Abaci structures of $(s, ms \pm 1)$-core partitions

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Abstract

We develop a geometric approach to the study of $(s, ms-1)$-core and $(s, ms+1)$-core partitions through the associated ms-abaci. This perspective yields new proofs for results of H. Xiong and A. Straub (originally proposed by T. Amdeberhan) on the enumeration of $(s, s+1)$ and $(s, ms-1)$-core partitions with distinct parts. It also enumerates the $(s, ms+1)$-cores with distinct parts. Furthermore, we calculate the size of the $(s, ms-1, ms+1)$-core partition with the largest number of parts. Finally we enumerate self-conjugate core partitions with distinct parts. The central idea throughout is that the ms-abaci of largest $(s, ms \pm 1)$-cores can be built up from s-abaci of $(s, s \pm 1)$-cores in an elegant way.

Keywords: Young diagrams; symmetric group; p-cores; abaci; triangular numbers

1 Introduction

1.1 Partitions and abacus diagrams

A *partition* λ of the positive integer n is a weakly decreasing sequence of positive integers which sum to n. We will call n the *size* of λ. Each of the integers which make up the partition is known as a *part* of the partition. For example, $(8, 6, 5, 5, 3, 2, 2, 2, 1)$ is a partition with size $n = 34$, and is alternatively written as $(8, 6, 5^2, 3, 2^3, 1)$.

A *Young diagram* is a pictorial representation of a partition. Simply put, it is a finite collection of boxes which are arranged in left-justified rows with the row lengths weakly decreasing (since each row of the Young diagram corresponds to a part in the partition). To each box in the Young diagram of λ we assign a *hook*, which is the set of boxes in the same row and to the right, and in the same column and below, as well as the box itself, which is called the *corner* of the hook. We use matrix notation to label the hooks: h_{ij} is the hook whose corner is in the i-th row and the j-th column. The number of boxes

$|h_{ij}|$ is the *hook length* of h_{ij}. The *first-column hook lengths* are those that appear in the left-most column of the Young diagram.

The first-column hook lengths uniquely determine a partition λ. We can generalize the set of first column hooks using the notion of a *bead set* X corresponding to λ, where $X = \{0, \dots, k-1, |h_{11}| + k, |h_{21}| + k, |h_{31}| + k, \dots\}$ for some non-negative integer k. It can also be seen as a finite set of non-negative integers, represented by *beads* at integral points of the x-axis, i.e., a bead at position x for each x in X and *spacers* at positions not in X. Then $|X|$ is the number of beads that occur after the zero position, wherever that may fall. The *minimal* bead-set X of λ is one where 0 labels the first spacer, and is exactly the set of first-column hook lengths.

Example 1. Suppose $\lambda = (4, 3, 2)$. Then $\{h_{\iota 1}\} = \{2, 4, 6\}$, where $1 \leqslant \iota \leqslant 3$ is the set of first column hook lengths, and a minimal bead set. Note that $X' = \{0, 2+1, 4+1, 6+1\} = \{0, 3, 5, 7\}$ and $X'' = \{0, 1, 2, 3, 2+4, 4+4, 6+4\} = \{0, 1, 2, 3, 6, 8, 10\}$ are two bead sets that also correspond to λ.

The set of hooks $\{h_{\iota\gamma}\}$ of λ correspond bijectively to pairs (x, y) where $x \in X$, $y \notin X$ and $x > y$; that is, a bead in a bead-set X of λ and a spacer to the left of it. Hooks of length s are those such that $x - y = s$.

The following result (Lemma 2.4, [15]) allows us to recover the size of the part from its corresponding bead.

Lemma 2. *Let X be a bead-set of a partition λ. The size of the part λ_α of λ corresponding to the bead $x' \in X$ is the number of spacers to the left of the bead, that is, $\lambda_\alpha = |y \notin X : y < x'|$.*

Given a fixed integer s, we can arrange the nonnegative integers into an *s-grid*, an array of s columns labeled from $0 \leqslant i \leqslant s-1$, and consider the columns as runners, on which beads are placed in their respective positions. This organizes a given bead-set by their values modulo s.

Definition 3 (***s***-abacus)**.** Consider a bead-set X. Placing a bead in each position on the s-grid where there is a value $x \in X$ gives the ***s-abacus diagram*** $\mathcal{S}$ of X. Positions not occupied by beads are *spacers*. A ***minimal*** s-abacus $\mathcal{S}$ corresponds to a minimal bead-set X (where the first spacer labels the zero position).

Example 4. Consider the partition $\lambda = (10, 5^2, 3^3, 1^4)$ with a minimal bead-set $X = \{1, 2, 3, 4, 7, 8, 9, 13, 14, 19\}$. Then Figure 1 displays the minimal 5-abacus associated to X.

Definition 5 (***s***-abacus position)**.** Let $\mathcal{S}$ be the s-abacus associated to a bead-set X. We say that a bead $x \in X$ has ***s-abacus position*** $(i, j) \in \mathcal{S}$, where $0 \leqslant i \leqslant s-1$ and $j \geqslant 0$ if and only if $i + js = x \in X$.

Definition 6. A ***sub-abacus*** $\mathcal{S}'$ of an s-abacus $\mathcal{S}$ is a set of s-abacus positions (i, j) that obey the property that if $(i, j) \in \mathcal{S}'$, then $(i, j) \in \mathcal{S}$.

1.2 s-core and simultaneous (s,t)-core partitions

A *s-core partition* (or simply s-core) of n is a partition in which no hook of length s appears in the Young diagram. Note that a bead x in runner i with a spacer y one row below, but also in runner i, corresponds to an s-hook of λ. A partition λ is a s-core if and only if its s-abacus has the property that no spacer occurs below a bead in a given runner. This is expressed in the following lemma.

Lemma 7. *An s-abacus $\mathcal{S}$ corresponds to an s-core partition if and only $(i,j) \in \mathcal{S}$ and $j > 0$ implies that $(i, j-1) \in \mathcal{S}$.*

We then have the following result.

Corollary 8. *An s-core partition is an ms-core partition for all $m > 1$.*

Proof. An ms-hook on an s-abacus $\mathcal{S}$ is expressed as a bead in abacus position (i,j) and a spacer in position $(i, j-m)$. Either there are no beads in positions $(i, j-1), \ldots, (i, j-m+1)$ or there is at least one. In the either case, we violate the condition of Lemma 7. ☐

A result of Sylvester from 1884 gives us the size of the largest possible first-column hook length of a simultaneous (s,t)-core.

Proposition 9. *If gcd(s,t)=1, the largest possible hook of an (s,t)-core has length $st - s - t$.*

In recent years, the study of core partitions has expanded to include partitions which are simultaneously cores for various integers. Anderson [5] first enumerated (s,t)-cores in the case when s and t are relatively prime. Subsequently, the work of Olsson and Stanton (and others) showed that, when $\gcd(s,t) = 1$, there is a unique (s,t)-core with largest size, denoted by $\kappa_{s,t}$. We call such a simultaneous core *largest*.

Theorem 10 (J. Olsson and D. Stanton, Theorem 4.1, [16])**.** *Let gcd$(s,t) = 1$. Then there is a unique largest (s,t)-core $\kappa_{s,t}$ such that*

$$|\kappa_{s,t}| = \frac{(s^2-1)(t^2-1)}{24}.$$

Using the notation $\kappa_{s,t}$ we restate a canonical result of J. Anderson (Proposition 1,[5]).

