On a vertex-minimal triangulation of $\mathbb{RP}^4$

Sonia Balagopalan
Department of Mathematics and Statistics,
Maynooth University,
Maynooth, Co. Kildare, Ireland

sonia.balagopalan@nuim.ie

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Abstract

We give three constructions of a vertex-minimal triangulation of 4-dimensional real projective space $\mathbb{RP}^4$. The first construction describes a 4-dimensional sphere on 32 vertices, which is a double cover of a triangulated $\mathbb{RP}^4$ and has a large amount of symmetry. The second and third constructions illustrate approaches to improving the known number of vertices needed to triangulate n-dimensional real projective space. All three constructions deliver the same combinatorial manifold, which is also the same as the only known 16-vertex triangulation of $\mathbb{RP}^4$. We also give a short, simple construction of the 22-point Witt design, which is closely related to the complex we construct.

1 Introduction

How many vertices does it take to (simplicially) triangulate real projective n-space, $\mathbb{RP}^n$? It is well known that the answer is 6 when $n = 2$, with the triangulation realized as the antipodal quotient of the icosahedron. In 1969, D.W. Walkup proved that a vertex-minimal triangulation of $\mathbb{RP}^3$ requires 11 vertices, and described all such triangulations [20]. P. Arnoux and A. Marin, in 1991, proved that the minimum number of vertices needed to triangulate $\mathbb{RP}^n$, $n \geqslant 3$, is at least $\binom{n+2}{2} + 1$ [1].

In 1987, W. Kühnel gave a triangulation of $\mathbb{RP}^n$ using $2^{n+1} - 1$ vertices, which takes the barycentric subdivision of the boundary of the $n+1$-simplex and quotients it by an antipodal map [13]. The Kühnel construction gives the smallest known explicit triangulations of $\mathbb{RP}^n$ for $n > 5$. For a survey of these and other results on minimal triangulations, and also all relevant definitions, see [7].

The `BISTELLAR` program of F.H. Lutz uses a heuristic search algorithm to reduce the f-vector of a given complex using bistellar flips [3, 15]. Among the several combinatorial manifolds found by this program was a 16-vertex triangulation of $\mathbb{RP}^4$, called $\mathbb{RP}^4_{16}$ [15, p.77]. This complex was obtained by applying the `BISTELLAR` program to the 31-vertex

$\mathbb{RP}^4$ due to Kühnel. The automorphism group of this particular triangulation was also computed in [15, p.77], and was found to be S_6, which acts on the 16-element vertex set by splitting it into orbits of size 6 and 10, and on the set of 150 facets by splitting it into orbits of size 30 and 120. No other triangulation of $\mathbb{RP}^4$ on 16 vertices is known. Apart from the information described above, there does not seem to be anything else known about a 16-vertex $\mathbb{RP}^4$ in the literature. We believe that this extremal object exhibiting such a high degree of symmetry deserves to be better understood, both for its own sake and for furthering our understanding of the general problem of triangulating $\mathbb{RP}^n$.

We present three constructions of a triangulated $\mathbb{RP}^4$ on 16 vertices, all of which turn out to be isomorphic to $\mathbb{RP}^4_{16}$, and make some observations about the remarkable combinatorial structure of this complex.

Construction 1 describes $\mathbb{RP}^4_{16}$ as the quotient of a triangulated 4-sphere on 32 vertices. In order for such a construction to be possible, the S^4 we construct needs to be antipodal. We say that a simplicial complex K is *antipodal* if it is invariant under an involution σ such that the (graph) distance between vertices v and $\sigma(v)$ in the 1-skeleton of K is at least 3. In particular, the links of v and $\sigma(v)$ are disjoint, and isomorphic to the link of $\overline{v}$ in the quotient complex $K/\langle\sigma\rangle$, and we say that σ is *link-separating* on k. If K is a combinatorial manifold, then it follows that $K/\langle\sigma\rangle$ is also a combinatorial manifold. We call σ the *antipodal map*. To be precise, when we refer to an antipodal complex, we are implicitly refering to a pair (K, σ).

We also point out a connection between the triangulated $\mathbb{RP}^4$ of Construction 1, and the Witt design on 22 points, W_{22}. We note that the smaller orbit of the facets of $\mathbb{RP}^4_{16}$ under its automorphism group is closely related to a well known symmetric 2-design or *biplane* on 16 points. Also, the partition of the vertex set into orbits suggests the construction of a "dual" biplane by the introduction of six new points. The resulting configuration can be extended to a 3-design on 22 points, with 77 blocks of size 6, of which W_{22} is unique up to isomorphism.

Our construction of W_{22} is short and elementary, and does not seem to appear as such in the literature. Nevertheless, no construction of an object so well understood can be said to be entirely new, and ours has many features in common to two previously known constructions, which we briefly note. We justify our choice to present our construction in full detail, more for what it illuminates about $\mathbb{RP}^4_{16}$ and its automorphism group, than for what it says about W_{22}.

Construction 1, though it exposits the remarkable symmetries of $\mathbb{RP}^4_{16}$, seems to rely on exceptional properties of the number 6, and is not a very encouraging as a model for analogous constructions in higher dimensions. We give two more constructions which are more promising in this direction. These constructions view $\mathbb{RP}^4$ as a 4-dimensional ball with antipodal simplices on its boundary identified. We start with a suitable convex 4-polytope, and place it inside its dual, and triangulate the regions thus formed, till we get a 3-sphere on the boundary, which we can glue to itself to give an $\mathbb{RP}^4$. Our second and third constructions follow this strategy, starting from a 16-cell and a suspended cube respectively. We also describe a way of looking at Walkup's $\mathbb{RP}^3_{11}$ and even $\mathbb{RP}^2_6$ as 3 and 2-dimensional analogues of our constructions. Both these constructions give the same complex as the first. A simple observation about the automorphism groups of these complexes allows us to identify all three constructions with each other. This supports the conjectured uniqueness of $\mathbb{R}\mathrm{P}^4_{16}$ as the vertex-minimal triangulation of $\mathbb{R}\mathrm{P}^4$.

2 First construction and combinatorial properties

We construct $\mathbb{R}\mathrm{P}^4_{16}$ by starting with the standard 4-sphere and constructing an antipodal S^4 on 32 vertices, by successive transformations.

Construction 1. Let Δ^5 denote the standard 5-simplex in $\mathbb{R}^6$. Let i, where $1 \leqslant i \leqslant 6$, denote $\mathbf{e}_i$, the i^{th} elementary vector. The set $V_1 = \{i | 1 \leqslant i \leqslant 6\}$ is the vertex set of Δ^5. The boundary of Δ^5 is a triangulated 4-sphere on these six vertices, and each of its facets is a 4-simplex containing five elements of V_1. Call this complex X_6.

Let $\mathbf{1} = \sum_{i=1}^{6} i$. Then the barycenter of the facet Δ_i with vertex set $V_1 \setminus i$ of X_6 is the point $\bar{i} = \frac{1}{5}(\mathbf{1} - i) \in \mathbb{R}^6$, where $1 \leqslant i \leqslant 6$. Call the set of these points V_5. Introducing these points allows us to subdivide each facet of X_6 as the union of five 4-simplices, by replacing Δ_i with the cone over each of its tetrahedra at the point $\bar{i}$. This gives a 12-vertex triangulated S^4 with vertex set $V_1 \cup V_5$. The facets of this complex are all the 4-simplices of the form $[i, j, k, l, \bar{m}]$, where $m \notin \{i, j, k, l\}$. Call this complex X_{12}.

Let $V_3 = \{ijk = \frac{1}{3}(i + j + k) | 1 \leqslant i < j < k \leqslant 6\}$ denote the set of barycenters of the triangles of Δ^5. We use these points to further subdivide each facet of X^4_{12} in the following way. The tetrahedron $[i, j, k, l]$ of the facet $[i, j, k, l, \bar{m}]$ can be decomposed into eleven tetrahedra. These are, from the outside in, six tetrahedra of the form $[i, j, ijk, ijl]$ corresponding to every pair of elements of $\{i, j, k, l\}$, four tetrahedra of the form $[i, ijk, ijl, ikl]$ corresponding to every element of $\{i, j, k, l\}$, and the tetrahedron $[ijk, ijl, ikl, jkl]$. See Figure 1 for an illustration of this subdivision. Here, ijk denotes $ijk \in V_3$.

We take the join of $\bar{m}$ with each of these tetrahedra to obtain a decomposition of the facet. This gives us a triangulated S^4 on 32 vertices, X_{32}, with three kinds of facets, containing two, three, and four vertices of V_3 respectively. The vertex set $V_1 \cup V_3 \cup V_5$ of X_{32} suggests a natural choice of antipodal map, the one that swaps i with $\bar{i}$ and ijk with $i'j'k'$, where $\{i, j, k, i', j', k'\} = \{1, \ldots, 6\}$.

In order to obtain an antipodal complex from X_{32}, we need to transform the complex to one that is invariant under the above map, and also separate antipodal vertices, till they are far enough apart. We do this using bistellar flips.

In X_{32}, any two vertices of V_1 form an edge, but no two vertices of V_5 do. Also, any vertex in V_1 is adjacent to any vertex in V_5 other than its antipode. By separating each of the edges within V_1, we can reduce the asymmetry of the complex, and also increase the distance between would-be antipodal pairs in $V_1 \cup V_5$. To achieve this, first note that each edge $[i, j]$ is contained in four triangles of the form $[i, j, \bar{k}]$, corresponding to each of its four neighbouring facets Δ_k in X_6. The link of $[i, j, \bar{k}]$ in X_{32} is the boundary of the triangle $[ijl, ijl', ijl'']$, where $\{i, j, k, l, l', l''\} = \{1, \ldots, 6\}$, each edge of said triangle corresponding to a tetrahedron containing $[i, j]$ joined with $\bar{k}$ in X_{12}. Since any triangle in X_{32} with vertices from V_3 is obtained by subdividing a tetrahedron, a triangle of the above type is not a face of X_{32}.

