A square root map on Sturmian words

Jarkko Peltomäki

Turku Centre for Computer Science TUCS & Department of Mathematics and Statistics
University of Turku
Turku, Finland

`r@turambar.org`

Markus A. Whiteland

Department of Mathematics and Statistics
University of Turku
Turku, Finland

`mawhit@utu.fi`

Abstract

We introduce a square root map on Sturmian words and study its properties. Given a Sturmian word of slope α, there exists exactly six minimal squares in its language (a minimal square does not have a square as a proper prefix). A Sturmian word s of slope α can be written as a product of these six minimal squares: $s = X_1^2 X_2^2 X_3^2 \cdots$. The square root of s is defined to be the word $\sqrt{s} = X_1 X_2 X_3 \cdots$. The main result of this paper is that $\sqrt{s}$ is also a Sturmian word of slope α. Further, we characterize the Sturmian fixed points of the square root map, and we describe how to find the intercept of $\sqrt{s}$ and an occurrence of any prefix of $\sqrt{s}$ in s. Related to the square root map, we characterize the solutions of the word equation $X_1^2 X_2^2 \cdots X_n^2 = (X_1 X_2 \cdots X_n)^2$ in the language of Sturmian words of slope α where the words X_i^2 are minimal squares of slope α.

We also study the square root map in a more general setting. We explicitly construct an infinite set of non-Sturmian fixed points of the square root map. We show that the subshifts Ω generated by these words have a curious property: for all $w \in \Omega$ either $\sqrt{w} \in \Omega$ or $\sqrt{w}$ is periodic. In particular, the square root map can map an aperiodic word to a periodic word.

Keywords: Sturmian word; standard word; optimal squareful word; word equation; continued fraction

1   Introduction

Kalle Saari studies in [17, 18] optimal squareful words which are aperiodic words containing the least number of minimal squares (that is, squares with no proper square prefixes) such that every position starts a square. Saari proves that an optimal squareful word always contains exactly six minimal squares, and he characterizes these squares; less than six minimal squares forces a word to be ultimately periodic. Moreover, he shows that Sturmian words are a proper subclass of optimal squareful words.

We propose a square root map for Sturmian words. Let s be a Sturmian word of slope α, and write it as a product of the six minimal squares in its language $\mathcal{L}(\alpha)$: $s = X_1^2 X_2^2 X_3^2 \cdots$. The square root of s is defined to be the word $\sqrt{s} = X_1 X_2 X_3 \cdots$. The main result of this paper is that the word $\sqrt{s}$ is also a Sturmian word of slope α. More precisely, we prove that the square root of the Sturmian word $s_{x,\alpha}$ of intercept x and slope α is $s_{\psi(x),\alpha}$ where $\psi(x) = \frac{1}{2}(x + 1 - \alpha)$. In addition to proving that the square root map preserves the language of a Sturmian word s, we show how to locate any prefix of $\sqrt{s}$ in s. We also characterize the Sturmian words of slope α which are fixed points of the square root map; they are the two Sturmian words $01c_\alpha$ and $10c_\alpha$ where c_α is the infinite standard Sturmian word of slope α. The majority of the proofs of results on Sturmian words rely heavily on the interpretation of Sturmian words as rotation words. Continued fractions and results from Diophantine approximation theory play a key role in several proofs.

Solutions of the word equation $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$ where the words X_i^2 are among the six minimal squares in $\mathcal{L}(\alpha)$ for some fixed irrational α are closely linked to the square root map. The study of these solutions to this word equation arises naturally from the study of fixed points of the square root map. The Sturmian fixed points of the square root map are fixed because they have arbitrarily long prefixes $X_1^2 \cdots X_n^2$ which satisfy the word equation. We characterize these specific solutions, i.e., those primitive words w such that $w^2 \in \mathcal{L}(\alpha)$ and w^2 can be written as a product of minimal squares $X_1^2 \cdots X_n^2$ satisfying the word equation. On the circle $[0, 1)$, the interval $[w]$ of such a word w can be seen to satisfy the square root condition $\psi([w^2]) \subseteq [w]$, so we instead study and characterize the primitive words satisfying this square root condition. The result is that the specific solutions to the word equation (or, equivalently, the primitive words satisfying the square root condition) are the reversals of standard and semistandard words of slope α (see Subsection 2.3 for a definition) and the reversed standard words with the first two letters exchanged. In particular, all of these specific solutions are nonperiodic. It was known that the word equation $(X_1^2 \cdots X_n^2) = X_1^2 \cdots X_n^2$ has nonperiodic solutions [7], but according to our knowledge no large families of nonperiodic solutions have been identified until our result. Word equations of the type $X_1^k \cdots X_n^k = (X_1 \cdots X_n)^k$ have been considered by Štěpán Holub [6, 7, 8].

The final central topic of this paper concerns the square root map in a more general setting. The square root map can be defined not only for Sturmian words but for any optimal squareful word. We construct an infinite family of non-Sturmian, linearly recurrent optimal squareful words Γ with properties similar to Sturmian words. The words Γ are

fixed points of the square root map. They are constructed by finding non-Sturmian solutions of the word equation $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$ and by building infinite words having arbitrarily long squares of such solutions as prefixes. The subshifts Ω generated by the words Γ exhibit behavior similar to Sturmian subshifts. The square root map preserves the language of several but not every word in Ω. Curiously, if the language of a word in Ω is not preserved under the square root map, then the image must be periodic. This result is very surprising since it is contrary to the plausible hypothesis that the square root of an aperiodic word is aperiodic.

The paper is organized as follows. In Section 3 we prove that the square root map preserves the language of a Sturmian word. As a corollary we obtain a description of those Sturmian words which are fixed points of the square root map. In Section 3 we observe that the intervals of the minimal squares in $\mathcal{L}(\alpha)$ satisfy the square root condition. In Section 4 we characterize all words $w^2 \in \mathcal{L}(\alpha)$ satisfying the square root condition. The result is that w^2 with w primitive satisfies the square root condition if and only if w is a reversed standard or semistandard word or a reversed standard word with the first two letters exchanged. Section 5 contains a proof of the characterization of the specific solutions of the word equation $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$ mentioned earlier. We show that a primitive word w satisfies the square root condition if and only if w^2 can be written as a product of minimal squares satisfying the word equation. In Section 6 we show how to locate prefixes of $\sqrt{s}$ in s. As an important step in proving this, we provide necessary and sufficient conditions for a Sturmian word to be a product of squares of reversed standard and semistandard words. We give a formula describing the square root of the Fibonacci word in Section 7. Section 8 is devoted to constructing the non-Sturmian fixed points Γ mentioned above and to demonstrating that the languages of the words in their subshifts are preserved or they are mapped to periodic words. We conclude the paper by giving some remarks on possible generalizations in Section 9 and by discussing a few open problems in Section 10.

A short version of this paper was published as an extended abstract in the proceedings of WORDS 2015 [14]. This work is also a part of the first author's Ph.D. dissertation [13].

2   Notation and Preliminary Results

In this section we review notation and basic concepts and results of word combinatorics, optimal squareful words, continued fractions, and Sturmian words. Most of the definitions and results provided here about words can be found in Lothaire's book [11].

An alphabet A is a finite non-empty set of *letters*, or *symbols*. A (finite) *word* over A is a finite sequence of letters of A obtained by concatenation. The concatenation of two words $u = a_0 \cdots a_{n-1}$ and $v = b_0 \cdots b_{m-1}$ is the word $u \cdot v = uv = a_0 \cdots a_{n-1}b_0 \cdots b_{m-1}$. In this paper we consider only *binary words*, that is, words over an alphabet of size two. Most of the time we take A to be the set $\{0, 1\}$. *The set of nonempty words over A is* denoted by A^+. We denote *the empty word* by ε and set $A^* = A^+ \cup \{\varepsilon\}$. A nonempty subset of A^* is called a *language*. Let $w = a_0 a_1 \cdots a_{n-1}$ be a word of n letters. We denote *the length n of w* by $|w|$; by convention $|\varepsilon| = 0$. The set of proper powers of a word w is

denoted by w^+.

An *infinite word* w over the alphabet A is a function from the nonnegative integers to A. We write concisely $w = a_0 a_1 a_2 \cdots$ with $a_i \in A$. The set of infinite words over A is denoted by A^ω. An infinite word w is said to be *ultimately periodic* if we can write it in the form $w = uv^\omega = uvvv\cdots$ for some words $u, v \in A^*$. If $u = \varepsilon$, then w is said to be *periodic*, or *purely periodic*. An infinite word which is not ultimately periodic is *aperiodic*. The *shift operator* T acts on infinite words as follows: $T(a_0 a_1 a_2 \ldots) = a_1 a_2 \cdots$.

A finite word u is a *factor* of the finite or infinite word w if we can write $w = vuz$ for some $v \in A^*$ and $z \in A^* \cup A^\omega$. If $v = \varepsilon$, then the factor u is called a *prefix* of w. If $z = \varepsilon$, then we say that u is a *suffix* of w. The set of factors of w, the *language of w*, is denoted by $\mathcal{L}(w)$. If $w = a_0 a_1 \cdots a_{n-1}$, then we let $w[i, j] = a_i \cdots a_j$ whenever the choices of positions i and j make sense. This notion is extended to infinite words in a natural way. An *occurrence* of u in w is a position i such that $w[i, i + |u| - 1] = u$. If such a position exists, then we say that u *occurs* in w.

A positive integer p is a *period* of $w = a_0 \cdots a_{n-1}$ if $a_i = a_{i+p}$ for $0 \leqslant i \leqslant n - p - 1$. If the finite word w has period p and $|w|/p \geqslant \alpha$ for some real α such that $\alpha \geqslant 1$, then w is called an *α-repetition*. An α-repetition is *minimal* if it does not have an α-repetition as a proper prefix. If $w = u^2$, then w is a *square* with *square root* u. A square is *minimal* if it does not have a square as a proper prefix. A word w is *primitive* if it is of the form z^n if and only if $n = 1$. Equivalently, a word w is primitive if and only if w occurs in w^2 exactly twice. The *primitive root of w* is the unique primitive word u such that $w = u^n$ for some $n \geqslant 1$. Let $w = v^\omega$ be a periodic infinite word. The *minimal period of w* is defined to be the primitive root of v.

Let $w = a_0 a_1 \cdots a_{n-1}$ be a word. *The reversal $\widetilde{w}$ of w* is the word $a_{n-1} \cdots a_1 a_0$. If $w = \widetilde{w}$, then we call w a *palindrome*. Let C be the *cyclic shift operator* defined by the formula $C(a_0 a_1 \cdots a_{n-1}) = a_1 \cdots a_{n-1} a_0$. The words $w, C(w), C^2(w), \ldots, C^{|w|-1}(w)$ are the *conjugates of w*. If u is a conjugate of w, then we say that u is *conjugate to w*.

An infinite word w is *recurrent* if each of its factors occurs in it infinitely often. Let $(i_n)_{n \geqslant 1}$ be the sequence of consecutive occurrences of a factor u in a recurrent word w. The *return time* of u is the quantity

$$\sup\{i_{j+1} - i_j \colon j \in \{1, 2, \ldots\}\},$$

which can be infinite. The factors $w[i_j, i_{j+1} - 1]$, $j \geqslant 1$ are the *returns to u in w*. If the return time of each factor of w is finite, then the word w is *uniformly recurrent*. Equivalently, w is uniformly recurrent if for each factor u of w there exists an integer R such that every factor of w of length R contains an occurrence of u. If there exists a global constant K such that the return time of any factor u of w is at most $K|u|$, then we say that w is *linearly recurrent*. Clearly a linearly recurrent word is uniformly recurrent. The *index* of a factor u of an infinite word w is defined to be

$$\sup\{n \colon u^n \in \mathcal{L}(w)\}.$$

If w is uniformly recurrent and aperiodic, then the index of every factor of w is finite.

A *subshift* Ω is a subset of A^ω such that

$$\Omega = \{w \in A^\omega \colon \mathcal{L}(w) \subseteq \mathcal{L}\}$$

for some language $\mathcal{L}$ such that $\mathcal{L} \subseteq A^*$. If we set above $\mathcal{L} = \mathcal{L}(w)$ where w is an infinite word, then we say that the subshift Ω is generated by w. Subshifts are clearly shift-invariant. If every word in a subshift is aperiodic, then we call the subshift *aperiodic*. A subshift is *minimal* if it does not contain nonempty subshifts as proper subsets. A nonempty subshift is minimal if and only if it is generated by a uniformly recurrent word.

2.1 Optimal Squareful Words

In [18] Kalle Saari considers α-repetitive words. An infinite word is α-*repetitive* if every position in the word starts an α-repetition and the number of distinct minimal α-repetitions occurring in the word is finite. If $\alpha = 2$, then α-repetitive words are called *squareful words*. This means that every position of a squareful word begins with a minimal square. Saari proves that if the number of distinct minimal squares occurring in a squareful word is at most 5, then the word must be ultimately periodic. On the other hand, if a squareful word contains at least 6 distinct minimal squares, then aperiodicity is possible. Saari calls the aperiodic squareful words containing exactly 6 minimal squares *optimal squareful words*. Further, he shows that optimal squareful words are always binary and that the six minimal squares must take a very specific form:

Proposition 1. *Let w be an optimal squareful word. If $10^i 1$ occurs in w for some $i > 1$, then the roots of the six minimal squares in w are*

$$
\begin{aligned}
&S_1 = 0, &&S_4 = 10^{\mathfrak{a}}, \\
&S_2 = 010^{\mathfrak{a}-1}, &&S_5 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}}, \\
&S_3 = 010^{\mathfrak{a}}, &&S_6 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1},
\end{aligned}
\tag{1}
$$

for some $\mathfrak{a} \geqslant 1$ and $\mathfrak{b} \geqslant 0$.

The optimal squareful words containing the minimal square roots of (1) are called *optimal squareful words with parameters $\mathfrak{a}$ and $\mathfrak{b}$*. For the rest of this paper we reserve this meaning for the symbols $\mathfrak{a}$ and $\mathfrak{b}$. Furthermore, we agree that the symbols S_i always refer to the minimal square roots (1).

Saari completely characterizes optimal squareful words [18, Theorem 17].

Proposition 2. *An aperiodic infinite word w is optimal squareful if and only if (up to renaming of letters) there exists integers $\mathfrak{a} \geqslant 1$ and $\mathfrak{b} \geqslant 0$ such that w is an element of the language*

$$0^*(10^{\mathfrak{a}})^*(10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}} + 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1})^\omega = S_1^* S_4^* (S_5 + S_6)^\omega.$$

2.2 Continued Fractions and Rational Approximations

In this section we review results on continued fractions and best rational approximations of irrational numbers needed in this paper. Good references on these subjects are the books of Khinchin [9] and Cassels [2].

Every irrational real number α has a unique infinite continued fraction expansion

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \tag{2}$$

with $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ for all $k \geqslant 1$. The numbers a_i are called *the partial quotients* of α. We focus here only on irrational numbers, but we note that with small tweaks much of what follows also holds for rational numbers, which have finite continued fraction expansions.

The *convergents* $c_k = \frac{p_k}{q_k}$ of α are defined by the recurrences

$$\begin{aligned} p_0 = a_0, \qquad p_1 = a_1 a_0 + 1, \qquad p_k = a_k p_{k-1} + p_{k-2}, \qquad k \geqslant 2, \\ q_0 = 1, \qquad q_1 = a_1, \qquad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \geqslant 2. \end{aligned}$$

The sequence $(c_k)_{k \geqslant 0}$ converges to α. Moreover, the even convergents are less than α and form an increasing sequence and, on the other hand, the odd convergents are greater than α and form a decreasing sequence.

If $k \geqslant 2$ and $a_k > 1$, then between the convergents c_{k-2} and c_k there are *semiconvergents* (called intermediate fractions in Khinchin's book [9]) which are of the form

$$\frac{p_{k,\ell}}{q_{k,\ell}} = \frac{\ell p_{k-1} + p_{k-2}}{\ell q_{k-1} + q_{k-2}}$$

with $1 \leqslant \ell < a_k$. When the semiconvergents (if any) between c_{k-2} and c_k are ordered by the size of their denominators, the sequence obtained is increasing if k is even and decreasing if k is odd.

Note that we make a clear distinction between convergents and semiconvergents, i.e., convergents are not a specific subtype of semiconvergents.

A rational number $\frac{a}{b}$ is a *best approximation* of the real number α if for every fraction $\frac{c}{d}$ such that $\frac{c}{d} \neq \frac{a}{b}$ and $d \leqslant b$ it holds that

$$|b\alpha - a| < |d\alpha - c|.$$

In other words, any other multiple of α with a coefficient at most b is further away from the nearest integer than $b\alpha$ is. The next important proposition shows that the best approximations of an irrational number are connected to its convergents (for a proof see Theorems 16 and 17 of [9]).

Proposition 3. *The best rational approximations of an irrational number are exactly its convergents.*

We identify the unit interval $[0, 1)$ with the unit circle $\mathbb{T}$. Let $\alpha \in (0, 1)$ be irrational. The map
$$R \colon [0, 1) \to [0, 1), \ x \mapsto \{x + \alpha\},$$
where $\{x\}$ stands for the fractional part of the number x, defines a rotation on $\mathbb{T}$. The circle partitions into the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. Points in the same interval of the partition are said to be on the same side of 0 and points in different intervals are said to be on the opposite sides of 0. (We are not interested in the location of the point $\frac{1}{2}$.) The points $\{q_k \alpha\}$ and $\{q_{k-1} \alpha\}$ are always on the opposite sides of 0. The points $\{q_{k,\ell} \alpha\}$ with $0 < \ell \leqslant a_k$ always lie between the points $\{q_{k-2} \alpha\}$ and $\{q_k \alpha\}$; see (4).

We measure the shortest distance to 0 on $\mathbb{T}$ by setting

$$\|x\| = \min\{\{x\}, 1 - \{x\}\}.$$

We have the following facts for $k \geqslant 2$ and for all ℓ such that $0 < \ell \leqslant a_k$:

$$\|q_{k,\ell} \alpha\| = (-1)^k (q_{k,\ell} \alpha - p_{k,\ell}), \tag{3}$$
$$\|q_{k,\ell} \alpha\| = \|q_{k,\ell-1} \alpha\| - \|q_{k-1} \alpha\|. \tag{4}$$

We can now interpret Proposition 3 as

$$\min_{0 < n < q_k} \|n\alpha\| = \|q_{k-1} \alpha\|, \quad \text{for } k \geqslant 1. \tag{5}$$

Note that rotating preserves distances; a fact we will often use without explicit mention. In particular, the distance between the points $\{n\alpha\}$ and $\{m\alpha\}$ is $\||n-m|\alpha\|$. Thus by (5) the minimum distance between the distinct points $\{n\alpha\}$ and $\{m\alpha\}$ with $0 \leqslant n, m < q_k$ is at least $\|q_{k-1} \alpha\|$. Formula (5) tells what is the point closest to 0 among the points $\{n\alpha\}$ for $1 \leqslant n \leqslant q_k - 1$. We are also interested in knowing the point closest to 0 on the side opposite to $\{q_{k-1} \alpha\}$. The next result is very important and concerns this; see [12, Proposition 2.2.].

Proposition 4. *Let α be an irrational number. Let n be an integer such that $0 < n < q_{k,\ell}$ with $k \geqslant 2$ and $0 < \ell \leqslant a_k$. If $\|n\alpha\| < \|q_{k,\ell-1} \alpha\|$, then $n = m q_{k-1}$ for some integer m such that $1 \leqslant m \leqslant \min\{\ell, a_k - \ell + 1\}$.*

2.3 Sturmian Words

Sturmian words are a well-known class of infinite, aperiodic binary words with minimal factor complexity. They are defined as the infinite words having $n + 1$ factors of length n for every $n \geqslant 0$. For our purposes it is more convenient to view Sturmian words as the infinite words obtained as codings of orbits of points in an irrational circle rotation with two intervals; see [15, 11]. Let us make this more precise. The frequency α of letter 1 (called the *slope*) in a Sturmian word exists, and it is irrational. Divide the circle $\mathbb{T}$ into two intervals I_0 and I_1 defined by the points 0 and $1 - \alpha$, and define the coding function ν by setting $\nu(x) = 0$ if $x \in I_0$ and $\nu(x) = 1$ if $x \in I_1$. The coding of the orbit of a point x

is the infinite word $s_{x,\alpha}$ obtained by setting its nth, $n \geqslant 0$, letter to equal $\nu(R^n(x))$ where R is the rotation by angle α. This word is Sturmian with slope α, and conversely every Sturmian word with slope α is obtained this way. To make the definition proper, we need to define how ν behaves in the endpoints 0 and $1 - \alpha$. We have two options: either take $I_0 = [0, 1 - \alpha)$ and $I_1 = [1 - \alpha, 1)$ or $I_0 = (0, 1 - \alpha]$ and $I_1 = (1 - \alpha, 1]$. The difference is seen in the codings of the orbits of the special points $\{-n\alpha\}$, and both options are needed to be able to obtain every Sturmian word of slope α as a coding of a rotation. However, in this paper we are not concerned about this choice. We make the convention that $I(x, y)$ with $x \neq y$ and $x, y \neq 0$ is whichever of the half-open intervals of $\mathbb{T}$ separated by the points x and y (taken modulo 1 if necessary) that does not contain the point 0 as an interior point. The interval $I(x, 0) = I(0, x)$ is whichever of the half-open intervals separated by the points 0 and x having smallest length (the case $x = \frac{1}{2}$ is not important in this paper). Since the sequence $(\{n\alpha\})_{n \geqslant 0}$ is dense in $[0, 1)$—as is well-known—every Sturmian word of slope α has the same language (that is, the set of factors); this language is denoted by $\mathcal{L}(\alpha)$. Further, all Sturmian words are uniformly recurrent.

For every factor $w = a_0 a_1 \cdots a_{n-1}$ of length n there exists a unique subinterval $[w]$ of $\mathbb{T}$ such that $s_{x,\alpha}$ begins with w if and only if $x \in [w]$. Clearly

$$[w] = I_{a_0} \cap R^{-1}(I_{a_1}) \cap \ldots \cap R^{-(n-1)}(I_{a_{n-1}}).$$

We denote the length of the interval $[w]$ by $|[w]|$. The points $0, \{-\alpha\}, \{-2\alpha\}, \ldots, \{-n\alpha\}$ partition the circle into $n + 1$ intervals, which have one-to-one correspondence with the words of $\mathcal{L}(\alpha)$ of length n. Among these intervals the interval containing the point $\{-(n + 1)\alpha\}$ corresponds to the right special factor of length n. A factor w is *right special* if both $w0, w1 \in \mathcal{L}(\alpha)$. Similarly a factor is *left special* if both $0w, 1w \in \mathcal{L}(\alpha)$. In a Sturmian word there exists a unique right special and a unique left special factor of length n for all $n \geqslant 0$. The language $\mathcal{L}(\alpha)$ is mirror-invariant, that is, for every $w \in \mathcal{L}(\alpha)$ also $\widetilde{w} \in \mathcal{L}(\alpha)$. It follows that the right special factor of length n is the reversal of the left special factor of length n. Sturmian words are also *balanced*; that is, the number of occurrences of the letter 1 in any two factors of the same length differ at most by 1.

Given the continued fraction expansion of an irrational $\alpha \in (0, 1)$ as in (2), we define the corresponding *standard sequence* $(s_k)_{k \geqslant 0}$ of words by

$$s_{-1} = 1, \qquad s_0 = 0, \qquad s_1 = s_0^{a_1 - 1} s_{-1}, \qquad s_k = s_{k-1}^{a_k} s_{k-2}, \qquad k \geqslant 2.$$

As s_k is a prefix of s_{k+1} for $k \geqslant 1$, the sequence (s_k) converges to a unique infinite word c_α called the infinite standard Sturmian word of slope α, and it equals $s_{\alpha,\alpha}$. Inspired by the notion of semiconvergents, we define *semistandard* words for $k \geqslant 2$ by

$$s_{k,\ell} = s_{k-1}^{\ell} s_{k-2}$$

with $1 \leqslant \ell < a_k$. Clearly $|s_k| = q_k$ and $|s_{k,\ell}| = q_{k,\ell}$. Instead of writing "standard or semistandard", we often simply write "(semi)standard". The set of standard words of slope α is denoted by $Stand(\alpha)$, and the set of standard *and* semistandard words of slope α is denoted by $Stand^+(\alpha)$. (Semi)standard words are left special as prefixes of the word

c_α. Every (semi)standard word is primitive [11, Proposition 2.2.3]. An important property of standard words is that the words s_k and s_{k-1} almost commute; namely $s_k s_{k-1} = wxy$ and $s_{k-1} s_k = wyx$ for some word w and distinct letters x and y. For more on standard words see [11, 1].