Proposition 11. *Suppose $s, t > 1$ are distinct integers such gcd$(s,t) = 1$, and let ν be a positive integer greater than 1. Then minimal ν-abacus $\mathcal{V}$ of any (s,t)-core will be a sub-abacus of the minimal ν-abacus $\mathcal{K}$ of $\kappa_{s,t}$ the largest (s,t)-core partition. Furthermore, if $(i,j) \in \mathcal{V}$ then $(i, j-1) \in \mathcal{V}$ and $(i-t, j) \in \mathcal{V}$ if $i > t$. If $i < t$ then $(i,j) \in \mathcal{V}$ implies $(s-t-1, j-1) \in \mathcal{V}$.*

As a consequence of Proposition 9, Theorem 10, and Proposition 11, we have the following useful result.

Corollary 12. *$\kappa_{s,t}$ is the unique (s,t)-core with a hook of length $st-s-t$.*

We note that the special case of $(s, ms+1)$-cores has attracted particular interest. Early examples of the now-resolved Armstrong conjecture (cf. [6] [10] [19]) included the $(s, s+1)$ and $(s, ms+1)$ cases, done by F. Zanello and R. Stanley [17] and A. Aggarwal [2] respectively. S. Fischel and M. Vazirani [9], have studied a bijection between $(s, ms+1)$-cores and dominant Shi regions.

Self-conjugate simultaneous core partitions are also of interest. B. Ford, H. Mai and L. Sze [8] have, in a manner analogous to Olsson-Stanton, enumerated the self-conjugate (s,t)-core partitions.

In Section 4, we study $(s, ms \pm 1)$-cores with distinct parts; in Section 5 we apply our methods to analogize the results of Xiong and Straub and enumerate the self-conjugate simultaneous $(s, s+1)$-core and $(s, ms \pm 1)$-core partitions with distinct parts. Before we do, we give an overview of existing results on simultaneous core partitions with distinct parts.

1.3 Simultaneous (s,t)-cores with distinct parts

Simultaneous core partitions with distinct parts were first introduced as an object of study by T. Amdeberhan. One of the conjectures proposed by T. Amdeberhan (Conjecture 11.9,[3]) has lead to new results by H. Xiong and A. Straub in this area. More recently A. Zaleski [22] has obtained results on moments of the generating functions of $(s, s+1)$-core partitions with distinct parts, building on work by S. Ekhad and D. Zeilberger [7].

Theorem 13 (H. Xiong, Theorem 1.1(1), [20])**.** *Let $s \geqslant 1$ and F_{s+1} be the $(s+1)$st Fibonacci number. Then F_{s+1} is the number of $(s, s+1)$-core partitions with distinct parts.*

Theorem 14 (A. Straub, Theorem 4.1, [18])**.** *Let $m, s \geqslant 1$. The number $E_m^-(s)$ of $(s, ms-1)$-core partitions with distinct parts is characterized by $E_m^-(1) = 1$ and $E_m^-(2) = m$ and, for $s \geqslant 3$,*

$$E_m^-(s) = E_m^-(s-1) + mE_m^-(s-2).$$

Our paper develops a framework from which results of H. Xiong and A. Straub in this direction follow naturally. That is, we use the geometry of the s-abacus of the largest $(s, s+1)$-core, and that of the ms-abacus of the largest $(s, ms-1)$-core and $(s, ms+1)$-core partitions, to prove Theorems 13 and 14 in a uniform manner. Before proving Theorem 14, however, we enumerate $(s, ms+1)$-core partitions with distinct parts (Theorem 15). In doing so, we provide a partition-theoretic meaning to a numerical relation first observed by Straub (see Lemma 4.3, [18]). This lays the groundwork for the proof of Theorem 14.

Theorem 15. *Let $m, s \geqslant 1$. The number $E_m^+(s)$ of $(s, ms+1)$-core partitions into distinct parts is characterized by $E_m^+(1) = 1$, $E_m^+(2) = m+1$ and, for $s \geqslant 3$,*

$$E_m^+(s) = E_m^+(s-1) + mE_m^+(s-2).$$

These proofs appear in Section 4.

1.4 Simultaneous $(s, ms-1, ms+1)$-core partitions

Suppose s, t, u are positive integers such that $\gcd(s, t, u) = 1$. Enumerating and calculating the size of simultaneous (s, t, u)-cores is more complicated than simultaneous (s, t)-cores, in part because no analogous result to Sylvester's characterization of the maximum possible hook length exists. However, for special cases, progress has been made. T. Amdeberhan and E. Leven [4], R. Nath and J. Sellers, [14], Xiong [20], and Yang-Zhang-Zhou [21] investigated $(s-1, s, s+1)$-cores, and both the size of the largest core and the number of such cores is known. V. Wang [19] has enumerated $(s, s+d, s+2d)$-cores. A. Aggarwal [1] has also studied containment properties of (s, t, u)-cores.

The methods described herein also allow us to study another family of triply simultaneous cores: in particular, we calculate the size of the **longest** $(s, ms-1, ms+1)$-core partition (that is, the core partition with the largest number of parts).

Theorem 16. *The size of the longest* $(s, ms-1, ms+1)$*-core is*

1. $\frac{m^2 t(t-1)(t^2-t+1)}{6}$ *if* $s = 2t-1$;
2. $\frac{m^2(t-1)^2(t^2-2t+3)}{6} - \frac{m(t-1)^2}{2}$ *if* $s = 2t-2$.

We also conjecture that this is the size of any largest $(s, ms-1, ms+1)$-core.

The key observation we utilize in proving all of our results is the way in which the ms-abaci of largest $(s, ms \pm 1)$-cores are built up from the s-abaci of $(s, s \pm 1)$-cores and other objects. Hence, we now transition to a detailed description of the relevant s-abaci and ms-abaci.

Note: For the remainder of the paper, $(s, ms\pm1)$-core partitions will refer to either an $(s, ms-1)$-core partition *or* an $(s, ms+1)$-core partition. The notation $(s, ms-1, ms+1)$-core will indicate a core that is simultaneously a s-core, an $(ms-1)$-core, *and* an $(ms+1)$-core.

2 s-abaci of $(s, s \pm 1)$-cores

The following two lemmas follow from Definition 5.

Lemma 17. *An* s*-abacus* $\mathcal{S}$ *is an* $(s+1)$*-core if* $(i, j) \in \mathcal{S}$ *implies*

1. $(i-1, j-1) \in \mathcal{S}$ *when* $0 < i \leqslant s-1$ *and* $j \geqslant 1$*, and*
2. $(s-1, j-2) \in \mathcal{S}$ *when* $i = 0$ *and* $j \geqslant 2$.

Proof. Suppose $\mathcal{S}$ is the s-abacus of an $(s+1)$-core. Then $(i, j) \in \mathcal{S}$ if and only if there if a bead in a position $s+1$ steps to the left, wrapping down-and-around-to-the-right the abacus when necessary. This is exactly the statement of the lemma. □

Lemma 18. *An s-abacus $\mathcal{S}$ represents an $(s-1)$-core partition if $(s-1,0) \notin \mathcal{S}$ and $(i,j) \in \mathcal{S}$ with $j > 0$ implies*

1. $(i+1, j-1) \in \mathcal{S}$ *when* $0 < i < s-1$
2. $(0, j) \in \mathcal{S}$ *when* $(s-1, j)$ *is.*

Proof. The argument is identical to that of Lemma 17 replacing $s+1$ by $s-1$, with the caveat that a bead in position $(s-1,0)$ is not permitted. □

A crucial part of our argument below will involve the following two abaci constructions.

