

On 1212-avoiding restricted growth functions

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Abstract

Restricted growth functions (RGFs) avoiding the pattern 1212 are in natural bijection with noncrossing partitions. Motivated by recent work of Campbell et al., we study five classical statistics bk , ls , lb , rs and rb on 1212-avoiding RGFs. We show the equidistribution of $(\text{ls}, \text{rb}, \text{lb}, \text{bk})$ and $(\text{rb}, \text{ls}, \text{lb}, \text{bk})$ on 1212-avoiding RGFs by constructing a simple involution. To our surprise, this result was already proved by Simion 22 years ago via an involution on noncrossing partitions. Our involution, though turns out essentially the same as Simion's, is defined quite differently and has the advantage that makes the discussion more transparent. Consequently, a multiset-valued extension of Simion's result is discovered. Furthermore, similar approach enables us to prove the equidistribution of $(\text{mak}, \text{rb}, \text{rs}, \text{bk})$ and $(\text{rb}, \text{mak}, \text{rs}, \text{bk})$ on 1212-avoiding RGFs, where “mak” is a set partition statistic introduced by Steingrímsson.

Through two bijections to Motzkin paths, we also prove that the triple of classical permutation statistics $(\text{exc} + 1, \text{den}, \text{inv} - \text{exc})$ on 321-avoiding permutations is equidistributed with the triple $(\text{bk}, \text{rb}, \text{rs})$ on 1212-avoiding RGFs, which generalizes another result of Simion. In the course, an interesting q -analog of the γ -positivity of Narayana polynomials is found.

Keywords: restricted growth function; pattern avoidance; noncrossing partitions; partition statistics; Narayana polynomials

1 Introduction

A *restricted growth function* (RGF) of length n is a word $w = w_1 w_2 \dots w_n$ on positive integers such that

$$w_1 = 1 \quad \text{and} \quad w_{i+1} \leq 1 + \max\{w_1, w_2, \dots, w_i\} \text{ for all } i \geq 1.$$

A *set partition* of $[n] := \{1, 2, \dots, n\}$ is a family of nonempty subsets (or blocks) B_1, \dots, B_k whose disjoint union is $[n]$. RGFs are of interest because they are in natural bijection with set partitions in the following way. We first write a set partition σ of $[n]$ as $B_1 / \dots / B_k$ in the *standard form* where $\min B_1 < \min B_2 < \dots < \min B_k$. Then the associated RGF $w(\sigma) = w_1 \dots w_n$ is defined as

$$w_i = j \text{ if and only if } i \in B_j.$$

For example, we have $w(13/24) = 1212$. Note that the number of *blocks* of a set partition becomes the greatest integer in its associated RGF. Therefore, we denote $\text{bk}(w)$ the greatest integer in w .

Let R_n be the set of all RGFs of length n . Wachs and White [16] investigated four statistics ls (*left smaller*), lb (*left bigger*), rs (*right smaller*) and rb (*right bigger*) on RGFs defined for each $w \in R_n$ by:

$$\begin{aligned} \text{ls}(w) &:= \sum_{i=1}^n \#\{w_j : j < i, w_j < w_i\}, & \text{lb}(w) &:= \sum_{i=1}^n \#\{w_j : j < i, w_j > w_i\}, \\ \text{rs}(w) &:= \sum_{i=1}^n \#\{w_j : j > i, w_j < w_i\} & \text{and} & \quad \text{rb}(w) := \sum_{i=1}^n \#\{w_j : j > i, w_j > w_i\}. \end{aligned}$$

For example, if $w = 12334155 \in R_8$, then $\text{ls}(w) = 16, \text{lb}(w) = 3, \text{rs}(w) = 4, \text{rb}(w) = 13$. Recently, a systematic study of RGF patterns with respect to $\text{ls}, \text{lb}, \text{rs}$ and rb was carried out by Campbell, Dahlberg, Dorward, Gerhard, Grubb, Purcell and Sagan in [1], where interesting connections with integer partitions and Motzkin paths were exhibited. At the end of their paper, unaware of the work by Simion [10], they conjectured that “ rb ” and “ ls ” are equidistributed on $R_n(1212)$, where $R_n(1212)$ is the set of 1212-avoiding RGFs of length n . Here an RGF $w \in R_n$ *avoids the pattern* 1212 means that there does not exist some indices $i < j < k < l$ such that $w_i = w_k < w_j = w_l$.

Motivated by Campbell et al.’s conjecture above, we have been able to rediscover and reprove the following quadruple equidistribution that was first established by Simion [10] two decades ago.

Theorem 1 (Simion 1994). *For any $n \geq 0$, the two four-variable statistics*

$$(\text{ls}, \text{rb}, \text{lb}, \text{bk}) \text{ and } (\text{rb}, \text{ls}, \text{lb}, \text{bk})$$

are equidistributed on $R_n(1212)$.

Our proof of Theorem 1 is by constructing an involution $\phi : R_n(1212) \rightarrow R_n(1212)$, which interchanges the statistics “ls” and “rb” but preserves the statistics “lb” and “bk”. It is not hard to show that a set partition is *noncrossing* (see [10] for the definition) if and only if its associated RGF is 1212-avoiding. It turns out that, under the natural bijection between set partitions and RGFs, our involution becomes Simion’s [10], which was defined on noncrossing partitions (see Remark 1). Our involution ϕ is defined directly on 1212-avoiding RGFs, which is based on some kind of direct sum decomposition and has the advantage that makes the discussion more transparent. In particular, our approach leads to the discovery of a *multiset-valued extension* of Theorem 1 (see Theorem 9). Furthermore, similar approach can be applied to prove the following quadruple equidistribution involving the set partition statistic “mak” introduced by Steingrímsson [13] (see the definition of “mak” on RGFs in Section 3).

Theorem 2. *For any $n \geq 0$, the two four-variable statistics*

$$(\text{mak}, \text{rb}, \text{rs}, \text{bk}) \text{ and } (\text{rb}, \text{mak}, \text{rs}, \text{bk})$$

are equidistributed on $R_n(1212)$.

Our next result builds a relation between statistics on 1212-avoiding RGFs and classical Permutation Statistics, including excedances, inversions and Denert’s statistic, on 321-avoiding permutations. We first review the involved statistics on permutations. Let \mathfrak{S}_n be the set of permutations of $[n]$. For each $\pi = \pi_1\pi_2 \dots \pi_n$ the *inversion* number of π is

$$\text{inv}(\pi) := \#\{(i, j) : i < j, \pi_i > \pi_j\},$$

the *set of excedances* (resp. *excedance number*) of π is

$$\text{EXC}(\pi) := \{i \in [n-1] : \pi_i > i\} \quad (\text{resp. } \text{exc}(\pi) := \#\text{EXC}(\pi))$$

and the *set of non-excedances* of π is

$$\text{NEXC}(\pi) := [n] - \text{EXC}(\pi).$$

Following Foata and Zeilberger [4, Th. 2], the *Denert’s statistic* of π may be defined by

$$\text{den}(\pi) := \text{inv}(\text{EXC}^*(\pi)) + \text{inv}(\text{NEXC}^*(\pi)) + \sum_{i \in \text{EXC}(\pi)} i,$$

where for a subset $S = \{i_1 < i_2 < \dots < i_k\} \subseteq [n]$, $S^*(\pi)$ is the subword $\pi_{i_1}\pi_{i_2} \dots \pi_{i_k}$ of π . For example, if $\pi = 2534716 \in \mathfrak{S}_7$, then $\text{inv}(\pi) = 8$, $\text{EXC}(\pi) = \{1, 2, 5\}$, $\text{exc}(\pi) = 3$ and $\text{den}(\pi) = \text{inv}(257) + \text{inv}(3416) + 1 + 2 + 5 = 10$. Note that “exc” is a *Eulerian* statistic, while both “inv” and “den” are *Mahonian*. Recall that a permutation $\pi \in \mathfrak{S}_n$ is said to be *321-avoiding* if there does not exist indices $i < j < k$ such that $\pi_i > \pi_j > \pi_k$. Let $\mathfrak{S}_n(321)$ be the set of all 321-avoiding permutations in \mathfrak{S}_n .