The only difference between the words c_α and $c_{\overline{\alpha}}$ where $\alpha = [0; 1, a_2, a_3, \ldots]$ and $\overline{\alpha} = [0; a_2 + 1, a_3, \ldots]$ is that the roles of the letters 0 and 1 are reversed. We may thus assume without loss of generality that $a_1 \geqslant 2$. For the rest of this paper we make the convention that α stands for an irrational number in $(0, 1)$ having the continued fraction expansion as in (2) with $a_1 \geqslant 2$, i.e., we assume that $0 < \alpha < \frac{1}{2}$. The numbers q_k and $q_{k,\ell}$ refer to the denominators of the convergents of α, and the words s_k and $s_{k,\ell}$ refer to the standard or semistandard words of slope α.

2.4 Powers in Sturmian Words

In this section we review some known results on powers in Sturmian words, and prove helpful results for the next section.

If a square w^2 occurs in a Sturmian word of slope α, then the length of the word w must be a really specific number, namely a denominator of a convergent or a semiconvergent of α. The proof can be found in [3, Theorem 1] or [12, Proposition 4.1].

Proposition 5. *If $w^2 \in \mathcal{L}(\alpha)$ with w nonempty and primitive, then $|w| = q_0$, $|w| = q_1$ or $|w| = q_{k,\ell}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$.*

Next we need to know when conjugates of (semi)standard words occur as squares in a Sturmian word.

Proposition 6. *The following holds:*

(i) A factor $w \in \mathcal{L}(\alpha)$ is conjugate to s_k for some $k \geqslant 0$ if and only if $|w| = |s_k|$ and $w^2 \in \mathcal{L}(\alpha)$.

(ii) Let w be a conjugate of $s_{k,\ell}$ with $k \geqslant 2$ and $0 < \ell < a_k$. Then $w^2 \in \mathcal{L}(\alpha)$ if and only if the intervals $[w]$ and $[s_{k,\ell}]$ have the same length.

(iii) Let $n = q_0$, $n = q_1$, or $n = q_{k,\ell}$ with $k \geqslant 2$ and $0 < \ell \leqslant a_k$, and let s be the (semi)standard word of length n. A factor $w \in \mathcal{L}(\alpha)$ of length n is conjugate to s if and only if w and s have equally many occurrences of the letter 0.

Proof. Claim *(i)* is a direct consequence of [3, Theorem 3] or alternatively [12, Theorem 4.5]. Claim *(ii)* can be inferred from Theorems 4.3 and 4.5 of [12]. Finally, claim *(iii)* is evident from the proof of [12, Theorem 4.3], but a short proof can be given: the idea is that every factor of length n except one exceptional factor v is conjugate to s since s^2 occurs in $\mathcal{L}(\alpha)$ by *(i)* and *(ii)*. As not every factor of length n may have the same number of letters 0 (a right special factor extends to two factors having different number of letters 0), it must be that v has a different number of letters 0 than any conjugate of s. $\qquad \square$

We also need to know the index of certain factors of Sturmian words. The following proposition follows directly from Theorems 3 and 4 of [3] or from [12, Theorem 4.5].

Proposition 7. *The index of the standard word s_k in $\mathcal{L}(\alpha)$ is $a_{k+1}+2$ for $k \geqslant 2$ and a_2+1 for $k = 1$. The index of the semistandard word $s_{k,\ell}$ in $\mathcal{L}(\alpha)$ with $k \geqslant 2$ and $0 < \ell < a_k$ is 2.*

3   The Square Root Map

In [18] Saari observed that every Sturmian word of slope $\alpha = [0; a_1, a_2, \ldots]$ is an optimal squareful word with parameters $\mathfrak{a} = a_1 - 1$ and $\mathfrak{b} = a_2 - 1$. The assumption $0 < \alpha < \frac{1}{2}$ implies that $0^2 \in \mathcal{L}(\alpha)$, so the six minimal squares in $\mathcal{L}(\alpha)$ are the same as in (1). In particular, Saari's result means that every Sturmian word can be (uniquely) written as a product of the six *minimal squares of slope α* (1). Thus the square root map introduced next is well-defined.

Definition 8. Let s be a Sturmian word of slope α, and factorize it as a product of minimal squares: $s = X_1^2 X_2^2 X_3^2 \cdots$. The *square root of s* is then defined to be the word $\sqrt{s} = X_1 X_2 X_3 \cdots$.

Let us consider as an example the famous Fibonacci word f. The Fibonacci word is a Sturmian word of slope $[0; 2, 1, 1, \ldots]$, so it has parameters $\mathfrak{a} = 1$ and $\mathfrak{b} = 0$. It is also the fixed point of the substitution $0 \mapsto 01, 1 \mapsto 0$. For more information, see for instance [11]. We have that

$$f = (010)^2(100)^2(10)^2(01)^2 0^2 (10010)^2 (01)^2 \cdots \quad \text{and}$$
$$\sqrt{f} = 010 \cdot 100 \cdot 10 \cdot 01 \cdot 0 \cdot 10010 \cdot 01 \cdots .$$

Note that a square root map can be defined for any optimal squareful word. However, now we only focus on Sturmian words; we study later the square root map for other optimal squareful words in Section 8.

We aim to prove the surprising fact that given a Sturmian word s the word $\sqrt{s}$ is also a Sturmian word having the same slope as s. Moreover, knowing the intercept of s, we can compute the intercept of $\sqrt{s}$.

In the proof we need a special function $\psi \colon \mathbb{T} \to \mathbb{T}$ defined as follows. For $x \in (0, 1)$ we set

$$\psi(x) = \frac{1}{2}(x + 1 - \alpha),$$

and we set

$$\psi(0) = \begin{cases} \frac{1}{2}(1 - \alpha), & \text{if } I_0 = [0, 1 - \alpha), \\ 1 - \frac{\alpha}{2}, & \text{if } I_0 = (0, 1 - \alpha]. \end{cases}$$

The mapping ψ moves a point x on the circle $\mathbb{T}$ towards the point $1 - \alpha$ by halving the distance between the points x and $1 - \alpha$. The distance to $1 - \alpha$ is measured in the interval I_0 or I_1 depending on which of these intervals the point x belongs to.
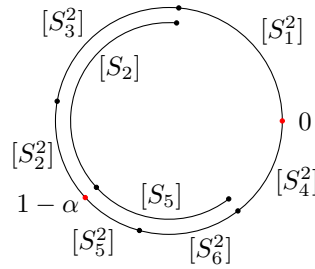
We can now state the result.

Figure 1: The positions of the intervals on the circle in the proof sketch of Lemma 11.

Theorem 9. *Let $s_{x,\alpha}$ be a Sturmian word with slope α and intercept x. Then $\sqrt{s_{x,\alpha}} = s_{\psi(x),\alpha}$. In particular, $\sqrt{s_{x,\alpha}}$ is a Sturmian word with slope α.*

For a combinatorial version of the above theorem see Theorem 29 in Section 6.

The main idea of the proof is to demonstrate that the square root map is actually the symbolic counterpart of the function ψ. We begin with a definition.

Definition 10. A square $w^2 \in \mathcal{L}(\alpha)$ satisfies the *square root condition* if $\psi([w^2]) \subseteq [w]$.

Note that if the interval $[w]$ in the above definition has $1 - \alpha$ as an endpoint, then w automatically satisfies the square root condition. This is because ψ moves points towards the point $1 - \alpha$ but does not map them over this point. Actually, if w satisfies the square root condition, then necessarily the interval $[w]$ has $1-\alpha$ as an endpoint (see Corollary 15).

We will only sketch the proof of the following lemma.

Lemma 11. *For every $i \in \{1, \ldots, 6\}$ the minimal square root S_i of slope α satisfies the square root condition and $\psi(\{x + 2|S_i|\alpha\}) = \{\psi(x) + |S_i|\alpha\}$ for all $x \in [S_i^2]$.*

Proof Sketch. It is straightforward to verify that

$$
\begin{aligned}
[S_1] &= I(0, 1 - \alpha), & [S_4] &= I(1 - \alpha, 1), \\
[S_2] &= I(-2\alpha, 1 - \alpha), & [S_5] &= I(1 - \alpha, -q_{2,1}\alpha), \\
[S_3] &= I(-2\alpha, 1 - \alpha), & [S_6] &= I(1 - \alpha, -q_{2,1}\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
[S_1^2] &= I(0, -2\alpha), & [S_4^2] &= I(-q_{2,1}\alpha, 1), \\
[S_2^2] &= I(-(q_{2,1} + 1)\alpha, 1 - \alpha), & [S_5^2] &= I(1 - \alpha, -(q_{3,1} + 1)\alpha), \\
[S_3^2] &= I(-2\alpha, -(q_{2,1} + 1)\alpha), & [S_6^2] &= I(-(q_{3,1} + 1)\alpha, -q_{2,1}\alpha),
\end{aligned}
$$

see Figure 1. Since ψ does not map points over the point $1 - \alpha$, it is evident that every minimal square root satisfies the square root condition.

Consider then the latter claim. Let $i \in \{1, \ldots, 6\}$. Suppose that $x \in [S_i^2] \setminus \{0\}$, $\{x + 2|S_i|\alpha\} \neq 0$, and $\lfloor x + 2|S_i|\alpha \rfloor = 2r$ for some $r \geqslant 0$. Then

$$
\psi(\{x + 2|S_i|\alpha\}) = \frac{1}{2}(x + 2|S_i|\alpha - 2r + 1 - \alpha) = \psi(x) + |S_i|\alpha - r = \{\psi(x) + |S_i|\alpha\} \quad (6)
$$

since ψ is a function from $\mathbb{T}$ to $\mathbb{T}$. We consider next the cases $i = 1$ and $i = 5$; the other cases are similar.

Suppose that $S_i = S_1$. Now $x + 2\alpha \geqslant 2\alpha > 0 = 2p_0$ and

$$x + 2\alpha \leqslant 1 - 2\alpha + 2\alpha = 1 = 2p_0 + 1,$$

so $x + 2\alpha \in (2p_0, 2p_0 + 1]$. The claim is thus clear as in (6) if $x \neq 0$ and $x \neq 1 - 2\alpha$. If $x = 0$, then $I_0 = [0, 1 - \alpha)$ and $\{\psi(x) + \alpha\} = \{\frac{1}{2}(1 - \alpha) + \alpha\} = \frac{1}{2}(1 + \alpha) = \psi(\{x + 2\alpha\})$. If $x = 1 - 2\alpha$, then $I_0 = (0, 1 - \alpha]$ and $\psi(\{x + 2\alpha\}) = 1 - \frac{\alpha}{2} = \{\psi(x) + \alpha\}$.

Assume then that $S_i = S_5$. Note that $|S_5| = q_2$. Using (4) we obtain that

$$\begin{aligned}
x + 2q_2\alpha &\leqslant \|(q_{3,1} + 1)\alpha\| + 2q_2\alpha \\
&= 1 - \alpha + \|q_{3,1}\alpha\| + 2p_2 + 2\|q_2\alpha\| \\
&= 1 - \alpha + \|q_1\alpha\| - \|q_2\alpha\| + 2p_2 + 2\|q_2\alpha\| \\
&= 1 - \alpha + \|q_1\alpha\| + \|q_2\alpha\| + 2p_2 \\
&\leqslant 2p_2 + 1,
\end{aligned}$$

where equality holds only if $x = \|(q_{3,1} + 1)\alpha\|$ and $a_2 = 1$. The length of the interval $[S_5^2]$ is $\|q_{3,1}\alpha\|$. Since $1 - \alpha \geqslant \alpha + \|q_1\alpha\|$ and $\alpha > \|q_1\alpha\| > \|q_2\alpha\|$, it follows from the preceding inequalities that $x + 2q_2\alpha > 2p_2$. Therefore $x + 2q_2\alpha \in (2p_2, 2p_2 + 1]$. If $a_2 > 1$ or $x \neq \|(q_{3,1} + 1)\alpha\|$, then the conclusion follows as in (6). Suppose finally that $a_2 = 1$ and $x = \|(q_{3,1} + 1)\alpha\|$. Now $I_0 = (0, 1 - \alpha]$, so $\psi(\{x + 2q_2\alpha\}) = \psi(0) = 1 - \frac{\alpha}{2}$. On the other hand,

$$\begin{aligned}
\psi(x) + q_2\alpha &= \frac{1}{2}(1 - \alpha + \|q_{3,1}\alpha\| + 1 - \alpha) + p_2 + \|q_2\alpha\| \\
&= \frac{1}{2}(1 - \alpha + \|q_1\alpha\| - \|q_2\alpha\| + 1 - \alpha + 2\|q_2\alpha\|) + p_2 \\
&= 1 - \frac{\alpha}{2} + p_2,
\end{aligned}$$

so the conclusion holds also in this case. $\qquad\square$

Proof of Theorem 9. Write $s_{x,\alpha} = X_1^2 X_2^2 X_3^2 \cdots$ as a product of minimal squares. Since the minimal square X_1^2 satisfies the square root condition by Lemma 11, we have that $\psi(x) \in [X_1]$. Hence both $\sqrt{s_{x,\alpha}}$ and $s_{\psi(x),\alpha}$ begin with X_1. Lemma 11 implies that $\psi(\{x + 2|X_1|\alpha\}) = \{\psi(x) + |X_1|\alpha\}$ for all $x \in [X_1^2]$. Thus by shifting $s_{x,\alpha}$ the amount $2|X_1|$ and by applying the preceding reasoning, we conclude that $s_{\psi(x),\alpha}$ shifted by the amount $|X_1|$ begins with X_2. Therefore the words $\sqrt{s_{x,\alpha}}$ and $s_{\psi(x),\alpha}$ agree on their first $|X_1| + |X_2|$ letters. By repeating this procedure, we conclude that $\sqrt{s_{x,\alpha}} = s_{\psi(x),\alpha}$. $\qquad\square$

Theorem 9 allows us to effortlessly characterize the Sturmian words which are fixed points of the square root map.

Corollary 12. *The only Sturmian words of slope α which are fixed by the square root map are the two words $01c_\alpha$ and $10c_\alpha$, both having intercept $1 - \alpha$.*

Proof. The only fixed point of the map ψ is the point $1 - \alpha$. Having this point as an intercept, we obtain two Sturmian words: either $01c_\alpha$ or $10c_\alpha$, depending on which of the intervals I_0 and I_1 the point $1 - \alpha$ belongs to. $\qquad\square$

The set $\{01c_\alpha, 10c_\alpha\}$ is not only the set of fixed points but also the unique attractor of the square root map in the set of Sturmian words of slope α. When iterating the square root map on a fixed Sturmian word $s_{x,\alpha}$, the obtained word has longer and longer prefixes in common with either of the words $01c_\alpha$ and $10c_\alpha$ because $\psi^n(x)$ tends to $1 - \alpha$ as n increases.

4 One Characterization of Words Satisfying the Square Root Condition

In the previous section we saw that the minimal squares, which satisfy the square root condition, were crucial in proving that the square root of a Sturmian word is again Sturmian with the same slope. The minimal squares of slope α are not the only squares in $\mathcal{L}(\alpha)$ satisfying the square root condition; in this section we will characterize combinatorially such squares. To be able to state the characterization, we need to define

$$RStand(\alpha) = \{\widetilde{w} \colon w \in Stand(\alpha)\},$$

the set of reversed standard words of slope α. Similarly we set

$$RStand^+(\alpha) = \{\widetilde{w} \colon w \in Stand^+(\alpha)\}.$$

We also need the operation L which exchanges the first two letters of a word (we do not apply this operation to too short words).

The main result of this section is the following.

Theorem 13. *A square $w^2 \in \mathcal{L}(\alpha)$ with w primitive satisfies the square root condition if and only if $w \in RStand^+(\alpha) \cup L(RStand(\alpha))$.*

As we remarked in Section 3, a square $w^2 \in \mathcal{L}(\alpha)$ trivially satisfies the square root condition if its interval $[w]$ has $1 - \alpha$ as an endpoint. Our aim is to prove that the converse is also true. We begin with a technical lemma.

Lemma 14. *Let $n = q_1$ or $n = q_{k,\ell}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$, and let i be an integer such that $1 < i \leqslant n$.*

(i) If $\{-i\alpha\} \in I_0$ and $\{-(i+n)\alpha\} < \{-i\alpha\}$, then $\psi(-(i+n)\alpha) > \{-i\alpha\}$.

(ii) If $\{-i\alpha\} \in I_1$ and $\{-(i+n)\alpha\} > \{-i\alpha\}$, then $\psi(-(i+n)\alpha) < \{-i\alpha\}$.

Proof. We prove *(i)*, the second assertion is symmetric. Suppose that $\{-i\alpha\} \in I_0$ and $\{-(i+n)\alpha\} < \{-i\alpha\}$. Note that the distance between the points $\{-i\alpha\}$ and $\{-(i+n)\alpha\}$

is less than α. It follows that $\{-n\alpha\} \in I_1$. Assume on the contrary that $\psi(-(i+n)\alpha) \leqslant \{-i\alpha\}$, that is,

$$\{-(i+n)\alpha\} + \frac{1}{2}(\{1-\alpha\} - \{-(i+n)\alpha\}) \leqslant \{-i\alpha\}.$$

Since $0 < \{-(i+n)\alpha\} < \{-i\alpha\}$, the distance between $\{-(i+n)\alpha\}$ and $\{-i\alpha\}$ is the same as the distance between 1 and $\{-n\alpha\}$. Thus by substituting

$$\{-(i+n)\alpha\} = \{-i\alpha\} - (1 - \{-n\alpha\})$$

to the above and rearranging, we have that

$$\{1-\alpha\} - \{-i\alpha\} \leqslant 1 - \{-n\alpha\}.$$

Since $\{-n\alpha\} \in I_1$, we obtain that

$$\| - (i-1)\alpha \| \leqslant \| - n\alpha \|. \tag{7}$$

Suppose now first that $n = q_{k,\ell}$ for some $k \geqslant 2$ with $0 < l \leqslant a_k$. Since $i - 1 < n$, Proposition 4 and (7) imply that $i - 1 = m q_{k-1}$ for some $1 \leqslant m \leqslant \min\{\ell, a_k - \ell + 1\}$. As $\{-n\alpha\} \in I_1$, the point $\{-q_{k-1}\alpha\}$ must lie on the opposite side of 0 in the interval I_0. Therefore $\{-(i-1)\alpha\} \in I_0$. Then by (7), the point $\{-i\alpha\}$ must lie in I_1. This is a contradiction. Suppose then that $n = q_1$. It is easy to see that (7) cannot hold for any i greater than 1. This concludes the proof. $\qquad \square$

Corollary 15. *If $w^2 \in \mathcal{L}(\alpha)$ with w primitive satisfies the square root condition, then the interval $[w]$ has $1 - \alpha$ as an endpoint.*

Proof. Let $n = |w|$. Proposition 5 implies that $n = q_0$, $n = q_1$, or $n = q_{k,\ell}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$. Say $n = q_0 = 1$. As the only factor of length 1 occurring as a square is 0, the claim holds as $[0] = I_0 = I(0, 1 - \alpha)$. Suppose then that $n = q_1$ or $n = q_{k,\ell}$.

Let $[w] = I(-i\alpha, -j\alpha)$. Then either $[w^2] = I(-i\alpha, -(j+|w|)\alpha)$ or $[w^2] = I(-(i+|w|)\alpha, -j\alpha)$. Suppose first that $[w] \subseteq I_0$. By symmetry we may assume that $\{-j\alpha\} > \{-i\alpha\}$. Now $[w^2] = [-(i+|w|)\alpha, -j\alpha)$ if and only if $j = 1$. Namely, if $j \neq 1$, then it is clear that it is possible to find a point $x \in I(-i\alpha, -j\alpha)$ close to $\{-j\alpha\}$ such that $\psi(x) > \{-j\alpha\}$, so the condition $\psi([w^2]) \subseteq [w]$ cannot be satisfied. If $[w^2] = [-i\alpha, -(j+|w|)\alpha)$ and $j \neq 1$, then by Lemma 14 $\psi(-(j+|w|)\alpha) > \{-j\alpha\}$, so the condition $\psi([w^2]) \subseteq [w]$ cannot be satisfied. Thus also in this case necessarily $j = 1$. The case where $[w] \subseteq I_1$ is proven symmetrically using the latter symmetric assertion of Lemma 14. $\qquad \square$

Next we study in more detail the properties of squares $w^2 \in \mathcal{L}(\alpha)$ whose interval has $1 - \alpha$ as an endpoint.

Proposition 16. *Consider the intervals of factors in $\mathcal{L}(\alpha)$ of length $n = q_1$ or $n = q_{k,\ell}$ with $k \geqslant 2$ and $0 < \ell \leqslant a_k$. Let u and v be the two distinct words of length n having intervals with endpoint $1 - \alpha$. Then the following holds.*

(i) *There exists a word w such that $u = xyw$ and $v = yxw = L(u)$ for distinct letters x and y.*

(ii) *Either u or v is right special.*

(iii) *If μ is the right special word among the words u and v, then $\mu^2 \in \mathcal{L}(\alpha)$.*

(iv) *If λ is the word among the words u and v which is not right special, then $\lambda^2 \in \mathcal{L}(\alpha)$ if and only if $n = q_1$ or $\ell = a_k$.*

Proof. Suppose first that $n = q_1$. Then it is straightforward to see that the factors u and v of length n having intervals with endpoint $1 - \alpha$ are $010^{a_1 - 2} = S_2$ and $10^{a_1 - 1} = S_4$. Clearly S_4 is right special and $L(S_4) = S_2$. Moreover $S_2^2, S_4^2 \in \mathcal{L}(\alpha)$.

Assume that $n = q_{k,\ell}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$. By Proposition 4 the point $\{-n\alpha\}$ is the point closest to 0 on the side opposite to the point $\{-q_{k-1}\alpha\}$. Thus either $\{-(n+1)\alpha\} \in [u]$ or $\{-(n+1)\alpha\} \in [v]$. Assume by symmetry that $\{-(n+1)\alpha\} \in [u]$. This means that the word u is right special, proving *(ii)*. Further, the endpoint of $[u]$ which is not $1 - \alpha$ must be after a rotation the next closest point to 0 on the side opposite to the point $\{-q_{k-1}\alpha\}$. Thus by Proposition 4 $[u] = I(-(q_{k,\ell-1} + 1)\alpha, 1 - \alpha)$ and consequently $[v] = I(1 - \alpha, -(q_{k-1} + 1)\alpha)$.

Since the points $x = \{(-(q_{k,\ell-1} + 1)\alpha\}$ and $y = \{-(q_{k-1} + 1)\alpha\}$ are on the opposite sides of the point $1 - \alpha$ and the points $\{x + \alpha\}$ and $\{y + \alpha\}$ are on the opposite sides of the point 0, it follows that u begins with cd and v begins with dc for distinct letters c and d. Assume on the contrary that $u = cdzeu'$ and $v = dczfv'$ for distinct letters e and f. In particular, $|z| \leqslant n - 3$. This means that the point $x' = \{x + (|z| + 2)\alpha\}$ is in $[e]$ and the point $y' = \{y + (|z| + 2)\alpha\}$ is in $[f]$. It must be that $e = c$ and $f = d$ as otherwise the point $x' - \alpha$ would be in $[c]$ and the point $y' - \alpha$ would be in $[d]$ contradicting the choice of z. Since α is irrational, either x' is closer to $1 - \alpha$ than x or y' is closer to $1 - \alpha$ than y.