Definition 19. Let $\mathcal{A}(s)$ be the s-abacus with beads in abacus positions (i,j) for every (i,j) such that $0 < i \leqslant s-1$ and $0 \leqslant j \leqslant i-1$.

Example 20. $\mathcal{A}(5) = \{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(4,3)\}$. [See Figure 1.]

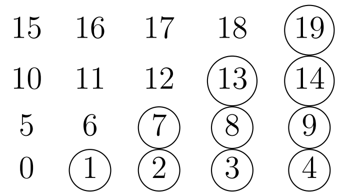

Figure 1: $\mathcal{A}(5)$

Lemma 21. *$\mathcal{A}(s)$ is the minimal s-abacus of $\kappa_{s,s+1}$, the largest $(s,s+1)$-core.*

Proof. $\mathcal{A}(s)$ is minimal by construction. To show $\mathcal{A}(s)$ is the s-abacus of an $(s,s+1)$-core we have to show that Lemma 7 and Lemma 17 are satisfied. Suppose $j > 0$. If $(i,j) \in \mathcal{A}(s)$ then, since $j \leqslant i-1$, it follows by Definition 19 that when $1 \leqslant i \leqslant s-1$, we have $(i, j-1) \in \mathcal{A}(s)$ and $(i-1, j-1) \in \mathcal{A}(s)$.

Let X be the underlying bead-set of $\mathcal{A}(s)$. Since $(s-1, s-2) \in \mathcal{A}(s)$, by Definition 5 we have $s-1+(s-2)s = s^2+s-1 \in X$. By Corollary 12 this implies that $\mathcal{A}(s)$ is the s-abacus of the largest $(s,s+1)$-core, since $s(s+1)-s-(s+1) = s^2-s-1$. □

Definition 22. Let $\mathcal{B}_k(s)$ be the s-abacus with beads in abacus positions (i,j) for every (i,j) such that $0 < i \leqslant s-1-k$ and $0 \leqslant j \leqslant s-i-k-1$.

The proofs of Lemmas 23 and 24 follow from Definition 22. Details are left to the reader.

Lemma 23. *$\mathcal{B}_k(s)$ has the following properties. If $(i,j) \in \mathcal{B}_k(s)$ then*

1. $(i, j-1) \in \mathcal{B}_k(s)$ *and*

2. $(i+1, j-1) \in \mathcal{B}_k(s)$.

Lemma 24. *$\mathcal{B}_1(s)$ is obtained from $\mathcal{B}_0(s)$ by removing beads in abacus-positions $(i, s-1-i)$ as $1 \leqslant i \leqslant s-1$.*

Example 25. $\mathcal{B}_0(5) = \{(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(3,0),(3,1),(4,0)\}$. [See Figure 2.]

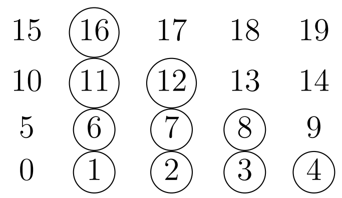

Figure 2: $\mathcal{B}_0(5)$: 5-abacus of a $(5,9)$-core

Example 26. $\mathcal{B}_1(5) = \{(1,0),(1,1),(1,2),(2,0),(2,1),(3,0)\}$. [See Figure 3.]

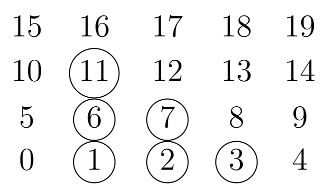

Figure 3: $\mathcal{B}_1(5)$: 5-abacus of the largest $(4,5)$-core

For the remainder of the paper, we focus our attention on $\mathcal{B}_0(s)$ and $\mathcal{B}_1(s)$.

Lemma 27. *$\mathcal{B}_1(s)$ is the minimal s-abacus of the largest $(s-1, s)$-core partition.*

Proof. $\mathcal{B}_1(s)$ is minimal by construction. By Lemma 7, Lemma 18 and Lemma 23, $\mathcal{B}_1(k)$ is a $(s-1, s)$-core.

Let X be the underlying bead-set of $\mathcal{B}_1(s)$. Since $(1, s-3) \in \mathcal{B}_1(s)$, $1 + (s-3)s = (s-1)s - s - (s-1) \in X$. By Corollary 12, we are done. □

Corollary 28. *$\mathcal{B}_1(s)$ is a sub-abacus of $\mathcal{B}_0(s)$.*

Proof. Follows from Lemma 24. □

Example 29. $\mathcal{B}_1(5)$ is a sub-abacus of $\mathcal{B}_0(5)$. [See Figures 2 and 3.]

3 ms-abaci of $(s, ms \pm 1)$-cores

We now generalize the results of the previous section by moving to $(s, ms \pm 1)$-cores.

Lemma 30. *Let $\mathcal{M}$ be an ms-abacus, where $m > 1$. Then $\mathcal{M}$ corresponds to an s-core partition if*

1. *$(i, j) \in \mathcal{M}$ then $(i - s, j)$ when $s \leqslant i \leqslant ms - 1$*
2. *$(i, j) \in \mathcal{M}$ then $(s - i - 1, j - 1) \in \mathcal{M}$ if $0 \leqslant i < s$.*

Proof. Part (1) is immediate. Part (2) ensures that when moving s positions to the left of (i, j) wraps around-and-down the ms-abacus, a bead occupies the relevant abacus position. □

The next corollary follows from Corollary 8 and the definition of an ms-abacus.

Corollary 31. *Let $\mathcal{M}$ be an ms-abacus. If $\mathcal{M}$ is an s-core, then, if $(i, j) \in \mathcal{M}$, we have $(i, j - 1) \in \mathcal{M}$.*

Definition 32. We define the following two special ms-abaci.

1. Let $\mathcal{E}_m^-(s)$ be the ms-abacus with beads in abacus positions $(i + \ell s, j)$, where $0 \leqslant \ell \leqslant m - 2$ for $1 \leqslant i \leqslant s - 1$ and $1 \leqslant j \leqslant s - i - 1$ and $(i + (m - 1)s, s - i - 2)$ for $1 \leqslant i \leqslant s - 2$ and $0 \leqslant j \leqslant s - i - 2$.
2. Let $\mathcal{E}_m^+(s)$ be the ms-abacus defined by $(i + \ell s, j)$ where $1 \leqslant i \leqslant s - 1$ and $0 \leqslant j \leqslant i - 1$ and $0 \leqslant \ell \leqslant m - 1$.

