider the number of walks and closed walks of length no more than four, then we can prove the following necessary condition for $\mathbb{R}$-cospectral graphs.

Theorem 5. *For the adjacency matrix, the following can be deduced form the generalized spectrum:*

(1) the number of subgraphs C_4.

(2) the number of subgraphs P_3.

(3) the number of subgraphs P_4, all the subgraphs C_4, P_3, P_4 are not necessarily induced.

Proof. Suppose G and H are $\mathbb{R}$-cospectral, then G and H have the same number of closed walks and walks of any given length k. It is well known that the numbers of closed walks of length $2, 3, 4$ are equal to twice number of edges; six times the number of triangles; twice number of edges plus four times the numbers P_3 and eight times the number of cycles C_4 in a graph; respectively. Thus G and H have the same number of edges, triangles. For (closed or non-closed) walks, it is clear that the numbers of walks of length $2, 3$ are equal to twice number of edges and P_3 contained in a graph; the sum of twice number of edges, four times the number of P_3, twice number of P_4 and six times the number of triangles, respectively, in a graph. Therefore, $\mathbb{R}$-cospectral graphs have the same number of cycles C_4 and paths P_3, P_4. $\square$

Theorem 5 can be easily used to determine some cospectral graphs that are not $\mathbb{R}$-cospectral, see an example below.

Example 6. Let W_n ($n \geqslant 2$) be the tree obtained from the path P_{n+2} by appending a pendent vertex to the two end 2-degree vertices, respectively, and $W_1 \cong K_{1,4}$. Let X_n ($n \geqslant 1$) be the disjoint union of the cycle C_4 and path P_n. It is known from [4] that W_n and X_n are cospectral. Since W_n contains no cycle C_4 and X_n has one, it follows from Theorem 5 that W_n and X_n are non-$\mathbb{R}$-cospectral.

Next we present another main result of the note.

Theorem 7. *Let G and H be a pair of cospectral graphs with a common simple eigenvalue λ_0. If $A(G)$ possesses a irrational normal eigenvector $\mathbf{x}_0$ while $A(H)$ has a rational normal eigenvector $\mathbf{y}_0$ corresponding to λ_0, then there exists no rational orthogonal matrix Q such that $A(G) = Q^T A(H) Q$.*

Proof. Assume to the contrary that there exists a rational orthogonal matrix $Q = (q_{ij})$, where $q_{ij} \in \mathbb{Q}$, such that $A(G) = Q^T A(H) Q$, then $QA(G) = A(H)Q$. It follows that

$$\lambda_0 Q\mathbf{x}_0 = Q(A(G)\mathbf{x}_0) = A(H)(Q\mathbf{x}_0), \tag{1}$$

namely, $Q\mathbf{x}_0$ is also an eigenvector of H corresponding to the simple eigenvalue λ_0. So

$$\mathbf{y}_0 = kQ\mathbf{x}_0 \quad (k \neq 0,\ k \in \mathbb{R}). \tag{2}$$

From $1 = \mathbf{y}_0^T\mathbf{y}_0 = k^2\mathbf{x}_0^T\mathbf{x}_0 = k^2$, we see that $k = \pm 1$ and $\mathbf{x}_0 = \pm Q^T\mathbf{y}_0$. Since $\mathbf{x}_0$ is an irrational eigenvector while $\pm Q^T\mathbf{y}_0$ is rational, we get a contradiction. $\square$

In order to exemplify Theorem 7, we give an example below. In fact, one may find other such cospectral pairs.

Example 8. If $n \neq k^2 - 1$ $(k = 2, 3, \dots)$, then there exists no rational orthogonal matrix Q such that $A(W_n) = Q^T A(X)_n Q$.