Suppose that x' is closer to $1 - \alpha$ than x. Since x' is on the same side of the point $1 - \alpha$ as x, it follows that

$$\|x' + \alpha\| = \|(q_{k,\ell-1} - |z| - 2)\alpha\| < \|q_{k,\ell-1}\alpha\| = \|x + \alpha\|.$$

Since $q_{k,\ell-1} - |z| - 2 < q_{k,\ell-1}$, by Proposition 4 it must be that $q_{k,\ell-1} - |z| - 2 \leqslant 0$. However, as $\|q_{k,\ell-1}\alpha\| = \| - q_{k,\ell-1}\alpha\|$, it follows by Proposition 4 that $|z| + 2 - q_{k,\ell-1} = mq_{k-1}$ for some $m \geqslant 1$. Thus $|z| + 2 \geqslant q_{k,\ell-1} + q_{k-1} = q_{k,\ell} = n$. This is, however, a contradiction as $|z| \leqslant n - 3$.

Suppose then that y' is closer to $1 - \alpha$ than y. Similar to above, it follows that

$$\|y' + \alpha\| = \|(q_{k-1} - |z| - 2)\alpha\| < \|q_{k-1}\alpha\| = \|y + \alpha\|.$$

Again, it must be that $q_{k-1} - |z| - 2 \leqslant 0$. Since $\|q_{k-1}\alpha\| = \| - q_{k-1}\alpha\|$, it follows from (5) that $|z| + 2 - q_{k-1} \geqslant q_k$. Therefore $|z| + 2 \geqslant q_k + q_{k-1} > n$. This is again a contradiction with the fact that $|z| \leqslant n - 3$.

Thus we conclude that $u = cdw$ and $v = dcw$ for some word w proving *(i)*. As $n = q_{k,\ell}$, it must be that the right special word of length n equals $\widetilde{s}_{k,\ell}$. Since u and v are conjugate

by Proposition 6 (iii), Proposition 6 implies that if $\ell = a_k$, then $u^2, v^2 \in \mathcal{L}(\alpha)$. Suppose that $\ell \neq a_k$. By Proposition 6, the word $s_{k,\ell}$ occurs as a square in $\mathcal{L}(\alpha)$. Since $\mathcal{L}(\alpha)$ is mirror-invariant, also $u^2 = \widetilde{s}_{k,\ell}^2 \in \mathcal{L}(\alpha)$. Therefore from Proposition 6 it follows that $|[u]| = \|q_{k,\ell-1}\alpha\| = |[s_{k,\ell}]|$. Now $[v] = I(1 - \alpha, -(q_{k-1} + 1)\alpha)$, so $|[v]| = \|q_{k-1}\alpha\| \neq |[u]|$. Thus Proposition 6 implies that $v^2 \notin \mathcal{L}(\alpha)$. Hence *(iii)* and *(iv)* are proved. $\qquad\square$

Proof of Theorem 13. If $|w| = 1$, then clearly $w = 0 = \widetilde{s}_0$, so the claim holds. We may thus focus on the case that $|w| > 1$.

Suppose that $w^2 \in \mathcal{L}(\alpha)$ satisfies the square root condition. By Corollary 15 the interval $[w]$ has $1 - \alpha$ as an endpoint. Moreover, Proposition 5 implies that $|w| = q_1$ or $|w| = q_{k,l}$ for some $k \geqslant 2$ with $0 < l \leqslant a_k$. Thus from Proposition 16 it follows that $w = \widetilde{s}$ or $w = L(\widetilde{s})$ where s is the (semi)standard word of length $|w|$. By Proposition 16 we have that $\widetilde{s}^2 \in \mathcal{L}(\alpha)$. Moreover, by Proposition 16 we have that $L(\widetilde{s})^2 \in \mathcal{L}(\alpha)$ if and only if $|w| = q_k$ for some $k \geqslant 1$. Thus $w \in RStand^+(\alpha) \cup L(RStand(\alpha))$.

Suppose then that $w \in RStand^+(\alpha) \cup L(RStand(\alpha))$. Note first that $L(w)$ has the same number of letters 0 as w, so w is conjugate to $L(w)$ by Proposition 6. Thus it follows from Proposition 6 that $w^2 \in \mathcal{L}(\alpha)$. Let u and v be the factors of length $|w|$ having endpoint $1 - \alpha$. By Proposition 16 the word u must be right special and $v = L(u)$. Since the right special factor of length $|w|$ is unique, either $w = u$ or $L(w) = u$. Thus the interval $[w]$ has $1 - \alpha$ as an endpoint. Then clearly w^2 satisfies the square root condition. $\qquad\square$

5 Characterization by a Word Equation

It turns out that the squares of slope α satisfying the square root condition have also a different characterization in terms of specific solutions of the word equation

$$X_1^2 X_2^2 \cdots X_n^2 = (X_1 X_2 \cdots X_n)^2 \tag{8}$$

in the language $\mathcal{L}(\alpha)$. We are interested only in the solutions of (8) where all words X_i are *minimal square roots* (1), i.e., primitive roots of minimal squares. Thus we give the following definition.

Definition 17. A nonempty word w is a *solution to* (8) if w can be written as a product of minimal square roots $w = X_1 X_2 \cdots X_n$ which satisfy the word equation (8). The solution is *trivial* if $X_1 = X_2 = \ldots = X_n$ and *primitive* if w is primitive. The word w is a *solution to* (8) *in* $\mathcal{L}(\alpha)$ if w is a solution to (8) and $w^2 \in \mathcal{L}(\alpha)$.

All minimal square roots of slope α are trivial solutions to (8). One example of a nontrivial solution is $w = S_2 S_1 S_4$ in the language of the Fibonacci word (i.e., in the language of slope $[0; 2, 1, 1, \ldots]$) since $w^2 = (01010)^2 = (01)^2 \cdot 0^2 \cdot (10)^2 = S_2^2 S_1^2 S_4^2$. Note that in the language of any Sturmian word there are only finitely many trivial solutions as the index of every factor is finite.

Note that the factorization of a word as product of minimal squares is unique. Indeed, if $X_1^2 \cdots X_n^2 = Y_1^2 \cdots Y_m^2$, where the squares X_i^2 and Y_i^2 are minimal, then either X_1^2 is

a prefix of Y_1^2 or vice versa. Therefore by minimality $X_1^2 = Y_1^2$, that is, $X_1 = Y_1$. The uniqueness of the factorization follows.

Our aim is to complete the characterization of Theorem 13 as follows.

Theorem 18. *Let $w \in \mathcal{L}(\alpha)$ with w primitive. The following are equivalent:*

(i) w is a primitive solution to (8) in $\mathcal{L}(\alpha)$,

(ii) w^2 satisfies the square root condition,

(iii) $w \in RStand^+(\alpha) \cup L(RStand(\alpha))$.

For later use in Section 8 we define the language $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$.

Definition 19. The language $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$ consists of all factors of the infinite words in the language
$$(10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}} + 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1})^{\omega} = (S_5 + S_6)^{\omega}.$$

Observe that by Proposition 2 every factor in $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$ is a factor of some optimal squareful word with parameters $\mathfrak{a}$ and $\mathfrak{b}$. Moreover, if $\alpha = [0; \mathfrak{a} + 1, \mathfrak{b} + 1, \ldots]$, then $\mathcal{L}(\alpha) \subseteq \mathcal{L}(\mathfrak{a}, \mathfrak{b})$.

Definition 20. The language $\Pi(\mathfrak{a}, \mathfrak{b})$ consists of all nonempty words in $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$ which can be written as products of the minimal squares (1).

Let $w \in \Pi(\mathfrak{a}, \mathfrak{b})$, that is, $w = X_1^2 \cdots X_n^2$ for minimal square roots X_i. Then we can define the square root of w by setting $\sqrt{w} = X_1 \cdots X_n$.

We need two technical lemmas. Their proofs are straightforward case-by-case analysis. The statement of Lemma 21 has a technical condition for later use in Section 8, which is perhaps better understood if the reader first reads the proof of Lemma 22 up to the point where Lemma 21 is invoked.

Lemma 21. *Let u and v be words such that*

- *u is a nonempty suffix of S_6,*

- *$|v| \geqslant |S_5 S_6|$,*

- *v begins with xy for distinct letters x and y,*

- *$uv \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$ and $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$.*

Suppose there exists a minimal square X^2 such that $|X^2| > |u|$ and X^2 is a prefix of uv or $uL(v)$. Then there exist minimal squares $Y_1^2, \ldots, Y_n^2$ such that X^2 and $Y_1^2 \cdots Y_n^2$ are prefixes of uv and $uL(v)$ of the same length and $X = Y_1 \cdots Y_n$.

Proof. Let Z^2 be a minimal square such that $|Z^2| > |u|$ and Z^2 is a prefix of uv or $uL(v)$. It is not obvious at this point that Z exists but its existence becomes evident as this proof progresses. By symmetry we may assume that Z^2 is a prefix of uv. To prove the claim we consider different cases depending on the word Z.

Case A. $Z = S_1 = 0$. Since u is a nonempty suffix of S_6 and $|Z^2| > |u|$, it must be that $u = 0$. As v begins with 0, we have that v begins with 01 by assumption. Since $v \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$ and $|v| \geqslant |S_6|$, the word v begins with either $010^{\mathfrak{a}}10^{\mathfrak{a}}$ or $010^{\mathfrak{a}+1}10^{\mathfrak{a}}$. In the latter case $L(v)$ would begin with $10^{\mathfrak{a}+2}1$ contradicting the assumption $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Hence v begins with $010^{\mathfrak{a}}10^{\mathfrak{a}}$. It follows that uv has $0010^{\mathfrak{a}}10^{\mathfrak{a}}$ as a prefix, that is, uv begins with $S_1^2 S_4^2$. On the other hand, the word $uL(v)$ has the word $S_3^2 = 010^{\mathfrak{a}+1}10^{\mathfrak{a}}$ as a prefix. Since $S_3 = S_1 S_4$, the conclusion of the claim holds.

Case B. $Z = S_2 = 010^{\mathfrak{a}-1}$. If $u = 0$, then v has $10^{\mathfrak{a}}10^{\mathfrak{a}}$ as a prefix and, consequently, $L(v)$ has $010^{\mathfrak{a}-1}10^{\mathfrak{a}}$ as a prefix contradicting the fact that $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Therefore by the assumptions that u is a nonempty suffix of S_6 and $|Z^2| > |u|$, it follows that $u = 010^{\mathfrak{a}}$. Thus v has $10^{\mathfrak{a}}$ as a prefix. Using the fact that $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$, we see that v begins with $10^{\mathfrak{a}+1}$ and $L(v)$ begins with $010^{\mathfrak{a}}$. Hence uv has $S_2^2 S_1^2$ as a prefix, and $uL(v)$ has S_3^2 as a prefix. Since $S_2 S_1 = S_3$, we conclude, as in the previous case, that the conclusion holds.

Case C. $Z = S_3 = 010^{\mathfrak{a}}$. Using again the fact that u is a suffix of S_6 and $|Z^2| > |u|$, we see that either $u = 0$ or $u = 010^{\mathfrak{a}}$. In the first case v begins with $10^{\mathfrak{a}+1}10^{\mathfrak{a}}$ and $L(v)$ begins with $010^{\mathfrak{a}}10^{\mathfrak{a}}$. Hence the word $uL(v)$ has $S_1^2 S_4^2$ as a prefix. As $S_1 S_4 = S_3$, the conclusion follows. Let us then consider the other case. Now $L(v)$ begins with $10^{\mathfrak{a}+1}$, so the word $uL(v)$ has $S_2^2 S_1^2$ as a prefix. Again, the conclusion follows since $S_2 S_1 = S_3$.

Case D. $Z = S_4 = 10^{\mathfrak{a}}$. Now the only option is that $u = 10^{\mathfrak{a}}$. Using the fact that $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$, we see that v cannot begin with $10^{\mathfrak{a}}1$, so v must have $10^{\mathfrak{a}+1}$ as a prefix. Further, since $|v| \geqslant |S_6|$, it must be that S_6 is a prefix of v. If $S_6 1$ would be a prefix of v, then the word $L(v)$ would have the word $(10^{\mathfrak{a}})^{\mathfrak{b}+2}1$ as a factor contradicting the fact that $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Thus $S_6 0$ is a prefix of v. Since $v \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$ and $|v| \geqslant |S_5 S_6|$, we have that $S_6 0 (10^{\mathfrak{a}})^{\mathfrak{b}+1} = S_5^2 10^{\mathfrak{a}}$ is a prefix of v. Consequently, the word $L(v)$ begins with $0(10^{\mathfrak{a}})^{\mathfrak{b}+1}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$, so $uL(v)$ has S_6^2 as a prefix. Assume first that $\mathfrak{b}$ is odd. It is straightforward to see that in this case

$$0(10^{\mathfrak{a}})^{\mathfrak{b}}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1} = (S_2^2)^{(\mathfrak{b}+1)/2} S_1^2 (S_4^2)^{(\mathfrak{b}+1)/2}.$$

Thus for the prefix $10^{\mathfrak{a}} S_5 10^{\mathfrak{a}}$ of uv we have that

$$10^{\mathfrak{a}} S_5^2 10^{\mathfrak{a}} = S_4^2 (S_2^2)^{(\mathfrak{b}+1)/2} S_1^2 (S_4^2)^{(\mathfrak{b}+1)/2}.$$

As $S_6 = S_4 S_2^{(\mathfrak{b}+1)/2} S_1 S_4^{(\mathfrak{b}+1)/2}$, the conclusion follows as before. Assume then that $\mathfrak{b}$ is even. It is now easy to show that

$$0(10^{\mathfrak{a}})^{\mathfrak{b}}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1} = (S_2^2)^{\mathfrak{b}/2} S_3^2 (S_4^2)^{\mathfrak{b}/2}.$$

Therefore

$$10^{\mathfrak{a}} S_5^2 10^{\mathfrak{a}} = S_4^2 (S_2^2)^{\mathfrak{b}/2} S_3^2 (S_4^2)^{\mathfrak{b}/2}.$$

Since $S_6 = S_4 S_2^{\mathfrak{b}/2} S_3 S_4^{\mathfrak{b}/2}$, the conclusion again follows.

Case E. $Z = S_5 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}}$. Now either $u = 10^{\mathfrak{a}}$ or $u = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. In the first case v must begin with $0(10^{\mathfrak{a}})^{\mathfrak{b}}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}}$. However, this implies that $L(v)$ begins with $10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}-1}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}}$ contradicting the fact that $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Consider then the latter case where v begins with $0(10^{\mathfrak{a}})^{\mathfrak{b}}$. As $L(v) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$ and $|v| \geqslant |S_6|$, it must be that $L(v)$ begins with $10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. Hence the word $uL(v)$ has S_6^2 as a prefix. Since the word v begins with $0(10^{\mathfrak{a}})^{\mathfrak{b}+2}$, the word uv has $S_5^2 S_4^2$ as a prefix. The conclusion follows as $S_5 S_4 = S_6$.

Case F. $Z = S_6 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. Now there are two possibilities: either $u = 10^{\mathfrak{a}}$ or $u = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. In the first case v begins with $0(10^{\mathfrak{a}})^{\mathfrak{b}+1}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$, so $L(v)$ begins with $10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. The word $uL(v)$ has $S_4^2 0(10^{\mathfrak{a}})^{\mathfrak{b}}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$ as a prefix. Proceeding as in the Case D depending on the parity of $\mathfrak{b}$, we see that the conclusion holds. Consider then the latter case $u = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. The word v must begin with u, so $L(v)$ has $0(10^{\mathfrak{a}})^{\mathfrak{b}+2}$ as a prefix. Clearly the word $uL(v)$ has $S_5^2 S_4^2$ as a prefix. As $S_6 = S_5 S_4$, the conclusion follows. $\square$

A more intuitive way of stating Lemma 21 is that under the assumptions of the lemma swapping two adjacent and distinct letters which do not occur as a prefix of a minimal square affects a product of minimal square only locally and does not change its square root.

Lemma 22. *Let w be a primitive solution to (8) having the word $S_6 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$ as a suffix such that $w^2, L(w) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Then $wL(w) \in \Pi(\mathfrak{a}, \mathfrak{b})$ and $\sqrt{wL(w)} = w$.*

Proof. If $w = S_6$, then it is easy to see that $wL(w) = S_5^2 S_4^2$ and $w = S_5 S_4$, so the claim holds. We may thus suppose that S_6 is a proper suffix of w.

Since w is a solution to (8), we have that $w^2 = X_1^2 \cdots X_n^2$ and $w = X_1 \cdots X_n$ for some minimal square roots X_i. It must be that $n > 1$ as if $n = 1$ then $w = X_1$, and it is not possible for S_6 to be a proper suffix of w. Assume for a contradiction that $X_1 = S_1$. Since $X_1 X_2$ is a prefix of w^2, it follows that X_2 begins with the letter 0. If $X_2 \neq S_1$, then $X_1 X_2$ begins with 001 but $X_1^2 X_2^2$ begins with 000, which is impossible. Hence $X_2 = S_1$, and by repeating the argument it follows that $X_k = S_1$ for all k such that $1 \leqslant k \leqslant n$. Thus w cannot have S_6 as a suffix, so we conclude that $X_1 \neq S_1$. Hence w always begins with 01 or 10.

We show that $|X_1^2| < |w|$. Assume on the contrary that $|X_1^2| \geqslant |w|$. Since w has the word S_6 as a suffix, it follows that S_6 is a factor of X_1^2. It follows that X_1 is one of the words S_5, S_6 or S_3 (if $\mathfrak{b} = 0$). If $X_1 = S_5$, then S_6 occurs in $X_1^2 = 10^{a+1}(10^a)^b 10^{a+1}(10^a)^b$ only as a prefix. Thus $w = S_6$ contradicting the fact that S_6 is a proper suffix of w. If $X_1 = S_6$, then S_6 occurs in $X_1^2 = 10^{a+1}(10^a)^{b+1}10^{a+1}(10^a)^{b+1}$ as a prefix and as a suffix. Since $w \neq S_6$, it must be that $w = X_1^2$ contradicting the primitivity of w. Let finally $\mathfrak{b} = 0$ and $X_1 = S_3$. Then S_6 occurs in $X_1^2 = 010^{a+1}10^a$ as a suffix. Hence $w = X_1^2$ contradicting again the primitivity of w.

Now there exists a maximal r such that $1 \leqslant r < n$ and $X_1^2 \cdots X_r^2$ is a prefix of w. Actually $X_1^2 \cdots X_r^2$ is a proper prefix of w, as otherwise $w^2 = (X_1^2 \cdots X_r^2)^2 =$

$(X_1 \cdots X_r X_1 \cdots X_r)^2$, so $w = (X_1 \cdots X_r)^2$ contradicting the primitivity of w. Thus when factorizing $wL(w)$ and w^2 as products of minimal squares, the first r squares are equal. Let u be the nonempty word such that $w = X_1^2 \cdots X_r^2 u$. By the definition of the number r, we have that u is a proper prefix of X_{r+1}^2. Suppose for a contradiction that $|u| > |S_6|$. It follows that u has S_6 as a proper suffix. This leaves only the possibilities that X_{r+1} is either of the words S_5 or S_6. However, if $X_{r+1} = S_5$, then S_6 cannot be a proper suffix of u, and if $X_{r+1} = S_6$, then r is not maximal. We conclude that $|u| \leqslant |S_6|$.

Next we show that w must satisfy $|w| \geqslant |S_5 S_6|$. Suppose first that w begins with the letter 0. Then as S_6 is a proper suffix of w and $w^2 \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$, it must be that w begins with $0(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. Suppose that this prefix overlaps with the suffix S_6. Then clearly $w = 0(10^{\mathfrak{a}})^{\mathfrak{b}} 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1} = (0(10^{\mathfrak{a}})^{\mathfrak{b}+1})^2$ contradicting the primitivity of w. If the prefix $0(10^{\mathfrak{a}})^{\mathfrak{b}+1}$ does not overlap with the suffix S_6, then $|w| \geqslant |S_5 S_6|$. Assume then that w begins with the letter 1. Similar to above, the word w must begin with $10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. In this case necessarily $|w| \geqslant |S_5 S_6|$.

Finally, we can apply Lemma 21 to the words u and w with $X = X_{r+1}$. We obtain minimal squares $Y_1^2, \ldots, Y_m^2$ such that $Y_1^2 \cdots Y_m^2$ is a prefix of $uL(w)$ and and $Y_1 \cdots Y_m = X_{r+1} \cdots X_{r+t}$ for some $t \geqslant 1$. Thus

$$wL(w) = X_1^2 \cdots X_r^2 Y_1^2 \cdots Y_m^2 X_{r+t+1}^2 \cdots X_n^2 \quad \text{and}$$
$$w = X_1 \cdots X_n = X_1 \cdots X_r Y_1 \cdots Y_m X_{r+t+1} \cdots X_n.$$

The claim is proved. □

Proposition 23. *Let $w \in RStand^+(\alpha) \cup L(RStand(\alpha))$. Then the word w is a primitive solution to (8) in $\mathcal{L}(\alpha)$.*

Proof. Note that Proposition 6 implies that $w^2 \in \mathcal{L}(\alpha)$. Suppose first that $|w| < |S_6|$ where $S_6 = \widetilde{s}_{3,1} = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. Clearly the minimal square root $S_1, \ldots, S_5$ are solutions to (8), so we are left with the case where $w = \widetilde{s}_{2,\ell} = 0(10^{\mathfrak{a}})^{\ell}$ for some ℓ such that $1 < \ell \leqslant \mathfrak{b} + 1$. It is straightforward to see that if ℓ is even, then

$$w^2 = (S_2^2)^{\ell/2} S_1^2 (S_4^2)^{\ell/2} \quad \text{and} \quad w = S_2^{\ell/2} S_1 S_4^{\ell/2}.$$

If ℓ is odd, then

$$w^2 = (S_2^2)^{(\ell+1)/2} S_3^2 (S_4^2)^{(\ell+1)/2} \quad \text{and} \quad w = S_2^{(\ell+1)/2} S_3 S_4^{(\ell+1)/2}.$$

Hence w is a solution to (8).

We may thus suppose that $|w| \geqslant |S_6|$, so w has S_6 as a suffix. We proceed by induction. Now either $w = \widetilde{s}_{k,\ell}$ for some $k \geqslant 3$ with $0 < \ell \leqslant a_k$ or $L(w) = \widetilde{s}_k$ for some $k \geqslant 3$. We assume that the claim holds for every word satisfying the hypotheses which are shorter than w. Consider first the case $w = \widetilde{s}_{k,\ell}$ for some $k \geqslant 3$ with $0 < \ell \leqslant a_k$. By the fact that $\widetilde{s}_{k-1}\widetilde{s}_{k-2} = L(\widetilde{s}_{k-2})\widetilde{s}_{k-1}$ we obtain that

$$w^2 = \widetilde{s}_{k-2}\widetilde{s}_{k-1}^{\ell}\widetilde{s}_{k-2}\widetilde{s}_{k-1}^{\ell} = \widetilde{s}_{k-2}\widetilde{s}_{k-1}^{\ell-1}L(\widetilde{s}_{k-2})\widetilde{s}_{k-1}^{\ell-1} \cdot \widetilde{s}_{k-1}^2 = \widetilde{s}_{k,\ell-1}L(\widetilde{s}_{k,\ell-1}) \cdot \widetilde{s}_{k-1}^2.$$

Now if $k = 3$ and $\ell = 1$, then the conclusion holds as $\widetilde{s}_{3,1} = S_6$ is a minimal square root. Hence we may assume that either $k > 3$ or $k = 3$ and $\ell > 1$. Since $\widetilde{s}_{k-1}$ is a solution to (8), we have that $\widetilde{s}_{k-1}^2 = X_1^2 \cdots X_n^2$ and $\widetilde{s}_{k-1} = X_1 \cdots X_n$ for some minimal square roots X_i. In other words,

$$\widetilde{s}_{k-1}^2 \in \Pi(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad \sqrt{\widetilde{s}_{k-1}^2} = \widetilde{s}_{k-1}.$$

Since $|\widetilde{s}_{k,\ell-1}| \geqslant |S_6|$, with an application of Lemma 22 we obtain that

$$\widetilde{s}_{k,\ell-1} L(\widetilde{s}_{k,\ell-1}) \in \Pi(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad \sqrt{\widetilde{s}_{k,\ell-1} L(\widetilde{s}_{k,\ell-1})} = \widetilde{s}_{k,\ell-1}.$$

Thus $w^2 \in \Pi(\mathfrak{a}, \mathfrak{b})$ and

$$\sqrt{w^2} = \sqrt{\widetilde{s}_{k,\ell-1} L(\widetilde{s}_{k,\ell-1})} \sqrt{\widetilde{s}_{k-1}^2} = \widetilde{s}_{k,\ell-1} \widetilde{s}_{k-1} = w,$$

so w is a solution to (8).