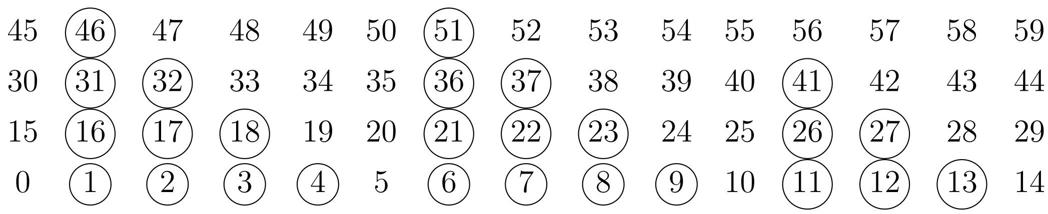

Figure 4: $\mathcal{E}_3^-(5)$: 15-abacus of the largest $(5, 14)$-core

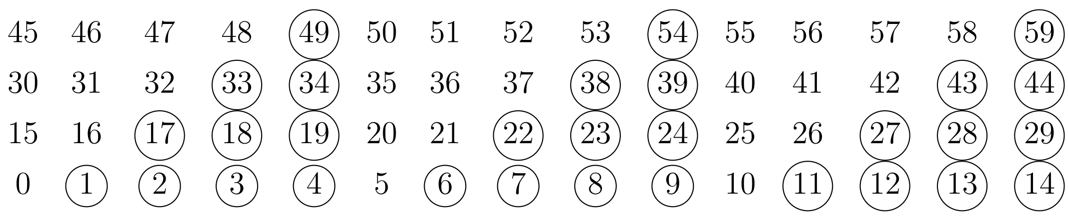

Figure 5: $\mathcal{E}_3^+(5)$: 15-abacus of the largest $(5, 16)$-core

Theorem 33. *The following relations hold for $\mathcal{E}_m^-(s)$ and $\mathcal{E}_m^+(s)$.*

1. *$\mathcal{E}_m^-(s)$ is the minimal ms-abacus of the largest $(s, ms-1)$-core partition.*

2. *$\mathcal{E}_m^+(s)$ is the minimal ms-abacus of the largest $(s, ms+1)$-core partition.*

Before we can prove Theorem 33, we need to define a new operation on abaci.

Definition 34. Suppose $\mathcal{A}$ and $\mathcal{B}$ are s-abaci and t-abaci respectively. We denote by $\mathcal{A} \wedge \mathcal{B}$ the $(s+t)$-abacus whose $0 \leqslant i \leqslant s-1$ runners correspond to the $s-1$ runners of $\mathcal{A}$, and whose $s \leqslant i \leqslant s+t-1$ runners of correspond to the $t-1$ runners of $\mathcal{B}$. This will be called **appending** $\mathcal{B}$ to $\mathcal{A}$ on the right. When we append $\mathcal{A}$ to itself m times, we will use the notation $\wedge_m \mathcal{A} = \underbrace{\mathcal{A} \wedge \cdots \wedge \mathcal{A}}_{\text{m}}$.

Lemma 35. *The following relations hold.*

1. *$\mathcal{E}_m^-(s) = (\wedge_{m-1}\mathcal{B}_0(s)) \wedge \mathcal{B}_1(s)$;*

2. *$\mathcal{E}_m^+(s) = \wedge_m \mathcal{A}(s)$.*

Proof. Fix an $\ell \in [0, m-1]$. Consider the projection map π_ℓ that takes $(i+\ell s, j)$ to (i, j), where $1 \leqslant i \leqslant s-1$. We prove each case separately.

1. Fix a $\ell \in [0, m-2]$. Then, under π_ℓ, the beads with abacus positions $(i+\ell s, j)$ $1 \leqslant i \leqslant s-1$, and $1 \leqslant j \leqslant s-i-1$ are in bijection with those in $\mathcal{B}_0(s)$. For $\ell = m-1$, under π_{m-1}, beads in positions $(i+(s-1)m, s-i-2)$ where $1 \leqslant i \leqslant s-2$ are in bijection with $\mathcal{B}_1(s)$.

2. Fix an $\ell \in [0, m-1]$. Then, under π_ℓ, the beads with abacus positions $(i+\ell s, j)$ $1 \leqslant i \leqslant s-1$, and $1 \leqslant j \leqslant s-i-1$ are in bijection with those in $\mathcal{A}(s)$. □

Example 36. $\mathcal{E}_3^-(5) = \wedge_2 \mathcal{B}_0(5) \wedge \mathcal{B}_1(5)$ and $\mathcal{E}_3^+(5) = \wedge_3 \mathcal{A}_0(5)(3)$. [See Figures 4 and 5.]

We are now in a position to prove Theorem 33, which is critical for the rest of our results.

Proof of Theorem 33. The abaci above are minimal, by construction. It remains to show they satisfy the relevant core properties and that they are of largest size. We consider each case separately.

1. Suppose $(i, j) \in \mathcal{E}_m^-(s)$, with $j > 0$. By Lemma 35, $(i, j) \in \wedge_{m-1}(\mathcal{B}_0(s)) \wedge \mathcal{B}_1(s)$. To see that $\mathcal{E}_m^-(s)$ is an s-core, we have to satisfy the conditions of Lemma 30. Suppose $j > 0$. If (i, j) is in the copy of $\mathcal{B}_1(s)$, then $(i-s, j) \in \mathcal{E}_m^-(s)$, since $\mathcal{B}_1(s)$ is a sub-abacus of $\mathcal{B}_0(s)$ by Lemma 28. If (i, j) is in one of the rightmost $m-2$ copies of $\mathcal{B}_1(s)$, then, $(i-s, j) \in \mathcal{E}_m^-(s)$. If (i, j) is in the leftmost copy $\mathcal{B}_1(s)$, notice that $\pi_0(i) = i$. It is enough to see that there exists an $(i+(m-1)s, j-1)$ in the rightmost copy of $\mathcal{B}_1(s)$, using π_{m-1} and Lemma 24.

To show that $\mathcal{E}_m^-(s)$ is an $(ms-1)$-core, we map an abacus position $(i+ks, j)$, where $j > 0$, to its local coordinates (i, j) via π_k. By Lemma 18, we know that $(i+1, j-1)$ is in $\mathcal{B}_\ell(s)$ where ℓ is either 0 or 1. Mapping this local abacus position back to the ms-abacus we conclude $(i+1+ks, j-1) \in \mathcal{E}_m^-(s)$. Since $(ms-1, 0) \notin \mathcal{E}_m^-(s)$ by construction, $\mathcal{E}_m^-(s)$ satisfies the criteria for an $(ms-1)$-core.

Finally we consider position $((m-2)s+1, s-2) \in \mathcal{E}_m^-(s)$. By Definition 5 this corresponds to the bead-value $s(ms-1) - s - ms + 1$, which by Corollary 12 means this partition is the $(s, ms-1)$-core of largest size.

2. The proof is analogous to (1); the abacus position $(ms-1, s-2)$ corresponds to the largest bead value in the underlying bead-set.

□

4 ms-abaci of $(s, ms \pm 1)$-cores with distinct parts

We now wish to turn our attention to simultaneous cores with distinct parts. This will allow us to provide unified proofs of Theorems 13, 14 and 15.

Lemma 37. *The partition λ with minimal s-abaci $\mathcal{S}$ has distinct parts if and only if $(i, j) \in \mathcal{S}$ implies that*

1. $(i-1, j) \notin \mathcal{S}$ *and* $(i+1, j) \notin \mathcal{S}$ *if* $1 < i < s-1$, *and*
2. $(i-1, j) \notin \mathcal{S}$ *if* $i = s-1$.