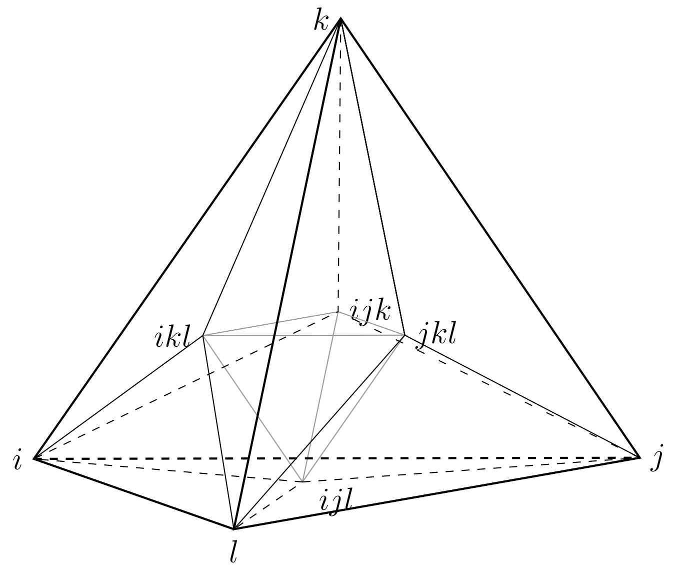

Figure 1: Tetrahedron subdivided by barycenters of its 2-faces

On the other hand, every triple of vertices of the form $\{ijl, ijl', ijl'')\}$ is the vertex set of the link of a unique triangle of the form $[i, j, \bar{k}]$, since i, j, k, l, l', l'' are all distinct. So we can apply simultaneous bistellar flips to all triangles of the form $[i, j, \bar{k}]$, replacing the facets of the form

$$[i, j, \bar{k}] * \partial[ijl, ijl', ijl'']$$

with the facets of the form

$$[ijl, ijl', ijl''] * \partial[i, j, \bar{k}].$$

After this round of flips, the link of $[i, j]$ is the boundary of the tetrahedron

$$[ijl, ijl', ijl'', ijl''']$$

where i, j, l, l', l'', l''' are distinct, each face of said tetrahedron being one introduced in the previous round of flips. As above, we can perform simultaneous bistellar flips to replace

$$[i, j] * \partial[ijl, ijl', ijl'', ijl''']$$

with

$$[ijl, ijl', ijl'', ijl'''] * \partial[i, j]$$

The facets of the resulting complex are of four types (or two types under the action of $C_2 \times S_6$, where S_6 permutes the axes of $\mathbb{R}^6$ and C_2 is generated by the antipodal involution):

$$[i, ijl, ijl', ijl'', ijl''']$$

($6 \times 5 = 30$ in number),

$$[\bar{m}, ijk, ijl, ikl, jkl]$$

(i, j, k, l, m distinct, 30 in number),

$$[i, \bar{k}, ijl, ijl', ijl'']$$

(i, j, k, l, l', l'' distinct, $6 \times 5 \times 4 = 120$ in number),

$$[i, \bar{m}, ijk, ijl, ikl]$$

(i, j, k, l, m distinct, $6 \times 5 \times 4 = 120$ in number).

This complex, call it S^4_{32}, is antipodal under the involution that takes i to $\bar{i}$ and ijk to $\frac{1}{3}\mathbf{1} - i - j - k$), and commutes with the S_6 action induced from the permutations of V_1. This involution is link-separating on S^4_{32}, and so quotients it to give a triangulated $\mathbb{R}\mathrm{P}^4$ on 16 vertices. □

If, in the above construction, we deform the 5-simplex to choose the elements of V_3 and V_5 to be of the form $i + j + k$ and $\mathbf{1} - i$ respectively, the antipodal map is just $\mathbf{x} \mapsto \mathbf{1} - \mathbf{x}$ on $V_1 \cup V_3 \cup V_5$, which also acts on the analogous geometric carrier of the complex above. This gives a closer analogy to the usual geometric notion of the antipodal map on S^n.

It is clear from the above construction that the triangulated $\mathbb{R}\mathrm{P}^4$ we constructed is invariant under the action of S_6 on $\{1, \dots, 6\}$. The action induced on the vertex set of this complex splits the vertices into two orbits, of size 6 and 10, corresponding to the quotients of $V_1 \cup V_5$ and V_3 respectively. We refer to this group of automorphisms as $\tilde{S}$, and it splits the 150 facets of this complex into two orbits, of size 30 and 120.

A quick comparison of the orbit representatives shows that the complex we have constructed is the same as the complex $\mathbb{R}\mathrm{P}^4_{16}$ in [15, p.77], available at [16].

B. Datta [6] has also calculated using `POLYMAKE` [10] that if we take a 4-sphere and choose points on it corresponding to a 5-simplex and the barycenters of its facets and triangles, the convex hull of these points is a simplicial polytope whose boundary is S^4_{32}. Even though this construction fails to generalize to higher dimensions, this gives a direct demonstration of the polytopality of S^4_{32}, and it would be interesting to shed further light on the effectiveness of this construction in four dimensions.

2.1 A connection to the Witt design on 22 points

A convenient mnemonic to represent the facets of $\mathbb{R}\mathrm{P}^4$ in Construction 1 is as follows.

Label the vertices of the complete graph K_6 with the points of the 6-vertex orbit under $\tilde{S}$. Now the remaining ten vertices correspond to the ten pairs of disjoint triangles, or *bisections*, in K_6. We use these points to label the edges of K_6 as follows. Each edge is contained in four triangles, each of which is in turn contained in exactly one bisection. Give each edge a label consisting of the four bisections it is contained in. Henceforth, when we refer to K_6, we mean the K_6 labelled thus. We denote the set of elements of the 6-vertex orbit by $\{A, \dots, F\}$, and that of the 10-vertex orbit by $\{0, \dots, 9\}$. See Figure 2.

Now the small $\tilde{S}$-orbit of the set of facets can be read off Figure 2 as a vertex-edge pair (v, e) of K_6, where $v \in \{A, \dots, F\}$ and e is an edge-label of size 4. The $\tilde{S}$-orbit of size 120 is given by a triple $(v, v', e(v, v'') \setminus e(v, v'))$, where $v, v', v'' \in \{A, \dots, F\}$, and $e(u, v)$

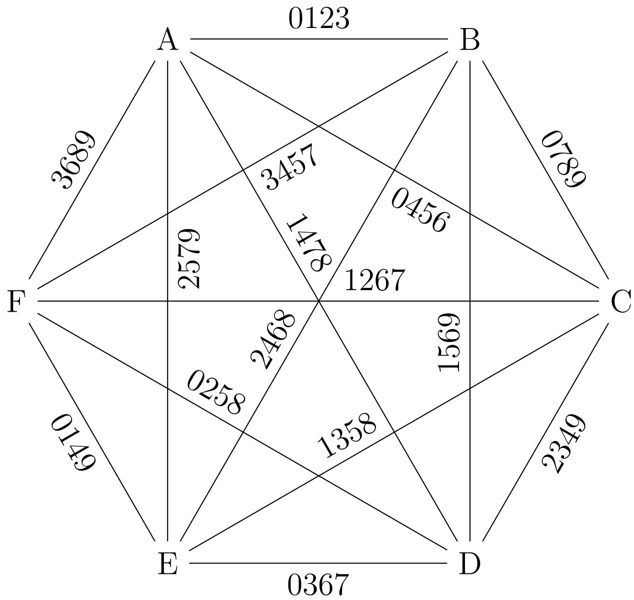

Figure 2: K_6 mnemonic for 16-vertex $\mathbb{R}\mathrm{P}^4$

denotes the 4-label of the edge between u and v. The correspondence between the facets of S^4_{32} listed above and the facets read off K_6 above is straightforward.

It is fruitful to consider the properties of K_6 and its labels, from the point of view of combinatorial designs. Recall that a t-(v, k, λ) *design* $\mathbb{D}$ is a pair $(\mathbb{V}, \mathbb{B})$, where $\mathbb{V}$ is a set of size v, whose elements are called *points*, and $\mathbb{B}$ is a set of k-subsets of $\mathbb{V}$ called *blocks*, such that any t-subset of $\mathbb{V}$ is contained in exactly λ elements of $\mathbb{B}$.

Let $\mathcal{V}$ (for vertices) denote the set $\{A, \dots, F\}$, and $\mathcal{B}$ (for bisections) denote the set $\{0, \dots, 9\}$. Let $\mathcal{E}$ denote the set of fifteen edge-labels of K_6.

Our first observation is that $(\mathcal{B}, \mathcal{E})$ is a 2-$(10, 4, 2)$ design. To see that this is a 2-design with $\lambda = 2$, note that any pair of distinct bisections intersect in exactly two edges. Note also that this design is *quasi-symmetric*, i.e., the intersections of any two blocks have two possible sizes, namely 1 and 2. To see this, first note that the vertex-set of a pair of adjacent edges in K_6 forms a triangle, so it is contained in a unique bisection. So the labels of these edges intersect in one element, the bisection containing the triangle formed by their edges. Now if two edges are not adjacent, they are contained in exactly two bisections, and their labels intersect in those two bisections.[1] Also note that this design naturally extends to a 2-$(16, 6, 2)$ design, with point-set $\mathcal{V} \cup \mathcal{B}$, and block-set $\widetilde{\mathcal{E}} := \{\{u, v\} \cup e(u, v) | u, v \in \mathcal{V}\} \cup \{\mathcal{V}\}$. This design is also symmetric, as the number of blocks is the same as the number of points, and any two blocks intersect in $\lambda = 2$ points.

[1] This also shows that $\tilde{S}$ is the full automorphism group of $\mathbb{R}\mathrm{P}^4_{16}$. Counting the number of facets containing each vertex in $\mathcal{V}$ and $\mathcal{B}$, we see that the full automorphism group G preserves these orbits. But if $G > \tilde{S} \simeq S_6$, then G can not act faithfully on $\mathcal{V}$, so there exists $g \in G \setminus \tilde{S}$ which fixes each block of $\mathcal{E}$. Then g also fixes the intersections of these blocks, which implies g fixes $\mathcal{B}$ pointwise. A contradiction.

Further, recall that a *perfect matching* or 1-factor of a graph is a partition (if it exists) of its vertex set, into edges of the graph, and that being a complete graph on an even number of vertices, K_6 has perfect matchings, which we henceforth simply call matchings. Every edge of K_6 is contained in exactly three matchings. The number of matchings in K_6 is $\frac{6!}{2!2!2!3!} = 15$. The three pairwise disjoint edges in any matching have labels which intersect pairwise in two elements of $\mathcal{B}$ each. But the bisections containing one pair of disjoint edges do not contain the third of these edges. So the union of three elements of $\mathcal{E}$ in a given matching is a subset of $\mathcal{B}$ of size 6. So we can label each matching by the four elements of $\mathcal{B}$ not in the labels of any of its three edges. The set $\mathcal{M}$ (for matching) of these 4-labels is the block set of another quasi-symmetric 2-$(10, 4, 2)$ design, whose blocks intersect in one point if the corresponding matchings are disjoint and in two points if the matchings intersect in an edge. To see that $(\mathcal{B}, \mathcal{M})$ is a 2-design, note that deleting a pair of bisections from K_6 leaves the disjoint union of an edge and a 4-cycle, which contains exactly two matchings. Also, to see that the intersection sizes are 2 and 1, note that the union of a pair of intersecting matchings is the disjoint union of an edge and a 4-cycle, the complement of a pair of bisections, and that the union of two disjoint matchings is a hexagon, whose complement in K_6 contains exactly one bisection.