Theorem 3. *There exists a bijection $\psi : \mathfrak{S}_n(321) \rightarrow R_n(1212)$ such that for each $\pi \in \mathfrak{S}_n(321)$, we have*

$$(\text{exc} + 1, \text{den}, \text{inv} - \text{exc})(\pi) = (\text{bk}, \text{rb}, \text{rs})(\psi(\pi)).$$

Consequently, we have the equidistribution

$$\sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{exc}(\pi)+1} p^{\text{den}(\pi)} q^{\text{inv}(\pi)-\text{exc}(\pi)} = \sum_{w \in R_n(1212)} t^{\text{bk}(w)} p^{\text{rb}(w)} q^{\text{rs}(w)}. \quad (1.1)$$

The special $q = 1$ case of equidistribution (1.1) was proved by Simion [10, Theorem 5.7] via a bijection from noncrossing partitions to $\mathfrak{S}_n(321)$. Note that Simion's bijection can not be applied to show (1.1). Consider for instance, $w = 12334155$ associated to the noncrossing partition $16/2/34/5/78$, at the end of her paper. The RGF w is mapped to $\sigma = 24153867 \in \mathfrak{S}_n(321)$ under her bijection which satisfies $\text{bk}(w) = 5 = \text{exc}(\sigma) + 1$ and $\text{rb}(w) = 13 = \text{den}(\sigma)$, but meanwhile $\text{rs}(w) = 4 \neq 2 = \text{inv}(\sigma) - \text{exc}(\sigma)$. Our bijection ψ is a function composition of two bijections to the Motzkin paths, one from $R_n(1212)$ to two-colored Motzkin paths of length $n - 1$ due to Campbell et al. [1] and the other is from the latter object to $\mathfrak{S}_n(321)$, which is new to the best of our knowledge.

Let

$$N_n(t, p, q) := \sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{exc}(\pi)} p^{\text{den}(\pi)} q^{\text{inv}(\pi)-\text{exc}(\pi)}. \quad (1.2)$$

Since counting noncrossing partitions of $[n]$ by the number of blocks minus 1 gives the n -th Narayana polynomial (cf. [11, Sec. 2.6])

$$N_n(t) = \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k+1} \binom{n}{k} t^k,$$

the polynomial $N_n(t, p, q)$ is a (p, q) -analog of the Narayana polynomial. It is well known that $N_n(t)$ is γ -positive, i.e., it can be expanded as $N_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k}$ with $\gamma_{n,k} \in \mathbb{N}$. Many combinatorial interpretations for $\gamma_{n,k}$ are known in the literature, see the excellent exposition by Petersen [11]. For instance, Postnikov, Reiner and Williams [12] showed that $\gamma_{n,k}$ enumerates the 231-avoiding permutations of length n , with k descents and without initial and double descents. As one application of our bijection ψ , the following neat q - γ -positivity expansion for the q -Narayana polynomial $N_n(t, 1, q)$ is derived.

Theorem 4. *Let*

$$\widetilde{\mathfrak{S}}_{n,k}(321) := \{\pi \in \mathfrak{S}_n(321) : \text{exc}(\pi) = k \text{ and whenever } i < \pi_i, (\pi^{-1})_{i+1} > i\}.$$

Then,

$$\sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{exc}(\pi)} q^{\text{inv}(\pi)-\text{exc}(\pi)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left(\sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(321)} q^{\text{inv}(\pi)-\text{exc}(\pi)} \right) t^k (1+t)^{n-1-2k}. \quad (1.3)$$

For example, we have $\widetilde{\mathfrak{S}}_{4,0}(321) = \{1234\}$ and $\widetilde{\mathfrak{S}}_{4,1}(321) = \{1423, 3124, 4123\}$. It then follows from (1.3) that

$$N_4(t, 1, q) = 1 + (3 + 2q + q^2)t + (3 + 2q + q^2)t^2 + t^3 = (1 + t)^3 + (2q + q^2)t(1 + t).$$

Note that Theorem 4 implies the sequence of coefficients of the polynomial (in t) $N_n(t, 1, q)$ is *palindromic* and *unimodal* in the sense that if $N_n(t, 1, q) = \sum_{i=0}^{n-1} a_{n,i}(q)t^i$, then

- $a_{n,i}(q) = a_{n,n-1-i}(q)$ for $0 \leq i \leq n-1$ and
- $a_{n,j+1}(q) - a_{n,j}(q) \in \mathbb{N}[q]$ for $0 \leq j < (n-1)/2$.

This result can be compared with the q - γ -positivity expansion of the (inv, exc)- q -Eulerian polynomials due to Shin and Zeng [9, Theorem 1].

The rest of this paper is organized as follows. In Section 2, we define the direct sum decomposition of RGFs and introduce the involution ϕ to prove Theorem 1. Based again on the direct sum decomposition, we construct in Section 3 a new involution τ on $R_n(1212)$ in the same spirit as ϕ (but more involved) to prove Theorem 2. Theorem 3 is proved in Section 4 by constructing the bijection ψ as function composition of two bijections to Motzkin paths. Section 5 is devoted to further applications of ψ . Indeed, we first establish Theorem 4 via a group action on Motzkin paths, then we continue to analyze Motzkin paths so as to obtain a recursion for $N_n(t, p, q)$ and a continued fraction expansion for the generating function of $N_n(t, 1, q)$. We conclude our paper with some further remarks.

2 Direct sum and the involution ϕ

We first recall one operation that is usually used to combine shorter words into longer ones. Suppose $w = w_1w_2 \dots w_{l_1}$ and $v = v_1v_2 \dots v_{l_2}$ are two words on positive integers with w 's (resp. v 's) maximum being m_1 (resp. m_2). We define the *direct sum* of w and v , denoted $w \oplus v$, to be a word with maximum being $m_1 + m_2$ and length being $l_1 + l_2$, and point-wisely it satisfies

$$(w \oplus v)_i = \begin{cases} w_i, & \text{for } i \in [1, l_1]; \\ v_{i-l_1} + m_1, & \text{for } i \in [l_1 + 1, l_1 + l_2]. \end{cases}$$

For example, we have $121 \oplus 12 = 12134$. If a word can be written as the direct sum of two non-empty words, we call it *direct sum decomposable* or simply *decomposable*. It is clear that the direct sum operation is *associative*, i.e., $u \oplus (v \oplus w) = (u \oplus v) \oplus w$. The following two fundamental lemmas are crucial for constructing our involution ϕ , the first of which should be immediate.

Lemma 5. (i) A word u is in $R_{n-1}(1212)$ if and only if $u1$ is in $R_n(1212)$. (ii) The direct sum $w \oplus v$ is a 1212-avoiding RGF if and only if both w and v are 1212-avoiding RGFs.

Now with the operation of direct sum in mind, we can give a new characterization of RGFs avoiding 1212.

Lemma 6. *For any RGF $w = w_1 \dots w_n \in R_n$, the following are equivalent:*

1. *The RGF w avoids 1212.*
2. *If $w_i = w_{i'}$ for some $i < i'$, then for all $j' > i'$, either $w_{j'} \leq w_{i'}$ or $w_{j'} > \max\{w_1, \dots, w_{i'}\}$.*
3. *w is one of the following two types.*
 - *Type I (indecomposable): w ends with 1, say $w = \hat{w}1 \dots 1$, where \hat{w} (possibly empty) does not end with 1 and avoids 1212.*
 - *Type II (decomposable): w does not end with 1, and there exists some non-empty 1212-avoiding RGFs u and v , such that $w = u \oplus v$.*

Proof. The equivalence of (1) and (2) is Lemma 5.2 in [1]. We show here their equivalence with (3). When w satisfies (3), then w avoids 1212 by Lemma 5. It remains to show that if the RGF w avoids 1212, then it must be one of the two types in (3).