Proof. A partition has distinct parts if and only if its minimal bead-set X satisfies the following property: if $x, y \in X$ and $x > y$, then $x - y \neq 1$. This is exactly the statement of the lemma when translated into s-abaci. □

The combination of Lemma 37 and Theorem 33 allow us to study the abaci of certain simultaneous core partitions with distinct parts.

Lemma 38. *Let $\mathcal{A}(s)$, $\mathcal{E}_m^-(s)$, and $\mathcal{E}_m^+(s)$ be as above.*

1. *The minimal s-abacus $\mathcal{S}$ of any $(s, s+1)$-core with distinct parts will be a sub-abacus of $\mathcal{A}(s)$ consisting of beads taken only from its first row.*
2. *The minimal ms-abacus $\mathcal{M}^-$ of any $(s, ms-1)$-core partition with distinct parts will be a sub-abacus of $\mathcal{E}_m^-(s)$ consisting of beads taken only from its first row.*
3. *The minimal ms-abacus $\mathcal{M}^+$ of any $(s, ms+1)$-core partition will be a sub-abacus of $\mathcal{E}_m^+(s)$ consisting only of beads taken only from its first row.*

Proof. We know by Proposition 11 that $\mathcal{S}$, $\mathcal{M}^-$ and $\mathcal{M}^+$ are sub-abaci of $\mathcal{A}(s)$, $\mathcal{E}_m^+(s)$ and $\mathcal{E}_m^-(s)$.

1. Suppose $(i, j) \in \mathcal{S}$ such that $j > 0$. Then $(i, j-1) \in \mathcal{S}$ and $(i-1, j-1) \in \mathcal{S}$ by Proposition 11, Lemma 17, Lemma 18, Lemma 21. This is a contradiction.

2. Suppose $(i, j) \in \mathcal{M}^-$ such that $j > 0$. Then $(i+1, j-1) \in \mathcal{M}^-$ by Lemma 18 and Theorem 33(1). However, $(i, j-1) \in \mathcal{M}^-$ by Proposition 11, Corollary 8, Lemma 30. This is a contradiction.

3. Suppose $(i, j) \in \mathcal{M}^+$ such that $j > 0$. Then $(i-1, j-1) \in \mathcal{M}^+$ by Lemma 17 and Theorem 33(2). However $(i, j-1) \in \mathcal{M}^+$ for the same reason as in (2). □

We now possess all of the necessary tools to prove Theorems 13–15 in a unified, combinatorial fashion. As noted earlier, we prove Theorems 13 and 15 first; Theorem 14 then follows from Theorem 39 and some manipulation.

Proof of Theorem 13. There is only one simultaneous $(1, 2)$-core partition with distinct parts, the empty partition. There are two simultaneous $(2, 3)$-core partitions with distinct parts: the empty partition, and $\lambda = (1)$. This gives us the initial conditions, $F_2 = 1$ and $F_3 = 2$.

By Lemma 38(1), for any s-abacus $\mathcal{S}$ of an $(s, s+1)$-core with distinct parts, if $(i, j) \in \mathcal{S}$ then $j = 0$ where $0 \leqslant i \leqslant s-1$. We divide the count into two cases, depending on whether or not $(s-1, 0) \in \mathcal{S}$.

If $(s-1, 0) \in \mathcal{S}$, then by Lemma 37, $(s-2, 0) \notin \mathcal{S}$. By considering only the runners $0 \leqslant i \leqslant s-3$, we conclude there are F_{s-1} possible s-abacus arrangements for $\mathcal{S}$ with $(s-1, 0) \in \mathcal{S}$. If $(s-1, 0) \notin \mathcal{S}$, then by considering only the runners $0 \leqslant i \leqslant s-2$, we can conclude that there are F_s possible s-abacus arrangements for $\mathcal{S}$ with $(s-1, 0) \notin \mathcal{S}$. Hence the total number of acceptable s-abacus arrangements for an $(s, s+1)$-core with distinct parts is $F_{s+1} = F_s + F_{s-1}$. This completes the proof. □

Proof of Theorem 15. There is only one simultaneous $(1, m+1)$-core; the empty partition. By Lemma 35 and Lemma 38(3), there are m simultaneous $(2, 2m+1)$-core partitions with distinct parts; the empty partition, and one partition for each set of abacus positions $\{\cup_{\ell=0}^{m'}(1+2\ell, 0)\}$ where $m' \in [0, m-1]$.

By Lemma 38(3) for any s-abacus $\mathcal{M}^+$ of a $(s, ms-1)$-core with distinct parts, if $(i, j) \in \mathcal{M}^+$, then $(i, j) = (i+\ell s, 0)$ where $0 < i \leqslant s-1$ when $0 \leqslant \ell \leqslant m-1$. We divide the count into two cases: where $(s-1, 0)$ is in $\mathcal{M}^+$, or where it is not.

Suppose first that $(s-1, 0) \in \mathcal{M}^+$. Then $(ks-2, 0) \notin \mathcal{M}^-$ for $0 \leqslant k \leqslant m-1$, by Lemma 37. So we can consider only the $i+\ell s$ where $0 < i \leqslant s-3$ for $0 \leqslant \ell \leqslant m-1$: the number of such acceptable $m(s-2)$-abacus arrangements is $E_m^+(s-2)$. However there are $m-1$ additional positions $(2s-1, 0), (3s-1, 0), (4s-1, 0), \ldots, (ms-1, 0)$ that can also be included without violating Lemma 37. So the total number of acceptable ms-abaci from this case is $mE_m^+(s-2)$.

Suppose $(s-1, 0) \notin \mathcal{M}_m^+(s)$ then $(s-1+\ell s, 0) \notin \mathcal{M}^+$ for $0 \leqslant \ell \leqslant m-2$. We consider only the $i+\ell s$ where $0 < i \leqslant s-2$ and $0 \leqslant \ell \leqslant m-1$: there are $E_m^+(s-1)$ possible abacus arrangements. The result follows. □

Theorem 39. $E_m^-(s) = E_m^+(s-1) + (m-1)E_m^+(s-2)$.

Proof. There is only one simultaneous $(1, m-1)$-core, the empty partition. By Lemma 35 and Lemma 38(2), there are $m-1$ simultaneous $(2, 2m-1)$-core partitions with distinct parts; namely the empty partition, plus one partition for each set of abacus positions $\cup_{\ell=0}^{m'}(1+2\ell, 0)$ where $m' \in [0, m-2]$.

By Lemma 38(2) for any s-abacus $\mathcal{M}^-$ of a $(s, ms-1)$-core with distinct parts, if $(i,j) \in \mathcal{M}^-$, then $(i,j) = (i+\ell s, 0)$ where $0 < i \leqslant s-1$ when $0 \leqslant \ell \leqslant m-2$, and $0 < i \leqslant s-2$ when $\ell = m-1$. We divide our count into two cases, depending on whether or not $(s-1, 0) \in \mathcal{M}^-$.

Suppose first that $(s-1, 0) \in \mathcal{M}^-$. Then $(ks-2, 0) \notin \mathcal{M}^-$ for $0 \leqslant k \leqslant m-1$, by Lemma 37. So we can consider only the $i+\ell s$ runners, where $0 < i \leqslant s-3$ and $0 \leqslant \ell \leqslant m-1$: the number of possible abacus arrangements is $E_m^-(s-2)$. However there are $m-2$ additional positions $(2s-1, 0), (3s-1, 0), (4s-1, 0), \ldots, (m-1)s-1, 0)$ that can also be included without violating Lemma 37. So the total number of acceptable ms-abaci from this case is $(m-1)E_m^+(s-2)$.