We introduce one final set of objects, the 1-factorizations of K_6. Recall that a 1-*factorization* of a graph is a partition of its edge-set, where each block in the partition is a matching of the graph. The graph K_6 has six 1-factorizations, which can be thought of as 5-edge-colourings where the matchings are the colour-classes. We label these from the set $\{U, V, \dots, Z\} = \mathcal{F}$ (for factorization).

It can also be seen that any matching is contained in exactly two 1-factorizations, and that any two disjoint matchings determine a unique 1-factorization. Now since any two 1-factorizations can have at most one matching in common, and there are fifteen pairs of 1-factorizations, any two 1-factorizations, say f, g intersect in a unique matching $m(f, g)$. This gives an extension of the design $(\mathcal{B}, \mathcal{M})$ to a symmetric 2-$(16, 6, 2)$ design $(\mathcal{B} \cup \mathcal{F}, \widetilde{\mathcal{M}})$, where $\widetilde{\mathcal{M}} = \{\{f, g\} \cup m(f, g) | f, g \in \mathcal{F}\} \cup \{\mathcal{F}\}$.

This can be represented by a "dual K_6", K_6^* with vertices labelled by $\mathcal{F}$ and edges labelled by the elements of $\mathcal{M}$, with $m(f, g)$ labelling the edge joining f and g. Figure 3 illustrates the two copies of K_6 with their edges labelled by the elements of $\mathcal{B}$. Note that the set $\mathcal{B}$ also indexes the bisections of K_6^*. This gives a correspondence between the $(3, 3)-$partitions of $\mathcal{V}$ and those of $\mathcal{F}$.

We note the following type of triple-incidence between the sets $\mathcal{V}, \mathcal{B}$, and $\mathcal{F}$. Given any two bisections b, b', each triangle in b intersects one triangle in b' in an edge e. This gives a partition of $\mathcal{V}$ into two edges and two points. The edges each correspond to the intersections of two triangles, one from b and one from b'. The two points left over determine a third edge disjoint from the other two. Moreover, this edge, as a 4-set in $\mathcal{E}$ contains neither b nor b'. Also, since the pair b, b' is contained in the labels of the other two edges, the 4-label in $\mathcal{M}$ of the 1-factor m composed of the three edges above is disjoint from $\{b, b'\}$. Since $e \in m$, the 4-labels of e and m are disjoint.

Similarly, given an incident edge-1-factor pair (e, m), their 4-labels are disjoint, and subtracting the union of these 4-labels from $\mathcal{B}$ leaves two bisections such that the triangles

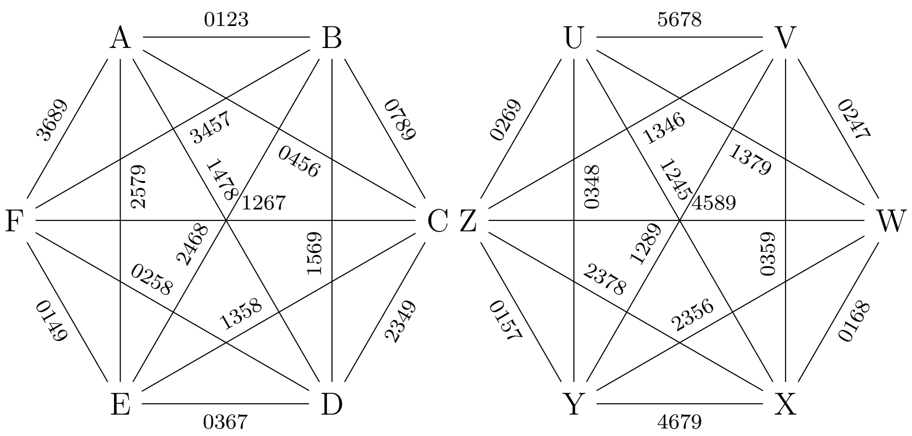

Figure 3: K_6 and K_6^*

of one intersects the triangles of the other in the two edges of $f \setminus e$. So we have a correspondence between pairs of bisections and incident edge-1-factor pairs of K_6.

We are now ready to describe the 22-point Witt design, the unique 3-(22, 6, 1) design first independently constructed by R.D. Carmichael and E. Witt in the 1930s [5, 21]. To be accurate, we construct *a* 3-(22, 6, 1) design, and make no claims as to its uniqueness. See, for example, [18] for a proof of the uniqueness of W_{22}, which relies on the embeddability in it of a 16-point biplane, which can be seen to be isomorphic to either $(\mathcal{V} \cup \mathcal{B}, \widetilde{\mathcal{E}})$ or $(\mathcal{B} \cup \mathcal{F}, \widetilde{\mathcal{M}})$ described above. For our point set of W_{22}, we take $\mathcal{V} \cup \mathcal{B} \cup \mathcal{F}$. The blocks are of three types. The first two sets of blocks are $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{M}}$, the block-sets of the two 2-(16, 6, 2) designs we described earlier. The third set of blocks is obtained as follows. For any incident edge-matching pair (e, m) in K_6, let $v, v' \in \mathcal{V}$ be the vertices that make up e, and let $f, f' \in \mathcal{F}$ be the 1-factorizations that intersect in m. Let $\underline{e}$ and $\underline{m}$ be the 4-sets corresponding to e and m in $\mathcal{E}$ and $\mathcal{M}$ respectively. Now for each of the 45 incident pairs (e, m), we take the block $\{v, v', f, f'\} \cup (\mathcal{B} \setminus (\underline{e} \cup \underline{m}))$. Call the set of such blocks $\mathcal{EM}$.

Theorem 1. *The pair* $(\mathcal{V} \cup \mathcal{B} \cup \mathcal{F}, \widetilde{\mathcal{E}} \cup \widetilde{\mathcal{M}} \cup \mathcal{EM})$ *defined above is a* 3-(22, 6, 1) *design.*

Proof. There are 77 blocks, each of size 6. As there are 22 points, and since $\binom{22}{3} = 77 \times \binom{6}{3}$, it is enough to show that every 3-subset of the point set appears in some block.

Consider the 3-subsets of $\mathcal{V} \cup \mathcal{B} \cup \mathcal{F}$. As $\mathcal{V}$ and $\mathcal{F}$ are themselves blocks, any 3-subset of either is in a block. There are $\binom{10}{3} = 120$ subsets of $\mathcal{B}$ of size 3. Consider the 3-subsets of $\mathcal{E}$ or $\mathcal{M}$. If we can show that no two sets of these forms have a 3-subset in common, it will follow that there are $2 \times 15 \times 4 = 120$ such sets, and that every 3-subset of $\mathcal{B}$ is in exactly one block. Two elements of $\mathcal{E}$ intersect in at most two points of $\mathcal{B}$. Similarly, two elements of $\mathcal{M}$ intersect in at most two points of $\mathcal{B}$. Now consider the intersection of an element $\underline{e} \in \mathcal{E}$ with $\underline{m} \in \mathcal{M}$. If the corresponding edge and matching of K_6 are incident, they do not intersect at all. Otherwise, the edge corresponding to $\underline{e}$ has its vertices in two (disjoint) edges of $\underline{m}$, say uu' and $vv' \subset \mathcal{V}$. (We write a set as a string from here on, for convenience and brevity.) Now since the labels of incident edges have one element in common, $|e(u,v) \cap e(u,u')| = |e(u,v) \cap e(v,v')| = 1$. Also since $u' \neq v'$, $e(u,v) \cap e(u,u') \neq e(u,v) \cap e(v,v')$. Since the labels of the third edge in the matching corresponding to $\underline{m}$ are contained in $e(u,u') \cup e(v,v')$, we have $|\underline{e} \cap \underline{m}| = |\underline{e} \cap (\mathcal{B} \setminus (e(u,u') \cup e(v,v')))| = 2$. So all the 3-subsets of the 4-labels of edges and matchings are distinct, and there are 120 such sets, so each 3-subset of $\mathcal{B}$ is in exactly one block.

Now consider a 3-set consisting of two elements of $\mathcal{V}$ and one element of $\mathcal{F}$, say $vv'f$. Since f contains exactly one matching containing the edge vv', this 3-set is contained in exactly one block of $\mathcal{EM}$. Similarly, given a 3-set of the form vff', the 1-factorizations f and f' intersect in a unique matching, which is a partition of the vertex set of K_6. So vff' is in exactly one element of $\mathcal{EM}$.

A 3-set of the form $vv'b$ is of one of the following types. If $b \in e(v,v')$, then $vv'b$ is contained (again, in fact, exactly once) in a block of $\widetilde{\mathcal{E}}$. If $b \notin e(v,v')$, then v, v' are in different triangles of b, and v, v' together with the remaining edges of these triangles forms a matching whose label does not contain b. So $vv'b$ is contained in a block of $\mathcal{EM}$.

Similarly, if $b \in m(f,f')$, then $ff'b$, is contained in exactly one block of $\mathcal{M}$. If $b \notin m(f,f')$, then it is an element of a 4-label of two of the edges in the matching m that f and f' intersect in. Let e be the third edge of this matching. Then $ff'b$ is in the block of $\mathcal{EM}$ corresponding to (e,m).

Now if a 3-set is of the form vbb', we have two possibilities. Consider the two blocks of $\mathcal{E}$ containing bb'. If v is in either of the corresponding edges of K_6, then vbb' is in $\widetilde{\mathcal{E}}$. The two vertices not in either of these edges, form the third edge of the matching containing the two edges whose labels contain bb'. So if v is on this edge, vbb' is in a block of $\mathcal{EM}$. Similarly a 3-set of the form fbb' is in a block of either $\widetilde{\mathcal{M}}$ or $\mathcal{EM}$.