Find the rightmost 1 in w , say at position i . If $i = n$, then clearly w is type I. Otherwise, $1 \leq i < n$. Let $u = w_1 w_2 \dots w_i$. Since w_i is the rightmost 1 of w , according to condition (2), we have $w_j > \max\{w_1, \dots, w_i\}$ for all $j > i$. Hence, $w = u \oplus v$, where $v_j = w_{i+j} - \text{bk}(u)$ for all $1 \leq j \leq n - i$. By Lemma 5, both u and v are RGFs and avoiding 1212. This shows w is type II in this case and the proof is complete. \square

We are ready to define our key map ϕ for Theorem 1.

Definition 1. *For any RGF w that avoids 1212, we define the map ϕ recursively and according to the type of w (set $\phi(\emptyset) = \emptyset$):*

- *Type I. Suppose $w = \hat{w}1^k$, where $k \geq 1$ and \hat{w} (possibly empty) does not end with 1. Define*

$$\phi(w) = \phi(\hat{w})1^k.$$

- *Type II. Suppose $w = u \oplus v$, then let*

$$\phi(w) = \phi(v) \oplus \phi(u).$$

Note that Lemma 5 ensures $\phi(w)$ is a 1212-avoiding RGF. In general, the direct sum decomposition of a 1212-avoiding RGF is not unique. For example, $12134 = 121 \oplus 12 = 1213 \oplus 1$. However, the following lemma shows that ϕ is independent of the decomposition of w , hence well-defined.

Lemma 7. *Suppose $w \in R_n(1212)$ has two different decompositions $w = u \oplus v = u' \oplus v'$. Then, $\phi(v) \oplus \phi(u) = \phi(v') \oplus \phi(u')$.*

Proof. We prove this result by induction on the length of w . Without loss of generality, we suppose $u \in R_k(1212)$ and $u' \in R_l(1212)$ with $1 \leq k < l < n$. Then, there exists a unique $c \in R_{l-k}(1212)$ such that $u' = u \oplus c$ and $v = c \oplus v'$. Hence, by induction on the length of w and by the associativity of the direct sum,

$$\begin{aligned}\phi(v) \oplus \phi(u) &= \phi(c \oplus v') \oplus \phi(u) = (\phi(v') \oplus \phi(c)) \oplus \phi(u) \\ &= \phi(v') \oplus (\phi(c) \oplus \phi(u)) = \phi(v') \oplus \phi(u'),\end{aligned}$$

as desired. \square

The following result provides a proof of Theorem 1.

Theorem 8. *The map ϕ is an involution on $R_n(1212)$ such that*

$$(\text{ls}, \text{rb}, \text{lb}, \text{bk})(w) = (\text{rb}, \text{ls}, \text{lb}, \text{bk})(\phi(w)) \quad (2.1)$$

for each $w \in R_n(1212)$.

Proof. This result can be checked easily by induction on n . Notice that ϕ preserves the type of 1212-avoiding RGFs. We discuss in two cases according to the type of w .

Type I: suppose $w = \hat{w}1^k$, where $k \geq 1$ and \hat{w} (possibly empty) does not end with 1. Thus, $\phi^2(w) = \phi(\phi(\hat{w})1^k) = \phi^2(\hat{w})1^k = \hat{w}1^k = w$. When $\hat{w} = \emptyset$, clearly $\phi(w) = w = 1^k$, $\text{ls}(w) = \text{rb}(w) = 0$ so (2.1) holds trivially. For the following we assume $\hat{w} \neq \emptyset$ and we have

$$\begin{aligned}\text{rb}(\phi(w)) &= \text{rb}(\phi(\hat{w})1^k) = \text{rb}(\phi(\hat{w})) = \text{ls}(\hat{w}) = \text{ls}(w), \\ \text{bk}(\phi(w)) &= \text{bk}(\phi(\hat{w})1^k) = \text{bk}(\phi(\hat{w})) = \text{bk}(\hat{w}) = \text{bk}(w) \quad \text{and} \\ \text{lb}(\phi(w)) &= \text{lb}(\phi(\hat{w})1^k) = \text{lb}(\phi(\hat{w})) + k \times (\text{bk}(\phi(\hat{w})) - 1) \\ &= \text{lb}(\hat{w}) + k \times (\text{bk}(\hat{w}) - 1) = \text{lb}(w).\end{aligned}$$

Type II: suppose $w = u \oplus v$. Then, $\phi^2(w) = \phi(\phi(v) \oplus \phi(u)) = \phi^2(v) \oplus \phi^2(u) = u \oplus v = w$. If v has length l with $1 \leq l < n$, then

$$\begin{aligned}\text{rb}(\phi(w)) &= \text{rb}(\phi(v) \oplus \phi(u)) = \text{rb}(\phi(v)) + \text{rb}(\phi(u)) + l \times \text{bk}(\phi(u)) \\ &= \text{ls}(v) + \text{ls}(u) + l \times \text{bk}(u) = \text{ls}(u \oplus v) = \text{ls}(w).\end{aligned}$$

Moreover, we have

$$\text{bk}(\phi(w)) = \text{bk}(\phi(v)) + \text{bk}(\phi(u)) = \text{bk}(v) + \text{bk}(u) = \text{bk}(w)$$

and

$$\text{lb}(\phi(w)) = \text{lb}(\phi(v)) + \text{lb}(\phi(u)) = \text{lb}(v) + \text{lb}(u) = \text{lb}(w).$$

In either case the statement is true, which completes the proof of Theorem 8. \square

For example, if $w = 1112322145541$, then $\phi(w) = \phi(111232214554)1 = (\phi(1221) \oplus \phi(11123221))1 = (1121 \oplus (\phi(111 \oplus 1211)1))1 = (1121 \oplus (\phi(1211) \oplus \phi(111)1))1 = (1121 \oplus 12113331)1 = 1121343355531$. We have

$$(\text{ls}, \text{rb}, \text{lb}, \text{bk})(w) = (19, 24, 9, 5) = (\text{rb}, \text{ls}, \text{lb}, \text{bk})(\phi(w)).$$

Remark 1. We recall Simion's involution ϕ' in [10] that proves Theorem 1. Given a noncrossing partition Π of $[n]$ written in standard form $B_1/B_2/\dots/B_k$. Let $f_i = \min\{a : a \in B_i\}$ for $1 \leq i \leq k$ and $f_{k+1} = n + 1$. Relabel the elements of $[n]$ as follows: the elements of the last block B_k are relabeled $1, 2, \dots, |B_k|$ in increasing order; next, the elements of the penultimate block B_{k-1} are relabeled $|B_k| + 1, |B_k| + 2, \dots, |B_k| + |B_{k-1}|$ in increasing order; and so on. Now, the i -th block of $\phi'(\Pi)$ consists of the set of new labels given to the elements in the original interval $\{f_{k-i+1}, f_{k-i+1} + 1, \dots, f_{k-i+2} - 1\}$.

For example, for $\Pi = 1\ 2\ 3\ 8\ 13/4\ 6\ 7/5/9\ 12/10\ 11$, $\phi'(\Pi) = 1\ 2\ 4\ 13/3/5\ 7\ 8\ 12/6/9\ 10\ 11$. Note that $w(\Pi) = w = 1112322145541$ and $\phi(w) = 1121343355531$ as computed in the previous example. Now we see that $\phi(w(\Pi)) = w(\phi'(\Pi))$ for this particular Π . In fact, this coincidence is true in general. One possible way to see this is by induction on n and by distinguishing two types of noncrossing partitions of $[n]$: (i) n is in the first block and (ii) n is not in the first block, which corresponds to type I and II of RGFs under the natural bijection. The details are left to the interested reader.