Suppose $(s-1, 0) \notin \mathcal{M}_m^-(s)$. Then we can consider only the $i+\ell s$ runners, where $0 \leqslant i \leqslant s-2$ for $0 \leqslant \ell \leqslant m-1$: the number of such acceptable $m(s-2)$-abaci arrangements is $E_m^+(s-1)$. Then the total number of acceptable ms-abaci arrangements is $E_m^+(s-1) + (m-1)E_m^+(s-2)$. □

Theorem 14 follows using a purely algebraic manipulation first employed by Straub, towards the end of the proof of Theorem 4.1 in [18].

Proof of Theorem 14. By Theorem 39, we know $E_m^-(s) = E_m^+(s-1) + (m-1)E_m^+(s-2)$. By Theorem 15 we have $E_m^+(s-1) = E_m^+(s-2) + mE_m^+(s-3)$ and $E_m^+(s-2) = E_m^+(s-3) + mE_m^+(s-4)$. Substituting, we get

$$E_m^-(s) = E_m^+(s-2) + mE_m^+(s-3) + (m-1)(E_m^+(s-3) + mE_m^+(s-4)),$$

which, when expanded and rearranged, gives

$$E_m^+(s-2) + (m-1)E_m^+(s-3) + m(E_m^+(s-3) + (m-1)E_m^+(s-4)).$$

Substituting again, we arrive at $E_m^-(s) = E_m^-(s-1) + mE_m^-(s-2)$. □

5 ms-abacus of the longest $(s, ms-1, ms+1)$-core

We now move to discuss triply simultaneous core partitions. Lemmas 42, 44, 47 and Corollary 46 follow from the relevant definitions. In the interest of brevity their proofs are omitted.

Definition 40. Let A and B each be s-abaci. Then the intersection of A and B, denoted $A \cap B$, is the sub-abacus of all beads in both A and B.

Definition 41. Let $\mathcal{S}$ be an s-abacus. We say $\mathcal{S}$ is an $s-$**pyramid** with base $[\gamma, \gamma']$ if when the first row consists of abacus positions $(i, 0)$, where $\gamma \leqslant i \leqslant \gamma'$, then the second row consists of positions $(i, 1)$ where $\gamma' + 1 \leqslant i \leqslant \gamma - 1$, and the third row consists of beads in abacus position $(i, 2)$ where $\gamma + 2 \leqslant i \leqslant \gamma' - 2$, and so on.

We let $\mathcal{C}_k(s) = \mathcal{A}(s) \cap \mathcal{B}_k(s)$.

Lemma 42. *Let $\mathcal{C}_0(s) = A(s) \cap B_0(s)$. Then $\mathcal{C}_0(s)$ contains beads at all positions (i, j) where (i, j) is such that*

1. $0 \leqslant j \leqslant i - 1$ *if* $0 < i \leqslant \left\lfloor \frac{s-1}{2} \right\rfloor$
2. $0 \leqslant j \leqslant s - i - 1$ *if* $\left\lfloor \frac{s+1}{2} \right\rfloor \leqslant i \leqslant s - 1$.

Proof. Follows by construction. □

Example 43. $\mathcal{C}_0(5) = \{(1, 0), (2, 0), (3, 0), (4, 0), (2, 1), (3, 1)\}$. [See Figure 6.]

15	16	17	18	19
10	11	12	13	14
5	6	$\textcircled{7}$	$\textcircled{8}$	9
0	$\textcircled{1}$	$\textcircled{2}$	$\textcircled{3}$	$\textcircled{4}$

Figure 6: $\mathcal{C}_0(5) = \mathcal{A}(5) \cap \mathcal{B}_0(5)$

Lemma 44. *Let $\mathcal{C}_1(s) = \mathcal{A}(s) \cap \mathcal{B}_1(s)$. Then $\mathcal{C}_1(s)$ contains beads at all positions (i, j) where (i, j) is such that*

1. $0 \leqslant j \leqslant i - 1$ *if* $0 < i \leqslant \left\lfloor \frac{s-1}{2} \right\rfloor$ *and*
2. $0 \leqslant j \leqslant s - i - 2$ *if* $\left\lfloor \frac{s+1}{2} \right\rfloor \leqslant i < s - 1$.

Example 45. $\mathcal{C}_1(5) = \{(1, 0), (2, 0), (3, 0), (2, 1)\}$. [See Figure 7.]

15	16	17	18	19
10	11	12	13	14
5	6	$\textcircled{7}$	8	9
0	$\textcircled{1}$	$\textcircled{2}$	$\textcircled{3}$	4

Figure 7: $\mathcal{C}_1(5) = \mathcal{A}(5) \cap \mathcal{B}_1(5)$

Corollary 46. *Let $C_0(s)$ and $C_1(s)$ as above. Then*

1. *$C_0(s)$ is a pyramid with base* $[1, s - 1]$.
2. *$C_1(s)$ is a pyramid with base* $[1, s - 2]$.

Lemma 47. *Let $\mathcal{S}$ be an s-abacus. If $\mathcal{S}$ is a pyramid, and $(i, j) \in \mathcal{S}$, where $j > 0$ then the following holds:*

1. $(i+1, j-1) \in \mathcal{S}$

2. $(i-1, j-1) \in \mathcal{S}$.

Example 48. $C_0(5)$ is a pyramid with base $[1,4]$. $C_1(5)$ is a pyramid with base $[1,3]$. They both satisfy Lemma 47. [See Figures 6 and 7.]

Lemma 49. *Suppose A and B are s-abaci and A' and B' are t-abaci. Then*

$$(A \wedge A') \cap (B \wedge B') = (A \cap B) \wedge (A' \cap B').$$

Proof. This follows from Definitions 34 and 40. □

Lemma 50. $\mathcal{E}_m^-(s) \cap \mathcal{E}_m^+(s) = (\wedge_{m-1}\mathcal{C}_0(s)) \wedge \mathcal{C}_1(s)$.

Proof. By a repeated use of Lemma 49, it is enough to look at the intersection of each of the constituent s-abaci of $\mathcal{E}_m^{\pm}(s)$ at each of the wedge positions ℓ, as $0 \leqslant \ell \leqslant m$. The result follows from Lemmas 42 and 44. □

Lemma 51. *$\mathcal{C}_1(s)$ is a sub-abacus of $\mathcal{C}_0(s)$.*

Proof. This follows from Corollary 28 and the definitions of $\mathcal{C}_1(s)$ and $\mathcal{C}_0(s)$. □

A simultaneous (s,t,u)-core partition with the greatest number parts (if it exists) will be called the **longest** one.

Proposition 52. *Suppose $s,t,u > 1$ are distinct positive integers such that gcd$(s,t) = 1$. Then there exists a longest (s,t,u)-core.*

Before we can prove Proposition 52, we need a new definition and a lemma.

Definition 53. Suppose $s,t,u > 1$ be distinct positive integers. Let $\mathcal{C}$ and $\mathcal{C}'$ correspond to the minimal ν-abaci of two distinct (s,t,u)-core partitions for some positive integer $\nu > 1$. Then $\mathcal{C} \cup \mathcal{C}'$ is the ν-abacus diagram consisting of the union of ν-abacus positions in both $\mathcal{C}$ and $\mathcal{C}'$.