The only remaining type of 3-set is of the form vfb. These can only be contained in blocks of $\mathcal{EM}$. Now there are 45 blocks in $\mathcal{EM}$, each corresponding to a pair of elements in $\mathcal{B}$. So each element b of $\mathcal{B}$ is in nine blocks of $\mathcal{EM}$, corresponding to the nine edges of K_6 not labelled by b. Let $v \in \mathcal{V}$ and let the bisection of K_6 corresponding to b be $vv'v'', uu'u''$. Then $v \in \mathcal{V}$ is in three edges whose labels do not contain b, each of which determine exactly one matching whose label does not contain b. All three matchings intersect in the edge $v'v''$. Since a 1-factorization is a partition of the edge set of K_6, no 1-factorization contains more than one of these matchings. So the pairs of 1-factors which intersect to give each of these matchings partition the vertex set of K_6^*. In other words, the two elements of $\mathcal{F}$ in each of the three blocks of $\mathcal{EM}$ containing a pair vb are all distinct. So any triple vfb is in a block of $\mathcal{EM}$. This completes the proof. $\square$

The arguments above also illustrate the duality between K_6 and K_6^*. All properties of the vertices and edges of K_6 have analogues in its 1-factorizations and matchings, or the vertices and edges of K_6^*. Applying the dual construction on K_6^* simply recovers K_6. Let S_6' be the group of permutations of $\mathcal{F}$ induced by the permutation group S_6 of $\mathcal{V}$ via permutations of $\mathcal{B}$. The actions of S_6 and S_6' on $\mathcal{V}$ and $\mathcal{F}$ respectively are *dual* or nonconjugate to each other, i.e., there is no one-to-one map between the elements of these two sets which send either group to the other. Abstractly, any isomorphism between these two sets corresponds to an outer automorphism of S_6. See [4, Chapter 6] for a more thorough treatment and several applications. In particular, we note that our construction of W_{22} has interesting parallels with the construction of the 5-$(12, 6, 1)$ Witt design on [4, p.86].

The reader familiar with the Witt designs and their automorphism groups will recognise that our description of W_{22} can be recovered from its usual forms by fixing any pair of its disjoint blocks and mapping it to $(\mathcal{V}, \mathcal{F})$. The ten remaining points correspond exactly to $(3, 3)$-partitions of $\mathcal{V}$, and also of $\mathcal{F}$.

We take this occasion to note early, if not original, appearances of the various objects that feature in our construction in the literature. The remarkable duality between the vertices and edges of K_6 and its 1-factors and 1-factorizations was noted by J.J. Sylvester in a paper from 1844, republished in [19]. The origins of the (isomorphism class of) 16-point biplanes are historic, and can be traced back to Kummer's 16_6 configuration. The combinatorial aspects of this configuration are studied in great depth in the classic treatise of R.W.H.T. Hudson, [11, Sections 5,9[2],23,24,25,26]. The observation that the actions of S_6 and S_6' on $\mathcal{V}$ and $\mathcal{F}$ respectively induce the same group of permutations on the set $\mathcal{B}$ of bisections of K_6 and K_6^* appears in [8].

We also point out some similarities and differences between our construction and two earlier ones. In [17, Table 8.2], D.M. Mesner, at the time unaware of the preexistence of W_{22} in the group theoretic literature, lists the blocks of a 3-$(22, 6, 1)$ design discovered by empirical search. In his proof of the uniqueness of this design [17, Theorem 8.7], he notes and relies on the facts that every block is disjoint from 16 other blocks, that every pair of points is contained in five blocks, and that the possible intersection sizes of blocks are 0 or 2. These observations enable him to fix an initial block, list up to relabellings the 60 blocks that intersect the initial block, then show that the remaining 16 blocks disjoint from the initial block are also fixed, noting that these are the blocks of a 2-design. See [12] for a historical account and survey of these and further implications of Mesner's work.

In [18], N.N. Roghelia and S.S. Sane, independently of Mesner, but familiar with the work of Witt as expounded by H. Lüneberg [14], prove the existence and uniqueness of W_{22} based on the classification of 16-point biplanes by the number of their *ovals*, or sets of four points, no three of which are in a block. They show that the unique 16-point biplane with 60 ovals is uniquely embeddable in W_{22}, by way of the 2-$(16, 4, 3)$ design with these ovals as blocks, and six new points. We also note that though Roghelia and Sane's construction refer to the complete graph K_6, this graph is indexed by blocks, unlike our K_6.

We would like to think of the construction of Mesner as starting with $6 + 16$ points and constructing $1 + 60 + 16$ blocks on them. Roghelia and Sane proceed in the opposite

[2] We would like to draw special attention to this section, which demonstrates, for any choice of bisection b of K_6, a correspondence between the set $\mathcal{V} \cup \mathcal{B} \setminus \{b\}$ consisting of its vertices and the remaining bisections, and the set of the edges of K_6. When the 1-factors of K_6 are added to both sets, the latter set corresponds to the point-set of the 21-point projective plane Π_4. The bisection b now corresponds to the one point which is used to extend Π_4 in the Witt-Lüneberg construction of W_{22}.

direction, starting with $16 + 6$ points and constructing $16 + 60 + 1$ blocks. Within this point of view, our construction starts with $6 + 10 + 6$ points and constructs $16 + 45 + 16$ blocks on these. At the time of discovering our construction, this author was only aware of the better known construction of W_{22} via the extension of the projective plane Π_4, and as embedded in the larger design W_{24}. Despite some objects in our construction pre-existing in the literature, we find our rather more unified view of W_{22}, where two biplanes coexist, mediated by the bisections of the dual pair of K_6 and K_6^*, to be of some interest.

We also remark that the connection between the 16-vertex $\mathbb{RP}^4$ we constructed earlier and the above 3-design is not simply restricted to the 2-(16, 6, 2) design $(\mathcal{V} \cup \mathcal{B}, \widetilde{\mathcal{E}})$. The $\tilde{S}$-orbit of the facets of our $\mathbb{RP}^4$ split naturally into fifteen sets of size eight, each containing a pair of vertices v, v' in $\mathcal{V}$. The closure of any of these 8-sets as facets, is the join of the edge $[v, v']$ with its link in our $\mathbb{RP}^4$, an octahedron with vertices from $\mathcal{B}$. Any octahedron is determined by three pairs of opposite vertices on each "axis", say $b_1b_1', b_2b_2', b_3b_3'$ in this case. Then each of the $15 \times 3 = 45$ quadruples $vv'b_ib_i', 1 \leqslant i \leqslant 3$ is contained in a block of $\mathcal{EM}$.[3] Also if we take the blocks of W_{22} and delete the elements of one block from all the others, the remaining blocks split into a set of sixteen blocks of size 6, and sixty blocks of size 4. The set $\mathbb{B}_6$ with sixteen blocks of size 6 form the block-set of a symmetric 2-(16, 6, 2) design, which corresponds to the block-set $\widetilde{\mathcal{E}}$ we started with. Now if we fix a block B of $\mathbb{B}_6$, this further splits the set $\mathbb{B}_4$ of 4-sets into 15 sets that are disjoint from it, (i.e., $\mathcal{M}$), and 45 sets that intersect B in two points, in correspondence with $\mathcal{EM}$. So we can think of each block of W_{22} as sitting in the centre of a configuration of 16 copies of $\mathbb{RP}^4_{16}$.

3 Further Constructions

It would be of interest to know if there is a general "algorithm" to construct minimal, or even smaller-than-known triangulations of real projective spaces. Our first construction of triangulated $\mathbb{RP}^4$, though short and straightforward, has some exceptional properties which may well be the results of numerical coincidences, and do not offer much hope of analogous constructions in higher dimensions. We provide two more ways of constructing a 16-vertex $\mathbb{RP}^4$ which are easier to generalise.

Remark 2. It must be borne in mind here that even though our choice of notation in the following constructions represents the vertices of n-dimensional complexes as points in $\mathbb{R}^n$, the objects we construct are purely abstract simplicial complexes, which we do not need to view as embedded in $\mathbb{R}^N$ for any N. Indeed, they most definitely do not embed in $\mathbb{R}^n$. Our choice of notation is motivated by ease of handling and conceptual visualization.

3.1 Constructions using cross-polytopes and hypercubes

We explore the possibility of constructing triangulated real projective n-space, $\mathbb{RP}^n$ in the following way. Take an n-dimensional cross-polytope and triangulate its interior, possibly

[3] These are also ovals of the biplane $(\mathcal{V} \cup \mathcal{B}, \widetilde{\mathcal{E}})$ used in [18]. The remaining 15 ovals are the blocks of the design $(\mathcal{B}, M)$.

by adding an extra point $\mathbf{0}$. Say we denote the vertices of the cross-polytope C^n by the vectors $\pm\mathbf{e}_i \in \mathbb{R}^n$, where $\mathbf{e}_i$ is the i^{th} elementary vector, $1 \leqslant i \leqslant n$.

Then we add 2^n simplices of the form $[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \ldots, \varepsilon_n\mathbf{e}_n, \sum_{i=1}^n \varepsilon_i\mathbf{e}_i]$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{\pm 1\}^n$. Let q_ε denote $\sum_{i=1}^n \varepsilon_i\mathbf{e}_i$, the new vertex in each of the above simplices, and let $Q = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^n\}$. The set of vertices in our complex is now $V = C \sqcup Q$, where C is the vertex set of the triangulated C^n.

Now we consider subsimplices of ∂C^n going down in dimension, and triangulate the links of each without adding any more vertices. We do this subject to the following conditions. First, for every facet

$$[\varepsilon_{i_1}\mathbf{e}_{i_1}, \varepsilon_{i_2}\mathbf{e}_{i_2}, \ldots, \varepsilon_{i_k}\mathbf{e}_{i_k}, q_{\varepsilon^{j_1}}, q_{\varepsilon^{j_2}}, \ldots, q_{\varepsilon^{j_{n-k+1}}}]$$

in the complex, its "opposite" facet

$$[-\varepsilon_{i_1}\mathbf{e}_{i_1}, -\varepsilon_{i_2}\mathbf{e}_{i_2}, \ldots, -\varepsilon_{i_k}\mathbf{e}_{i_k}, q_{-\varepsilon^{j_1}}, q_{-\varepsilon^{j_2}}, \ldots, q_{-\varepsilon^{j_{n-k+1}}}]$$

is also in the complex. Second, no vertex q_ε in Q is joined to its "opposite" vertex $q_{-\varepsilon} = -q_\varepsilon$. In other words $[q_\varepsilon, q_{-\varepsilon}]$ is not an edge of the complex. Third, if a vertex $u \in V$ is joined to another vertex $v \in V$, i.e, if $[u, v]$ is an edge of the complex, then $[u, -v]$ is not an edge of the complex. These conditions are equivalent to the existence of a link-separating involution on the complex.