Consider next the case $w = L(\widetilde{s}_k)$ for some $k \geqslant 3$. Similar to above,

$$\begin{aligned} w^2 &= L(\widetilde{s}_{k-2}) \widetilde{s}_{k-1}^{a_k} L(\widetilde{s}_{k-2}) \widetilde{s}_{k-1}^{a_k} = L(\widetilde{s}_{k-2}) \widetilde{s}_{k-1}^{a_k+1} \widetilde{s}_{k-2} \widetilde{s}_{k-1}^{a_k-1} \\ &= L(\widetilde{s}_{k-2}) \widetilde{s}_{k-1} \widetilde{s}_{k-3} \widetilde{s}_{k-2}^{a_{k-1}} \widetilde{s}_{k-1}^{a_k-1} \widetilde{s}_{k-2} \widetilde{s}_{k-1}^{a_k-1} = L(\widetilde{s}_{k-2}) \widetilde{s}_{k-1} \widetilde{s}_{k-3} \widetilde{s}_{k-2}^{a_{k-1}-1} \cdot \widetilde{s}_{k,a_k-1}^2 \\ &= \widetilde{s}_{k-1} \widetilde{s}_{k-2} \widetilde{s}_{k-3} \widetilde{s}_{k-2}^{a_{k-1}-1} \cdot \widetilde{s}_{k,a_k-1}^2 = \widetilde{s}_{k-1} L(\widetilde{s}_{k-1}) \cdot \widetilde{s}_{k,a_k-1}^2. \end{aligned}$$

If $k > 3$, then the claim follows using the induction hypothesis and Lemma 22 as above. In the case $k = 3$ we have that

$$\widetilde{s}_{k-1} L(\widetilde{s}_{k-1}) \in \Pi(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad \sqrt{\widetilde{s}_{k-1} L(\widetilde{s}_{k-1})} = \widetilde{s}_{k-1}.$$

Namely, it is not difficult to see that if $\mathfrak{b}$ is even, then

$$\widetilde{s}_{k-1} L(\widetilde{s}_{k-1}) = (S_2^2)^{1+\mathfrak{b}/2} S_1^2 (S_4^2)^{\mathfrak{b}/2} \quad \text{and} \quad \widetilde{s}_{k-1} = S_2^{1+\mathfrak{b}/2} S_1 S_4^{\mathfrak{b}/2}.$$

If $\mathfrak{b}$ is odd, then

$$\widetilde{s}_{k-1} L(\widetilde{s}_{k-1}) = (S_2^2)^{(\mathfrak{b}+1)/2} S_3^2 (S_4^2)^{(\mathfrak{b}-1)/2} \quad \text{and} \quad \widetilde{s}_{k-1} = S_2^{(\mathfrak{b}+1)/2} S_3 S_4^{(\mathfrak{b}-1)/2}.$$

Thus w is a solution to (8) also in the case $k = 3$. $\qquad\square$

Note that a word w in the set $L(RStand^+(\alpha)) \setminus L(RStand(\alpha))$ is a solution to (8) but not in the language $\mathcal{L}(\alpha)$. Rather, w is a solution to (8) in $\mathcal{L}(\beta)$ where β is a suitable irrational such that $L(w)$ is a reversed standard word of slope β.

From Proposition 23 we conclude the following interesting fact:

Corollary 24. *There exist arbitrarily long primitive solutions of* (8) *in* $\mathcal{L}(\alpha)$.

We can now prove Theorem 18.

Proof of Theorem 18. By Proposition 23 and Theorem 13 it is sufficient to prove that *(i)* implies *(ii)*.

Suppose that w is a solution to (8) in $\mathcal{L}(\alpha)$. Write w^2 as a product of minimal squares: $w^2 = X_1^2 X_2^2 \cdots X_n^2$. Let $x \in [w^2]$. Then the word $s_{x,\alpha}$ begins with $X_1^2 X_2^2 \cdots X_n^2$, so by Theorem 9 the word $\sqrt{s_{x,\alpha}} = s_{\psi(x),\alpha}$ begins with $X_1 X_2 \cdots X_n$. Therefore $\psi(x) \in [X_1 X_2 \cdots X_n] = [w]$. Thus w^2 satisfies the square root condition. $\qquad\square$

6 A More Detailed Combinatorial Description of the Square Root Map

Recall from Section 3 that the square root $\sqrt{s}$ of a Sturmian word s has the same factors as s. The proofs were dynamical; we used the special mapping ψ on the circle. In this section we describe combinatorially why the language is preserved; we give a location for any prefix of $\sqrt{s}$ in s. As a side product, we are able to describe when a Sturmian word is uniquely factorizable as a product of squares of reversed (semi)standard words.

Let us begin with an introductory example. Recall from Section 3 the square root of the Fibonacci word f:

$$f = (010)^2(100)^2(10)^2(01)^2 0^2 (10010)^2 (01)^2 \cdots ,$$
$$\sqrt{f} = 010 \cdot 100 \cdot 10 \cdot 01 \cdot 0 \cdot 10010 \cdot 01 \cdots .$$

Obviously the square root $X_1 = 010$ of $(010)^2$ occurs as a prefix of f. Equally clearly the word $010 \cdot 100 = \sqrt{(010)^2(100)^2}$ occurs, not as a prefix, but after the prefix X_1 of f. Thus the position of the first occurrence of $010 \cdot 100$ shifted $|X_1| = 3$ positions from the position of the first occurrence of X_1. However, when comparing the position of the first occurrence of $\sqrt{(010)^2(100)^2(10)^2}$ with the first occurrence of $010 \cdot 100$, we see that there is no further shift. By further inspection, the word $\sqrt{(010)^2(100)^2(10)^2(01)^2 0^2 (10010)^2}$ occurs for the first time at position $|X_1|$ of f. This is no longer true for the first seven minimal squares; the first occurrence of $X_1 X_2 = 010 \cdot 100 \cdot 10 \cdot 01 \cdot 0 \cdot 10010 \cdot 01$ is at position $|X_1 X_2| = 16$ of f. The amount of shift from the previous position $|X_1| = 3$ is $|X_2| = 13$; observe that both of these numbers are Fibonacci numbers. Thus the amount of shift was exactly the length of the square roots added after observing the previous shift. As an observant reader might have noticed, both of the words X_1 and X_2 are reversed standard words, or equivalently, primitive solutions to (8). Repeating similar inspections on other Sturmian words suggests that there is a certain pattern to these shifts and that knowing the pattern would make it possible to locate prefixes of $\sqrt{s}$ in the Sturmian word s. Thus it makes very much sense to "accelerate" the square root map by considering squares of solutions to (8) instead of just minimal squares. Next we make these somewhat vague observations more precise.

Every Sturmian word has a solution of (8) as a square prefix. Next we aim to characterize Sturmian words having infinitely many solutions of (8) as square prefixes. The next two lemmas are key results towards such a characterization.

Lemma 25. *Consider the reversed (semi)standard word $\widetilde{s}_{k,\ell}$ of slope α with $k \geqslant 2$ and $0 < \ell \leqslant a_k$. The set $[\widetilde{s}_{k,\ell}] \setminus \{1 - \alpha\}$ equals the disjoint union*

$$\left(\bigcup_{i=0}^{\infty} \bigcup_{j=1}^{a_{k+2i}} [\widetilde{s}_{k+2i,j}^2] \right) \setminus \bigcup_{i=1}^{\ell-1} [\widetilde{s}_{k,i}^2].$$

Analogous representations exist for the sets $[\widetilde{s}_0] \setminus \{1 - \alpha\}$ and $[\widetilde{s}_1] \setminus \{1 - \alpha\}$.

To put it more simply: for each $x \neq 1-\alpha$ there exists a unique reversed (semi)standard word w such that $x \in [w^2]$. To illustrate the proof, we begin by giving a proof sketch.

Proof Sketch. Consider as an example the interval $[0] = I(0, 1-\alpha)$. It is easy to see that $[0^2] = I(0, -2\alpha) = I(0, -(q_0 + 1)\alpha)$, so $[0] = [0^2] \cup I(-(q_0 + 1)\alpha, 1 - \alpha)$. The interval $I(-(q_0 + 1)\alpha, 1 - \alpha)$ is the interval of the factor $\widetilde{s}_{2,1}$. Therefore $[0] = [\widetilde{s}_0^2] \cup [\widetilde{s}_{2,1}]$. Since $\widetilde{s}_{2,1}^2 \in \mathcal{L}(\alpha)$, the interval $[\widetilde{s}_{2,1}]$ splits into two parts: $[\widetilde{s}_{2,1}] = [\widetilde{s}_{2,1}^2] \cup J$. It is straightforward to show that $J = I(-(q_{2,1} + 1)\alpha, 1 - \alpha)$. Again, the interval J is the interval of the factor w which equals either $\widetilde{s}_{2,2}$ or $\widetilde{s}_{4,1}$ depending on the number a_2. Therefore $[0] = [\widetilde{s}_0^2] \cup [\widetilde{s}_{2,1}^2] \cup [w]$. This process can be repeated for the interval $[w]$ and indefinitely after that. The very same idea can be applied to any interval $[\widetilde{s}_{k,\ell}]$. $\square$

Proof of Lemma 25. Consider the lengths of the reversed (semi)standard words beginning with the same letter as $\widetilde{s}_{k,\ell}$. Out of these lengths we can form the unique increasing sequence (b_n) such that $b_1 = q_{k,\ell-1}$. If we set $s_1 = \widetilde{s}_{k,\ell}$ and $J_1 = I(-(b_1 + 1)\alpha, 1 - \alpha)$, then based on the observations in the proof of Proposition 16 we see that $J_1 = [s_1]$. The interval J_1 is split by the point $\{-(q_{k,\ell} + 1)\alpha\} = \{-(b_2 + 1)\alpha\}$. It must be that $[s_1^2] = I(-(b_1 + 1)\alpha, -(b_2 + 1)\alpha)$. Otherwise $[s_1^2] = [s_1] \cap R^{-b_2}([s_1]) = I(-(b_2 + 1)\alpha, 1 - \alpha)$, so the points $\{-(b_1 + b_2)\alpha\}$ and $\{-b_1\alpha\}$ are on the opposite sides of 0. Furthermore, $\|(b_1 + b_2)\alpha\|$ equals the distance between the points $\{-b_1\alpha\}$ and $\{-b_2\alpha\}$, so $\|(b_1 + b_2)\alpha\| = \|q_{k-1}\alpha\|$. Since also the point $\{-q_{k-1}\alpha\}$ is on the side opposite to $\{-b_1\alpha\}$, it follows that $q_{k-1} = b_1 + b_2$ which is obviously false. Thus $J_2 = J_1 \setminus [s_1^2] = I(-(b_2 + 1)\alpha, 1 - \alpha)$ is the interval of s_2, the unique reversed (semi)standard word of length b_3 beginning with the same letter as s_1. By repeating this when $n > 1$, we see that the interval J_n is split by the point $\{-(b_{n+1} + 1)\alpha\}$ and that $[s_n^2] = I(-(b_n + 1)\alpha, -(b_{n+1} + 1)\alpha)$. Then there is a unique reversed (semi)standard word s_{n+1} such that $[s_{n+1}] = I(-(b_{n+1} + 1)\alpha, 1 - \alpha) = J_n \setminus [s_n^2]$; we set $J_{n+1} = [s_{n+1}]$. By the definition of the sequence (b_n), the words s_{n+1} and s_1 begin with the same letter. This yields a well-defined sequence (J_n) of nested subintervals of J_1. It is clear that $|J_n| \to 0$ as $n \to \infty$. It follows that

$$[\widetilde{s}_{k,\ell}] \cup \{1 - \alpha\} = J_1 \cup \{1 - \alpha\} = \bigcup_{n=1}^{\infty} [s_n^2] \cup \{1 - \alpha\}.$$

The sets $[s_n^2]$ are by definition disjoint. The claim follows since the indexing in the claim is just another way to express the reversed (semi)standard words having lengths from the sequence (b_n).

The above proof works as it is for the cases $\widetilde{s}_0$ and $\widetilde{s}_1$; only minor adjustments in notation are needed. $\square$

Lemma 26. *Let $u \in RStand^+(\alpha)$ and $v \in RStand^+(\alpha) \cup L(RStand^+(\alpha))$. Then u^2 is never a proper prefix of v^2.*

Proof. If $v \in RStand^+(\alpha)$ and $|u| \neq |v|$, then by Lemma 25 (see the remark after its statement), the intervals $[u^2]$ and $[v^2]$ are disjoint. Hence u^2 can never be a proper prefix of v^2. Assume then that $v \in L(RStand^+(\alpha))$. If $|v| \leqslant |\widetilde{s}_1|$, then v^2 is a minimal

square, so it is not possible for u^2 to be a proper prefix of v^2. Suppose that $|v| = |\widetilde{s}_{k,\ell}|$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$. As in the proof of Proposition 16, we have that $[v] = I(-(q_{k-1}+1)\alpha, 1-\alpha)$. If u begins with the same letter as v and $|u| < |v|$, then $|u| \leqslant |\widetilde{s}_{k-1}|$. It follows, as in the proof of Lemma 25, that the distance between $1-\alpha$ and either of the endpoints of the interval $[u^2]$ must be at least $\|q_{k-1}\alpha\|$. Hence the intervals $[v]$ and $[u^2]$ are disjoint, so u^2 is not a proper prefix of v^2. $\square$

Let s be a fixed Sturmian word of slope α. Since the index of a factor of a Sturmian word is finite, Lemma 26 and Theorem 18 imply that if s has infinitely many solutions of (8) as square prefixes then no word in $RStand^+(\alpha)$ is a square prefix of s. We have now the proper tools to prove the following:

Proposition 27. *Let $s_{x,\alpha}$ be a Sturmian word of slope α and intercept x. Then $s_{x,\alpha}$ begins with a square of a word in $RStand^+(\alpha)$ if and only if $x \neq 1-\alpha$.*

Proof. If $x \neq 1-\alpha$, then $x \in I_0 \setminus \{1-\alpha\} = [\widetilde{s}_0] \setminus \{1-\alpha\}$ or $x \in I_1 \setminus \{1-\alpha\} = [\widetilde{s}_1] \setminus \{1-\alpha\}$. Thus by applying Lemma 25 to $I_0 \setminus \{1-\alpha\}$ or $I_1 \setminus \{1-\alpha\}$, we see that the word $s_{x,\alpha}$ begins with a square of a word in $RStand^+(\alpha)$.

Suppose then that $x = 1-\alpha$. Then $s_{x,\alpha} \in \{01c_\alpha, 10c_\alpha\}$. It is a well-known fact that $s_{2k} = P_{2k}10$ and $s_{2k+1} = Q_{2k+1}01$ for some palindromes P_{2k} and Q_{2k+1} for every $k \geqslant 1$ (see e.g. [11, Lemma 2.2.8]). As $c_\alpha = \lim_{k\to\infty} s_k$, it follows that $01c_\alpha = \lim_{k\to\infty} \widetilde{s}_{2k}$ and $10c_\alpha = \lim_{k\to\infty} \widetilde{s}_{2k+1}$. Hence by Lemma 26, the word $s_{x,\alpha}$ cannot have as a prefix a square of a word in $RStand^+(\alpha)$. $\square$

It follows that if s has infinitely many solutions of (8) as square prefixes, then $s \in \{01c_\alpha, 10c_\alpha\}$.

Next we take one extra step and characterize when s can be written as a product of squares of words in $RStand^+(\alpha)$.

Theorem 28. *A Sturmian word s of slope α can be written as a product of squares of words in $RStand^+(\alpha)$ if and only if s is not of the form $X_1^2 X_2^2 \cdots X_n^2 c$ where $X_i \in RStand^+(\alpha)$ and $c \in \{01c_\alpha, 10c_\alpha\}$. If s is a product of squares in $RStand^+(\alpha)$, then this product is unique.*

Proof. This is a direct consequence of Proposition 27 and Lemma 26. $\square$

Suppose that $s \notin \{01c_\alpha, 10c_\alpha\}$. Then the word s has only finitely many solutions of (8) as square prefixes. We call the longest solution *maximal*. Observe that the maximal solution is not necessarily primitive since any power of a solution to (8) is also a solution. Sturmian words of slope α can be classified into two types.

Type A. Sturmian words s of slope α which can be written as products of maximal solutions to (8). In other words, it can be written that $s = X_1^2 X_2^2 \cdots$ where X_i is the maximal solution occurring as a square prefix of the word $T^{h_i}(s)$ where $h_i = |X_1^2 X_2^2 \cdots X_{i-1}^2|$.

Type B. Sturmian words s of slope α which are of the form $s = X_1^2 X_2^2 \cdots X_n^2 c$ where $c \in \{01c_\alpha, 10c_\alpha\}$ and the words X_i are maximal solutions as above.

Proposition 27 and Lemma 26 imply that the words X_i in the above definitions are uniquely determined and that the primitive root of a maximal solution is in $RStand^+(\alpha)$. Consequently, a maximal solution is always right special. When finding the factorization of a Sturmian word as a product of squares of maximal solutions, it is sufficient to detect at each position the shortest square of a word in $RStand^+(\alpha)$ and take its largest even power occurring in that position.

Keeping the Sturmian word s of slope α fixed, we define two sequences (μ_k) and (λ_k). We set $\mu_0 = \lambda_0 = \varepsilon$. Following the notation above, we define depending on the type of s as follows.

(A) If s is of type A, then we set for all $k \geqslant 1$ that

$$\mu_k = X_1^2 X_2^2 \cdots X_k^2 \quad \text{and}$$
$$\lambda_k = X_1 X_2 \cdots X_k.$$

(B) If s is of type B, then we set for $1 \leqslant k \leqslant n$ that

$$\mu_k = X_1^2 X_2^2 \cdots X_k^2 \quad \text{and}$$
$$\lambda_k = X_1 X_2 \cdots X_k,$$

and we let

$$\mu_{n+1} = X_1^2 X_2^2 \cdots X_n^2 c \quad \text{and}$$
$$\lambda_{n+1} = X_1 X_2 \cdots X_n c.$$

Compare these definitions with the example in the beginning of this section; the words X_1 and X_2 are maximal solutions in the Fibonacci word (which is of type A).

We are finally in a position to formulate precisely the observations made in the beginning of this section and state the main result of this section.

Theorem 29. *Let s be a Sturmian word with slope α.*
(A) If s is of type A, then
$$\sqrt{s} = \lim_{k \to \infty} T^{|\lambda_k|}(s).$$
Moreover, the first occurrence of the prefix λ_{k+1} of $\sqrt{s}$ is at position $|\lambda_k|$ of s for all $k \geqslant 0$.
(B) If s is of type B, then
$$\sqrt{s} = T^{|\lambda_n|}(s).$$
Moreover, the first occurrence of the prefix λ_{k+1} with $0 \leqslant k \leqslant n-1$ is at position $|\lambda_k|$ of s, and the first occurrence of any prefix of $\sqrt{s}$ having lenght greater than $|\lambda_n|$ is at position $|\lambda_n|$ of s.
In particular $\sqrt{s}$ is a Sturmian word with slope α.

The theorem only states where the prefixes λ_k of $\sqrt{s}$ occur for the first time. For the first occurrence of other prefixes of $\sqrt{s}$ we do not have a guaranteed location.

To illustrate the theorem, consider next τ, the eighth shift of the Fibonacci word. If we write under the word τ each of the corresponding words λ_k at the position of their first
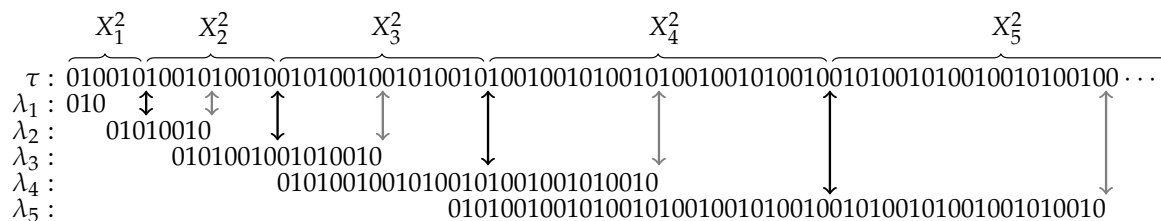
Figure 2: The first occurrences of the words λ_k in τ. The eighth shift of the Fibonacci word was used since for the Fibonacci word the lengths $|\lambda_k|$ grow very rapidly.

occurrence we get the picture in Figure 2. Theorem 29 shows that the nice pattern where the words λ_k overlap continues indefinitely and, moreover, that if we replace τ with any other Sturmian word (of type A) we obtain a similar picture. Most of the results of this paper were motivated by the discovery of this pattern.

Before proving the theorem we need one more result.

Proposition 30. *Suppose that s is a Sturmian word of type A. Then the word λ_k is right special and a suffix of the word μ_k for all $k \geqslant 0$.*

Proof. This proof might be tricky to follow. We advise the reader to keep the picture of Figure 3 in mind while reading the proof. This picture depicts only the Case A below but is surely helpful.

The assertion is evident when $k = 0$. Suppose that $k \geqslant 0$ and assume that λ_k is right special and that λ_k is a suffix of the word μ_k. It is equivalent to say that $\{-(|\lambda_k|+1)\alpha\} \in [\lambda_k]$ and $[\mu_k] \subseteq R^{-|\lambda_k|}([\lambda_k])$ (evidently $2|\lambda_k| = |\mu_k|$). We write simply $\lambda = \lambda_k$, $\mu = \mu_k$, and $X = X_{k+1}$. This proof utilizes only the facts that $\mu X^2 \in \mathcal{L}(\alpha)$ and that λ is right special and a suffix of the word μ, not the structure of the words λ and μ implied by their definitions. Thus without loss of generality, we may assume that X is primitive. Consequently, $X \in RStand^+(\alpha)$. It follows that

$$[X] = I(-(q+1)\alpha, 1-\alpha) \text{ and} \tag{9}$$
$$[X^2] = [X] \cap R^{-|X|}([X]) = I(-(q+1)\alpha, -(|X|+1)\alpha).$$

for some nonnegative integer q. Let $x = \{-(|\mu|+1)\alpha\}$. It follows from the hypothesis $\{-(|\lambda|+1)\alpha\} \in [\lambda]$ that $x \in R^{-|\lambda|}([\lambda])$. By (9) the point x is an endpoint of the interval $R^{-|\mu|}([X])$.

Let then $y = \{-(|\mu X|+1)\alpha\}$. By (9) the point y is an endpoint of the interval $R^{-|\mu X|}([X])$ and an interior point of the interval $R^{-|\mu|}([X])$. Suppose for a contradiction that $y \notin R^{-|\lambda|}([\lambda])$. As $x \notin R^{-|\mu X|}([X])$ (otherwise it would follow that $1-\alpha \in R^{-|X|}([X])$ which contradicts (9)), it follows that $R^{-|\lambda|}([\lambda]) \cap R^{-|\mu X|}([X]) = \emptyset$. Since

$$[\mu_{k+1}] = [\mu] \cap R^{-|\mu|}([X]) \cap R^{-|\mu X|}([X]),$$

we have that $[\mu_{k+1}] \subseteq R^{-|\mu X|}([X])$. By assumption $[\mu_{k+1}] \subseteq [\mu] \subseteq R^{-|\lambda|}([\lambda])$. Thus $[\mu_{k+1}] \subseteq R^{-|\lambda|}([\lambda]) \cap R^{-|\mu X|}([X])$, so by the above we are forced to conclude that $[\mu_{k+1}] = \emptyset$.
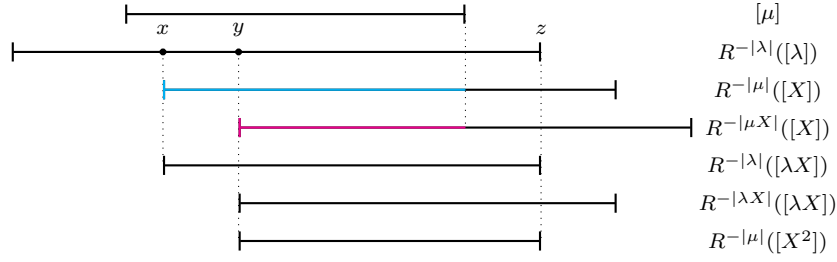
Figure 3: A possible arrangement for the intervals in the Case A of the proof of Proposition 30. The blue color marks the interval $[\mu X]$ and magenta marks the interval $[\mu X^2] = [\mu_{k+1}]$.
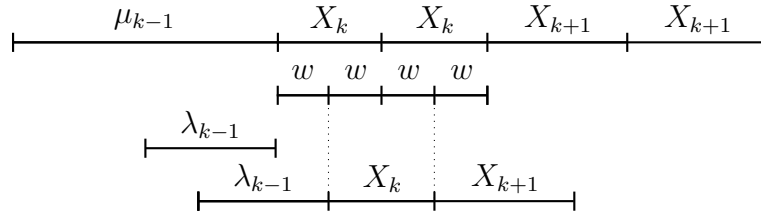

Figure 4: Possible locations for factors in the proof of Theorem 29.