For any $w \in R_n$, let us define the multiset-valued extension of $\text{ls}(w)$ as the multiset

$$\text{LS}(w) := \{\text{ls}(w_1), \text{ls}(w_2), \dots, \text{ls}(w_n)\},$$

where $\text{ls}(w_i) = \#\{w_j : j < i, w_j < w_i\}$. Similarly, we define $\text{LB}(w)$, $\text{RS}(w)$ and $\text{RB}(w)$. Actually, the involution ϕ can be applied to show the following multiset-valued extension of Theorem 1 without any difficulty.

Theorem 9. *For any $n \geq 0$, the two mixed four-variable statistics (both integer-valued and multiset-valued)*

$$(\text{LS}, \text{RB}, \text{LB}, \text{bk}) \text{ and } (\text{RB}, \text{LS}, \text{LB}, \text{bk})$$

are equidistributed on $R_n(1212)$.

3 Steingrímsson's set partition statistic “mak”

In [13], Steingrímsson introduced many statistics on (ordered or unordered) set partitions, of which the statistic “mak” is quite interesting. This statistic was inspired by the permutation statistic under the same name “mak” introduced by Foata and Zeilberger [4]. Here we will define “mak” on RGFs directly as follows.

Let $w \in R_n$ be a RGF. For each $i \in [n]$, define

$$\text{lcs}_i(w) := \#\{w_j : j < i, w_j < w_i \text{ and } w_k \neq w_j \text{ for all } k > j\}.$$

Let us denote $\text{lcs}(w) = \sum_i \text{lcs}_i(w)$. For example, if $w = 12334155 \in R_8$, then $\text{lcs}(w) = 0 + 0 + 1 + 1 + 2 + 0 + 4 + 4 = 12$. Under the natural bijection, this statistic agrees with

the “left closer smaller”, also denoted “lcs”, defined on set partitions in [13]. Now the statistic “mak” can be defined by

$$\text{mak}(w) := \text{lb}(w) + \text{lcs}(w).$$

Continuing with our running example, we have $\text{mak}(w) = 3 + 12 = 15$. It turns out that “mak” is equidistributed with “rb” on R_n and even more is true.

Theorem 10 (See [6, 7, 16]). *For any $n \geq 0$, we have the quadruple equidistribution*

$$\sum_{w \in R_n} p_1^{\text{mak}(w)} p_2^{\text{rb}(w)} q^{\text{rs}(w)} t^{\text{bk}(w)} = \sum_{w \in R_n} p_1^{\text{rb}(w)} p_2^{\text{mak}(w)} q^{\text{rs}(w)} t^{\text{bk}(w)} \quad (3.1)$$

Moreover, there holds

$$\sum_{\substack{w \in R_n \\ \text{bk}(w)=k}} p^{\text{rb}(w)} q^{\text{rs}(w)} = S_{p,q}(n, k) = \sum_{\substack{w \in R_n \\ \text{bk}(w)=k}} p^{\text{mak}(w)} q^{\text{rs}(w)}, \quad (3.2)$$

where $S_{p,q}(n, k)$ is the (p, q) -Stirling number of the second kind defined by the recursion:

$$S_{p,q}(n, k) = \begin{cases} p^{k-1} S_{p,q}(n-1, k-1) + \left(\sum_{i=0}^{k-1} p^i q^{k-1-i} \right) S_{p,q}(n-1, k) & \text{if } 0 < k \leq n, \\ \delta_{n,k} & \text{if } n = 0 \text{ or } k = 0. \end{cases}$$

Note that the first identity of (3.2) is due to Wachs and White [16], while the second was proved by Ksavlouf and Zeng [7] in answering a conjecture of Steingrímsson [13]. The quadruple equidistribution (3.1), as well as two different generalizations to ordered set partitions, were proved by Kasraoui and Zeng in [6] via the intermediate structure of path diagrams. Our quadruple equidistribution stated in Theorem 2 is a restricted version of (3.1).

Based on the direct sum decomposition structure showed in Lemma 6, we will construct recursively a new involution $\tau : R_n(1212) \rightarrow R_n(1212)$, which interchanges the statistics “mak” and “rb” but preserves the statistics “rs” and “bk”. This leads to a proof of Theorem 2.

Definition 2. For any $w = w_1 \dots w_n \in R_n(1212)$, we define $\tau(w) = \tau(w)_1 \tau(w)_2 \dots \tau(w)_n$ recursively according to the type of w (see Example 1). Set $\tau(\emptyset) = \emptyset$ and suppose τ is already defined for all $R_m(1212)$, $0 \leq m < n$.

- *Type I.* This means w must end with 1. Let \tilde{w} be the word obtained from w by subtracting each letter of w by 1 and then omitting all the 0s, and for any k with $w_k \neq 1$, define

$$\epsilon(k) := |\{j \geq k : w_j \neq 1\}|.$$

Then,

$$\tau(w)_i := \begin{cases} 1 & \text{if } w_{n+1-i} = 1, \\ \tau(\tilde{w})_{\epsilon(n+1-i)} + 1 & \text{if } w_{n+1-i} \neq 1. \end{cases}$$

For example, given $w = 1112322145541$, we have $\tilde{w} = 12113443$ and $\epsilon(4) = 8, \epsilon(5) = 7, \epsilon(9) = 4$, etc.

- *Type II.* Suppose $w = u \oplus v$, then let

$$\tau(w) = \tau(v) \oplus \tau(u).$$

When w is of type II, with the aid of Lemma 5 and 7, it is clear that $\tau(w)$ also avoids 1212 and is independent of the decomposition of w . But for those w of type I, we need the following lemma to see that τ is indeed well-defined.

Lemma 11. *For any $w \in R_n(1212)$:*

$$\text{if } \tau(w)_i = \tau(w)_j, \quad \text{then } w_{n+1-i} = w_{n+1-j}. \quad (3.3)$$

Consequently, $\tau(w) \in R_n(1212)$ and τ is well-defined.

Proof. We prove (3.3) by induction on n . It holds trivially for $n = 1$ since $\tau(1) = 1$. Now suppose $w \in R_n(1212)$ is of type I. If $\tau(w)_i = \tau(w)_j = 1$, then by definition $w_{n+1-i} = w_{n+1-j} = 1$. Otherwise $\tau(w)_i = \tau(w)_j = a > 1$, hence $\tau(\tilde{w})_{\epsilon(n+1-i)} = \tau(\tilde{w})_{\epsilon(n+1-j)} = a - 1$. Now since \tilde{w} is of shorter length than w , say $\tilde{w} \in R_m(1212)$, $m < n$, by inductive hypothesis we get $\tilde{w}_{m+1-\epsilon(n+1-i)} = \tilde{w}_{m+1-\epsilon(n+1-j)}$. A moment of reflection reveals that the $(m + 1 - \epsilon(n + 1 - i))$ -th letter in \tilde{w} is indeed the $(n + 1 - i)$ -th letter in w . We get $w_{n+1-i} = w_{n+1-j}$ as desired. The case with w being of type II is similar by induction, thus omitted.

Next we use (3.3) to show that $\tau(w) \in R_n(1212)$ for any w of type I. Again we use induction on n . Suppose on the contrary that $\tau(w) \notin R_n(1212)$, by induction we can assume such occurrence of pattern 1212 involves letter 1. More precisely, suppose there exists some indices $i < j < k$ such that

$$\tau(w)_1 = \tau(w)_j = 1, \quad \tau(w)_i = \tau(w)_k = a > 1.$$

Then by (3.3) we have

$$w_n = w_{n+1-j} = 1, \quad w_{n+1-i} = w_{n+1-k} = a' > 1.$$

Now the subword $w_1 w_{n+1-k} w_{n+1-j} w_{n+1-i}$ forms a pattern 1212 in w , a contradiction. This completes the proof. \square

The following result parallels Theorem 8 and proves Theorem 2.