Our goal is to continue this till the boundary of the link of every vertex in C (with two possible exceptions) is a triangulated 2^{n-1}-vertex $(n-2)$-sphere. We then apply the identification map $q_\varepsilon \sim q_{-\varepsilon}$ on Q, leaving us with $2n$ (or $2(n-1)$) suspended S^{n-2}. We then triangulate the interiors of these $(2^{n-1}+2)$-vertex $(n-1)$-spheres, to get a triangulation of $\mathbb{R}\mathrm{P}^n$.

Example 3. It is easily seen that the paradigm outlined above can be used to construct $\mathbb{R}\mathrm{P}^2_6$ as follows. See Figure 4. We triangulate the square with vertices $\pm\mathbf{e}_1, \pm\mathbf{e}_2$ by joining $+\mathbf{e}_1$ with $-\mathbf{e}_1$, into triangles $[\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2], [\mathbf{e}_1, -\mathbf{e}_1, -\mathbf{e}_2]$.

Next we add the triangles $\pm[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2], \pm[\mathbf{e}_1, -\mathbf{e}_2, \mathbf{e}_1-\mathbf{e}_2]$. The links of the vertices $\pm\mathbf{e}_1$ are 5-vertex 1-spheres, and the links of $\pm\mathbf{e}_2$ have boundaries $\pm\{[\mathbf{e}_1+\mathbf{e}_2], [-\mathbf{e}_1+\mathbf{e}_2]\}$ respectively. Now we apply the map $q_\varepsilon \sim q_{-\varepsilon}$. We triangulate the 1-sphere containing $\pm\mathbf{e}_2$ by adding the triangles $[+\mathbf{e}_2, -\mathbf{e}_2, \overline{\mathbf{e}_1+\mathbf{e}_2}], [+\mathbf{e}_2, -\mathbf{e}_2, \overline{\mathbf{e}_1-\mathbf{e}_2}]$. Since the link of $[+\mathbf{e}_1, -\mathbf{e}_1]$ is already a 0 sphere in our complex, we triangulate the remaining square as $[+\mathbf{e}_1, \overline{\mathbf{e}_1+\mathbf{e}_2}, \overline{\mathbf{e}_1-\mathbf{e}_2}], [-\mathbf{e}_1, \overline{\mathbf{e}_1+\mathbf{e}_2}, \overline{\mathbf{e}_1-\mathbf{e}_2}]$. This gives us $\mathbb{R}\mathrm{P}^2_6$.

Example 4. In order to construct $\mathbb{R}\mathrm{P}^3$ by the same approach, start with the octahedron C^3 spanned by the points $\pm\mathbf{e}_i$, where $i = 1, 2, 3$. We can triangulate the interior of the octahedron by taking the cone over its boundary at the point $\mathbf{0}$. This gives us eight tetrahedra of the form

$$[\mathbf{0}, \varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3].$$

The boundary of the octahedron consists of the eight triangles of the form $[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3]$, where $\varepsilon_i = \pm 1,\ 1 \leqslant i \leqslant 3$. Now add a set Q of eight new "outer" vertices $q_\varepsilon = \sum_{i=1}^3 \varepsilon_i\mathbf{e}_i$ for each $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$, by taking the eight tetrahedra of the form

$$[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3, \varepsilon_1\mathbf{e}_1 + \varepsilon_2\mathbf{e}_2 + \varepsilon_3\mathbf{e}_3].$$

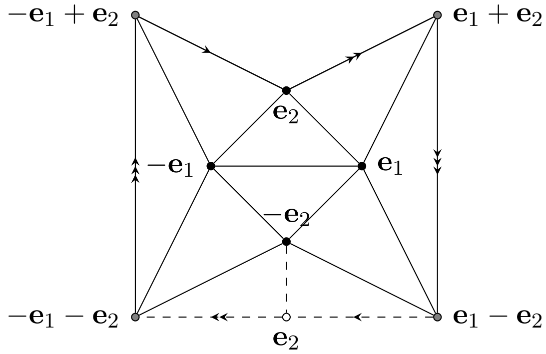

Figure 4: $\mathbb{RP}^2$ triangulated with squares

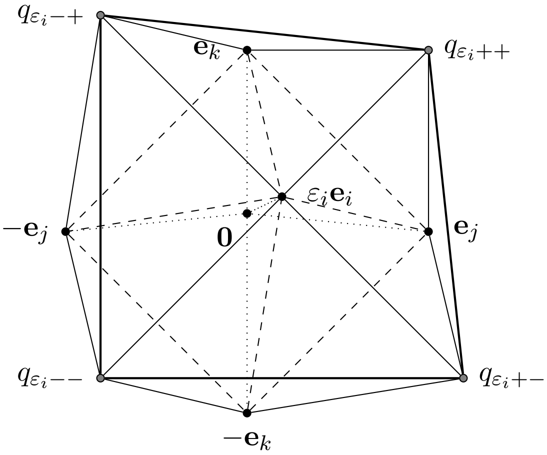

Figure 5: Neighbourhood of $\varepsilon_i\mathbf{e}_i$ in a 3-ball before quotienting.

The boundary of this complex is a triangulated S^2 with f-vector $[14, 36, 24]$.

Now consider the link of an edge of C^3. The link of $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j]$ is the path $[\varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j - \mathbf{e}_k, -\mathbf{e}_k], [-\mathbf{e}_k, \mathbf{0}], [\mathbf{0}, \mathbf{e}_k], [\mathbf{e}_k, \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \mathbf{e}_k]$, where $\{i, j, k\} = \{1, 2, 3\}$. Its boundary consists of the two points $\varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j \pm \mathbf{e}_k$. Close the boundary of $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j]$ by adding the tetrahedron

$$[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j, \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j - \mathbf{e}_k, \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \mathbf{e}_k].$$

This gives twelve new tetrahedra, and the boundary of the new complex is still a triangulated S^2 with fourteen vertices. But the boundary of the link of a vertex $\varepsilon_i\mathbf{e}_i$ of C^3 is now the boundary of a square, whose vertices are all its neighbours in Q. See Figure 5 for an illustration of the neighbourhood of $\varepsilon_i\mathbf{e}_i$. Also note that the subcomplex spanned by Q is the set of edges of the 3-dimensional cube Q^3.

We can now identify $\sum_{i=1}^{3} \varepsilon_i\mathbf{e}_i$ with the point $-\sum_{i=1}^{3} \varepsilon_i\mathbf{e}_i$, and close the boundary of the link of $\varepsilon_i\mathbf{e}_i$ by taking its cone at the point $-\varepsilon_i\mathbf{e}_i$. That is, for each pair $\pm\mathbf{e}_i$, we take the four tetrahedra

$$[\varepsilon_i\mathbf{e}_i, -\varepsilon_i\mathbf{e}_i, \pm\overline{\mathbf{e}_j+\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}, \overline{\mathbf{e}_j+\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}]$$

The link of the vertex $\varepsilon_i\mathbf{e}_i$ is now the triangulated 8-vertex S^2 with facets

$$[\mathbf{0}, \pm\mathbf{e}_j, \pm\mathbf{e}_k], \pm[\mathbf{e}_j, \mathbf{e}_k, \overline{\mathbf{e}_j+\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}], \pm[\mathbf{e}_j, -\mathbf{e}_k, \overline{\mathbf{e}_j-\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}],$$
$$[\pm\mathbf{e}_j, \overline{\mathbf{e}_j+\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}, \overline{\mathbf{e}_j-\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}], [\overline{\mathbf{e}_j+\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}, \overline{\mathbf{e}_j-\mathbf{e}_k+\varepsilon_i\mathbf{e}_i}, -\varepsilon_i\mathbf{e}_i]$$

The above complex is an 11-vertex triangulation of $\mathbb{R}\mathrm{P}^3$. This complex is the same as the minimal $\mathbb{R}\mathrm{P}^3_{11}$ described by Walkup in [20], as the antipodal quotient of a 22-vertex S^3.

We now tackle $\mathbb{R}\mathrm{P}^4$.

Construction 2. We start with a 4-dimensional (solid) hyperoctahedron C^4, given by the convex hull of $\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3, \pm\mathbf{e}_4\}$. We triangulate C^4 by joining the vertices $+\mathbf{e}_1$ and $-\mathbf{e}_1$. The resulting complex is a set of eight 4-simplices which can be visualized as the join of the line segment $[-\mathbf{e}_1, +\mathbf{e}_1]$ with the boundary of the octahedron spanned by $\{\pm\mathbf{e}_2, \pm\mathbf{e}_3, \pm\mathbf{e}_4\}$.

The boundary of this triangulated C^4 is just the boundary ∂C^4 of C^4, which is a triangulated 3-sphere with f-vector $[8, 24, 32, 16]$. Now we take the cone over each of the 16 facets of this boundary with a different point. That is, for each facet $[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3, \varepsilon_4\mathbf{e}_4]$ of ∂C^4, take the cone over this facet at the point $q_\varepsilon = \sum_{i=1}^4 \varepsilon_i\mathbf{e}_i$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm1\}^4$. This gives sixteen such 4-simplices, and the boundary now has 24 vertices, $16 \times 4 + 24 = 88$ edges, $16 \times \binom{4}{2} + 32 = 128$ triangles and $16 \times 4 = 64$ tetrahedra. Denote this triangulation of S^3 by $X^{(1)}$.

Now consider the link of each triangle $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j, \varepsilon_k\mathbf{e}_k]$ of ∂C^4 in $X^{(1)}$. These are of two kinds. If $1 \notin \{i, j, k\}$, then the link of $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j, \varepsilon_k\mathbf{e}_k]$ is

$$[-\mathbf{e}_1 + \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \varepsilon_k\mathbf{e}_k, -\mathbf{e}_1], [-\mathbf{e}_1, +\mathbf{e}_1], [+\mathbf{e}_1, +\mathbf{e}_1 + \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \varepsilon_k\mathbf{e}_k].$$

Now suppose $1 \in \{i, j, k\}$, then the link of the triangle is

$$[-\mathbf{e}_l + \sum_{i,j,k} \varepsilon_\alpha\mathbf{e}_\alpha, -\mathbf{e}_l], [-\mathbf{e}_l, -\varepsilon_1\mathbf{e}_1], [-\varepsilon_1\mathbf{e}_1, +\mathbf{e}_l], [+\mathbf{e}_l, +\mathbf{e}_l + \sum_{i,j,k} \varepsilon_\alpha\mathbf{e}_\alpha],$$

where l is the coordinate in $\{1, 2, 3, 4\} \setminus \{i, j, k\}$. In either case the endpoints of the link of the triangle are the two points $q_{\varepsilon^{+l}}$ and $q_{\varepsilon^{-l}}$ corresponding to the two tetrahedra containing it in C^4.