This is a contradiction since X is chosen in such a way that $[\mu_{k+1}] = [\mu X^2] \neq \emptyset$. We conclude that $y \in R^{-|\lambda|}([\lambda])$.

Now $R^{-|\lambda|}([\lambda X]) = R^{-|\lambda|}([\lambda]) \cap R^{-|\mu|}([X])$. Since $y \in R^{-|\lambda|}([\lambda]), R^{-|\mu|}([X])$, it follows that $y = \{-(|\mu X| + 1)\alpha\} \in R^{-|\lambda|}([\lambda X])$. Thus $R^{|\lambda|}(y) = \{-(|\lambda X| + 1)\alpha\} \in [\lambda X]$, so the word λX is right special. We have two cases depending on the length of the interval $R^{-|\mu|}([X])$ compared to the length of the interval $R^{-|\lambda|}([\lambda])$.

Case A. $R^{-|\mu|}([X]) \nsubseteq R^{-|\lambda|}([\lambda])$. In this case $R^{-|\lambda|}([\lambda X]) = I(x, z)$, where z is an endpoint of $R^{-|\lambda|}([\lambda])$. Since y is an interior point of $R^{-|\lambda|}([\lambda X])$, $R^{-|X|}(x) = y$, and $x \notin R^{-|\lambda X|}([\lambda X])$, we obtain that $I(y, z) \subseteq R^{-|\lambda X|}([\lambda X])$. Since y is also an interior point of $R^{-|\lambda|}([\lambda])$, we obtain similarly that $R^{-|\lambda|}([\lambda]) \cap R^{-|\mu|}([X^2]) = I(y, z)$. Thus

$$[\mu_{k+1}] = [\mu] \cap R^{-|\mu|}([X^2]) \subseteq R^{-|\lambda|}([\lambda]) \cap R^{-|\mu|}([X^2]) = I(y, z) \subseteq R^{-|\lambda X|}([\lambda X]).$$

This proves that $\lambda X = \lambda_{k+1}$ is a suffix of μ_{k+1}.

Case B. $R^{-|\mu|}([X]) \subseteq R^{-|\lambda|}([\lambda])$. It follows that $R^{-|\lambda|}([\lambda X]) = R^{-|\mu|}([X])$, so we have $R^{-|\lambda X|}([\lambda X]) = R^{-|\mu X|}([X])$. Since $R^{-|\mu|}([X^2]) \subseteq R^{-|\mu X|}([X])$, we get that

$$[\mu_{k+1}] = [\mu] \cap R^{-|\mu|}([X^2]) \subseteq [\mu] \cap R^{-|\mu X|}([X]) \subseteq [\mu] \cap R^{-|\lambda X|}([\lambda X])$$

proving that also in this case $\lambda X = \lambda_{k+1}$ is a suffix of μ_{k+1}. $\square$

Note that even though λ_k is right special and always a suffix of μ_k, it is not necessary for μ_k to be right special.

Proof of Theorem 29. Since Sturmian words of type B differ from Sturmian words of type A essentially only by the fact that the sequence of maximal solutions is finite, it is in this proof enough to consider the case that s is of type A.

Proposition 30 says that λ_k is always a suffix of μ_k for all $k \geqslant 0$. Since $|\mu_k| = 2|\lambda_k|$, it follows that the word $T^{|\lambda_k|}(s)$ has the word λ_k as a prefix. Therefore $\sqrt{s} = \lim_{k \to \infty} T^{|\lambda_k|}(s)$.

It remains to prove that the first occurrence of λ_{k+1} in s is at position $|\lambda_k|$ of s for all $k \geqslant 0$. It is clear that the first occurrence of $\lambda_1 = X_1$ is at position $|\lambda_0| = 0$. Assume that $k > 0$, and suppose for a contradiction that λ_{k+1} occurs before the position $|\lambda_k|$. Since λ_k is a prefix of λ_{k+1}, by induction we see that λ_{k+1} cannot occur before the position $|\lambda_{k-1}|$. This means that an occurrence of $X_k X_{k+1}$ begins in s at position ν such that $|\mu_{k-1}| \leqslant \nu < |\mu_{k-1} X_k|$; see Figure 4. Observe that s has at position $|\mu_{k-1}|$ an occurrence of X_k^2. Write now $X_k = w^t$ with $w \in RStand^+(\alpha)$. Since w is primitive, we must have that $\nu = |\mu_{k-1}| + r|w|$ with $0 \leqslant r < t$. Thus X_{k+1} occurs in s at position $\nu + |X_k| = |\mu_{k-1}| + (r+t)|w|$. Since $r < t$, it follows that either w is a prefix of X_{k+1} or X_{k+1} is a prefix of w.

Suppose first that w is a prefix of X_{k+1}. If $w = X_{k+1}$, then the prefix $\mu_{k-1} X_k^2$ of s is followed by w^2. Now w^{2t+2} is a solution to (8) implying that X_k is not a maximal solution to (8). Since this is contradictory, we infer that $|w| < |X_{k+1}|$. Since X_{k+1} occurs at position $|\mu_{k-1}| + (r+t)|w| < |\mu_k|$ and X_{k+1} has w as a prefix, it must be that X_{k+1} begins with wa where a is the first letter of w. Since w is right special and $w^2 \in \mathcal{L}(\alpha)$, it follows that X_{k+1}^2 begins with w^2. Like above, this implies that X_k is not maximal. This is a contradiction.

Suppose then that X_{k+1} is a proper prefix of w. First of all, X_{k+1} must be primitive as otherwise X_{k+1}, and consequently also w, would have as a prefix a square of some word in $RStand^+(\alpha)$ contradicting Lemma 26. The assumption that X_{k+1} is a prefix of w implies that X_{k+1} and w begin with the same letter. Like above, since w is right special and $w^2 \in \mathcal{L}(\alpha)$, it must be that w occurs after the prefix μ_k of s. Since also X_{k+1}^2 occurs after the prefix μ_k, by Lemma 26 we conclude that the word w must be a proper prefix of X_{k+1}^2. Observe now that the assumption that X_{k+1} is a proper prefix of w excludes the possibilities that $w = \widetilde{s}_0 = 0$ or $w = \widetilde{s}_1 = 10^{\mathsf{a}}$. Therefore $w = \widetilde{s}_{h,\ell}$ for some $h \geqslant 2$ with $0 < \ell \leqslant a_h$. Because $|w| < 2|X_{k+1}|$, we must have that $|X_{k+1}| > |\widetilde{s}_{h-2}|$. On the other hand, since $|X_{k+1}| < |w|$ and X_{k+1} and w begin with the same letter, the only option is that $X_{k+1} = \widetilde{s}_{h,\ell'}$ with $0 < \ell' < \ell$. Now

$$X_{k+1}^2 = (\widetilde{s}_{h-2}\widetilde{s}_{h-1}^{\ell'})^2 = \widetilde{s}_{h-2}\widetilde{s}_{h-1}^{\ell'} L(\widetilde{s}_{h-1})\widetilde{s}_{h-2}\widetilde{s}_{h-1}^{\ell'-1},$$

so as w is a prefix of X_{k+1}^2, it must be that $\widetilde{s}_{h-1} = L(\widetilde{s}_{h-1})$. This is a contradiction. This final contradiction ends the proof. $\qquad \square$

As a conclusion of this section, we study the lengths of the maximal solutions of (8). Namely, let $s = X_1^2 X_2^2 \cdots$ be a Sturmian word of type A factorized as a product of maximal solutions X_i. Computer experiments suggest that typically the sequence $(|X_i|)$ is strictly increasing. However, there are examples where $|X_i| > |X_{i+1}|$ for some $i \geqslant 1$. It is natural to ask if the lengths can decrease significantly or if oscillation is possible. It turns out that neither is possible. In Corollary 33 we prove that $\liminf_{i \to \infty} |X_i| = \infty$.

First we need a result on certain periods of (semi)standard words.

Lemma 31. *Let $u, v \in Stand^+(\alpha)$ and $|u| > |v|$. If u is a prefix of some word in v^+, then $u = s_{k,\ell}$ and $v = s_{k-1}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$.*

Proof. Suppose that u is a prefix of some word in v^+. If $u = s_1 = 0^a 1$, then necessarily $v = s_0 = 0$. Then obviously u is not a prefix of any word in v^+. Therefore $u = s_{k,\ell}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$. Suppose that $k = 2$. Then $u = (0^a 1)^\ell 0$. It is straightforward to show that v must equal to $s_1 = 0^a 1$; u cannot be a prefix of a word in v^+ if $v = s_0 = 0$ or $v = s_{2,\ell'}$ for some ℓ' such that $0 < \ell' < \ell$. Thus we may assume that $k > 2$.

Suppose first that $|v| > |s_{k-1}|$. Then by the assumption $|u| > |v|$, it must be that $v = s_{k,\ell'}$ for some ℓ' such that $\ell' < \ell$. Since u is a prefix of some word in v^+, it follows that the word $w = s_{k-1}^{\ell-\ell'} s_{k-2}$ is a prefix of some word in $s_{k-2} v^+$. Since the word w begins with $s_{k-1} s_{k-2}$, we obtain that $s_{k-2} v$ begins with $s_{k-1} s_{k-2}$, so $s_{k-1} s_{k-2} = s_{k-2} s_{k-1}$. This is a contradiction.

Assume then that $|v| < |s_{k-1}|$. Now the prefix s_{k-1} of u is a prefix of some word in v^+, so by induction $v = s_{k-2}$. Now $u = (s_{k-2}^{a_{k-1}} s_{k-3})^\ell s_{k-2}$, so as u is a prefix of some word in v^+, it follows that $z = s_{k-3} s_{k-2}$ is a prefix of some word in v^+. This means that z ends with a prefix of s_{k-2} of length $|s_{k-3}|$. As the prefix of s_{k-2} of length $|s_{k-3}|$ is s_{k-3}, the word z ends with s_{k-3}. Consequently $s_{k-3} s_{k-2} = s_{k-2} s_{k-3}$; a contradiction.

The only remaining option is that $v = s_{k-1}$. This is certainly possible. $\qquad\square$

The next proposition describes precisely under which conditions it is possible that $|X_i| > |X_{i+1}|$. Moreover, it rules out the possibility that the lengths decrease significantly or oscillate.

Proposition 32. *Let $s = X_1^2 X_2^2 X_3^2 \cdots$ be a Sturmian word of type A with slope α factorized as a product of maximal solutions X_i. If $|X_1| > |X_2|$, then $X_1 = \widetilde{s}_{k,\ell}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k - 1$, the primitive root of X_2 is $\widetilde{s}_{k-1}$, and $|X_3| > |X_1|$.*

Proof. Assume that $|X_1| > |X_2|$. Let us first make the additional assumption that X_1 is primitive. In particular, $X_1 \in RStand^+(\alpha)$. Let u be the primitive root of X_2. Then $u \in RStand^+(\alpha)$ and, moreover, by the assumption $|X_1| > |X_2|$ it holds that $|u| < |X_1|$. By Proposition 30 the word $\lambda_2 = X_1 X_2$ is a suffix of the word $\mu_2 = X_1^2 X_2^2$. Therefore X_1 is a proper suffix of $X_1 X_2$, so $X_1 X_2 = Z X_1$ for some nonempty word Z. A standard argument shows that X_1 is a suffix of some word in X_2^+ (see e.g., [10, Proposition 1.3.4]). Consequently, $\widetilde{X}_1$ is a prefix of a word in $\widetilde{u}^+$. As $|u| < |X_1|$, Lemma 31 implies that $X_1 = \widetilde{s}_{k,\ell}$ and $u = \widetilde{s}_{k-1}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k$.

Suppose now that $\ell = a_k$. Then the word $X_1^2 X_2^2$ contains $\widetilde{s}_{k-1} \widetilde{s}_{k-2} \widetilde{s}_{k-1}^{a_k+2}$ as a factor. Thus $s_{k-1}^{a_k+2} s_{k-2} s_{k-1} \in \mathcal{L}(\alpha)$. As s_{k-1} is a prefix of $s_{k-2} s_{k-1}$, it follows that $s_{k-1}^{a_k+3} \in \mathcal{L}(\alpha)$ contradicting Proposition 7. Therefore $\ell \leqslant a_k - 1$.

Let us then relax the assumption that X_1 is primitive. Let v be the primitive root of X_1, so that $X_1 = v^j$ for some $j \geqslant 1$. Consider now the Sturmian word $T^{(2j-2)|v|}(s) = v^2 X_2^2 \cdots$. By the above arguments $v = \widetilde{s}_{k,\ell}$ for some $k \geqslant 2$ with $0 < \ell \leqslant a_k - 1$ and the primitive

root of X_2 is $\widetilde{s}_{k-1}$. Further, as $\ell \neq a_k$, it follows from Proposition 7 that $v^3 \notin \mathcal{L}(\alpha)$. Thus $j = 1$, that is, $X_1 = \widetilde{s}_{k,\ell}$.

It remains to show that $|X_3| > |X_1|$. Assume for a contradiction that $|X_3| \leqslant |X_1|$. It is not possible that $|X_3| < |X_2|$ as the preceding arguments show that then X_2 must be a reversed semistandard word; however, X_2 is a power of the reversed standard word $\widetilde{s}_{k-1}$. Hence by the maximality of X_2 we have that $|X_3| > |X_2|$. Let $X_3 = w^t$ with $w \in RStand^+(\alpha)$ and $t \geqslant 1$. As $|X_2| < |X_3| \leqslant |X_1|$, we have that $|s_{k-1}| < t|w| \leqslant |s_{k,\ell}|$.

Assume for a contradiction that $|w| < |s_{k-1}|$. If w is a reversed semistandard word, then Proposition 7 implies that $t = 1$, so $t|w| > |s_{k-1}|$ cannot hold. Thus w is a reversed standard word. If $w = \widetilde{s}_0 = 0$, then clearly $t|w| > |s_{k-1}| \geqslant |s_1|$ cannot hold as the index of the factor 0 in $\mathcal{L}(\alpha)$ is $a_1 + 1$. Thus $w \neq \widetilde{s}_0$. Suppose first that $w = \widetilde{s}_{k-2}$. Now

$$t|w| > |s_{k-1}| = a_{k-1}|s_{k-2}| + |s_{k-3}|,$$

so $t > a_{k-1}$. Since $X_3^2 \in \mathcal{L}(\alpha)$, Proposition 7 implies that $2t \leqslant a_{k-1} + 2$. Therefore

$$a_{k-1} + 2 \geqslant 2t > 2a_{k-1}$$

implying that $a_{k-1} = 1$. However, if $a_{k-1} = 1$, then $a_{k-1} + 2$ is odd, so actually $2t < a_{k-1} + 2$. Then $a_{k-1} + 2 > 2t > 2a_{k-1}$, so $a_{k-1} < 1$; a contradiction. Suppose then that $w = \widetilde{s}_{k-3}$. Now

$$t|w| > |s_{k-1}| \geqslant |s_{k-2}s_{k-3}| = |s_{k-3}^{a_{k-2}} s_{k-4} s_{k-3}| > (a_{k-2} + 1)|s_{k-3}|,$$

so $t > a_{k-2} + 1$. Like previously, as $X_3^2 \in \mathcal{L}(\alpha)$, Proposition 7 implies that $2t \leqslant a_{k-2} + 2$. Like above, we obtain that $a_{k-2} < 0$; a contradiction. Similar to above

$$|s_{k-1}| \geqslant (a_{k-2} + 1)|s_{k-3}| + |s_{k-4}| \geqslant 2|s_{k-3}| + |s_{k-4}| > (2a_{k-3} + 1)|s_{k-4}|.$$

As $2a_{k-3} + 1 \geqslant a_{k-3} + 2$, we conclude that $|s_{k-4}^{a_{k-3}+2}| < |s_{k-1}|$. Therefore by Proposition 7 it is not possible that $|w| \leqslant |s_{k-4}|$. In conclusion, it is not possible that $t|w| > |s_{k-1}|$. This is a contradiction.

Now $|w| > |\widetilde{s}_{k-1}|$ (by the maximality of X_2 it must be that $w \neq \widetilde{s}_{k-1}$). Because $|w| \leqslant |\widetilde{s}_{k,\ell}|$, we have that $w = \widetilde{s}_{k,\ell'}$ for some ℓ' such that $0 < \ell' \leqslant \ell$. Since $\ell \neq a_k$, the word w is a reversed semistandard word, so by Proposition 7 we have that $t = 1$. By Proposition 30 the word $\lambda_3 = X_1 X_2 X_3$ is a suffix of the word $\mu_3 = X_1^2 X_2^2 X_3^2$. It follows that $\widetilde{s}_{k-2}\widetilde{s}_{k-1}^{\ell+r} = \widetilde{s}_{k-1}^{\ell+r-\ell'}\widetilde{s}_{k-2}\widetilde{s}_{k-1}^{\ell'}$ where r is such that $X_2 = \widetilde{s}_{k-1}^r$. Therefore the words $\widetilde{s}_{k-2}$ and $\widetilde{s}_{k-1}$ commute; a contradiction. This final contradiction proves that $|X_3| > |X_1|$. $\square$

Corollary 33. *Let $s = X_1^2 X_2^2 \cdots$ be a Sturmian word of type A with slope α factorized as a product of maximal solutions X_i. Then $\liminf_{i \to \infty} |X_i| = \infty$.*

Proof. This follows from Proposition 32: if $|X_{i+1}| < |X_i|$ for some $i \geqslant 1$, then $|X_{i+2}| > |X_i|$. $\square$

7 The Square Root of the Fibonacci Word

In this section we prove a formula for the square root of the Fibonacci word. To do this we factorize the Fibonacci word as a product of maximal solutions to (8).

We denote by Φ the slope of the Fibonacci word, that is, $\Phi = [0; 2, 1, 1, \ldots]$. Further, we set

$$
t_k = \begin{cases} 01, & \text{if } k \text{ is even}, \\ 10, & \text{if } k \text{ is odd}. \end{cases}
$$

We need two lemmas specific to the slope Φ.

Lemma 34. *For the standard words of slope Φ it holds that $t_k s_k s_{k+1} s_{k+2} = \widetilde{s}_{k+2}^2 t_{k+1}$ for all $k \geqslant 0$.*

Proof. The case $k = 0$ is verified directly: $t_0 s_0 s_1 s_2 = 01 \cdot 0 \cdot 01 \cdot 010 = (010)^2 \cdot 10 = \widetilde{s}_2^2 t_1$. Let then $k \geqslant 1$. There exists a palindrome P_k such that $s_k = P_k \widetilde{t}_k$ for all $k \geqslant 1$ (see e.g. [11, Lemma 2.2.8]). Now

$$
\begin{aligned}
t_k s_k s_{k+1} s_{k+2} &= t_k P_k \widetilde{t}_k s_{k+1} s_{k+2} = \widetilde{s}_k \widetilde{t}_k P_{k+1} \widetilde{t}_{k+1} s_{k+2} = \widetilde{s}_k t_{k+1} P_{k+1} \widetilde{t}_{k+1} s_{k+2} \\
&= \widetilde{s}_k \widetilde{s}_{k+1} \widetilde{t}_{k+1} P_{k+2} \widetilde{t}_{k+2} = \widetilde{s}_k \widetilde{s}_{k+1} \widetilde{s}_{k+2} \widetilde{t}_{k+2} = \widetilde{s}_{k+2}^2 t_{k+1},
\end{aligned}
$$

which proves the claim. $\qquad\qquad\square$

Lemma 35. *For the standard words of slope Φ it holds that $s_{3k+4} = \prod_{i=0}^{k} \widetilde{s}_{3i+2}^2 \cdot t_{k+1}$ for all $k \geqslant 0$.*

Proof. If $k = 0$, then $s_4 = 01001010 = \widetilde{s}_2^2 t_1$. Let then $k \geqslant 1$. Now

$$
\begin{aligned}
s_{3k+4} &= s_{3k+3} s_{3k+2} = s_{3k+2} s_{3k+1} s_{3k+2} = s_{3k+1} s_{3k} s_{3k+1} s_{3k+2} \\
&= s_{3(k-1)+4} s_{3k} s_{3k+1} s_{3k+2} = \prod_{i=0}^{k-1} \widetilde{s}_{3i+2}^2 \cdot t_k s_{3k} s_{3k+1} s_{3k+2}
\end{aligned}
$$

where the last equality follows by induction. By applying Lemma 34 we obtain that

$$
s_{3k+4} = \prod_{i=0}^{k-1} \widetilde{s}_{3i+2}^2 \cdot \widetilde{s}_{3k+2}^2 t_{k+1} = \prod_{i=0}^{k} \widetilde{s}_{3i+2}^2 \cdot t_{k+1},
$$

which proves the claim. $\qquad\qquad\square$

As an immediate corollary to Lemma 35 we obtain a formula for the square root of the Fibonacci word.

Theorem 36. *For slope Φ we have that*

$$
c_\Phi = \prod_{i=0}^{\infty} \widetilde{s}_{3i+2}^2 \quad \text{and} \quad \sqrt{c_\Phi} = s_{\frac{1}{2}, \Phi} = \prod_{i=0}^{\infty} \widetilde{s}_{3i+2}.
$$

a_3 / a_4	1	2	3
1	2, 5, 8	2, 4, 7	2, 5, 8
2	2, 3, 6	2, 4, 7	2, 3, 6
3	2, 3, 5	2, 4, 7	2, 3, 5

Table 1: How X_1, X_2, and X_3 are affected when a_3 and a_4 vary in the case that $a_1 = 2$ and $a_2 = 1$.

a_3 / a_4	1	2	3
1	1	0	1
2	1	0	1
3	0	0	0

Table 2: How the first letter of X_4 varies when a_3 and a_4 vary in the case that $a_1 = 2$ and $a_2 = 1$.

The preceding arguments are very specific to the Fibonacci word. The reader might wonder if formulas for the square roots of other standard Sturmian words exist. Surely, for some specific words such formulas can be derived, but we believe no general factorization for the square roots of standard Sturmian words can be given. Let us give some arguments supporting our belief.

Let $s = X_1^2 X_2^2 \cdots$ be a standard Sturmian word of slope α factorized as a product of maximal solutions to (8). The word s begins with the word $0^{\mathfrak{a}}1$. Therefore if $\mathfrak{a} > 1$, then $X_1 = 0^{\lfloor \mathfrak{a}/2 \rfloor}$. Thus if $\mathfrak{a} > 1$, then X_2 begins with 0 if and only if $\mathfrak{a}$ is odd. Because of the asymmetry of the letters 0 and 1 in the minimal squares of slope α (1), the parity of the parameter $\mathfrak{a}$ greatly influences the remaining words X_i. Moreover, it is not just the partial quotient a_1 which influences the factorization. Suppose for instance that $a_1 = 2$ and $a_2 = 1$. Table 1 shows how the values of the partial quotients a_3 and a_4 affect the words X_i. The cell of the table tells which squares of reversed standard words the words X_1, X_2, and X_3 correspond to. For example if $a_3 = 2$ and $a_4 = 1$, then the standard Sturmian word of slope $[0; 2, 1, 2, 1, \ldots]$ begins with $\widetilde{s}_2^2 \widetilde{s}_4^2 \widetilde{s}_7^2$. Table 2 tells the first letter of the corresponding word X_4^2. As can be observed from Table 2, the first letter of X_4^2 varies when a_3 and a_4 vary. Because of the asymmetry, it is thus expected that slight variation in partial quotients drastically changes the factorization as a product of maximal solutions to (8). Since similar behavior is expected from the rest of the partial quotients, it seems to us that no nice formula (like e.g., the formula of Theorem 36) can be given for the square root of a standard Sturmian word in terms of reversed standard words.