Theorem 12. *The map τ is an involution on $R_n(1212)$ such that*

$$(\text{mak}, \text{rb}, \text{rs}, \text{bk})(w) = (\text{rb}, \text{mak}, \text{rs}, \text{bk})(\tau(w)) \quad (3.4)$$

for each $w \in R_n(1212)$.

Proof. That τ is an involution and $\text{bk}(\tau(w)) = \text{bk}(w)$ can be easily checked by definition and induction. Notice that τ also preserves the type of 1212-avoiding RGFs, so we deal with two types respectively.

Type II is simpler so we consider it first. Given any $w = u \oplus v \in R_n(1212)$, suppose $u \in R_m(1212)$, $1 \leq m < n$, then $v \in R_{n-m}(1212)$. We compute directly the remaining three statistics.

$$\begin{aligned}
\text{rb}(\tau(w)) &= \text{rb}(\tau(v) \oplus \tau(u)) = (n-m) \times \text{bk}(u) + \text{rb}(\tau(v)) + \text{rb}(\tau(u)) \\
&= (n-m) \times \text{bk}(u) + \text{mak}(v) + \text{mak}(u) \\
&= (\text{lb}(u) + \text{lb}(v)) + ((n-m) \times \text{bk}(u) + \text{lcs}(u) + \text{lcs}(v)) \\
&= \text{lb}(w) + \text{lcs}(w) = \text{mak}(w), \\
\text{mak}(\tau(w)) &= \text{mak}(\tau(v) \oplus \tau(u)) = \text{lb}(\tau(v) \oplus \tau(u)) + \text{lcs}(\tau(v) \oplus \tau(u)) \\
&= \text{lb}(\tau(v)) + \text{lb}(\tau(u)) + \text{lcs}(\tau(v)) + \text{lcs}(\tau(u)) + m \times \text{bk}(v) \\
&= \text{mak}(\tau(v)) + \text{mak}(\tau(u)) + m \times \text{bk}(v) \\
&= \text{rb}(v) + \text{rb}(u) + m \times \text{bk}(v) = \text{rb}(w), \\
\text{rs}(\tau(w)) &= \text{rs}(\tau(v) \oplus \tau(u)) = \text{rs}(\tau(v)) + \text{rs}(\tau(u)) = \text{rs}(v) + \text{rs}(u) = \text{rs}(w).
\end{aligned}$$

Type I is more involved. First note that since τ is an involution, we can strengthen (3.3) to the following form:

$$\tau(w)_i = \tau(w)_j \quad \text{if and only if} \quad w_{n+1-i} = w_{n+1-j}. \quad (3.5)$$

This is the key to the ensuing argument. Given any w of type I, for each a , $1 < a \leq \text{bk}(w)$, we consider the maximal subword of w formed purely by 1 and a , it must be of the following form since w avoids 1212

$$w^{(a)} = w_{i_1} \dots w_{i_r} w_{j_1} \dots w_{j_s} w_{k_1} \dots w_{k_t} = 1^r a^s 1^t,$$

where the sub-indices are the original ones in w . As a result of property (3.5), we see that

$$\begin{aligned}
\tau(w)^{(b)} &= \tau(w)_{n+1-k_t} \dots \tau(w)_{n+1-k_1} \tau(w)_{n+1-j_s} \dots \tau(w)_{n+1-j_1} \tau(w)_{n+1-i_r} \dots \tau(w)_{n+1-i_1} \\
&= 1^t b^s 1^r
\end{aligned}$$

must be the maximal subword of $\tau(w)$ composed of 1 and b only, for certain $1 < b \leq \text{bk}(w)$. And when a runs over $2, 3, \dots, \text{bk}(w)$, so does b . Now we analyse their contributions to each of the three statistics.

- $\text{mak} = \text{lb} + \text{lcs}$: $w^{(a)}$ adds t to lb , 0 to lcs , while $\tau(w)^{(b)}$ adds r to lb , 0 to lcs ;
- rb : $w^{(a)}$ adds r to rb , while $\tau(w)^{(b)}$ adds t to rb ;
- rs : both $w^{(a)}$ and $\tau(w)^{(b)}$ add s to rs .

We sum up all the contributions when a and b traverse $2, 3, \dots, \text{bk}(w)$ respectively, and use induction on \tilde{w} , this amounts to establish (3.4), and the proof ends here. \square

Example 1. For instance, if $w = 1112322145541$, then we need to first compute

$$\tau(\tilde{w}) = \tau(1211 \oplus 1221) = \tau(1221) \oplus \tau(1211) = 1221 \oplus 1121 = 12213343,$$

which in turn gives us $\tau(w) = 1233214454111$. And we check that

$$(\text{mak}, \text{rb}, \text{rs}, \text{bk})(w) = (17, 24, 11, 5) = (\text{rb}, \text{mak}, \text{rs}, \text{bk})(\tau(w)).$$

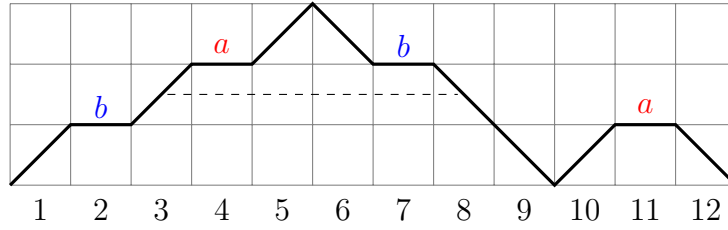


Figure 1: A two-colored Motzkin path in $\mathcal{M}_{12}^{(2)}$

4 The bijection ψ and a proof of Theorem 3

The Motzkin paths are at the heart of our proof of Theorem 3. Recall that a *Motzkin path* of length n is a lattice path in \mathbb{N}^2 starting at $(0, 0)$, ending at $(n, 0)$, with three possible steps:

$(1, 1) = U$ (up step), $(1, -1) = D$ (down step) and $(1, 0) = H$ (horizontal step).

For our purpose, we color each horizontal step of a Motzkin path by color *a* or *b*. We call such a Motzkin path a *two-colored Motzkin path*. See Fig. 1 for a display of the two-colored Motzkin path $U**b**U**a**U**D****b**DDU**a**D$. Denote by $\mathcal{M}_n^{(2)}$ the set of all two-colored Motzkin paths of length n . Note that $\mathcal{M}_{n-1}^{(2)}$ is enumerated by the famous n -th *Catalan number*

$$C_n = \frac{1}{n+1} \binom{2n}{n},$$

as well as $R_n(1212)$ and $\mathfrak{S}_n(321)$. There is a bijection from $\mathcal{M}_{n-1}^{(2)}$ to $R_n(1212)$ due to Campbell et al. [1] which forms one step of our bijection ψ that we now describe.

Let $M = s_1 s_2 \dots s_{n-1}$ be a two-colored Motzkin path in $\mathcal{M}_{n-1}^{(2)}$. Given a step s_i in M , we realize s_i geometrically as the line segment in the plane that connects two lattice points in the obvious way. If $s_i = U$ then we pair it with the first D -step s_j to its right, i.e. $j > i$, whose midpoint has the same height as the midpoint of s_i . Continuing with our running example path, s_3 is paired with s_8 (see the dashed line in Fig. 1). The associated RGF $\psi_1(M) = w = w_1 \dots w_n$ of M is then defined as follows. Let $w_1 = 1$ and

$$w_{i+1} = \begin{cases} 1 + \max\{w_1, \dots, w_i\} & \text{if } s_i = U \text{ or } s_i = \textcolor{blue}{b}, \\ w_i & \text{if } s_i = \textcolor{red}{a}, \\ w_j & \text{if } s_i = D \text{ is paired with the } U\text{-step } s_j. \end{cases}$$

For instance, for the two-colored Motzkin path M in Fig. 1, we have $\psi_1(M) = w = 1234454631771$ and $\text{rb}(w) = 28, \text{rs}(w) = 16, \text{bk}(w) = 7$. Campbell et al. [1, Th. 5.7] showed that $\psi_1 : \mathcal{M}_{n-1}^{(2)} \rightarrow R_n(1212)$ is a bijection that satisfies the property

$$\text{area}(M) = \text{rs}(w),$$

where $w = \psi_1(M)$ and $\text{area}(M)$ is the area between M and the x -axis. More precisely, they proved that $\text{rs}(w_i)$ equals $h(s_i)$ (resp. $h(s_i) + 1$) if $s_i = \textcolor{red}{a}, \textcolor{blue}{b}$ or U (resp. $s_i = D$). But what are the two statistics “rb” and “bk” on RGFs corresponding to?