We can now close the links of the triangle $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j, \varepsilon_k\mathbf{e}_k]$ by adding the 4-simplices

$$[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j, \varepsilon_k\mathbf{e}_k, \sum_{i,j,k} \varepsilon_\alpha\mathbf{e}_\alpha - \mathbf{e}_l, \sum_{i,j,k} \varepsilon_\alpha\mathbf{e}_\alpha + \mathbf{e}_l]$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We have added 32 such 4-simplices, and the boundary of the new complex is a triangulated S^3 with 24 vertices, $88+32 = 120$ edges, $128-32+(32\times3) = 192$ triangles, and $64 + 32 = 96$ tetrahedra. Call the boundary $X^{(2)}$.

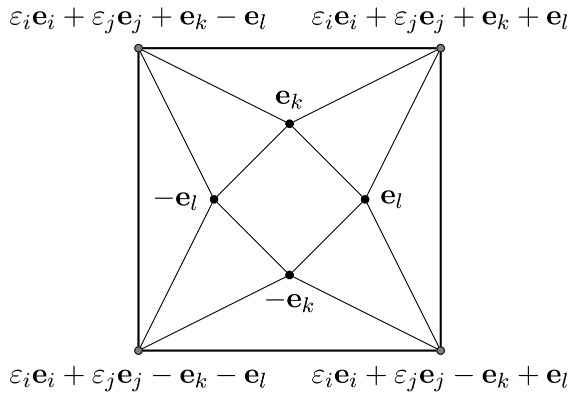

Figure 6: Antiprism in the link of $[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j]$ in $X^{(2)}$

Observe now that the subcomplex spanned by the subset of vertices $Q = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^4\}$ is the 1-skeleton of a 4-dimensional hypercube, which is the dual of C^4.

Now consider the links of the edges of C^4. The link of the edge $[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j]$, when $1 \notin \{i, j\}$ has eight vertices, namely $\pm\mathbf{e}_k, \pm\mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j \pm \mathbf{e}_k \pm \mathbf{e}_1$, where $\{i, j, k\} = \{2, 3, 4\}$. These vertices can be seen as forming the corners of a 4-sided antiprism whose opposite squares are $(+\mathbf{e}_1, +\mathbf{e}_k, -\mathbf{e}_1, -\mathbf{e}_k)$ and $(\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k - \mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k - \mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k + \mathbf{e}_1)$. All edges and triangles of this antiprism are faces of $X^{(2)}$. The (cyclically ordered) square $(+\mathbf{e}_1, +\mathbf{e}_k, -\mathbf{e}_1, -\mathbf{e}_k)$ is triangulated by the edge $[\mathbf{e}_1, -\mathbf{e}_1]$, and no triangle with vertices from the other square is a face of $X^{(2)}$. The link of the edge $[\varepsilon_1 \mathbf{e}_1, \varepsilon_j \mathbf{e}_j]$ is almost the same, with the only difference being that the square $(+\mathbf{e}_k, +\mathbf{e}_l, -\mathbf{e}_k, -\mathbf{e}_l)$ is triangulated by taking the cone of its boundary at $-\varepsilon_1 \mathbf{e}_1$. See Figure 6.

In either case, the boundary of the link of the edge $[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j]$ is the boundary of the square

$$(\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k - \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k - \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k + \mathbf{e}_l).$$

We triangulate each of these squares by joining one pair of non-adjacent vertices by a diagonal. Prima facie, we seem to have some amount of choice in this situation. All we have to ensure here is that if we introduce an edge $[q_\varepsilon, q_{\varepsilon'}]$, then we also include the edge $[-q_\varepsilon, -q_{\varepsilon'}]$.

Recall that the 4-dimensional hypercube Q^4 is bipartite, and any vertex-colouring partitions the vertices $\{q_\varepsilon | \varepsilon \in \{\pm 1\}^4\}$ into two sets,

$$Q_e = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^4, \prod_{i=1}^{4} \varepsilon_i = +1\} \text{ and } Q_o = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^4, \prod_{i=1}^{4} \varepsilon_i = -1\}.$$

Also note that q_ε and $-q_\varepsilon = q_{-\varepsilon}$ are always in the same block of the partition. Also any square in Q^4 contains exactly two vertices from Q_o and two from Q_e.

So we can triangulate each square with boundary

$$(\varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_l, \varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j+\mathbf{e}_k-\mathbf{e}_l, \varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j-\mathbf{e}_k-\mathbf{e}_l, \varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j-\mathbf{e}_k+\mathbf{e}_l)$$

by joining its vertices in either of Q_o or Q_e by a diagonal.

But, since the link of $[\mathbf{e}_1, -\mathbf{e}_1]$ is already a sphere, we need to triangulate the boundaries of the links of $\pm\mathbf{e}_1$ "internally". Additionally, this triangulation can not introduce a diagonal through the interiors of either sphere, as any point $u \in Q$ at (Hamming) distance 3 to a point v is at distance 1 to its antipode $-v$. There is one way of triangulating an 8-vertex 2-sphere with the given partial 1-skeleton without interior diagonals, i.e., the triangulation of the solid cube into five tetrahedra. So we triangulate each square in the link of $\pm\mathbf{e}_1$ by joining the elements of say, Q_o, by edges.

Now each element u of Q_o is joined to three other elements v_1, v_2, v_3 of Q_o at distance 2 from it. The other three elements of Q_o at distance 2 from u are $-v_1, -v_2$, and $-v_3$. So none of the elements of Q_o can be joined when triangulating the remaining squares. This forces us to triangulate the remaining squares by joining the vertices in Q_e by an edge. This is possible, since for each vertex of Q_e, the three vertices at distance 2 from it, which are across a square in the link of some $[\pm\mathbf{e}_1, \pm\mathbf{e}_i]$, have been ruled out in the previous step.

So for $\{i,j,k\} = \{2,3,4\}$, we replace the join of a line segment and the boundary of a square in $X^{(2)}$, i.e,

$$\begin{aligned}[][\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j] * (\partial[\varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j+\mathbf{e}_1+\varepsilon_i\varepsilon_j\mathbf{e}_k, \varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j-\mathbf{e}_1-\varepsilon_i\varepsilon_j\mathbf{e}_k]*\\ \partial[\varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j-\mathbf{e}_1+\varepsilon_i\varepsilon_j\mathbf{e}_k, \varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j+\mathbf{e}_1-\varepsilon_i\varepsilon_j\mathbf{e}_k]),\end{aligned}$$

with the join of a new line segment with the boundary of another square, i.e,

$$\begin{aligned}[][\varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j+\mathbf{e}_1+\varepsilon_i\varepsilon_j\mathbf{e}_k, \varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j-\mathbf{e}_1-\varepsilon_i\varepsilon_j\mathbf{e}_k] * (\partial[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j]*\\ \partial[\varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j-\mathbf{e}_1+\varepsilon_i\varepsilon_j\mathbf{e}_k, \varepsilon_i\mathbf{e}_i+\varepsilon_j\mathbf{e}_j+\mathbf{e}_1-\varepsilon_i\varepsilon_j\mathbf{e}_k])\end{aligned}$$

So we are adding two 4-simplices for every edge in C^4, $24 \times 2 = 48$ in total. The f-vector of the boundary remains $[24, 120, 192, 96]$, the same as that of $X^{(2)}$. Call the boundary of the current complex $X^{(3)}$.

Now consider the link of a vertex from C^4 in $X^{(3)}$. The vertex-set of the link of $\varepsilon_i\mathbf{e}_i$ in $X^{(3)}$ is the set of vertices of the hypercube Q^4 whose i^{th} co-ordinate is ε_i. These vertices span a 3-dimensional cube, and the triangles of the link of $\varepsilon_i\mathbf{e}_i$ in $X^{(3)}$ are "halves" of squares of Q^4.

Now consider opposite pairs of vertices of C^4. The boundaries of the links of $+\mathbf{e}_i$ and $-\mathbf{e}_i$ are opposite (cubical) faces of Q^4. Moreover, the map $\mathbf{x} \mapsto -\mathbf{x}$ swaps the triangulations of these cubes. So the boundary of X^3 is an antipodal S^3.

In order to triangulate the link of $\varepsilon_i\mathbf{e}_i, 2 \leqslant i \leqslant 4$, we could triangulate the interior of the 8-vertex 2-sphere (or triangulated cube) described above by taking its cone at the point $-\varepsilon_i\mathbf{e}_i$. In other words, we take the join of the edge $[\pm\mathbf{e}_i]$ with the 8-vertex S^2 which is now the link of both $\mathbf{e}_i$ and $-\mathbf{e}_i$.

This leaves the pair $\pm\mathbf{e}_1$. The boundaries of the links of either vertex is an 8-vertex S^2, or the boundary of a cube triangulated by joining "every other vertex by an edge". As mentioned above, we triangulate the links of each vertex by splitting it into five tetrahedra, the vertices of four of which have one element each of Q_e and its three neighbouring elements of Q_e. The vertices of the fifth are four elements of Q_o.

Now we apply the map $x \mapsto -x$ on Q. This gives a triangulated $\mathbb{R}\mathrm{P}^4$ with 16 vertices. □

We give one more way of triangulating $\mathbb{R}\mathrm{P}^4$ with 16 vertices. Here we work with polyhedral complexes instead of the usual simplicial complexes. Again, the idea is to construct a 4-dimensional ball with antipodal boundary, then to quotient via a restriction of the antipodal map. The description of the polyhedral complex used in this construction sacrifices rigour in the service of intuition. See [2, Appendix A] for a more rigorous treatment.

Construction 3. Start with a (solid, 3-dimensional) cube Q^3, embedded in $\mathbb{R}^4$ with vertices $(\pm1, \pm1, \pm1, 0)$. Consider its suspension SQ^3 at the points $(0, 0, 0, \pm1)$. The boundary of this object is a 3-dimensional polyhedral complex with ten vertices and $2 \times 6 = 12$ (square-)pyramidal faces. The base of each of these pyramids is a face $S(\pm i)$ of the cube Q^3 given by $x_i = \pm1, x_4 = 0$, where $1 \leqslant i \leqslant 3$. We avoid triangulating the interior of SQ^3 for the time being.