De Luca and Fici proved a nice formula for a certain shift of a standard Sturmian word [4, Theorem 18].

Proposition 37. *Let c_α be the standard Sturmian word of slope $\alpha = [0; \mathfrak{a} + 1, \mathfrak{b} + 1, \ldots]$.*

Then

$$c_\alpha = 0^\mathfrak{a} 10^{\mathfrak{a}-1} \prod_{k=1}^{\infty} \widetilde{s}_k^2.$$

As a corollary of this we see that the word $\sqrt{T^{2\mathfrak{a}}(c_\alpha)} = \prod_{k=1}^{\infty} \widetilde{s}_k$ is a Sturmian word of slope α with intercept $\psi(\{(2\mathfrak{a}+1)\alpha\}) = \mathfrak{a}\alpha$. We have thus shown that

$$c_\alpha = 0^{\mathfrak{a}-1} \prod_{k=1}^{\infty} \widetilde{s}_k.$$

In particular, we obtain the well-known result that the Fibonacci infinite word is a product of the reversed Fibonacci words.

8 A Curious Family of Subshifts

In this section we construct a family of linearly recurrent and optimal squareful words which are not Sturmian but are fixed points of the (more general) square root map. Moreover, we show that any subshift Ω generated by such a word has a curious property: for every $w \in \Omega$ either $\sqrt{w} \in \Omega$ or $\sqrt{w}$ is periodic.

It is evident from Proposition 2 that Sturmian words are a proper subclass of optimal squareful words. As Sturmian words have the exceptional property that their language is preserved under the square root map, it is natural to ask if other optimal squareful words can have this property. We show that, indeed, such words exist by an explicit construction. The idea behind the construction is to mimic the structure of the Sturmian words $01c_\alpha$ and $10c_\alpha$. The simple reason why these words are fixed points of the square root map (thus preserving the language) is that they have arbitrarily long squares of solutions to (8) as prefixes. Thus to obtain a fixed point of the square root map, it is sufficient to find a sequence (u_k) of solutions to (8) with the property that u_k^2 is a proper prefix of u_{k+1}^2 for all $k \geqslant 1$. Let us show how such a sequence can be obtained.

Let S be a fixed primitive solution to (8) in the language of some Sturmian word with slope $[0; \mathfrak{a}+1, \mathfrak{b}+1, \ldots]$ such that $|S| > |S_6|$. In particular, S has the word $S_6 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$ as a proper suffix. Recall from the proof of Lemma 22 that $|S| \geqslant |S_5 S_6|$. We denote the word $L(S)$ simply by L. Using the word S as a seed solution, we produce a sequence (γ_k) of primitive solutions to (8) defined by the recurrence

$$\gamma_1 = S, \quad \gamma_{k+1} = L(\gamma_k)\gamma_k^2 \quad \text{for } k \geqslant 2. \tag{10}$$

We need to prove that the sequence (γ_k) really is a sequence of primitive solutions to (8). Before showing this, let us define

$$\Gamma_1 = \lim_{k \to \infty} \gamma_{2k} \quad \text{and} \quad \Gamma_2 = \lim_{k \to \infty} \gamma_{2k+1}. \tag{11}$$

The limits exist as γ_k^2 is always a prefix of γ_{k+2}. Hence both Γ_1 and Γ_2 have arbitrarily long squares of words in the sequence (γ_k) as prefixes. Observe also that $\mathcal{L}(\Gamma_1) = \mathcal{L}(\Gamma_2)$.

As there is not much difference between Γ_1 and Γ_2 in terms of structure, we set Γ to be either of these words.

Taking for granted that the sequence (γ_k) is a sequence of solutions to (8), we see that $\sqrt{\Gamma} = \Gamma$. Note that we also need to ensure that the word Γ is optimal squareful for the square root map to make sense.

Next we aim to prove the following.

Proposition 38. *The word γ_k is a primitive solution to (8) in $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$ for all $k \geqslant 1$.*

Recall from Section 5 that the language $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$ consists of all factors of the infinite words in the language

$$(10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}} + 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1})^{\omega} = (S_5 + S_6)^{\omega}.$$

Before we can prove Proposition 38, we need to know that the words γ_k are primitive and that they are factors of some optimal squareful word with parameters $\mathfrak{a}$ and $\mathfrak{b}$.

Lemma 39. *The word γ_k is primitive for all $k \geqslant 1$.*

Proof. We proceed by induction. By definition γ_1 is primitive. Let $k \geqslant 1$, and suppose for a contradiction that γ_{k+1} is not primitive; that is, $\gamma_{k+1} = L(\gamma_k)\gamma_k^2 = z^n$ for some primitive word z and $n > 1$. If $n = 2$, then obviously $|\gamma_k|$ must be even, and the suffix of γ_k of length $|\gamma_k|/2$ must be a prefix of γ_k. This contradicts the primitivity of γ_k. The case $n = 3$ would imply that $\gamma_k = L(\gamma_k)$, which is not possible. Hence $n > 3$, and further $|z| < |\gamma_k|$. As γ_k^2 is a suffix of some word in z^+, it follows that $z = uv$ where vu is a suffix of γ_k. On the other hand, z is a suffix of γ_k, so $uv = vu$. Since z is primitive, the only option is that u is empty. Thus $\gamma_k \in z^+$; a contradiction with the primitivity of γ_k. $\quad\square$

Lemma 40. *We have that $\gamma_k, L(\gamma_k) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$ for all $k \geqslant 1$.*

Proof. For a suitable slope $\alpha = [0; \mathfrak{a}+1, \mathfrak{b}+1, \ldots]$, either of the words S and L is a reversed standard word of slope α. Thus by Theorem 18 both S^2 and L^2 are in $\mathcal{L}(\alpha)$, so $S^2, L^2 \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$.

We clearly have that $\gamma_1 \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Note that by the assumption $|S| > |S_6|$ both of the words S and L have the word $s = S_6 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$ as a proper suffix. Write $S = us$. Since s begins with $10^{\mathfrak{a}+1}$ and S^2 has sus as a suffix, it follows that $us \in (S_5 + S_6)^+$. Using the fact that $L \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$, we see that $\gamma_2 = LSS = L(u)s(us)^2 \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Clearly $L(\gamma_2) = S^3 = (us)^3 \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Proceeding by induction we may assume that $k \geqslant 2$ and $\gamma_k, L(\gamma_k) \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Since γ_k has either S or L as a prefix, it can be written that $\gamma_k = vszs$ with $|vs| = |S|$. It follows that $sz \in (S_5 + S_6)^+$. Since svs is a suffix of either S^2 or L^2, we have that $sv \in (S_5 + S_6)^+$. Therefore $svsz \in (S_5 + S_6)^+$. As $L(\gamma_k) = L(vsz)s \in \mathcal{L}(\mathfrak{a}, \mathfrak{b})$, we have that $L(vsz)$ is a suffix of some word in $(S_5 + S_6)^+$. Overall, the word $\gamma_{k+1} = L(vsz)(svsz)^2s$ is in $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$. Clearly then must the word $L(\gamma_{k+1}) = (vszs)^3 = vsz(svsz)^2s$ also be in $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$. $\quad\square$

Note that without the assumption $|S| > |S_6|$ the conclusion of the above lemma fails to hold. If $S = S_6 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$, then $L = 0(10^{\mathfrak{a}})^{\mathfrak{b}+2}$ and $LS = 0(10^{\mathfrak{a}})^{\mathfrak{b}+2}10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}$. Therefore $LS \notin \mathcal{L}(\mathfrak{a}, \mathfrak{b})$, and consequently $\gamma_2 = LS^2 \notin \mathcal{L}(\mathfrak{a}, \mathfrak{b})$.

Proof of Proposition 38. We proceed by induction. By Lemma 39 the word γ_k is primitive for all $k \geqslant 1$. Lemma 40 tells that both of the words γ_k and $L(\gamma_k)$ are in $\mathcal{L}(\mathfrak{a}, \mathfrak{b})$ for all $k \geqslant 1$. By definition both γ_1 and $L(\gamma_1)$ are solutions to (8). We may thus assume that $k \geqslant 1$ and both γ_k and $L(\gamma_k)$ are solutions to (8). It follows from Lemma 22 that

$$\gamma_k L(\gamma_k) \in \Pi(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad \sqrt{\gamma_k L(\gamma_k)} = \gamma_k.$$

Since $L(\gamma_k)$ is a solution to (8), Lemma 22 also implies that

$$L(\gamma_k)\gamma_k \in \Pi(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad \sqrt{L(\gamma_k)\gamma_k} = L(\gamma_k).$$

Because

$$\gamma_{k+1}^2 = L(\gamma_k)\gamma_k \cdot \gamma_k L(\gamma_k) \cdot \gamma_k^2,$$

we obtain that

$$\gamma_{k+1}^2 \in \Pi(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad \sqrt{\gamma_{k+1}^2} = \sqrt{L(\gamma_k)\gamma_k}\sqrt{\gamma_k L(\gamma_k)}\sqrt{\gamma_k^2} = L(\gamma_k)\gamma_k\gamma_k = \gamma_{k+1}.$$

This proves that γ_{k+1} is a solution to (8). Consider next the word $L(\gamma_{k+1}) = \gamma_k^3$. Because $(L(\gamma_{k+1}))^2 = (\gamma_k^2)^3$, it is evident that

$$(L(\gamma_{k+1}))^2 \in \Pi(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad \sqrt{(L(\gamma_{k+1}))^2} = \gamma_k^3 = L(\gamma_{k+1}).$$

Therefore also $L(\gamma_{k+1})$ is a solution to (8). The conclusion follows. $\qquad \square$

As we remarked earlier, we have now proved that Γ is a fixed point of the square root map. Next we show that the word Γ is aperiodic, linearly recurrent, and not Sturmian.

Lemma 41. *The word γ_2^2 is not a factor of any Sturmian word.*

Proof. By definition $\gamma_2 = LS^2$. Write $S = xyw$ and $L = yxw$ for some word w and distinct letters x and y. Now $\gamma_2^2 = yxw(xyw)^2yxw(xyw)^2$, so the word γ_2^2 has the factors xwx and ywy. Hence γ_2^2 is not balanced, so it cannot be a factor of any Sturmian word. $\qquad \square$

Lemma 42. *The word Γ is aperiodic and linearly recurrent.*

Proof. The recurrence (10) and the definition (11) of Γ show that for all $k \geqslant 1$ the word Γ is a product of the words $\gamma_{k+1} = L(\gamma_k)\gamma_k^2$ and $L(\gamma_{k+1}) = \gamma_k^3$ such that between two occurrences of $L(\gamma_{k+1})$ there is always γ_k^2 or γ_k^5. From this it follows that the return time of a factor of Γ of length $|\gamma_k|$ is at most the return time of the factor $L(\gamma_k)$, which is at most $6|\gamma_k|$. Let then w be a factor of Γ such that $|\gamma_k| < |w| \leqslant |\gamma_{k+1}|$. Since w is a factor of some factor of Γ of length $|\gamma_{k+1}|$, it follows that the return time of w is at most $6|\gamma_{k+1}|$. Now $6|\gamma_{k+1}| = 18|\gamma_k| < 18|w|$ proving that Γ is linearly recurrent.

The preceding shows that γ_k is followed in $\mathcal{L}(\Gamma)$ by both γ_k and $L(\gamma_k)$. As the first letters of γ_k and $L(\gamma_k)$ are distinct, the factor γ_k is right special. Thus $\mathcal{L}(\Gamma)$ contains arbitrarily long right special factors, so Γ must be aperiodic. $\qquad \square$

Since linearly recurrent words have linear factor complexity [5, Theorem 24], it follows from Lemma 42 that Γ has linear factor complexity.

We observed in the previous proof that the word Γ is a product of the words S and L such that between two occurrences of L in this product there is always S^2 or S^5. Since S and L are primitive, any word $w \in \mathcal{L}(\Gamma)$ which is a product of the words S and L such that $|w| \geqslant 6|S|$ must synchronize to the factorization of Γ as a product of the words S and L. That is, for any factorization $\Gamma = uw\Gamma'$ we must have that $|u|$ is a multiple of $|S|$.

Theorem 43. *The word Γ is a non-Sturmian, linearly recurrent optimal squareful word which is a fixed point of the square root map.*

Proof. The fact that Γ is optimal squareful and linearly recurrent follows from Lemmas 40 and 42. The argument outlined at the beginning of this section shows that Γ is a fixed point of the square root map as by Proposition 38 the words γ_k which occur as square prefixes in Γ are solutions to (8). Finally, Γ contains the factor γ_2^2, so Γ is not Sturmian by Lemma 41. $\square$

Denote by Ω the subshift consisting of the infinite words having language $\mathcal{L}(\Gamma)$. As Γ is linearly recurrent, it is uniformly recurrent, so the subshift Ω is minimal. The rest of this section is devoted to proving the result mentioned in the beginning of this section.

Theorem 44. *For all $w \in \Omega$ either $\sqrt{w} \in \Omega$ or $\sqrt{w}$ is (purely) periodic with minimal period conjugate to S. Moreover, there exists words $u, v \in \Omega$ such that $\sqrt{u} \in \Omega$ and $\sqrt{v}$ is periodic.*

This result is very surprising since it is contrary to the plausible hypothesis that an aperiodic word must map to an aperiodic word under the square root map.

It is not difficult to prove Theorem 44 for words in Ω which are products of the words S and L. We prove this special case next in Lemma 45. However, difficulties arise since a word in Ω can start in an arbitrary position of an infinite product of S and L. There are certain well-behaved positions in S and L which are easier to handle. Theorem 44 is proved for these special positions in Lemma 47. The rest of the effort is in demonstrating that all the other cases can be reduced to these well-behaved cases. We begin by proving the easier cases, and we conclude with the reductions.

Lemma 45. *If a word $w \in \Omega$ can be written as a product of the words S and L, then $\sqrt{w} \in \Omega$.*

Proof. Any word u which is a product of the words S and L can be naturally written as a binary word $\overline{u}$ over the alphabet $\{S, L\}$. If such a word $\overline{u}$ has even length, then it is a word over the alphabet $A = \{SS, SL, LS, LL\}$. Using the fact that $\sqrt{SS} = S$, $\sqrt{SL} = S$, $\sqrt{LS} = L$, and $\sqrt{LL} = L$ (see Lemma 22), we can define a square root for a word over A.

Without loss of generality we may assume that $\Gamma = \Gamma_1$, that is, $\Gamma = \lim_{k \to \infty} \gamma_{2k}$. The word γ_{2k}^2 is a prefix of Γ for all $k \geqslant 1$. Thus γ_{2k} has occurrences at positions 0 and $|\gamma_{2k}|$ of Γ. Clearly $|\gamma_k| = 3^{k-1}|S|$, so the word $\overline{\gamma}_{2k}$ occurs in $\overline{\Gamma}$ in an even and in an odd position for all $k \geqslant 1$.

Let $w \in \Omega$, and let v be a prefix of w of length $|v| = 2n|S|$ for some $n \geqslant 1$, so $\overline{v}$ is a word over A. The word v is a factor of γ_{2j} for some $j \geqslant 1$. Since $\overline{\gamma}_{2j}$ occurs in $\overline{\Gamma}$ in an even and in an odd position, the word $\overline{v}$ occurs in an even position in $\overline{\Gamma}$. Hence $\overline{\Gamma}$ can be factored as $\overline{\Gamma} = z\overline{v}t$ where z and t are finite or infinite words over A. Since Γ is a fixed point of the square root map, we have that $\overline{\Gamma} = \sqrt{z}\sqrt{\overline{v}}\sqrt{t}$. Hence $\sqrt{v} \in \mathcal{L}(\Gamma)$. It follows that $\mathcal{L}(\sqrt{w}) \subseteq \mathcal{L}(\Gamma)$, so $\sqrt{w} \in \Omega$. $\qquad\square$

Definition 46. Let w be a word and ℓ be an integer such that $0 < \ell < |w|$. If the factor of w^3 of length $|w^2|$ starting at position ℓ can be written as a product of minimal squares X_1^2, ..., X_n^2, then we say that the position ℓ of w is *repetitive*. If in addition $|X_1^2 \cdots X_m^2| \neq |w| - \ell, |w^2| - \ell$ for all m such that $1 \leqslant m \leqslant n$, then we say that the position ℓ is *nicely repetitive*.

For example if $\mathfrak{a} = 1$, $\mathfrak{b} = 0$, and $S = 1001001010010$, then the position 1 of S is repetitive as the factor $00100101001010010010100101$ of S^3 of length $|S^2| = 26$ starting at position 1 is in $\Pi(\mathfrak{a}, \mathfrak{b})$. This position is not nicely repetitive as $|0^2 \cdot (10010)^2| = 12 = |S| - 1$. The position 2 of S, however, can be checked to be nicely repetitive. The position 4 of S is not repetitive as the factor $00101001010010010100101001$ of length 26 starting at position 4 is not in $\Pi(\mathfrak{a}, \mathfrak{b})$.

In the upcoming proof of Theorem 44 we will show that if $w \in \Omega$ is a product of the words S and L and ℓ is a nicely repetitive position of S, then the word $\sqrt{T^\ell(w)}$ is always periodic. On the other hand, we show that if ℓ is not a nicely repetitive position then $\sqrt{T^\ell(w)}$ is always in Ω.

Next we identify some good positions in the suffix S_6 of S. As we observed in the proof of Lemma 22, the suffix S_6 of S restricts locally how a factorization of a word as a product of minimal squares continues after an occurrence of S_6. Consider a product $X_1^2 \cdots X_n^2$ of minimal squares which has an occurrence of S_6 at position ℓ. Then for some $m \in \{1, \ldots, n\}$ the minimal square X_m^2 must begin at some of the positions $\ell, \ell + 1, \ldots, \ell + |S_6| - 1$. Otherwise some minimal square would have S_6 as an interior factor; yet no such minimal square exists. Among the positions $\ell, \ell + 1, \ldots, \ell + |S_6| - 1$ we are interested in the largest position where a minimal square may begin. Let

$$\mathcal{B} = \{\ell \in \{0, \ldots, |S_6| - 1\} \colon$$
$$\text{no square of length at most } |S_6| - \ell \text{ begins at position } \ell \text{ of } S_6\}.$$

It is straightforward to see that

$$\mathcal{B} = \{|S_6| - |S_6|, |S_6| - |S_4|, |S_6| - |S_3|, |S_6| - |S_1|\}.$$

We are interested in those positions of the suffix S_6 of S where no minimal square begins. Hence we define

$$\mathcal{B}_S = \{\ell \colon \ell - |S| + |S_6| \in \mathcal{B}\} = \{|S| - |S_6|, |S| - |S_4|, |S| - |S_3|, |S| - |S_1|\}.$$

A consequence of the definitions is that if ℓ is a position of S such that $\ell \notin \mathcal{B}_S$, then there exists $\ell' \in \mathcal{B}_S \cup \{|S|\}$ such that $S[\ell, \ell' - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$. This fact is used later several times.

Lemma 47. *Suppose that $w \in \Omega$ can be written as a product of the words S and L. Assume that the position $\ell \in \mathcal{B}_S$ is nicely repetitive. Let the prefix of $T^\ell(w)$ of length $|S^2|$ be factorized as a product of minimal squares $X_1^2 \cdots X_n^2$. Then the word $\sqrt{T^\ell(w)}$ is periodic with minimal period $X_1 \cdots X_n$. Moreover, $X_1 \cdots X_n$ is conjugate to S.*

Proof Sketch. As ℓ is repetitive, the factor u of length $|S^2|$ of S^3 starting at position ℓ is in $\Pi(\mathfrak{a}, \mathfrak{b})$. If we substitute the middle S in S^3 with L, then an application of Lemma 21 shows that the factor of length $|S^2|$ of SLS starting at position ℓ is still in $\Pi(\mathfrak{a}, \mathfrak{b})$ and that the square root of this factor coincides with the square root of u (here we need that $\ell \in \mathcal{B}_S$). Further analysis shows that if we substitute the words S in S^3 in any way, then the square root of the factor of length $|S^2|$ beginning at position ℓ is unaffected. Since ℓ is repetitive, the prefix of $T^{\ell+|S^2|}(w)$ of length $|S^2|$ is again in $\Pi(\mathfrak{a}, \mathfrak{b})$ and has the same square root, and so on. Thus $\sqrt{T^\ell(w)}$ is periodic. Since both the square of the period and S^2 occur in a suitable Sturmian word; having equal lengths, they must be conjugate by Proposition 6. $\qquad\square$

Proof. We have that $|S| \geqslant |S_5 S_6|$, so $\ell > 1$. Let u be the suffix of S of length $|S| - \ell$. Since ℓ is repetitive, the factor v of S^3 of length $|S^2|$ starting at position ℓ can be factorized as a product of minimal squares $Y_1^2 \cdots Y_m^2$. We have that $|Y_1^2| > |u|$ because $\ell \in \mathcal{B}_S$.

Next we consider how the situation changes if any of the words S in S^3 is substituted with L. Substituting the first S with L does not affect the product as $\ell > 1$. Suppose then that the second word S is substituted with L. By applying Lemma 21 to the words u and S with $X = Y_1$, we see that the factor of length $|S^2|$ of SLS starting at position ℓ can still be factorized as a product of minimal squares and that the square root of this factor coincides with the square root of v. Consider next what happens when the third word S is substituted with L. Let

$$ r = \max\{i \in \{1, \ldots, m\} \colon |Y_1^2 \cdots Y_i^2| \leqslant |S^2| - \ell\}. $$

Set $\ell' = \ell + |Y_1^2 \cdots Y_r^2| - |S|$. Since ℓ is nicely repetitive, we have that $\ell' < |S|$. By the maximality of r and the definition of the set $\mathcal{B}_S$, we thus have that $\ell' \in \mathcal{B}_S$. Applying Lemma 21 to the suffix of S of length $|S| - \ell'$ and S with $X = Y_{r+1}$ we obtain, like above, that the product of minimal squares is affected but the square root is not. Substituting the second and third words S with L gives the same result: first proceed as above and substitute the second word S and then make the second substitution like above but apply Lemma 21 for the word L instead of S.

We have concluded that however we substitute the words S in S^3, the square root of the factor of length $|S^2|$ beginning at position ℓ never changes. The word w is obtained from the word S^ω by substituting some of the words S with L. By the preceding, the prefix of $T^\ell(w)$ of length $|S^2|$ can be factorized as a product of minimal squares $X_1^2 \cdots X_n^2$. Since ℓ is repetitive, the prefix of $T^{\ell+|S^2|}(w)$ of length $|S^2|$ can also be factorized as a product of some minimal squares (perhaps different) but the square root still equals $X_1 \cdots X_n$. By repeating this observation we see that

$$ \sqrt{T^\ell(w)} = (X_1 \cdots X_n)^\omega. $$

By our choice of S we have that $S \in \{\widetilde{s}_k, L(\widetilde{s}_k)\}$ where $\widetilde{s}_k$ is a reversed standard word of some slope $\alpha = [0; \mathfrak{a}+1, \mathfrak{b}+1, \ldots]$. Let $\beta = [0; b_1, b_2, \ldots]$ be a number such that $a_i = b_i$ for $1 \leqslant i \leqslant k$ and $b_{k+1} \geqslant 5$. Then by the definition of standard words $S^5 \in \mathcal{L}(\beta)$. By the preceding, the prefix of $T^\ell(S^5)$ of length $|S^4|$ can be written as a product of minimal squares, and the square root of these minimal squares equals $(X_1 \cdots X_n)^2$. Since the square root of a Sturmian word of slope β is a Sturmian word of slope β, we have that $(X_1 \cdots X_n)^2 \in \mathcal{L}(\beta)$. As $|X_1 \cdots X_n| = |S|$, it follows by Proposition 6 that $X_1 \cdots X_n$ is conjugate to S. Since S is primitive, so is $X_1 \cdots X_n$, and hence the period $X_1 \cdots X_n$ is minimal. $\qquad \square$

Lemma 48. *Every seed solution S has at least one nicely repetitive position ℓ such that $\ell \in \mathcal{B}_S$.*

Proof. Suppose that $S = \widetilde{s}_{k,i}$ for some $k \geqslant 3$ and $0 < i \leqslant a_k$. It is sufficient to show that $r = |\widetilde{s}_{k,i-1}|$ is a nicely repetitive position of S. If $r \notin \mathcal{B}_S$, then there exists $r' \in \mathcal{B}_S$ such that $S[r, r'-1] \in \Pi(\mathfrak{a}, \mathfrak{b})$. Since the position r is nicely repetitive, so must r' be. If $S = L(\widetilde{s}_{k,i})$, then as $r > 1$, an application of Lemma 21 shows that the conclusion holds also in this case.

