Pattern Avoidance and Young Tableaux

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Abstract

Motivated by [13, Theorem 4.1], this paper extends Lewis's bijection to a bijection between a more general class $\mathcal{L}(n,k,I)$ of permutations and the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$, so the cardinality

$$|\mathcal{L}(n,k,I)| = f^{\langle (k+1)^n \rangle},$$

is independent of the choice of $I \subseteq [n]$. As a consequence, we obtain some new combinatorial realizations and identities on Catalan numbers. In the end, we raise a problem on finding a bijection between $\mathcal{L}(n,k,I)$ and $\mathcal{L}(n,k,I')$ for distinct I and I'.

Keywords: Pattern avoidance, Young tableaux, Catalan numbers.

1 Introduction

Let S_n denote the permutation group on $[n] = \{1, 2, \ldots, n\}$. Write $\sigma \in S_n$ in the form $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$. For $m \leqslant n$, if $\sigma \in S_n$ and $\pi = \pi_1 \cdots \pi_m \in S_m$, we say that σ *contains the pattern* π if there is an index subsequence $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ such that $\sigma_{i_j} < \sigma_{i_k}$ iff $\pi_j < \pi_k$ for $1 \leqslant j, k \leqslant m$, that is, σ has a subsequence which is order isomorphic to π. Otherwise, σ *avoids the pattern* π, or say, σ is π-*avoiding*. Given a pattern π and a set S of permutations, we denote by $S(\pi)$ the set of elements of S that avoid π. For example, the permutation 354261 contains a subsequence 5421 order isomorphic to 4321 and no subsequence order isomorphic to 1234, that is, $354261 \notin S_6(4321)$ and $354261 \in S_6(1234)$. The study on pattern avoidance of permutations started from MacMahon [14]. He proved that the number of permutations which can be divided into

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two decreasing subsequences is the Catalan number, i.e., $|\mathcal{S}_n(123)| = C_n = \frac{1}{n+1}\binom{2n}{n}$. Later in $1970's$, Knuth [10, 11] proved that for any $\pi \in S_3$,

$$|S_n(\pi)| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In past decades, various articles considered permutations avoiding some patterns, see [3, 4, 5, 6, 9, 12, 15].

A *partition* of the nonnegative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of positive integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$ and $\sum_{i=1}^{l} \lambda_i = n$, written as $\lambda \vdash n$. Repetitions of equal elements use to be written by exponential notations, e.g., the partition $(6, 4, 4, 4, 2)$ is abbreviated $\langle 6, 4^3, 2\rangle$. Associated to a partition $\lambda \vdash n$, a *Young diagram* is a collection of boxes arranged in left-justified rows with λ_i boxes in row i. We usually identify each partition with its Young diagram and speak of them interchangeably. If λ is a Young diagram with n boxes, a *standard Young tableau* T of shape λ denoted by $\lambda = \mathrm{sh(T)}$ is a filling of numbers $1, 2, \ldots, n$ into those n boxes such that each number appears once and entries strictly increase along each row and down each column. We use matrix coordinates to identify boxes in a standard Young tableau, i.e., $\mathrm{T}(i, j)$ is the box in the i-th row and j-th column of T. The hook length formula expresses the number of standard Young tableaux of shape λ as

$$f^{\lambda} = \frac{n!}{\prod h_{ij}(\lambda)},$$

where $h_{ij}(\lambda)$ is the number of boxes $\mathrm{T}(x, y)$ with either $x \geqslant i, y = j$ or $x = i, y \geqslant j$. It is well-known that the *Robinson-Schensted-Knuth* (abbreviated RSK) correspondence establishes a one-to-one correspondence between permutations and ordered pairs of standard Young tableaux of the same shape, see [8, 16] for more details. Thus one can apply the hook length formula to enumerate permutations avoiding some patterns. To our knowledge, there are some results obtained in this way, such as [1, 2, 7, 13].

For positive integers n, k and an index set $I \subseteq [n]$, let $\mathcal{L}(n, k, I)$ be the set of permutations $\sigma = \sigma_{11}\sigma_{12}\cdots\sigma_{1j_1}\sigma_{21}\cdots\sigma_{2j_2}\cdots\sigma_{n1}\cdots\sigma_{nj_n} \in S_{kn+|I|}$ satisfying the following conditions

(C1). $j_i = k + 1$ if $i \in I$ and $j_i = k$ otherwise;

(C2). $\sigma_{i1} < \sigma_{i2} < \cdots < \sigma_{ij_i}$ for all $1 \leqslant i \leqslant n$;

(C3). σ avoids the pattern $12\cdots(k + 1)(k + 2)$.

By taking $I = [n]$ and $\varnothing$ respectively, the notation $\mathcal{L}(n, k, I)$ extends $\mathcal{L}_{n,k+1}(1, 2, \ldots, k+2)$ and $\mathcal{L}_{n,k}(1, 2, \ldots, k + 2)$ defined in [13]. In 2011, Lewis [13, Theorem 4.1] established a bijection between $\mathcal{L}(n, k, \varnothing)$ and the set of standard Young tableaux of shape $\langle(k + 1)^n\rangle$. The main result of this paper extends Lewis's bijection to a bijection between $\mathcal{L}(n, k, I)$ and the set of standard Young tableaux of shape $\langle(k + 1)^n\rangle$ for any index set $I \subseteq [n]$. As an obvious consequence, we obtain that the cardinality

$$|\mathcal{L}(n, k, I)| = f^{\langle(k+1)^n\rangle},$$

does not depend on the choice of I. Taking $k = 1$, we obtain a class of combinatorial realizations of Catalan numbers. Applying the inclusion-exclusion principle, some new combinatorial identities on Catalan numbers will be formulated. Finally, we raise an open problem on finding a bijection between $\mathcal{L}(n, k, I)$ and $\mathcal{L}(n, k, I')$ for distinct subsets I and I' of $[n]$, respectively.