Lemma 54. *Suppose $s,t,u > 1$ are distinct positive integers, let ν be a positive integer greater than 1, and let $\mathcal{C}$ and $\mathcal{C}'$ be as above. Then $\mathcal{C}'' = \mathcal{C} \cup \mathcal{C}'$ is the minimal ν-abacus of some (s,t,u)-core partition.*

Proof. It is enough to show that if a ν-abacus position (i,j) is in $\mathcal{C}''$ then the μ-abacus positions corresponding to bead-positions $(i+\mu\cdot j)-s$, $(i+\mu\cdot j)-t$ and $(i+\mu\cdot j)-u$ are also in $\mathcal{C}''$. But this follows from the definition. □

proof of Proposition 52. Any (s,t,u)-core partition is also by definition an (s,t)-core partition. Then, since gcd$(s,t) > 1$, we know by Proposition 11 that any (s,t,u)-core partition will have a minimal ν-abacus that is a sub-abacus of the minimal ν-abacus of $\kappa_{s,t}$. Since there are only finitely many such possibilities, there are finitely many (s,t,u)-core partitions. We consider the union of all their minimal ν-abaci. By the repeated use of Lemma 54, we see such a union corresponds to the minimal ν-abacus of an (s,t,u)-core partition. Clearly this minimal ν-abacus will have the most possible beads. Since each bead corresponds to a "part" of the associated (s,t,u)-core partition, this is the longest one. □

Lemma 55. *Let $\mathcal{L}_m(s)$ is the minimal ms-abacus of the longest $(s, ms-1, ms+1)$-core. Then*

$$\mathcal{L}_m(s) = (\wedge_{m-1}\mathcal{C}_0(s)) \wedge \mathcal{C}_1(s).$$

Proof. We know by Proposition 52 that a longest $(s, ms-1, ms+1)$-core partition exists. Furthermore, by construction, the abacus of any $(s, ms-1, ms+1)$-core will be a sub-abacus of the longest one. Hence, tt is enough to show that $\mathcal{L}_m(s)$ is the ms-abacus of an $(s, ms-1, ms+1)$-core, and that the inclusion of beads in any other abacus positions in $\mathcal{E}^-_m(s)$ or $\mathcal{E}^+_m(s)$ will violate one of the core conditions. To see it is an s-core, we consider a bead in three abacus positions: in the rightmost $\mathcal{C}_1(s)$, the leftmost $\mathcal{C}_0(s)$, or one of the $m-2$ wedge-copies of $\mathcal{C}_0(s)$ in the middle. For a bead in the rightmost $\mathcal{C}_1(s)$; by Lemma 51, there is a bead s-positions to the left and in the same row, since $\mathcal{C}_1(s)$ is a sub-abacus of $\mathcal{C}_0(s)$. The same argument applies to beads in the middle $m-2$ copies of $\mathcal{C}_0(s)$. Suppose a bead is in the leftmost copy of $\mathcal{C}_0(s)$ with abacus position (i, j), where $j > 0$. Then it is enough that $((ms-1)-i-1, j-1) \in \mathcal{L}_m(s)$. This follows by the construction of $\mathcal{C}_1(s)$, and the projection map π_{m-1}.

To see that $\mathcal{L}_m(s)$ is an $(ms-1, ms+1)$-core, it is enough to use the projection maps π_ℓ for $0 \leqslant \ell \leqslant m-1$, and the Lemma 47. Finally, to see that $\mathcal{L}_m(s)$ is longest such, consider the inclusion of a bead in an ms-abacus position in $\mathcal{E}^+_m(s)$ or $\mathcal{E}^-_m(s)$ but outside of $\mathcal{E}^-_m(s) \cap \mathcal{E}^+_m(s)$. In this case, either an $(ms-1)$-hook or an $(ms+1)$-hook will arise from a spacer in either the position down-and-to-the-right, or down-and-to-the-left. □

Example 56. $\mathcal{L}_m(s) = \wedge_2\mathcal{C}_0(5) \wedge \mathcal{C}_1(5)$. [See Figure 8.]

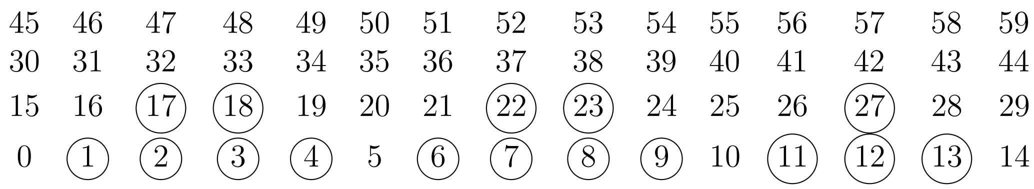

Figure 8: $\mathcal{L}_3(5) = \mathcal{E}^-_3(5) \cap \mathcal{E}^+_3(5)$

Proof of Theorem 16. By Corollary 46 and Lemma 49 we can describe $\mathcal{L}_m(s)$ as union of pyramid-abaci with bases $[1, s-1]$, $[s+1, 2s-1]$, $[2s+1, 3s-1], \ldots, [(m-1)s+1, ms-2]$. This uniquely determines the placement of beads in abacus positions each row j. In light of the structure described above, it is clear that the total number of spacers in the first $j+1$ rows (for a particular j) is given by

$$\begin{aligned}\sum_{i=0}^{j}((2i+1)m+1) &= \frac{2mj(j+1)}{2} + (m(j+1) + (j+1)) \\ &= m(j+1)^2 + (j+1)\end{aligned}$$

after elementary simplification.

Thus, the contribution to the size of this particular core at row j is given by

$$\begin{aligned}
&\sum_{\ell=0}^{m-2}((mj^2+j)+(j+1+\ell(2j+1))(s-2(j+1))+\\
&\sum_{\ell=m-1}^{m-1}((mj^2+j)+(j+1+\ell(2j+1))(s-2(j+2))\\
=\;&(m-1)(mj^2+2j+1)(s-(2j+1))+(2j+1)(s-(2j+1))\left(\frac{(m-2)(m-1)}{2}\right)\\
&+m(j+1)^2(s-(2j+2))\\
=\;&\frac{(s-(2j+1))(m-1)m}{2}\left(2j^2+2j+1\right)+m(j+1)^2(s-(2j+2)) \qquad (1)
\end{aligned}$$

using elementary summation properties and straightforward algebraic simplifications.

In order to determine the total size of this core, we simply sum (1) above over all relevant rows of the abacus. This yields

$$\begin{aligned}
&\sum_{j=0}^{t-2}\frac{(s-(2j+1))(m-1)m}{2}\left(2j^2+2j+1\right)+m(j+1)^2(s-(2j+2))\\
=\;&ms(t-1)\left(\frac{m(t-1)^2}{3}+\frac{m}{6}+\frac{t-1}{2}\right)-m(t-1)^2\left(\frac{m(t-1)^2+1}{2}+t-1\right)
\end{aligned}$$

using well–known results on sums of integer powers. Replacing s by $2t-1$ or $2t-2$ yields the results of this theorem after elementary simplification. □

The size of a *largest* $(s-1,s,s+1)$-core partitions was obtained by Amdeberhan-Leven (Theorem 4.3, [4]), Yang-Zhang-Zhou (Corollary 3.5, [21]) and Xiong (Corollary 1.2, [20]). When $m=1$, the size of a largest $(s,ms-1,ms+1)$-core partition, agrees with the size of the longest $(s,ms-1,ms+1)$. This agreement leads us to the following conjecture.