Observe that the word $\widetilde{s}_{k,i-1}$ is both a prefix and a suffix of S. Using the fact that $\widetilde{s}_{k-2}\widetilde{s}_{k-3} = L(\widetilde{s}_{k-3}\widetilde{s}_{k-2})$ we obtain that

$$S^3 = \widetilde{s}_{k,i-1}\widetilde{s}_{k-1}\widetilde{s}_{k-2}\widetilde{s}_{k-1}^i\widetilde{s}_{k,i} = \widetilde{s}_{k,i-1} \cdot \widetilde{s}_{k-1}\widetilde{s}_{k-2}\widetilde{s}_{k-3}\widetilde{s}_{k-2}^{a_{k-1}-1} \cdot \widetilde{s}_{k,i-1}\widetilde{s}_{k,i}$$
$$= \widetilde{s}_{k,i-1} \cdot \widetilde{s}_{k-1}L(\widetilde{s}_{k-1}) \cdot \widetilde{s}_{k,i-1}^2\widetilde{s}_{k-1}.$$

By Lemma 22 the word $\widetilde{s}_{k-1}L(\widetilde{s}_{k-1})$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$. Since $\widetilde{s}_{k,i-1}$ is a solution to (8), we have that $\widetilde{s}_{k,i-1}^2 \in \Pi(\mathfrak{a}, \mathfrak{b})$. Overall, the factor $\widetilde{s}_{k-1}L(\widetilde{s}_{k-1})\widetilde{s}_{k,i-1}^2$ of S^3 of length $|S^2|$ starting at position r is in $\Pi(\mathfrak{a}, \mathfrak{b})$. Thus the position r of S is repetitive.

Suppose for a contradiction that the suffix of S of length $|S| - r$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$, that is, $S = \widetilde{s}_{k,i-1}X_1^2 \cdots X_n^2$ for some minimal square roots X_j. It follows that $s_{k-1} = X_1^2 \cdots X_n^2$. Since s_{k-1} is a solution to (8), it follows that $s_{k-1} = (X_1 \cdots X_n)^2$. This contradicts the primitivity of s_{k-1}. Similarly if the suffix of S^2 of length $|S^2| - r$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$, then $\widetilde{s}_{k,i-1} \in \Pi(\mathfrak{a}, \mathfrak{b})$ contradicting the primitivity of $\widetilde{s}_{k,i-1}$. We conclude that the position r is nicely repetitive. $\qquad \square$

Lemma 47 and Lemma 48 now imply the following:

Corollary 49. *There exist uncountably many linearly recurrent optimal squareful words having (purely) periodic square root.*

Proof. We only need to show that there are uncountably many such words. Consider the words in Ω which can be written as a product of the words S and L. Viewed over the binary alphabet $\{S, L\}$, these words form an infinite subshift $\overline{\Omega}$. Let us show that $\overline{\Omega}$ is minimal. Then the conclusion follows by well-known arguments from topology: a minimal subshift is always finite or uncountable and an aperiodic subshift cannot be finite (use the fact that a perfect set is always uncountable).

Let $\overline{w} \in \overline{\Omega}$ (we use the notation of the proof of Lemma 45). Let $\overline{u} \in \mathcal{L}(\overline{w})$ be a factor such that $|\overline{u}| \geqslant 6$. As $|u| \geqslant 6|S|$, every occurrence of u in Γ must synchronize to the factorization of Γ as a product of S and L. It follows that every return to u in Γ is a product of S and L. Since the return time of u is finite in Γ, the return time of the word $\overline{u}$ in $\overline{w}$ is also finite. Hence $\overline{\Omega}$ is minimal. $\qquad\square$

We also prove the following weaker result, which we need later.

Lemma 50. *The position $|S| - |S_6|$ of S is repetitive.*

Proof. We prove first by induction that the prefix of the word $S_6 \widetilde{s}_{k,\ell}^2$ of length $2|\widetilde{s}_{k,\ell}| - |S_6|$ is a product of minimal squares for $k \geqslant 2$ and ℓ such that $0 < \ell \leqslant a_k$. Let us first establish the base cases.

Recall that $\widetilde{s}_2 = 0(10^{\mathfrak{a}})^{\mathfrak{b}+1}$ and $\widetilde{s}_{3,1} = S_6$. We have that

$$S_6 \widetilde{s}_2^2 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1}(0(10^{\mathfrak{a}})^{\mathfrak{b}+1})^2 = S_5^2 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1} = S_5^2 S_6.$$

In addition, for $0 < \ell \leqslant a_3$, we have that

$$S_6 \widetilde{s}_{3,\ell}^2 = S_6 \widetilde{s}_{3,1} \widetilde{s}_2^{\ell-1} \widetilde{s}_{3,\ell} = S_6^2 \widetilde{s}_2^{\ell-1} \widetilde{s}_{3,\ell} = S_6^2 \widetilde{s}_2^{\ell-1} \widetilde{s}_1 \widetilde{s}_2^{\ell}.$$

The case $\ell = 1$ is clear. So let us assume that $\ell > 1$. We have that

$$S_6 \widetilde{s}_{3,\ell}^2 = S_6^2 \widetilde{s}_2^{\ell-1} \widetilde{s}_1 \widetilde{s}_2^{\ell-2} \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}} S_6,$$

so it is sufficient to show that the word $\widetilde{s}_2^{\ell-1} \widetilde{s}_1 \widetilde{s}_2^{\ell-2} \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}}$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$.

Suppose first that $\ell - 1$ is even. Then as $\widetilde{s}_2$ is a solution to (8), it is enough to show that $\widetilde{s}_1 \widetilde{s}_2^{\ell-2} \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}} \in \Pi(\mathfrak{a}, \mathfrak{b})$. Since $\widetilde{s}_1 \widetilde{s}_2 = L(\widetilde{s}_2) \widetilde{s}_1$, we have that

$$\widetilde{s}_1 \widetilde{s}_2^{\ell-2} \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}} = L(\widetilde{s}_2)^{\ell-2} \widetilde{s}_1 \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}}.$$

Now $\widetilde{s}_1 \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}} = L(\widetilde{s}_2)$. The word $L(\widetilde{s}_2)$ is a solution to (8), so the conclusion follows as $\ell - 1$ is even.

Suppose next that $\ell - 1$ is odd. We need to show that $\widetilde{s}_2 \widetilde{s}_1 \widetilde{s}_2^{\ell-2} \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}} \in \Pi(\mathfrak{a}, \mathfrak{b})$. Using the facts $\widetilde{s}_1 \widetilde{s}_2 = L(\widetilde{s}_2) \widetilde{s}_1$ and $\widetilde{s}_1 \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}} = L(\widetilde{s}_2)$ we obtain that

$$\widetilde{s}_2 \widetilde{s}_1 \widetilde{s}_2^{\ell-2} \widetilde{s}_0 \widetilde{s}_1^{\mathfrak{b}} = \widetilde{s}_2 L(\widetilde{s}_2)^{\ell-1}.$$

By Lemma 22 the word $\widetilde{s}_2 L(\widetilde{s}_2)$ is a product of minimal squares. Since $\ell - 1$ is odd and $L(\widetilde{s}_2)$ is a solution to (8), the conclusion follows.

We have established the base cases. Now for $k \geqslant 4$ and $0 < \ell \leqslant a_k$, we have that

$$S_6 \widetilde{s}_{k,\ell}^2 = S_6 (\widetilde{s}_{k-2} \widetilde{s}_{k-1}^{\ell})^2.$$

By induction $S_6 \widetilde{s}_{k-2} = X_1^2 \cdots X_n^2 S_6$ and $S_6 \widetilde{s}_{k-1} = Y_1^2 \cdots Y_m^2 S_6$ for some minimal square roots $X_1, \ldots, X_n, Y_1, \ldots, Y_m$. Therefore

$$S_6 \widetilde{s}_{k,\ell}^2 = (X_1^2 \cdots X_n^2 (Y_1^2 \cdots Y_m^2)^{\ell})^2 S_6.$$

We have thus proved that the prefix of the word $S_6 \widetilde{s}_{k,\ell}^2$ of length $2|\widetilde{s}_{k,\ell}| - |S_6|$ is a product of minimal squares for $k \geqslant 2$ and ℓ such that $0 < \ell \leqslant a_k$.

Now if $S = \widetilde{s}_{k,\ell}$ for some $k \geqslant 2$ and ℓ such that $0 < \ell \leqslant a_k$, then the claim is clear by the above. Suppose that $S = L(\widetilde{s}_{k,\ell})$. Now if $S_6 \widetilde{s}_{k,\ell} \notin \Pi(\mathfrak{a}, \mathfrak{b})$, then two applications of Lemma 21 show that the claim holds. Assume that $S_6 \widetilde{s}_{k,\ell} \in \Pi(\mathfrak{a}, \mathfrak{b})$. Since the prefix of $S_6 \widetilde{s}_{k,\ell}^2$ of length $2|\widetilde{s}_{k,\ell}| - |S_6|$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$, this means that the prefix of $\widetilde{s}_{k,\ell}$ of length $|\widetilde{s}_{k,\ell}| - |S_6|$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$. It is sufficient to show that the prefixes of $\widetilde{s}_{k,\ell}$ and $L(\widetilde{s}_{k,\ell})$ of length $2|\widetilde{s}_2|$ are in $\Pi(\mathfrak{a}, \mathfrak{b})$. Since $\widetilde{s}_1 \widetilde{s}_2 = L(\widetilde{s}_2)\widetilde{s}_1$, the word $\widetilde{s}_{4,1} = \widetilde{s}_2 \widetilde{s}_3$ has $\widetilde{s}_2 L(\widetilde{s}_2)$ as a prefix. If $a_3 > 1$, then the word $\widetilde{s}_3 = \widetilde{s}_1 \widetilde{s}_2^{a_3}$ has $L(\widetilde{s}_2)\widetilde{s}_2$ as a prefix. Finally if $a_3 = 1$, then the word $\widetilde{s}_{5,1} = \widetilde{s}_3 \widetilde{s}_4 = \widetilde{s}_1 \widetilde{s}_2 \widetilde{s}_4$ has $L(\widetilde{s}_2)^2$ as a prefix. Lemma 22 shows that $\widetilde{s}_2 L(\widetilde{s}_2)$, $L(\widetilde{s}_2)\widetilde{s}_2$, and $L(\widetilde{s}_2)^2$ are all in $\Pi(a, b)$. The conclusion follows. $\qquad\square$

There is no clear pattern for other positions in $\mathcal{B}_S$; it depends on the word S if a position in $\mathcal{B}_S$ is repetitive or not. The position $|S| - |S_6|$ is not always nicely repetitive. Suppose that $\mathfrak{a} = 1$, $\mathfrak{b} = 0$, and $S = \widetilde{s}_{3,3} = 10(010)^3$. Then the factor beginning at position $|S| - |S_6| = 6$ of S^3 of length $|S^2|$ is a product of minimal squares: $(10010)^2 \cdot (010)^2 \cdot (100)^2$. As $|(10010)^2 \cdot (010)^2| = 16 = |S^2| - 6$, the position 6 is not nicely repetitive.

Since none of the minimal squares can be a proper prefix of another minimal square, it is easy to factorize words as products of minimal squares from left to right. Next we consider what happens if we start to backtrack from a given position to the left.

Lemma 51 (Backtracking Lemma). *Let X, Y_1, ..., Y_n be minimal square roots. Let w be a word having both of the words X^2 and $Y_1^2 \cdots Y_n^2$ as suffixes. If $|X| > |Y_n|$, then $|X| > |Y_1 \cdots Y_n|$ and the word $Y_1 \cdots Y_n$ is a suffix of X.*

Proof. Suppose that $|X| > |Y_n|$. We may assume that n is as large as possible. We prove the lemma by considering different options for the word X.

Clearly we cannot have that $X = S_1$. Let $X = S_4$. Now X^2 can have a proper minimal square suffix only if $\mathfrak{a} > 1$. If $\mathfrak{a}$ is even, then we must have that

$$X^2 = 10^{\mathfrak{a}}1(S_1^2)^{\mathfrak{a}/2} \quad \text{and} \quad Y_{n-\mathfrak{a}/2+1} = \ldots = Y_n = S_1.$$

The suffix $(S_1^2)^{\mathfrak{a}/2}$ of w cannot be preceded by S_2^2 as otherwise w would have $S_2 S_1^{\mathfrak{a}} = 010^{2\mathfrak{a}-1}$ as a suffix; this is not possible as $2\mathfrak{a} - 1 > \mathfrak{a}$. Therefore there is no choice for $Y_{n-\mathfrak{a}/2}$. Thus $|Y_1^2 \cdots Y_n^2| < |X^2|$ and $Y_1 \cdots Y_n$ is a suffix of X. If $\mathfrak{a}$ is odd, then similarly

$$X^2 = 10^{\mathfrak{a}}10(S_1^2)^{(\mathfrak{a}-1)/2} \quad \text{and} \quad Y_{n-(\mathfrak{a}-1)/2+1} = \ldots = Y_n = S_1.$$

Again there is no choice for $Y_{n-(\mathfrak{a}-1)/2}$, and the conclusion holds. Similar considerations show that the conclusion holds if $X \in \{S_2, S_3\}$.

Let then $X = S_5$. It is obvious that now $Y_n \in \{S_1, S_3, S_4\}$. If $Y_n = S_1$ or $\mathfrak{b} = 0$, then like above $Y_1 = \ldots = Y_n = S_1$ and $Y_1 \cdots Y_n$ is a suffix of X. We may thus suppose that $\mathfrak{b} > 0$. Say $Y_n = S_3$. Then we must have $\mathfrak{b} = 1$ and $X^2 = 10^{\mathfrak{a}+1}10^{\mathfrak{a}-1}Y_n^2$. Like above, the remaining minimal square roots Y_i with $i < n$ must equal to S_1 and there must be $\lfloor(\mathfrak{a}-1)/2\rfloor$ of them. Since there is no further choice, the conclusion holds as clearly

$Y_1 \cdots Y_n$ is a suffix of X. Suppose then that $\mathfrak{b} > 1$. The next case is $Y_n = S_4$. Assume first that $\mathfrak{b}$ is even. Then it is straightforward to see that necessarily

$$Y_{n-\mathfrak{b}/2+1} = \ldots = Y_n = S_4 \quad \text{and} \quad X^2 = 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}}10^{\mathfrak{a}+1}(S_4^2)^{\mathfrak{b}/2}.$$

Thus $Y_{n-\mathfrak{b}/2} = S_1$ and, further, it must be that

$$Y_{n-\mathfrak{b}} = \ldots = Y_{n-\mathfrak{b}/2-1} = S_2 \quad \text{and} \quad X^2 = 10^{\mathfrak{a}+1}10^{\mathfrak{a}-1}(S_2^2)^{\mathfrak{b}/2}S_1^2(S_4^2)^{\mathfrak{b}/2}.$$

Like before, the remaining minimal squares Y_i with $i < n - \mathfrak{b}$ must equal to S_1 and there must be $\lfloor (\mathfrak{a} - 1)/2 \rfloor$ of them. Therefore

$$Y_1 \cdots Y_n = S_1^{\lfloor (\mathfrak{a}-1)/2 \rfloor} S_2^{\mathfrak{b}/2} S_1 S_4^{\mathfrak{b}/2} = 0^{\lfloor (\mathfrak{a}-1)/2 \rfloor + 1}(10^{\mathfrak{a}})^{\mathfrak{b}}$$

is a suffix of X, so the conclusion holds. If $\mathfrak{b}$ is odd, then in a similar fashion

$$X^2 = 10^{\mathfrak{a}+1}10^{\mathfrak{a}-1}(S_2^2)^{(\mathfrak{b}-1)/2}S_3^2(S_4)^{(\mathfrak{b}-1)/2},$$

so $Y_{n-(\mathfrak{b}-1)/2} = S_3$ and

$$Y_{n-(\mathfrak{b}-1)/2+1} = \ldots = Y_n = S_4 \quad \text{and} \quad Y_{n-\mathfrak{b}+1} = \ldots = Y_{n-(\mathfrak{b}-1)/2-1} = S_2.$$

Again, the final $\lfloor (\mathfrak{a} - 1)/2 \rfloor$ minimal square roots must equal S_1. Since

$$Y_1 \cdots Y_n = S_1^{\lfloor (\mathfrak{a}-1)/2 \rfloor} S_2^{(\mathfrak{b}-1)/2} S_3 S_4^{(\mathfrak{b}-1)/2} = 0^{\lfloor (\mathfrak{a}-1)/2 \rfloor + 1}(10^{\mathfrak{a}})^{\mathfrak{b}}$$

is a suffix of X, the conclusion holds.

If $X = S_6$, then it is clear that $Y_n \neq S_5$. The conclusion follows as in the case $X = S_5$. $\qquad \square$

The next lemma is useful in the proof of Theorem 44.

Lemma 52. *Let w be an infinite product of the words S and L and ℓ_1, ℓ_2, ℓ_3 be positions of w such that $\ell_1 < \ell_2 < \ell_3$. Let r be the largest integer such that $\ell_1 \geqslant r|S|$. If*

- $w[\ell_1, \ell_3 - 1], w[\ell_2, \ell_3 - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$,

- $\ell_1 - r|S| \in \mathcal{B}_S$, *and*

- $\ell_2 \leqslant (r + 1)|S|$,

then for all $u \in \Pi(\mathfrak{a}, \mathfrak{b})$ such that $uw[\ell_2, \ell_3 - 1]$ is a suffix of $w[0, l_3 - 1]$ we have that $|uw[\ell_2, \ell_3 - 1]| < |w[\ell_1, \ell_3 - 1]|$.

Proof. Let $v = w[\ell_1, \ell_3 - 1]$ and $u = w[\ell_2, \ell_3 - 1]$. Since $v, u \in \Pi(\mathfrak{a}, \mathfrak{b})$, we may write $v = X_1^2 \cdots X_n^2$ and $u = Y_1^2 \cdots Y_m^2$ for some minimal square roots X_i and Y_i. Suppose that $n \geqslant m$. If $X_{n-m+i} = Y_i$ for all $i \in \{1, \ldots, m\}$, then as $|v| > |u|$, we must have that $n > m$. This means that the prefix X_1^2 of v ends before the position ℓ_2, that is, $\ell_1 + |X_1^2| <$

$\ell_2 \leqslant (r+1)|S|$. This contradicts the fact that $\ell_1 - r|S| \in \mathcal{B}_S$. Therefore as $|v| > |u|$, we we conclude that there exists a maximal $j \in \{1, \ldots, m\}$ such that $X_{n-m+j} \neq Y_j$. If $|Y_j| > |X_{n-m+j}|$, then by the Backtracking Lemma we have that $|X_1^2 \cdots X_{n-m+j}^2| < |Y_j^2|$. This is not possible as $|v| > |u|$. Therefore $|Y_j| < |X_{n-m+j}|$. Let $z \in \Pi(\mathfrak{a}, \mathfrak{b})$ be such that zu is a suffix of $w[0, \ell_3 - 1]$. Write $z = Z_1^2 \cdots Z_t^2$ for minimal square roots Z_i. Applying the Backtracking Lemma to the words X_{n-m+j}^2 and $Z_1^2 \cdots Z_t^2 Y_1^2 \cdots Y_j^2$ yields that $|Z_1^2 \cdots Z_t^2 Y_1^2 \cdots Y_j^2| < |X_{n-m+j}^2|$. It follows that $|zu| < |v|$. If $n < m$, then as $|v| > |u|$, there exists a maximal $j' \in \{1, \ldots, n\}$ such that $Y_{m-n+j'} \neq X_{j'}$. Proceeding as above, we see that the conclusion holds. $\qquad \square$

Finally we can give a proof of Theorem 44.

Proof of Theorem 44. Let $w \in \Omega$. Since Γ is uniformly recurrent and a product of the words S and L, there exists a word $w' \in \Omega$ such that w' is a product of S and L and $w = T^\ell(w')$ for some ℓ such that $0 \leqslant \ell < |S|$ (recall that a product of S and L occurring in Γ having length at least $6|S|$ must synchronize to the factorization of Γ as a product of S and L). If $\ell = 0$, then the conclusion holds by Lemma 45, so we can assume that $\ell > 0$. Write w as a product of minimal squares: $w = X_1^2 X_2^2 \cdots$. Let

$$r_1 = \max\{\{0\} \cup \{i \in \{1, 2, \ldots\} \colon |X_1^2 \cdots X_i^2| \leqslant |S| - \ell\}\}.$$

If $r_1 > 0$, then set $\ell_1 = \ell + |X_1^2 \cdots X_{r_1}^2|$. If $r_1 = 0$, then we set $\ell_1 = \ell$. By the maximality of r_1 and by the definition of the set $\mathcal{B}_S$, it follows that $\ell_1 \in \mathcal{B}_S \cup \{|S|\}$ (indeed, the word L also has S_6 as a suffix). See Figure 5.

To aid comprehension we have separated different parts of the proof as distinct claims with their own proofs. Any new definitions and assumptions given in one of the subproofs are valid only up to the end of the subproof.

Claim 53. *If $\ell_1 = |S|$, then $\sqrt{w} \in \Omega$.*

Proof. Suppose that $\ell_1 = |S|$. By the definition of the number r_1, we have that $r_1 > 0$ and the word $T^{|S|-\ell}(w) = T^{|S|}(w') = X_{r_1+1}^2 X_{r_2+2}^2 \cdots$ is a product of the words S and L. Now $zw' \in \Omega$ for some $z \in \{S, L\}$. Since zw' is a product of S and L, by Lemma 45 $\sqrt{zw'} \in \Omega$. By the choice of S as a solution to (8) and by Lemma 22, the first $|S^2|$ letters of zw' can be written as a product of minimal squares. Hence $zw' = Y_1^2 \cdots Y_n^2 X_{r_1+1}^2 X_{r_1+2}^2 \cdots$ for some minimal square roots $Y_1, \ldots, Y_n$. By the Backtracking Lemma, we have that $X_1 \cdots X_{r_1}$ is a suffix of $Y_1 \cdots Y_n$. Thus the word $\sqrt{w} = X_1 \cdots X_{r_1} X_{r_1+1} \cdots$ is a suffix of the word $\sqrt{zw'} = Y_1 \cdots Y_n X_{r_1} X_{r_1+1} \cdots$. Therefore $\mathcal{L}(\sqrt{w}) \subseteq \mathcal{L}(\sqrt{zw'}) = \mathcal{L}(\Gamma)$, so $\sqrt{w} \in \Omega$. $\qquad \square$

We assume that $\ell_1 \in \mathcal{B}_S$. Now either the position ℓ_1 of the word S is nicely repetitive or it is not.