To answer this question, we need to introduce a set-valued extension of “rb”, which is different from the multiset-valued statistic “RB” introduced earlier. For each RGF $w \in R_n(1212)$, a *shifted left-to-right maximum (position)* of w is an index $i \in [n - 1]$ such that $w_{i+1} > w_j$ for all $j < i + 1$. Let $\text{SLR}(w)$ be the set of all shifted left-to-right maximum of w . For example, $\text{SLR}(1234454631771) = \{1, 2, 3, 5, 7, 10\}$.

Lemma 13. *For $w \in R_n(1212)$, we have*

$$\text{bk}(w) = \#\text{SLR}(w) + 1 \quad \text{and} \quad \text{rb}(w) = \sum_{i \in \text{SLR}(w)} i.$$

Proof. For any integer $t \in \{2, 3, \dots, \text{bk}(w)\}$, from left to right, w_i is the first occurrence of letter t in w if and only if $i - 1$ is a shifted left-to-right maximum of w . Thus, the first equality follows.

Note that if $w_k = w_{k'}$ for some $k < k'$, then all the letters in between positions k and k' should be no less than w_k . Otherwise, say $w_l < w_k$ with $k < l < k'$, then there must exist a letter $w_{l'} = w_l$ that occurs before w_k , but this means $w_{l'}w_kw_lw_{k'}$ is a 1212 pattern, a contradiction. Therefore, for any fixed letter $t \in \{2, 3, \dots, \text{bk}(w)\}$, only the first occurrence of t in w could contribute one to $\text{rb}(w_i)$ for each letter w_i ($w_i < t$) to the left of it. The second equality then follows. \square

Let us introduce the statistic $\text{Ub}(M)$ for a two-colored Motzkin path $M \in \mathcal{M}_{n-1}^{(2)}$ as

$$\text{Ub}(M) := \{i \in [n - 1] : s_i = U \text{ or } s_i = \textcolor{blue}{b}\},$$

with its cardinality denoted $\text{ub}(M) := \#\text{Ub}(M)$. It is clear from the definition of ψ_1 that $\text{Ub}(M) = \text{SLR}(\psi_1(M))$. Hence,

Lemma 14. *For each $M \in \mathcal{M}_{n-1}^{(2)}$, we have*

$$\text{Ub}(M) = \text{SLR}(\psi_1(M)) \quad \text{and} \quad \text{area}(M) = \text{rs}(\psi_1(M)).$$

Next, we will introduce a bijection $\psi_2 : \mathfrak{S}_n(321) \rightarrow \mathcal{M}_{n-1}^{(2)}$ which transforms the pair (EXC, inv – exc) to (Ub, area). Our bijection ψ_2 is inspired by a bijection of Foata and Zeilberger [4] from permutations to *Laguerre histories* and a bijection of Cheng et al. [2] from 321-avoiding permutations to two-colored Motzkin paths.

We need two important vectors to keep track of the values and positions of excedances. For $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n(321)$, let

$$\mathbf{val}(\pi) = (v_1, \dots, v_n) \quad \text{and} \quad \mathbf{pos}(\pi) = (p_1, \dots, p_n),$$

where $v_i = \chi(i > \pi_i^{-1})$ and $p_i = \chi(\pi_i > i)$. For instance, for $\pi = 3\,7\,8\,1\,9\,2\,10\,4\,5\,13\,6\,11\,12$ in $\mathfrak{S}_{13}(321)$, we have

$$\mathbf{val}(\pi) = (0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1), \quad \mathbf{pos}(\pi) = (1, 1, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0).$$

Now, the associated two-colored Motzkin path $\psi_2(\pi) = M = s_1 s_2 \dots s_{n-1}$ is defined by

$$s_i = \begin{cases} U & \text{if } v_{i+1} = 0 \text{ and } p_i = 1, \\ \textcolor{blue}{b} & \text{if } v_{i+1} = p_i = 1, \\ D & \text{if } v_{i+1} = 1 \text{ and } p_i = 0, \\ \textcolor{red}{a} & \text{if } v_{i+1} = p_i = 0. \end{cases} \quad (4.1)$$

Continuing with our example, $\psi_2(37819210451361112) = U\textcolor{blue}{b}U\textcolor{red}{a}UD\textcolor{blue}{b}DDU\textcolor{red}{a}D$, which is the path in Fig. 1.

We first need to show that ψ_2 is well defined, that is the path M as produced above ends on x -axis and stays weakly above it during the process. In other words, we require the number of U 's in any prefix of M is at least as great as the number of D 's and with equality for M in its entirety. This follows from the simple fact that for any fixed $i \in [n-1]$,

$$|\{j \in \text{EXC}(\pi) : j \leq i\}| \geq |\{j \in \text{EXC}(\pi) : \pi_j \leq i+1\}|,$$

and with equality when $i = n-1$.

For each $i \in \text{EXC}(\pi)$ (resp. $i \in \text{NEXC}(\pi)$), we call π_i the *excedance* (resp. *non-excedance*) value of π . We will also need the following characterization of 321-avoiding permutations, which is folkloric in pattern avoidance.

Lemma 15. *A permutation is 321-avoiding if and only if both the subsequence formed by its excedance values and the one formed by the remaining non-excedance values are increasing. Equivalently, π is 321-avoiding if and only if $\text{inv}(\text{EXC}^*(\pi)) = \text{inv}(\text{NEXC}^*(\pi)) = 0$.*

We are ready for the key lemma of this section.

Lemma 16. *The mapping $\psi_2 : \mathfrak{S}_n(321) \rightarrow \mathcal{M}_{n-1}^{(2)}$ is a bijection such that for each $\pi \in \mathfrak{S}_n(321)$, we have*

$$\text{EXC}(\pi) = U\textcolor{blue}{b}(\psi_2(\pi)) \quad \text{and} \quad (\text{inv} - \text{exc})(\pi) = \text{area}(\psi_2(\pi)). \quad (4.2)$$

Proof. Given a two-colored Motzkin path M that is an image under ψ_2 , we have the information for the positions and values of all the excedances of π such that $\psi_2(\pi) = M$. Hence, by Lemma 15 such a π must be unique. This shows that ψ_2 is injective and thus bijective, since the cardinalities of $\mathcal{M}_{n-1}^{(2)}$ and $\mathfrak{S}_n(321)$ are both the Catalan number C_n .