Conjecture 57. The size of a largest $(s,ms-1,ms+1)$-core is

1. $\frac{m^2t(t-1)(t^2-t+1)}{6}$ if $s=2t-1$;
2. $\frac{m^2(t-1)^2(t^2-2t+3)}{6}-\frac{m(t-1)^2}{2}$ if $s=2t-2$.

There are two such largest partitions; one corresponding to $\mathcal{L}(s)$, and one corresponding to its conjugate.

If Conjecture 57 is true, then we have the following elegant corollary.

Corollary 58 (contingent on Conjecture 57)**.** *Let s be odd. The size of a largest $(s,ms-1,ms+1)$-core partition is divisible by m^2.*

6 ms-abaci of self-conjugate $(s, ms \pm 1)$-core partitions with distinct parts

We close this paper by applying our tools to prove results on self-conjugate simultaneous core partitions with distinct parts. The following lemm is well-known.

Lemma 59. *The non-zero 2-core partitions are exactly those of the form* $(k, k-1, k-2, \ldots, 1)$*. The minimal bead-sets of the 2-cores are of the form* $\{\cup_{\ell \leqslant k} 2\ell - 1\}$*.*

Lemma 60. *Let* X *be a bead set of a self-conjugate partition. Then there exists a half-integer* θ *such that if* $x \in X$ *and* $x > \theta$ *then there exists a* $y \notin X$ *such that* $|y - \theta| = |x - \theta|$*.*

Proof. See Corollary 3.4 in [13]. □

Lemma 61. *The self-conjugate partitions with distinct parts are exactly the* 2*-core partitions.*

Proof. Every 2-core partition is clearly a self-conjugate partitions with distinct parts. Now suppose we have a self-conjugate partition λ with distinct parts. Then it must have a bead-set X that consists of alternating spacer-and-beads. Suppose not. If two beads occur in a row, we know that it violates having distinct parts. Suppose two spacers occur in a row. If $y, y+1 \notin X$ and both $y, y+1 < \theta$ or both $y, y+1 > \theta$ then, by Lemma 60, there will be two beads in succession on the other side of θ. If $y < \theta$ and $\theta < y+1$, then by Lemma 60 λ is not self-conjugate. □

With the results of the previous sections and the lemmas above, we can consider self-conjugate simultaneous core partitions with distinct parts.

Proposition 62. *The number* $F_*(s)$ *of self-conjugate* $(s, s+1)$*-core partitions with distinct parts obeys the following relations:* $F_*(1) = 1, F_*(2) = 2$ *and* $F_*(2\alpha) = F_*(2\alpha+1) = \alpha+1$*, where* $\alpha \geqslant 1$*.*

Proof. There is only one $(1,2)$-core, the empty partition. There are two $(2,3)$-cores, the empty partition and the partition $\lambda = (1)$. Suppose $n = 2\alpha$. Then the self-conjugate $(2\alpha, (2\alpha)m+1)$-cores with distinct parts will be, by Lemma 61, the empty set plus the 2-cores that can be accommodated as sub-abaci of $\cup(i, 0)$ where $1 \leqslant i \leqslant 2\alpha - 1$. There are α such cores, and this number remains unchanged if $s = 2\alpha + 1$. □

Proposition 63. *The number* $E^-_{m,*}(s)$ *of self-conjugate* $(s, ms-1)$*-cores with distinct parts obeys the following relations:* $E^-_{m,*}(1) = 1$*,* $E^-_{m,*}(2) = m$ *and*

1. $E^-_{m,*}(2\alpha) = m\alpha$ *and*
2. $E^-_{m,*}(2\alpha+1) = \alpha + 1$

for all $m \geqslant 1$ *and* $\alpha \geqslant 1$*.*

Proof. The argument is similar to Proposition 62. There is only one $(1, m-1)$-core: the empty partition. There are m self-conjugate $(2, 2m-1)$-cores, the empty set and the the partitions corresponding to $\cup_{\ell=1}^{m'}(2\ell-1, 0)$, where $1 \leqslant m' \leqslant m-1$. This gives us the initial conditions. We consider separately the cases when s is odd or even.

1. Suppose $s = 2\alpha$, and $\alpha > 0$. Then the self-conjugate $(2\alpha, (2\alpha)m-1)$-cores with distinct parts will be, by Lemma 61 the empty set plus the 2-cores accommodated as sub-abaci of $\{\cup(i+(2\alpha)\ell, 0) \cup (i'+(2\alpha)(m-1), 0)\}$ as $0 \leqslant i \leqslant 2\alpha-1$, $0 \leqslant \ell \leqslant m-2$ and $0 \leqslant i' \leqslant 2\alpha-2$. There are $m\alpha - 1$ such 2-cores; when we count the empty partition we arrive at $m\alpha$.

2. Suppose $s = 2\alpha+1$. Then the number of self-conjugate $(2\alpha, (2\alpha+1)m-1)$-cores with distinct parts will be, by Lemma 61, the number of 2-cores that can be accommodated as sub-abaci of $\{\cup(i+(2\alpha+1)\ell, 0) \cup (i'+(2\alpha+1)(m-1), 0)\}$ as $1 \leqslant i \leqslant 2\alpha$, $0 \leqslant \ell \leqslant m-2$ and $1 \leqslant i' \leqslant 2\alpha-1$. However, since $(2\alpha+1, 0) \notin \mathcal{E}_m^-(2\alpha+1)$, there are only α such non-empty 2-cores; those that can be accommodated from abacus positions $(i, 0)$ where $1 \leqslant i \leqslant 2\alpha$. □

Example 64. $F_*(8) = F_*(9) = 5$. The set of self-conjugate $(8, 9)$-core partitions with distinct parts is $\{\varnothing, (1), (2, 1), (3, 2, 1), (4, 3, 2, 1)\}$. This is also the set of self-conjugate $(9, 10)$-core partitions with distinct parts.

Proposition 65. *The number $E_{m,*}^+(s)$ of self-conjugate $(s, ms+1)$-cores obeys the following relations: $E_{m,*}^+(1) = 1$, $E_{m,*}^+(2) = m+1$ and*

1. $E_{m,*}^+(2\alpha) = m\alpha + 1$ *and*

2. $E_{m,*}^+(2\alpha+1) = \alpha + 1$

for all $m \geqslant 1$ and $\alpha > 1$.

Proof. The argument in both cases is similar to ones above, with the added consideration that, for (1), the partition corresponding to $\{\cup_{\gamma=1}^{\alpha} \cup_{\ell=0}^{m-1} (2\gamma - 1 + \ell s, 0)\}$ must also be counted. □

Corollary 66. $E_{m,*}^-(2\alpha+1) = E_{m,*}^+(2\alpha+1)$.

Example 67. $E_{3,*}^-(5) = E_{3,*}^+(5) = 3$. The set of self-conjugate $(5, 14)$-cores with distinct parts is exactly $\{\varnothing, (1), (1, 3)\}$, which is also the set of self-conjugate $(5, 16)$-cores with distinct parts.

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