First, we construct a "dual" cell complex D outside SQ^3 by adding faces of increasing dimension, starting with points.

Corresponding to each face with base square $S(\pm i)$ and apex $(0, 0, 0, \varepsilon)$ of SQ^3, (where $\varepsilon \in \{\pm1\}$), take the point $3(\sigma_1, \sigma_2, \sigma_3, \varepsilon)$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector in $\mathbb{R}^3$ taking value $\pm i$ at the i-th coordinate and 0 elsewhere. So for example, the pyramid with apex $(0, 0, 0, 1)$ and base $S(-2)$ gives the point $(0, -3, 0, 3)$. Corresponding to each of the 12 faces of the boundary of the suspended cube, we get 12 points.

Now join each pair of points in D by an edge if the corresponding facets of SQ^3 intersect in a triangle or square. This gives six edges corresponding to each square of Q^3. Additionally, SQ^3 has $2 \times 12 = 24$ triangles corresponding to each point in $\{(0, 0, 0, \pm1)\}$ and each edge of Q^3. This gives 24 more edges.

Next, consider the edges of SQ^3, of which there are $12 + 2 \times 8$. The twelve edges of Q^3 give twelve rectangles in D, of which the long pair of edges corresponds to the squares of Q^3 which intersect in this edge, while the short pair corresponds to the two triangles of SQ^3 intersecting this edge. Each of the remaining sixteen edges has as endpoints a vertex v of Q^3 and a point in $\{(0, 0, 0, \pm1)\}$. Each such edge gives a triangle of D, whose edges correspond to the three squares in Q^3 intersecting in v.

Now for each of the $8 + 2 = 10$ vertices of SQ^3, we add a polyhedron to D. The eight vertices of Q^3 give triangular prisms, whose faces are the three rectangles in D corresponding to the three edges intersecting in this vertex, and the two triangles corresponding to the edges joining the vertex to each of $(0, 0, 0, \pm1)$. Corresponding to either of the vertices $(0, 0, 0, \pm1)$, we have an octahedron whose faces correspond to the 8 edges of SQ^3 intersecting at the chosen vertex.

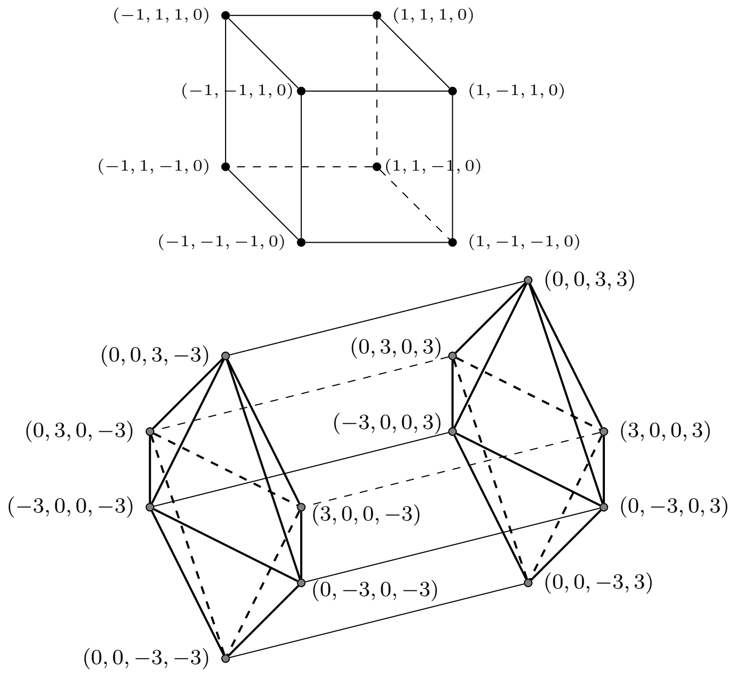

Figure 7: Cube Q^3 and octahedral prism D

We summarize the dualities described above in a table.

	SQ^3	D	
Dim	Faces	Faces	Dim
0	$8+2$ points	8 prisms+2 octahedra	3
1	$12+2\times 8$ edges	12 rectangles+2×8 triangles	2
2	6 squares+2×12 triangles	$6+2\times 12$ edges	1
3	2×6 pyramids	2×6 points	0

We can visualize the complex D as a prismed octahedron, as in Figure 7.

Now we can write down some of the 4-simplices in our triangulation. Join each triangle in SQ^3 to its corresponding edge in D. This gives 24 facets. Join each triangle in D to its corresponding edge in SQ^3. This gives 16 more facets.

Each vertex of D is adjacent to five other vertices in D. In the 40 simplices listed above, each vertex of D is joined to four vertices of Q^3 and one vertex of $\{(0,0,0,\pm 1)\}$. Also, each of the vertices $(0,0,0,\pm 1)$ is joined to the six vertices of the octahedron corresponding to it in D. Each vertex of Q^3 is joined to three other vertices of Q^3, both of $(0,0,0,\pm 1)$, and the six vertices of the triangular prism corresponding to it in D. Since the restricted antipodal map we wish to apply to the complex we are constructing takes $v \in D$ to $-v$, we can not add any more edges to our complex that contain a vertex of D.

Now we consider the links of the 24 triangles in SQ^3 in our complex so far. The triangle containing an edge whose i, j^{th} coordinates satisfy $x_i = \varepsilon_i, x_j = \varepsilon_j$, and the suspension point $(0,0,0,\varepsilon)$, is already joined to the (short) edge of D whose endpoints correspond to the squares $x_i = \varepsilon_i, x_4 = \varepsilon$, and $x_j = \varepsilon_j, x_4 = \varepsilon$. Now we join each tetrahedron, obtained by joining the above triangle to each of the vertices of the edge (in D) above, to a third point on the corresponding square in Q^3. Of the two remaining points on each square in Q^3, we choose the point the product of whose first three co-ordinates is 1. For example, the triangle $[(1,1,-1,0),(1,-1,-1,0),(0,0,0,-1)]$ has as link $[(3,0,0,-3),(0,0,-3,-3)]$. The corresponding simplices we add are the cones over tetrahedra

$$[(1,1,-1,0)(1,-1,-1,0),(0,0,0,-1),(3,0,0,-3)] * [(1,1,1,0)] \text{ and}$$
$$[(1,1,-1,0),(1,-1,-1,0),(0,0,0,-1),(0,0,-3,-3)] * [(-1,1,-1,0)].$$

Note that any such 4-simplex can be obtained by starting with either of the two edges of Q^3 contained in it. So we get $2 \times 12 \times 2/2 = 24$ simplices.

The link of each of the 24 triangles in SQ^3 is now a path of length 3 with endpoints from the vertices of Q^3. Moreover, if the triangle T is obtained by joining an edge E of Q^3 with a point in $(0,0,0,\pm1)$, with $v \in E$ such that the product of the first three co-ordinates is -1, then the endpoints of the link of T are the neighbours of v in the square in Q^3 containing v but not E. For the next set of 4-simplices, join the endpoints of the link of each triangle in SQ^3 by an edge. For example, the triangle $[(1,1,-1,0),(1,-1,-1,0),(0,0,0,-1)]$ is joined to the edge $[(1,1,1,0),(-1,1,-1,0)]$. So for each choice of $\{(0,0,0,\pm1)\}$ and each of the four vertices $v = (\varepsilon_1,\varepsilon_2,\varepsilon_3,0)$ of Q^3 such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, we have a simplex whose vertices are v, its three neighbours in Q^3, and one of $(0,0,0,\pm1)$. This gives eight more simplices. The link of every triangle in SQ^3 is now a circle.

In constructing the previous two sets of simplices, we added one diagonal to each square face of Q^3. Recall that each square in Q^3 given by the equation $x_i = \varepsilon_i$ corresponds to the edge of D spanned by $3(\sigma_1,\sigma_2,\sigma_3,\pm1)$, where $\sigma_j = \varepsilon_i$ if $i = j$ and 0 otherwise. Now consider a triangle in our complex with vertices in this square, say $[v_0,v_1,v_2]$, and let the product of the first three coordinates of v_0 be -1. The link of this triangle has four edges. Let v_0' denote the third vertex adjacent to v_0 in Q^3. In the link of this triangle, v_0' is joined to $(0,0,0,\pm1)$, and the latter vertices are respectively joined to $3(\sigma_1,\sigma_2,\sigma_3,\pm1)$. We join the triangle $[v_0,v_1,v_2]$ to the line segment $[3(\sigma_1,\sigma_2,\sigma_3,1),3(\sigma_1,\sigma_2,\sigma_3,-1)]$. The twelve triangles with vertices on a square in Q^3 give one simplex each, so we get twelve new simplices.

Also, in our last but one set of simplices, we introduced four new triangles, each consisting of three vertices of Q^3, such that the first three coordinates of each have product 1. The link of each such triangle consists of two edges, where the common neighbour of the three vertices is joined to each of $(0,0,0,\pm1)$. So we have four triangles forming the boundary of a tetrahedron, the boundaries of the links of each being the vertices $(0,0,0,\pm1)$. We add two new simplices, by taking the tetrahedron consisting of these four vertices of Q^3 and joining it to each of the points $(0,0,0,\pm1)$.

So now the links of all triangles with vertices from SQ^3 are circles. We consider the links of edges in SQ^3.

The link of an edge of the form $[(\varepsilon_1, \varepsilon_2, \varepsilon_3, 0), (0, 0, 0, \varepsilon)]$, where $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, is an octahedron. One of the faces of this octahedron consists of the three neighbours of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, 0)$ in Q^3 and its opposite face in this octahedron is the triangle corresponding to the chosen edge in D.

The link of an edge of the form $[(\varepsilon_1, \varepsilon_2, \varepsilon_3, 0), (0, 0, 0, \varepsilon)]$, where $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, is a 9-vertex S^2 with six vertices from Q^3 and the remaining three vertices from its corresponding triangle in D.

The boundary of the link of an edge in Q^3 is the boundary of its corresponding square in D.

Note that the boundary of this complex is now a triangulated S^3 with f-vector $[22, 102, 160, 80]$.