The first equality in (4.2) is clear from the definition of ψ_2 and we will prove the second one by calculating $(\text{inv} - \text{exc})(\pi)$ via the two associated vectors $\mathbf{val}(\pi)$ and $\mathbf{pos}(\pi)$. Indeed, using the characterization given in Lemma 15, we see that for any $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n(321)$, an excedance of π must also be a left-to-right maximum of π . Consequently, an excedance, say π_i , contributes $\pi_i - i$ to the value of $\text{inv}(\pi)$, while each non-excedance contributes nothing. This observation leads to the following computation of $\text{inv}(\pi)$ and $\text{exc}(\pi)$:

$$\begin{aligned} \text{inv}(\pi) &= \langle \mathbf{val}(\pi), \mathbf{n} \rangle - \langle \mathbf{pos}(\pi), \mathbf{n} \rangle, & \mathbf{n} &= (1, 2, \dots, n), \\ \text{exc}(\pi) &= \langle \mathbf{val}(\pi), \mathbf{e} \rangle = \langle \mathbf{pos}(\pi), \mathbf{e} \rangle, & \mathbf{e} &= (1, 1, \dots, 1), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual *inner product* for vectors in \mathbb{R}^n . Note that by definition, $\mathbf{val}(\pi)$ (resp. $\mathbf{pos}(\pi)$) always begins (resp. ends) with 0. Now suppose $\psi_2(\pi) = M = s_1 s_2 \dots s_{n-1}$, then using (4.1) and a little bit of linear algebra, we get

$$(\text{inv} - \text{exc})(\pi) = \sum_{s_i=D} i - \sum_{s_j=U} j = \text{area}(M),$$

where the last equality is due to the fact that all U -steps and D -steps are paired up, and the distance from U to D in any given pair is exactly the amount they contributes to the value of $\text{area}(M)$. The proof of the lemma is complete. \square

Combining Lemma 14 and Lemma 16, we arrive at the following result.

Theorem 17. *The composition $\psi = \psi_1 \circ \psi_2 : \mathfrak{S}_n(321) \rightarrow R_n(1212)$ is a bijection such that for each $\pi \in \mathfrak{S}_n(321)$, we have*

$$\text{EXC}(\pi) = \text{SLR}(\psi(\pi)) \quad \text{and} \quad \text{inv}(\pi) - \text{exc}(\pi) = \text{rs}(\psi(\pi)).$$

In view of Lemmas 13 and 15, Theorem 17 implies Theorem 3.

5 Further applications of ψ

5.1 Proof of Theorem 4

We will apply a simple group action on two-colored Motzkin paths which results in a proof of Theorem 4. Let $x \in [n-1]$ and $M = s_1 \dots s_{n-1} \in \mathcal{M}_{n-1}^{(2)}$. Define the action Θ_x on M by

$$\Theta_x(M) = \begin{cases} M & \text{if } s_x = U \text{ or } s_x = D, \\ s_1 \dots s_{x-1} \textcolor{blue}{b} s_{x+1} \dots s_{n-1} & \text{if } s_x = \textcolor{red}{a}, \\ s_1 \dots s_{x-1} \textcolor{red}{a} s_{x+1} \dots s_{n-1} & \text{if } s_x = \textcolor{blue}{b}. \end{cases}$$

In other words, Θ_x changes the color of the x -th step of M . Note that all actions commute with each other, i.e., $\Theta_y \circ \Theta_x(M) = \Theta_x \circ \Theta_y(M)$ for any $x, y \in [n-1]$. Moreover, each Θ_x is an involution on $\mathcal{M}_{n-1}^{(2)}$. For any subset $S \subseteq [n-1]$, we then define the function $\Theta_S : \mathcal{M}_{n-1}^{(2)} \rightarrow \mathcal{M}_{n-1}^{(2)}$ by $\Theta_S = \prod_{x \in S} \Theta_x$, the function compositions of all Θ_x with $x \in S$. Hence, the group \mathbb{Z}_2^{n-1} acts on $M \in \mathcal{M}_{n-1}^{(2)}$ via the function Θ_S ($S \subseteq [n-1]$). For example, $\Theta_{\{1,2,4,7\}}(U \textcolor{blue}{b} U \textcolor{red}{a} U D \textcolor{blue}{b} D D U \textcolor{red}{a} D) = U \textcolor{red}{a} U \textcolor{blue}{b} U D \textcolor{red}{a} D D U \textcolor{red}{a} D$.

This \mathbb{Z}_2^{n-1} -action divides the set $\mathcal{M}_{n-1}^{(2)}$ into disjoint orbits and each orbit has exactly one two-colored Motzkin path whose H -steps, if any, all have color $\textcolor{red}{a}$. For example, under the \mathbb{Z}_2^2 -action, the disjoint orbits of $\mathcal{M}_2^{(2)}$ are $\{UD\}$ and $\{\textcolor{red}{a}\textcolor{red}{a}, \textcolor{red}{a}\textcolor{blue}{b}, \textcolor{blue}{b}\textcolor{red}{a}, \textcolor{blue}{b}\textcolor{blue}{b}\}$ with the representatives being UD and $\textcolor{red}{a}\textcolor{red}{a}$, respectively. Note that this \mathbb{Z}_2^{n-1} -action preserves the area of any two-colored Motzkin path. Therefore, if we choose the set of representatives of all the disjoint orbits under the \mathbb{Z}_2^{n-1} -action on $\mathcal{M}_{n-1}^{(2)}$ as

$$\mathcal{O}_{n-1} := \{M \in \mathcal{M}_{n-1}^{(2)} : M \text{ has no } \textcolor{blue}{b}\text{-step}\},$$

then we have

Lemma 18. For $n \geq 1$, we have

$$\sum_{M \in \mathcal{M}_{n-1}^{(2)}} t^{\text{ub}(M)} q^{\text{area}(M)} = \sum_{M \in \mathcal{O}_{n-1}} q^{\text{area}(M)} t^{\text{ub}(M)} (1+t)^{n-1-2\text{ub}(M)}. \quad (5.1)$$

Since $M \in \mathcal{M}_{n-1}^{(2)}$ has no \mathbf{b} -step if and only if $\psi_2^{-1}(M) = \pi_1 \dots \pi_n$ has no exceedance, say at i , such that $\pi_{i+1}^{-1} \leq i$, applying ψ_2^{-1} to both sides of (5.1) then gives the q - γ -positivity expansion in Theorem 4.

Parallel to Theorem 4, there is also a similar expansion regarding 1212-avoiding RGFs. Let

$$\widetilde{R}_n(1212) := \{w \in R_n(1212) : \text{whenever } w_i < w_{i+1}, \text{ there exists } j > i+1 \text{ s.t. } w_j = w_i\}.$$

Applying ψ_1 to both sides of (5.1) we get

Theorem 19. Let $\widetilde{R}_{n,k}(1212) := \{w \in \widetilde{R}_n(1212) : \text{bk}(w) = k-1\}$. Then,

$$\sum_{w \in R_n(1212)} t^{\text{bk}(w)} q^{\text{rs}(w)} = t \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left(\sum_{\pi \in \widetilde{R}_{n,k}(1212)} q^{\text{rs}(w)} \right) t^k (1+t)^{n-1-2k}.$$

5.2 A recurrence for $N_n(t, p, q)$

It is known that the Catalan numbers C_n satisfy the recurrence relation

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^n C_k C_{n-k}.$$

Regarding our (p, q) -Narayana polynomials $N_n(t, p, q)$, we have the following *Catalan-like recursion*, which can be derived easily from Lemma 16 by using the usual decomposition of two-colored Motzkin paths.

Theorem 20. The (p, q) -Narayana polynomials satisfy the recurrence formula:

$$N_{n+1}(t, p, q) = (1+tp)N_n(tp, p, q) + tp \sum_{k=1}^{n-1} q^k N_k(tp, p, q) N_{n-k}(tp^{k+1}, p, q) \quad (5.2)$$

for $n \geq 0$ and $N_0(t, p, q) = 1$.