Label the vertices of W_n as follows: the vertices on the path P_{n+2} be consecutively marked as $u_1, \dots, u_{n+2}$. u_{n+3}, u_{n+4} are adjacent to u_{n+1} and u_2, respectively. Label the path P_n of X_n by $v_1 \dots, v_n$ and the four vertices on the cycle C_4 by $v_{n+1}, \dots, v_{n+4}$. Then the adjacency matrices $A(W_n)$ and $A(X_n)$ are of the form:

$$A(W_n) = \left[\begin{array}{ccccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{3}$$

$$A(X_n) = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 1 & 0 \end{matrix} & O \\ \hline O & \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix} \end{array}\right]. \tag{4}$$

Since graphs W_n and X_n share the common adjacency spectrum $\{2, 0^{(2)}, -2\} \cup \{2\cos\frac{j}{n+1}\pi$ $(j = 1, \dots, n)\}$, where the exponent indicates the multiplicity of the eigenvalue. We see that the spectral radius 2 is a simple eigenvalue. Solving the characteristic equation $A(W_n)\mathbf{x} = 2\mathbf{x}$ gives that $\mathbf{x} = (1, \overbrace{2, \dots, 2}^{n}, 1, 1, 1)^T$. Similarly, from $A(X_n)\mathbf{y} = 2\mathbf{y}$, we obtain $\mathbf{y} = (\overbrace{0, \dots, 0}^{n}, 1, 1, 1, 1)^T$. Normalizing the above two eigenvectors gives $\mathbf{x}_0 = 2\sqrt{n+1}\mathbf{x}$ and $\mathbf{y}_0 = \frac{1}{2}\mathbf{y}$. Obviously, $\mathbf{x}_0$ is irrational while $\mathbf{y}_0$ is rational, it follows our deduction by Theorem 7

Furthermore, we calculate the concrete irrational orthogonal matrix Q in Example 8 for $n = 1, 2, 3$.

Example 9. Suppose $A(W_n)$ and $A(X_n)$ be shown in Eqs. (3) and (4), define Q_1, Q_2

and Q_3 as follows:

$$Q_1 = \begin{bmatrix} -\frac{1}{2\sqrt{3}} & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} + \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} + \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2} + \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{2} + \frac{1}{2\sqrt{3}} & -\frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{2} + \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{2} + \frac{1}{2\sqrt{3}} & \frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 \\ -\frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{2} + \frac{1}{2\sqrt{3}} \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{5}} & 0 & \frac{1}{2\sqrt{5}} & 0 & -\frac{1}{2\sqrt{2}} - \frac{\sqrt{5}}{4} & -\frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{5}} & \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{5}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{5}} & 0 & -\frac{1}{2\sqrt{5}} & 0 & -\frac{1}{2\sqrt{2}} + \frac{\sqrt{5}}{4} & -\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{5}} & \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} & \frac{1}{2} & \frac{1}{\sqrt{5}} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{5}} & -\frac{1}{2\sqrt{5}} \\ \frac{3}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2\sqrt{5}} & \frac{1}{2} & -\frac{1}{\sqrt{5}} & \frac{1}{2} & 0 & \frac{1}{\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ -\frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

Then a direct calculation shows that $A(W_n) = Q_n^T A(X_n) Q_n$ $(n = 1, 2, 3)$.

Finally, we provide a pair of $\mathbb{R}$-cospectral graphs on eight vertices to answer the question of the statement (ii).

Example 10. [1] Let Γ_1 and Γ_2 be a pair of $\mathbb{R}$-cospectral graphs with adjacency matrices

$$A(\Gamma_1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \ A(\Gamma_2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Suppose that

$$\widetilde{Q}_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{Q}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

It is straightforward to verify $A(\Gamma_1) = \widetilde{Q}_i^T A(\Gamma_2)\widetilde{Q}_i$ $(i = 1, 2)$. In addition, simple calculation shows that $A(\Gamma_1)$ and $A(\Gamma_2)$ have the identical characteristic polynomial

$$P(x) = x^2(x^6 - 14x^4 - 14x^3 + 21x^2 + 14x - 11),$$

so it follows that Γ_1 (Γ_2) has a multiple eigenvalue 0 and only seven main eigenvalues.

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