Now consider each edge $[v_1, v_2]$ in Q^3 and its opposite edge in Q^3, $[-v_1, -v_2]$. If the boundary of the links of the first edge is $[w_1, w_2], [w_2, w_3], [w_3, w_4], [w_4, w_1]$, then the link of the opposite edge is $[-w_1, -w_2], [-w_2, -w_3], [-w_3, -w_4]$, $[-w_4, -w_1]$.

Now we apply the antipodal map $v \mapsto -v$ on the vertices of D, the two above squares will be identified. So will the two octahedra O^+ and O^- which are the boundaries of the links of $\pm(0, 0, 0, 1)$ respectively.

We close the links of the edges $\pm[v_1, v_2]$ by joining each of the tetrahedra containing $[v_1, v_2]$ to that vertex of $[-v_1, -v_2]$, the product of whose first three coordinates is -1.

We close the boundary by adding the simplices obtained by joining each of the triangles $[(1, 1, 1, 0),\ (1, 1, -1, 0),\ (-1, -1, -1, 0)]$ and $[(-1, -1, -1, 0),\ (-1, -1, 1, 0),\ (1, 1, -1, 0)]$ with each of the edges in S. This gives $2 \times 6 \times 4 = 48$ simplices.

Now we have joined each vertex v in Q^3 to its opposite vertex $-v$. The link of the edge $[v, -v]$ consists of twelve triangles. Suppose $v = (\varepsilon_1, \varepsilon_2, \varepsilon_3, 0)$ where $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, and let $\overline{P_v}$ be the image under the antipodal map on D of the prism(s) corresponding to v (and $-v$) in D. Then the faces of the link of $[v, -v]$ are the three square faces of $\overline{P_v}$, each subdivided by the corresponding neighbour of v. The boundary of this complex is the set of edges of two disjoint triangles in the image of O^+ (and O^-) under the map $x \mapsto -x$. We add the joins of $[v, -v]$ with each of these triangles. This gives $4 \times 2 = 8$ simplices.

We close the boundaries of $\pm(0, 0, 0, 1)$ by joining each of the faces of the octahedron to the edge $[(0, 0, 0, 1), (0, 0, 0, -1)]$. This gives eight more simplices.

This gives a 16-vertex triangulation of $\mathbb{R}\mathrm{P}^4$ with 150 simplices. □

4 Similarities, differences, and concluding remarks

The starting objects of our two latter constructions, such as the hyperoctahedron, the 4-cube, the suspended cube, and octahedral prism suggest that automorphism groups of the triangulations we constructed are very close to $C_2 \times S_4$. In fact the simplices we add in each step of each construction are typically orbits under this group. But on explicit computation, the automorphism groups of both complexes turn out to be isomorphic to S_6, acting on $10 + 6$ vertices.

In Construction 2, the vertex-orbit of S_6 of size 6 consists of the two points of the hyperoctahedron C^4 used to triangulate it internally, (namely $\pm\mathbf{e}_1$), and the vertices of

Q_e.

In Construction 3, the smaller S_6 orbit consists of the suspension points $(0, 0, 0, \pm 1)$, and the four vertices of the cube the product of whose first three coordinates is -1.

One construction starts with a suspended octahedron (hyperoctahedron) on the inside and a cubical prism (hypercube) on the outside of our 4-dimensional ball. The other starts with a suspended cube on the inside and an octahedral prism on the outside. This gives us a way of visualizing either construction as the other one "turned inside-out".

Also recall that the $C_2 \times S_4$ is the stabilizer of a 2-subset in S_6. If we consider any pair of elements of the orbit $\mathcal{O}_6$ of size 6 in either construction, we find that the link of the edge joining them is an octahedron consisting of six points of the longer orbit $\mathcal{O}_{10}$, and that the intersection of their links is a solid cube triangulated with five tetrahedra, where the vertices of the inner tetrahedron are the remaining vertices of $\mathcal{O}_{10}$ and the other four vertices are the remaining vertices of $\mathcal{O}_6$. This also gives a correspondence with the vertex set of the triangulated $\mathbb{RP}^4$ of Construction 1, which induces simplicial isomorphisms between all three complexes. This gives weight to our belief that the following is true.

Conjecture 5. [7] $\mathbb{RP}^4_{16}$ is the unique 16-vertex triangulation of $\mathbb{RP}^4$.

Their close connection not withstanding, Constructions 2 and 3 offer different perspectives on how to construct triangulations of $\mathbb{RP}^n$ for other values of n. The boundaries of the very final 4-balls we construct before applying the antipodal map are antipodal 3-spheres. In the first case, the 3-sphere has 24 vertices, and the quotient only gives a 12-vertex $\mathbb{RP}^3$, whereas in the second construction, we get the same 3-sphere constructed by Walkup as the double cover of his $\mathbb{RP}^3_{11}$.

Also note that Walkup's $\mathbb{RP}^3_{11}$ also fits into the paradigm outlined in Construction 3. Recall that the initial object of our construction was an octahedron, surrounded by a cube. Observe that the octahedron and cube are respectively a suspension of a square and a prismed (solid) square.

The two constructions in Section 3 point to different possible generalizations. Construction 2 suggests constructions using a hyperoctahedron placed within a hypercube to obtain an $\mathbb{RP}^n$ on $2^{(n-1)} + 2n$ or $2^{(n-1)} + 2n + 1$ vertices. This author has tested this approach to construct an $\mathbb{RP}^5$ on $2^{(5-1)} + 2 \times 5 + 1 = 27$ vertices and an $\mathbb{RP}^6$ on $2^5 + 12 + 1 = 45$ vertices.

Construction 3 suggests the possibility of taking a mixture of suspended and prismed polyhedra in higher dimensions to lower the number of vertices needed even further. If this can be realized by starting with a double suspension of the 3-cube, then it may be possible to triangulate $\mathbb{RP}^5$ with only $8 + 4 + \frac{24}{2} = 24$ vertices. F.H. Lutz has discovered an S_4-invariant 24-vertex triangulation of $\mathbb{RP}^5$, the facet-list of which is available at [16]. This author has not investigated the possibility that this complex is isomorphic to the one we postulate. The `BISTELLAR` program used to discover this triangulation is available through the `GAP` package `simpcomp` [9], along with a library of known triangulations and other tools for calculating with complexes.

Our observation that double covers of $\mathbb{RP}^{n-1}$ appear as boundaries of n-balls in the constructions of $\mathbb{RP}^n$ suggests the following question. Given an antipodal n-sphere, is it possible to "thicken" it to an $(n+1)$-ball while preserving a suitable restriction of the antipodal map, then glue the quotiented boundary to itself to get a triangulated $\mathbb{RP}^{n+1}$? If the answer is yes, it suggests the possibility of inductively triangulating $\mathbb{RP}^n$. In the case of the double cover of $\mathbb{RP}^4_{16}$ in Construction 1, it is easy to construct a B^5 on 32 vertices such that the antipodal map applies to the orbit of size 20. But the question of whether there exists a gluing of the quotiented boundary which produces a triangulated $\mathbb{RP}^5$ is harder to answer. If such a triangulation exists, it would have $12+\frac{20}{2}=22$ vertices, which would make it vertex-minimal by the theorem of Arnoux and Marin.

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References

[1] Pierre Arnoux and Alexis Marin. The Kühnel triangulation of the complex projective plane from the view point of complex crystallography. II. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 45(2):167–244, 1991.

[2] Sonia Balagopalan. *Some results on Vertex-Minimal Triangulations of Manifolds*. PhD thesis, National University of Ireland Maynooth, 2014. http://eprints.nuim.ie/5024.

[3] Anders Björner and Frank H. Lutz. Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere. *Experiment. Math.*, 9(2):275–289, 2000.

[4] P. J. Cameron and J. H. van Lint. *Designs, graphs, codes and their links*, volume 22 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1991.

[5] R. D. Carmichael. Tactical Configurations of Rank Two. *Amer. J. Math.*, 53(1):217–240, 1931.

[6] Basudeb Datta. Personal communication.

[7] Basudeb Datta. Minimal triangulations of manifolds. *J. Indian Inst. Sci.*, 87(4):429–449, 2007.

[8] W. L. Edge. Some implications of the geometry of the 21-point plane. *Math. Z.*, 87:348–362, 1965.

[9] Felix Effenberger and Jonathan Spreer. `simpcomp` - *a* `GAP` *toolkit for simplicial complexes, Version 1.6.1*, 2013.

[10] Ewgenij Gawrilow and Michael Joswig. polymake: a framework for analyzing convex polytopes. In Gil Kalai and Günter M. Ziegler, editors, *Polytopes — Combinatorics and Computation*, pages 43–74. Birkhäuser, 2000.

[11] R. W. H. T. Hudson. *Kummer's quartic surface*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1990. With a foreword by W. Barth, Revised reprint of the 1905 original.

[12] Mikhail H. Klin and Andrew J. Woldar. Dale Mesner, Higman & Sims, and the strongly regular graph with parameters $(100, 22, 0, 6)$. *Bull. Inst. Combin. Appl.*, 63:13–35, 2011. Expanded version, privately circulated.

[13] W. Kühnel. Minimal triangulations of Kummer varieties. *Abh. Math. Sem. Univ. Hamburg*, 57:7–20, 1987.

[14] Heinz Lüneburg. Über die Gruppen von Mathieu. *J. Algebra*, 10:194–210, 1968.

[15] Frank Hagen Lutz. *Triangulated manifolds with few vertices and vertex-transitive group actions*. Berichte aus der Mathematik. [Reports from Mathematics]. Verlag Shaker, Aachen, 1999. Dissertation, Technischen Universität Berlin, Berlin, 1999.

[16] Frank Hagen Lutz. The Manifold Page. http://page.math.tu-berlin.de/~lutz/stellar/, 2014.

[17] Dale M. Mesner. Negative latin square designs. Institute of Statistics, UNC, NC Mimeo series, 410, 1964.

[18] Natwar N. Roghelia and Sharad S. Sane. Classification of (16, 6, 2)-designs by ovals. *Discrete Math.*, 51(2):167–177, 1984.

[19] James Joseph Sylvester. Elementary researches in the analysis of combinatorial aggregation. In *Collected Mathematical Papers*, volume I, chapter 17, pages 91 – 102. Cambridge University Press, 1904.

[20] David W. Walkup. The lower bound conjecture for 3- and 4-manifolds. *Acta Math.*, 125:75–107, 1970.

[21] Ernst Witt. Über Steinersche Systeme. *Abh. Math. Sem. Univ. Hamburg*, 12(1):265–275, 1937.