Proof. It follows from Lemma 16 that (for $n \geq 0$)

$$N_{n+1}(t, p, q) = \sum_{M \in \mathcal{M}_n^{(2)}} t^{\text{ub}(M)} p^{\sum_{i \in \mathbf{U}\mathbf{b}(M)} i} q^{\text{area}(M)}. \quad (5.3)$$

Now for each $M \in \mathcal{M}_n^{(2)}$, we distinguish three cases according to the first step of M :

- It is an U -step, then M can be decomposed uniquely as $M = UM_1DM_2$ with $M_1 \in \mathcal{M}_{k-1}^{(2)}$ and $M_2 \in \mathcal{M}_{n-k-1}^{(2)}$ for some $1 \leq k \leq n-1$. In this case, we have $\text{ub}(M) = 1 + \text{ub}(M_1) + \text{ub}(M_2)$, $\sum_{i \in \text{Ub}(M)} i = 1 + \sum_{i \in \text{Ub}(M_1)} (i+1) + \sum_{i \in \text{Ub}(M_2)} (k+1+i)$ and $\text{area}(M) = k + \text{area}(M_1) + \text{area}(M_2)$.
- It is a b -step, then $M = bM'$ with $M' \in \mathcal{M}_{n-1}^{(2)}$. In this case, we have $\text{ub}(M) = 1 + \text{ub}(M')$, $\sum_{i \in \text{Ub}(M)} i = 1 + \sum_{i \in \text{Ub}(M')} (i+1)$ and $\text{area}(M) = \text{area}(M')$.
- It is an a -step, then $M = aM''$ with $M'' \in \mathcal{M}_{n-1}^{(2)}$. In this case, we have $\text{ub}(M) = \text{ub}(M'')$, $\sum_{i \in \text{Ub}(M)} i = \sum_{i \in \text{Ub}(M'')} (i+1)$ and $\text{area}(M) = \text{area}(M'')$.

Summarizing all the above three cases then gives (5.2). \square

This recursion is a new generalization of [2, Theorem 1.1]. For the reader's convenience, we list the first few values of $N_n(t, p, q)$ in the following:

$$\begin{aligned} N_1(t, p, q) &= 1; N_2(t, p, q) = 1 + pt; N_3(t, p, q) = 1 + (p + p^2 + pq)t + p^3t^2; \\ N_4(t, p, q) &= 1 + (p + p^2 + pq + p^3 + p^2q + pq^2)t(1 + tp^2) + p^6t^3; \\ N_5(t, p, q) &= 1 + (p + p^2 + pq + p^3 + p^2q + pq^2 + p^4 + p^3q + p^2q^2 + pq^3)t(1 + t^2p^5) \\ &\quad + (p^3 + p^4 + p^3q + 2p^5 + 2p^4q + 2p^3q^2 + p^6 + 2p^5q + p^4q^2 + p^3q^3 \\ &\quad + p^7 + p^6q + 2p^5q^2 + p^4q^3 + p^3q^4)t^2 + p^{10}t^4. \end{aligned}$$

Let Γ_+^d be the set of all polynomials in $\mathbb{N}[t, q]$ that have coefficients in $\mathbb{N}[q]$ when expanded in $\{t^k(1+t)^{d-2k}\}_{k=0}^{\lfloor d/2 \rfloor}$. Expansion (1.3) shows that $N_n(t, 1, q) \in \Gamma_+^{n-1}$, a fact which was already known in [8, Corollary 2.1]. We note that this fact also follows immediately from Theorem 20. Actually, setting $p = 1$ in (5.2) we get

$$N_{n+1}(t, 1, q) = (1+t)N_n(t, 1, q) + t \sum_{k=1}^{n-1} q^k N_k(t, 1, q) N_{n-k}(t, 1, q).$$

Since $\Gamma_+^n \cdot \Gamma_+^m := \{fg : f \in \Gamma_+^n \text{ and } g \in \Gamma_+^m\} \subseteq \Gamma_+^{n+m}$, we see $N_n(t, 1, q) \in \Gamma_+^{n-1}$ from the above recursion by induction on n .

5.3 Continued fraction expression for $N_n(t, 1, q)$

Let \mathcal{M}_n be the set of all Motzkin paths of length n . For a Motzkin path $M = s_1 s_2 \dots s_n \in \mathcal{M}_n$, we define the height $h(s_i)$ of the step s_i to be the y -coordinate of the starting point of s_i . According to the combinatorial theory of continued fractions developed by Flajolet [3], if we weight each up (resp. down, horizontal) step of a Motzkin path M with height h by u_h (resp. d_h and l_h) and define the weight of M as the product of its step weights, denoted by $w(M)$, then

$$1 + \sum_{n \geq 1} \sum_{M \in \mathcal{M}_n} w(M) x^n = \frac{1}{1 - l_0 x - \frac{u_0 d_1 x^2}{1 - l_1 x - \frac{u_1 d_2 x^2}{\dots}}}. \quad (5.4)$$

The following continued fraction expansion for $N_n(t, 1, q)$ is a direct consequence of Flajolet's result and our bijection ψ_2 .

Theorem 21. *The ordinary generating function $N(t, q; x) := \sum_{n \geq 1} N_n(t, 1, q)x^n$ has the continued fraction expansion*

$$N(t, q; x) = \frac{x}{1 - (1+t)x - \frac{tqx^2}{1 - (1+t)qx - \frac{tq^3x^2}{1 - (1+t)q^2x - \frac{tq^5x^2}{\dots}}}}.$$

Proof. By setting the weights $l_h = (1+t)q^h$, $u_h = tq^h$ and $d_h = q^h$ in (5.4) and using the interpretation of $N_n(t, 1, q)$ in (5.3), we get the desired continued fraction expansion. \square

6 Final remarks

Wachs and White [16] proved that “rb” and “ls” are equidistributed on the whole set R_n . In view of Theorem 1, one may wonder if the stronger property that the two pairs (rb, ls) and (ls, rb) are equidistributed on R_n would hold. This is not true, the first counterexample occurs when $n = 6$.

Let $(q; q)_n := \prod_{i=1}^n (1 - q^i)$ and $[n]_q := 1 + q + \dots + q^{n-1}$. Recall that the q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_{n-k}(q; q)_k}$ for $0 \leq k \leq n$. The Förlinger-Hofbauer q -Narayana polynomials [5] that we denote as $C_n(t, q)$ are defined by

$$C_n(t, q) := \sum_{k=0}^{n-1} q^{k(k+1)} \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q t^k.$$

This q -analog of Narayana polynomials is different from $N_n(t, 1, q)$ and has already been widely studied in the literature (cf. [2, 5, 14, 17]). A natural question that one may ask is if there are interpretations of $C_n(t, q)$ in terms of pattern avoiding permutations or RGFs that are similar to Theorem 3. This question is answered by works of Förlinger-Hofbauer [5], Stump [14, Section 3] and Zhao-Zhong [17, Theorem 4.2] that we organize as a result in the following.

Theorem 22 (See [5, 14, 17]). *The following equidistribution holds:*

$$C_n(t, q) = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des}(\pi)} q^{\text{maj}(\pi) + \text{maj}(\pi^{-1})} = \sum_{w \in R_n(1212)} t^{\text{bk}(w)-1} q^{\text{ls}(w) - \text{rb}(w) + n(\text{bk}(w)-1)},$$

where $\mathfrak{S}_n(231)$ is the set of 231-avoiding permutations in \mathfrak{S}_n , $\text{des}(\pi) := \sum_{\pi_i > \pi_{i+1}} 1$ and $\text{maj}(\pi) := \sum_{\pi_i > \pi_{i+1}} i$ for each permutation π .

Recently, Dilks, Krattenthaler and Wachs [15] proved via q -hypergeometric techniques the following q - γ -positivity expansion of $C_n(t, q)$:

$$C_n(t, q) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(q) t^k q^{k(k+1)} \prod_{i=k+1}^{n-k-1} (1 + tq^{2i}),$$

where $\gamma_{n,k}(q) = q^k \begin{bmatrix} n-1 \\ 2k \end{bmatrix}_q C_k(q^2)$ with $C_k(q) := \frac{1}{[k+1]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q = C_k(1, q)$. In view of Theorems 19 and 22, we would like to pose the following open problem for further investigation.

Problem 23. *Is there a combinatorial interpretation for the γ -coefficients $\gamma_{n,k}(q)$ in terms of 231-avoiding permutations or 1212-avoiding RGFs? Or more precisely, can one find certain statistic, say “st”, on $\widetilde{R}_n(1212)$ so that $\gamma_{n,k}(q) = \sum_{w \in \widetilde{R}_n(1212)} q^{\text{st}(w)}$?*

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