Claim 54. *If ℓ_1 is a nicely repetitive position of S, then $\sqrt{w}$ is periodic with minimal period conjugate to S.*

Proof. By Lemma 47 the word $\sqrt{T^{\ell_1}(w')}$ is periodic with minimal period z conjugate to S. If $\ell_1 = \ell$, then there is nothing more to prove, so assume that $\ell_1 \neq \ell$. There exists $u, v \in \{S, L\}$ such that $uvw' \in \Omega$. Since ℓ_1 is a nicely repetitive position of S, the prefix of $T^{\ell_1}(uvw')$ of length $|S^2|$ is a product of minimal squares and its square root equals z by Lemma 47. Since the factor $w'[\ell, \ell_1 - 1]$ is also a product of minimal squares, the Backtracking Lemma implies that $\sqrt{w'[\ell, \ell_1 - 1]}$ is a suffix of z. Now $\sqrt{w} = \sqrt{w'[\ell, \ell_1 - 1]}\sqrt{T^{\ell_1}(w')}$, so $\sqrt{w}$ is periodic with minimal period conjugate to S. $\qquad\square$

If the position ℓ_1 of S is not nicely repetitive, then either it is not repetitive or it is repetitive but not nicely repetitive.

Claim 55. *If ℓ_1 is a repetitive but not nicely repetitive position of S, then $\sqrt{w} \in \Omega$.*

Proof. Suppose that ℓ_1 is a repetitive but not a nicely repetitive position of S. This means that either $S^3[\ell_1, |S| - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$ or $S^3[\ell_1, |S^2| - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$ (they both cannot be in $\Pi(\mathfrak{a}, \mathfrak{b})$ as this would imply that S is not primitive). Thus either $w'[\ell_1, |S| - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$ or $w'[\ell_1, |S^2| - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$ (in the latter case Lemma 21 ensures that $w'[\ell_1, |S^2| - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$). The former case is, however, not possible as it would contradict the maximality of r_1. Thus only the latter option is possible. Since w' is a product of the words S and L, the prefix $w'[0, |S^2| - 1]$ of w' is a product of minimal squares. Since $w'[\ell, \ell_1 - 1], w'[\ell_1, |S^2| - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$, the Backtracking Lemma implies that $\sqrt{w'[\ell, |S^2| - 1]}$ is a suffix of $\sqrt{w'[0, |S^2| - 1]}$. Thus $\sqrt{w}$ is a suffix of $\sqrt{w'}$. As $\sqrt{w'} \in \Omega$ by Lemma 45, we conclude that $\sqrt{w} \in \Omega$. $\qquad\square$

Now we may suppose that ℓ_1 is not a repetitive position of S. We let

$$r_2 = \max\{i \in \{r_1 + 1, r_1 + 2, \ldots\} : |X_1^2 \cdots X_i^2| \leqslant |S^2| - \ell\},$$
$$r_3 = \max\{i \in \{r_2 + 1, r_2 + 2, \ldots\} : |X_1^2 \cdots X_i^2| \leqslant |S^3| - \ell\}, \text{ and}$$
$$r_4 = \max\{i \in \{r_3 + 1, r_3 + 2, \ldots\} : |X_1^2 \cdots X_i^2| \leqslant |S^4| - \ell\}.$$

The numbers r_2, r_3, and r_4 are well-defined as the words S and L are not minimal squares. We set

$$\ell_2 = \ell_1 + |X_{r_1+1}^2 \cdots X_{r_2}^2|,$$
$$\ell_3 = \ell_2 + |X_{r_2+1}^2 \cdots X_{r_3}^2|, \text{ and}$$
$$\ell_4 = \ell_3 + |X_{r_3+1}^2 \cdots X_{r_4}^2|.$$

Intuitively, the positions ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are the successive positions of w which are closest from the left to the boundaries of the words S and L in the factorization of w' as a product of the words S and L such that the prefix up to the position is a product of minimal squares; see Figure 5. Let $g_1 = \ell_1$, $g_2 = \ell_2 - |S|$, $g_3 = \ell_3 - |S^2|$, and $g_4 = \ell_4 - |S^3|$. By the maximality of the numbers r_2, r_3, and r_4, we see that $g_i \in \mathcal{B}_S \cup \{|S|\}$ for $i = 1, 2, 3, 4$.

Claim 56. *We have that $g_1, g_3 \neq |S|$. If g_2 or g_4 equals $|S|$, then $\sqrt{w} \in \Omega$.*

Proof. By our assumption that $\ell_1 \in \mathcal{B}_S$, we have that $g_1 \neq |S|$. If $g_2 = |S|$, then the factor $w'[\ell_1, |S^2| - 1]$ would be a product of minimal squares. This case was already considered in Claim 55 where we concluded that $\sqrt{w} \in \Omega$.

Suppose that $g_3 = |S|$. Consider the positions ℓ_1 and $|S|$ of w'. Both of the factors $u = w'[\ell_1, \ell_3 - 1]$ and $v = w'[|S|, \ell_3 - 1] = w'[\ell_3 - |S^2|, \ell_3 - 1]$ are in $\Pi(\mathfrak{a}, \mathfrak{b})$ (because $v \in \{SS, SL, LS, LL\}$). Now $zw' \in \Omega$ for some $z \in \{S, L\}$. Since $SS, SL, LS, LL \in \Pi(\mathfrak{a}, \mathfrak{b})$, the prefix of zw' of length $|S^2|$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$. Lemma 52 applied to the word $(zw')[0, |S| + \ell_3 - 1]$ implies that $|(zw')[0, |S| + \ell_3 - 1]| < |u| < |S^3|$ which is nonsense. Therefore $g_3 \neq |S|$.

Assume then that $g_4 = |S|$. Suppose for a contradiction that $g_2 \neq g_4$. Both of the factors $u' = w'[\ell_2, \ell_4 - 1]$ and $v' = w'[|S|^2, \ell_4 - 1] = w'[\ell_4 - |S^2|, \ell_4 - 1]$ are in $\Pi(\mathfrak{a}, \mathfrak{b})$. Since $g_2 \neq g_4$, also $\ell_2 \neq |S^2|$. Thus by the definition of ℓ_2, we have that $\ell_2 < |S_2|$. Lemma 52 applied to the word $w'[0, \ell_4 - 1]$ shows that $|w'[0, \ell_4 - 1]| < |u'| < |S^3|$ which is absurd. This contradiction shows that $g_2 = g_4 = |S|$, so $\sqrt{w} \in \Omega$. $\qquad\square$

We may now assume that $g_i \in \mathcal{B}_S$ for all $i \in \{1, 2, 3, 4\}$.

Claim 57. *The position g_2 of S is nicely repetitive.*

Proof. Assume on that neither of the positions g_2 and g_3 is a repetitive position of S. First note that as g_1 is not repetitive, we have that $g_3 \neq g_1$. Similarly $g_2 \neq g_4$. If $g_1 = g_2$, then it follows from Lemma 21 and the definitions of the positions ℓ_2 and ℓ_3 that $g_2 = g_3$; a contradiction. Hence $g_1 \neq g_2$. Similarly $g_2 \neq g_3$ as otherwise the position g_2 would be repetitive. Finally, $g_3 \neq g_4$ because g_3 is not repetitive. We have two cases: either $g_1 = g_4$ or $g_1 \neq g_4$.

Assume that $g_4 \neq g_1$. By Lemma 50 the position $|S| - |S_6|$ of S is repetitive, so $g_1, g_2, g_3 \in \mathcal{B}_S \setminus \{|S| - |S_6|\} = \{|S| - |S_1|, |S| - |S_3|, |S| - |S_4|\}$. Since all of the positions g_1, g_2, and g_3 are distinct, the only option is that $g_4 = |S| - |S_6|$. Since the position $|S| - |S_6|$ is repetitive, by Lemma 21 the factor $u = w'[\ell_4 - |S^2|, \ell_4 - 1]$ is in $\Pi(\mathfrak{a}, \mathfrak{b})$. By the definition of the positions ℓ_2, ℓ_3, and ℓ_4 also $v = w'[\ell_2, \ell_4 - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$. Since $g_2 \neq g_4$, also $\ell_2 \neq \ell_4 - |S^2|$. Since $|S| - |S_6|$ is the smallest element of the set $\mathcal{B}_S$, we have that $\ell_2 > \ell_4 - |S^2|$. As $w[\ell_1, \ell_2 - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$, we obtain by Lemma 52 that $|w[\ell_1, \ell_4 - 1]| < |u| = |S^2|$. This is a contradiction.

Hence we have that $g_1 = g_4$. Since the factor $w[\ell_1, \ell_2 - 1]$ is a product of minimal squares, the number $c_1 = \ell_2 - \ell_1$ is even. Similarly the numbers $c_2 = \ell_3 - \ell_2$ and $c_3 = \ell_4 - \ell_3$ are even. Thus the number $c_1 + c_2 + c_3 = 3|S|$ is even, so $|S|$ is even. It follows that the numbers $d_1 = g_2 - g_1$, $d_2 = g_3 - g_2$, and $d_3 = g_4 - g_3 = g_1 - g_3$ are even. However, exactly two of the numbers $|S_1|$, $|S_3|$, and $|S_4|$ have odd length. Hence exactly two of the numbers g_1, g_2, and g_3 are odd. Thus it is not possible that all of the numbers d_1, d_2, and d_3 are even. This is a contradiction.

The previous contradiction shows that one of the positions g_2 and g_3 is a repetitive position of S. Suppose for a contradiction that g_3 is repetitive. We have that $w'[\ell_1, \ell_3 - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$ and $w'[\ell_3 - |S^2|, \ell_3 - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$. Similar to the second paragraph of this subproof, using Lemma 52 we obtain a contradiction unless $g_1 = g_3$. Even this conclusion
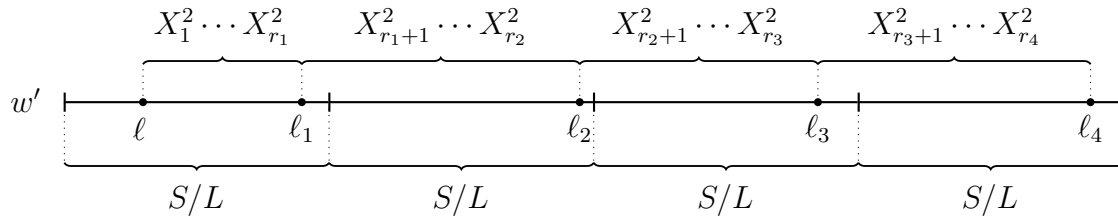
Figure 5: The positions ℓ, ℓ_1, ℓ_2, ℓ_3, and ℓ_4 of w' and the minimal squares between the positions.

is contradictory as g_1 is not repetitive. Therefore g_3 can not be repetitive, so g_2 is a repetitive position of S. Now if g_2 would not be nicely repetitive, we would have by the maximality of r_2 that $w'[\ell_2, |S^3| - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$, that is, $g_3 = |S|$. However, since $g_3 \in \mathcal{B}_S$, we have that g_2 is a nicely repetitive position of S. $\qquad\square$

We are now in the final stage of the proof. We will show that $\sqrt{w}$ is periodic with minimal period conjugate to S.

We can now argue as in the proof of Claim 54. Since g_2 is a nicely repetitive position of S, by Lemma 47 the word $\sqrt{T^{\ell_2}(w')}$ is periodic with minimal period z conjugate to S. We have that $uw' \in \Omega$ for some $u \in \{S, L\}$. Since g_2 is a nicely repetitive position of S, the prefix of $T^{g_2}(uw')$ of length $|S^2|$ is a product of minimal squares and its square root equals z by Lemma 47. Since $w'[\ell, \ell_2 - 1] \in \Pi(\mathfrak{a}, \mathfrak{b})$, the Backtracking Lemma implies that $\sqrt{w'[\ell, \ell_2 - 1]}$ is a suffix of z. Now $\sqrt{w} = \sqrt{w'[\ell, \ell_2 - 1]}\sqrt{T^{\ell_2}(w')}$, so $\sqrt{w}$ is periodic with minimal period conjugate to S.

By Lemma 48 the word S always has at least one nicely repetitive position. It therefore follows that there exists a word in Ω having a periodic square root. $\qquad\square$

9   Remarks on Generalizations

It is natural to think that the square root map could be generalized to obtain a cube root map and, further, a kth root map. However, in [17, Theorem 5.3.] Saari proves the following reformulation of a result of Mignosi, Restivo, and Salemi.

Proposition 58. *If w is an everywhere α-repetitive word with $\alpha \geqslant \phi + 1$, where ϕ is the golden mean, then w is ultimately periodic.*

Generalizing the square root map to a cube root map would require everywhere 3-repetitive words. By the above such words must be ultimately periodic, so we expect that this direction of research would not be fruitful.

Another way to generalize the square root map is to use abelian powers instead of ordinary powers. For abelian powers a result like Proposition 58 does not exist. For instance, by [16, Theorem 1.9.] every position in a Sturmian word begins with an abelian k^{th} power for all $k \geqslant 2$. Abelian square root can be defined for e.g. optimal squareful words

as we will see shortly. However, abelian cubes in Sturmian words do not work. Consider again the Fibonacci word f. The minimal abelian cube prefix of $T(f)$ is $10 \cdot 01 \cdot 01$. This abelian cube is followed by the factor 00, so the root of the next abelian cube must begin with 00. Hence if we define the abelian cube root of $T(f)$ to be the product of the roots of the abelian cubes, the resulting word begins with 1000 which is not a factor of f. Thus by defining an abelian cube root map in this way, we lose the main property that the mapping preserves the languages of Sturmian words.

In [18] Saari also considers optimal abelian squareful words. *Optimal abelian squareful words* are defined by replacing minimal squares with minimal abelian squares in the definition of optimal squareful words. Let $w = X_1 X_1' X_2 X_2' \cdots$ be a product of minimal abelian squares $X_i X_i'$. We define its *abelian square root* as the word $\sqrt[ab]{w} = X_1 X_2 \cdots$. It follows from [18, Theorem 18] that the six minimal squares are products of exactly five minimal abelian squares (this is straightforward to verify directly). Thus if w is an optimal squareful word, then $\sqrt{w} = \sqrt[ab]{w}$. Thus by Theorem 9 the abelian square root of a Sturmian word $s_{x,\alpha}$ is the Sturmian word $s_{\psi(x),\alpha}$. Also, by Theorem 44 there exists a minimal subshift Ω such that for all $w \in \Omega$ either $\sqrt[ab]{w} \in \Omega$ or $\sqrt[ab]{w}$ is periodic. Saari proves in [18, Theorem 19] that an optimal abelian squareful word must have at least five distinct minimal abelian squares, but he leaves the characterization of these sets of minimal abelian squares open. Thus it is possible that there exists optimal abelian squareful words which contain other minimal abelian squares than those given by [18, Theorem 18]. For such words the abelian square root map could exhibit different behavior than the square root map (if the square root map is even defined for such words). We have not extended our research to this direction.

We could also generalize the special function ψ. Divide the distance D between x and $1 - \alpha$ into k parts and choose the image of x to be $x + \frac{t}{k}D$ among the points

$$x + \frac{1}{k}D, x + \frac{2}{k}D, \ldots, x + \frac{k-1}{k}D$$

to obtain the function

$$\psi_{k,t} \colon \mathbb{T} \to \mathbb{T}, x \mapsto \frac{1}{k}(tx + (k-t)(1-\alpha)).$$

The map $\psi_{k,t}$ is a perfectly nice function on the circle $\mathbb{T}$, but to make things interesting we would need to find a symbolic interpretation for it. We have not figured out any such interpretation for these generalized functions.

10   Open Problems

In Section 8 we saw that there are non-Sturmian words whose language is preserved under the square root map. However, Sturmian words satisfy an even stronger property: by Theorem 9 for the Sturmian subshift Ω_α of slope α it holds that $\sqrt{\Omega_\alpha} \subseteq \Omega_\alpha$. This property is not satisfied by the aperiodic and minimal subshift Ω_Γ of the word Γ constructed in Section 8 since by Theorem 44 there is a word in Ω_Γ having periodic square root; since

Ω_Γ is aperiodic and minimal, it cannot contain such words. We are thus led to ask the following question we could not answer:

Question 59. If Ω is a subshift containing optimal squareful words satisfying $\sqrt{\Omega} \subseteq \Omega$, does the subshift Ω only contain Sturmian words?

Let us briefly see that if we do not require all words in Ω to be aperiodic then the above question has a negative answer.

Proposition 60. *There exists a non-minimal non-Sturmian subshift Ω containing square-ful words such that $\sqrt{\Omega} \subseteq \Omega$.*

Proof Sketch. Let S be a seed solution as in Section 8, and let Γ be a corresponding fixed point of the square root map generated by the seed S as in Section 8. Further, set $\Delta = S^\omega$, let Ω_Δ be the subshift generated by Δ, and let Ω_Γ be the subshift generated by Γ. If $w \in \Omega_\Gamma$, then by Theorem 44 either $\sqrt{w} \in \Omega_\Gamma$ or $\sqrt{w} \in \Omega_\Delta$. Hence if we are able to show that $\sqrt{\Omega_\Delta} \subseteq \Omega_\Delta$, then the non-minimal and non-Sturmian subshift $\Omega_\Gamma \cup \Omega_\Delta$ has the desired properties.

Let $w \in \Omega_\Delta$, so $w = T^\ell(\Delta)$ for some $0 \leqslant \ell < |S|$. Write w as a product of minimal squares: $w = X_1^2 X_2^2 \cdots$. We can now argue as in the proof of Theorem 44. If $|X_1^2 \cdots X_n^2| = |S| - \ell$ for some $n \geqslant 1$ or $|X_1^2 \cdots X_m^2| = |S^2| - \ell$ for some $m \geqslant 1$, then using the fact that $\sqrt{\Delta} = \Delta$ it is straightforward to see that $\sqrt{w} \in \Omega_\Delta$. Otherwise either ℓ is a nicely repetitive position of S or $\ell + |X_1^2 \cdots X_i^2| - |S|$ is a nicely repetitive position of S where

$$i = \max\{j \in \{1, 2, \ldots\} \colon |X_1^2 \cdots X_j^2| \leqslant |S^2| - \ell\}.$$

In both of these cases we deduce with the help of Lemma 47 that $\sqrt{w} \in \Omega_\Delta$. $\square$

There are other interesting related questions. Consider the limit set

$$\Omega \cap \sqrt{\Omega} \cap \sqrt{\sqrt{\Omega}} \cap \ldots.$$

We know very little about the limit set except in the Sturmian case when it contains the two fixed points $01c_\alpha$ and $10c_\alpha$. For the subshift generated by the word Γ of Section 8 we proved that the limit set contains at least two fixed points. We ask:

Question 61. When is the limit set nonempty? If it is nonempty, does it always contain fixed points? Can it contain points which are not fixed points?

It is a genuine possibility that the limit set is empty. Consider for instance the word $\zeta = \tau(\sigma^\omega(6))$, the morphic image of the fixed point of the morphism $\sigma \colon 6 \mapsto 656556, 5 \mapsto 5$ under $\tau \colon 6 \mapsto S_6^2, 5 \mapsto S_5^2$ where $S_5 = 100$ and $S_6 = 10010$ are minimal square roots of slope $\alpha = [0; 2, 1, \ldots]$. It is straightforward to verify that ζ is optimal squareful and uniformly recurrent and that the returns to the factor 101 in $\mathcal{L}(\zeta)$ are 10100, $101(001)^2 00$ and $101(001)^4 00$. By considering all possible occurrences of the factor $w = \tau(56565) \in \mathcal{L}(\zeta)$ in any product of the minimal squares of slope α, it can be shown that the square

root of the product always contains a return to the factor 101 which is not in $\mathcal{L}(\zeta)$. Since the factor w occurs in every point in the subshift Ω_ζ generated by ζ, we conclude that $\Omega_\zeta \cap \sqrt{\Omega_\zeta} = \emptyset$.

In Section 8 we constructed infinite families of primitive solutions to (8) using the recurrence $\gamma_{k+1} = L(\gamma_k)\gamma_k^2$. Why this construction worked was because the seed solution S and the word $L = L(S)$ satisfy $\sqrt{SS} = S$, $\sqrt{SL} = S$, $\sqrt{LS} = L$, and $\sqrt{LL} = L$, that is, $\sqrt{(LSS)^2} = \sqrt{LS \cdot SL \cdot SS} = LSS$. Similarly $\sqrt{(SLLLL)^2} = SLLLL$, so substituting for example $S = 01010010$ we obtain the primitive solution

$$S_2 S_1 S_4 S_3 S_5 S_4 S_3 S_5 S_6 S_5 S_4 S_3 S_5 S_4 S_3 = 01010010100100101001001010010010010010010010$$

to (8) in $\mathcal{L}(1,0)$. More solutions can be obtained with analogous constructions. Restricting to the languages of optimal squareful words, we ask:

Question 62. What are the primitive solutions w of (8) in $\mathcal{L}(a,b)$ such that w or w^2 is not Sturmian and w is not obtainable by the above construction?

Acknowledgments

References

[1] J. Berstel, On the index of Sturmian words. In *Jewels Are Forever*, pages 287–294. Springer-Verlag, 1999

[2] J. W. S. Cassels. *An Introduction to Diophantine Approximation*. Cambridge Tracts in Mathematics and Mathematical Physics, volume 45, Cambridge University Press, 1957.

[3] D. Damanik and D. Lenz. Powers in Sturmian sequences. *Eur. J. Combin.*, 24:377–390, 2003. doi:10.1016/S0195-6698(03)00026-X

[4] A. De Luca and G. Fici. Open and closed prefixes of Sturmian words. In *Combinatorics on Words*, volume 8079 of *Lecture Notes in Comput. Sci.*, pages 132–142. Springer, 2013. doi:10.1007/978-3-642-40579-2

[5] F. Durand, B. Host and C. Skau. Substitution dynamical systems, Bratteli diagrams and dimension groups. *Ergodic Theory Dynam. Systems*, 19.4:953–993, 1999.

[6] Š. Holub. A solution of the equation $(x_1^2 \cdots x_n^2)^3 = (x_1^3 \cdots x_n^3)^2$. In *Contributions to General Algebra*, 11, (Olomouc/Velké Karlovice, 1998), pages 105–111. Heyn, Klagenfurt, 1999.

[7] Š. Holub. In search of a word with special combinatorial properties. In *Computational and Geometric Aspects of Modern Algebra*, volume 275 of *London Mathematical Society Lecture Note Series*, pages 120–127. Cambridge University Press, 2000. `doi:10.1017/CBO9780511600609.011`

[8] Š. Holub. Local and global cyclicity in free semigroups. *Theor. Comput. Sci.*, 262.1–2:25–36, 2001. `doi:10.1016/S0304-3975(00)00156-0`

[9] A. Ya. Khinchin. *Continued Fractions*. Dover Publications, Mineola, New York, 1997.

[10] M. Lothaire. *Combinatorics on Words*. Encyclopedia of Mathematics and Its Applications, volume 17. Addison-Wesley, 1983.

[11] M. Lothaire. *Algebraic Combinatorics on Words*. Encyclopedia of Mathematics and Its Applications, volume 90. Cambridge University Press, 2002.

[12] J. Peltomäki. Characterization of repetitions in Sturmian words: A new proof. *Info. Proc. Letters*, 115.11:886–891, 2015. `doi:10.1016/j.ipl.2015.05.011`

[13] J. Peltomäki. *Privileged Words and Sturmian Words*. TUCS Dissertations 214. Turku Centre for Computer Science, University of Turku, 2016. `http://www.doria.fi/handle/10024/124473`

[14] J. Peltomäki and M. Whiteland. A square root map on Sturmian words (Extended abstract), In *Combinatorics on Words*, volume 9304 of *Lecture Notes in Comput. Sci.*, pages 197–209. Springer, 2015. `doi:10.1007/978-3-319-23660-5`

[15] N. Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics*. Lecture Notes in Mathematics, 1794. Springer, 2002. `doi:10.1007/b13861`

[16] G. Richomme, K. Saari and L. Q. Zamboni. Abelian complexity of minimal subshifts. *J. Lond. Math. Soc.*, 83.2:79–95, 2011. `doi:10.1112/jlms/jdq063`

[17] K. Saari. *On the Frequency and Periodicity of Infinite Words*. TUCS Dissertations 97. Turku Centre for Computer Science, University of Turku, 2008. `http://users.utu.fi/kasaar/pubs/phdth.pdf`

[18] K. Saari. Everywhere α-repetitive sequences and Sturmian words. *Eur. J. Combin.*, 31:177-192, 2010. `doi:10.1016/j.ejc.2009.01.004`