2 Main Theorem

As mentioned above, under the RSK correspondence, the study of pattern avoidance can be reduced to study standard Young tableaux. Below are some facts on the RSK correspondence which can be found in [8, 16, 17].

Lemma 2.1 (Row Bumping Lemma)**.** [8, p.9] *Two successive row-insertions, first row-inserting x in a tableau T and then row-inserting x' in the resulting tableau $(T \leftarrow x)$, give rise to two routes R and R', and two new boxes B and B'. If $x \leqslant x'$, then R is strictly left of R', and B is strictly left of and weakly below B'. If $x > x'$, then R' is weakly left of R, and B' is weakly left of and strictly below B.*

Lemma 2.2. [17, p.387-389] *If $\omega \in S_n$ and $\omega \xrightarrow{RSK} (P, Q)$, then the number of columns (rows) of P is the length of the longest increasing (decreasing) subsequence of ω.*

Now we introduce our main result whose proof is similar as Lewis's result [13, Theorem 4.1]. One may refer to [8] for some notations used in the proof.

Theorem 2.3. *For a fixed index set $I \subseteq [n]$, there is a bijection between $\mathcal{L}(n, k, I)$ and the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$ and so the cardinality*

$$|\mathcal{L}(n, k, I)| = f^{\langle (k+1)^n \rangle},$$

is independent of the choice of I.

Proof. For $\sigma = \sigma_{11} \cdots \sigma_{1j_1} \cdots \sigma_{n1} \cdots \sigma_{nj_n} \in \mathcal{L}(n, k, I)$, let $T_0 = \varnothing$ and $T_i = (T_{i-1} \leftarrow \sigma_{i1} \leftarrow \sigma_{i2} \leftarrow \cdots \leftarrow \sigma_{ij_i})$, the resulting tableau by row-insertions of $\sigma_{i1}, \ldots, \sigma_{ij_i}$ in T_{i-1} successively. Denote by $T_i - T_{i-1}$ the skew tableau obtained by removing from T_i the boxes of $sh(T_{i-1})$ and by $\mathrm{sh}(T_i)/\mathrm{sh}(T_{i-1})$ the shape of $T_i - T_{i-1}$. By Row Bumping Lemma 2.1, all boxes of $\mathrm{sh}(T_i)/\mathrm{sh}(T_{i-1})$ must lie in different columns of T_i.

Since σ avoids $12 \cdots (k+1)(k+2)$, by Lemma 2.2, each T_i has at most $k+1$ columns. Let $[n] - I = \{i_1, \ldots, i_{n-|I|} \mid i_1 < \cdots < i_{n-|I|}\}$. It is clear from the definition (C1) of $\mathcal{L}(n, k, I)$ that $j_{i_t} = k$ for all $t = 1, \ldots, n - |I|$. Hence there is a unique $v_t \in [k+1]$ such that no new box is added to the v_t-th column when performing the row insertions $(T_{i_t-1} \leftarrow \sigma_{i_t 1} \leftarrow \sigma_{i_t 2} \leftarrow \cdots \leftarrow \sigma_{i_t j_{i_t}})$. While if $i \in I$, then $j_i = k+1$ and each column will get a new box in the process.

Now we construct a map from $\mathcal{L}(n, k, I)$ to the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$ by the following algorithm. Given a $\sigma = \sigma_{11} \cdots \sigma_{1j_1} \cdots \sigma_{n1} \cdots \sigma_{nj_n} \in \mathcal{L}(n, k, I)$. To obtain a tableau T of shape $\langle (k+1)^n \rangle$, we fill all boxes of $\langle (k+1)^n \rangle$ with entries $[(k+1)n]$ step by step as i goes from 1 to n.

At the first step, if $j_1 = k$, we fill the left k boxes on the top row of $\langle (k+1)^n \rangle$ successively with numbers $\sigma_{11}, \ldots, \sigma_{1j_1}$ and the box at the bottom-right conner of $\langle (k+1)^n \rangle$ with the number $(k+1)n$, otherwise $j_1 = k+1$ and fill the top row of $\langle (k+1)^n \rangle$ successively with numbers $\sigma_{11}, \ldots, \sigma_{1j_1}$. Generally at the i-th step, notice that $\mathrm{sh}(T_i) \leqslant \langle (k+1)^n \rangle$ and the process $T_i = (T_{i-1} \leftarrow \sigma_{i1} \leftarrow \sigma_{i2} \leftarrow \cdots \leftarrow \sigma_{ij_i})$ produces j_i new boxes with entries. Then fill the corresponding j_i new boxes of $\langle (k+1)^n \rangle$ with those entries. Consequently at this step we get a copy of T_i on the top-left of $\langle (k+1)^n \rangle$. Additionally for $i = i_t \notin I$, fill the bottom box of the remaining empty boxes in the v_t-th column of $\langle (k+1)^n \rangle$ with the number $(k+1)n + 1 - t$. Thus at this step every column has a new box filled with an entry and all numbered boxes of T are separated to two components. One component is a copy of T_i on the top-left of T, the other on the bottom-right. Moreover, vertically moving the bottom component up to the top component and joining both components together yields a Young tableau of shape $\langle (k+1)^i \rangle$. Hence, the outcome of the n-th step is a standard Young tableau T of shape $\langle (k+1)^n \rangle$ with entries $[(k+1)n]$, which completes the construction of a map sending a permutation $\sigma \in \mathcal{L}(n,k,I)$ to a standard Young tableau T of shape $\langle (k+1)^n \rangle$.

To prove the map is bijective, since the RSK correspondence is a bijection, it is enough to construct a pair (P,Q) of standard Young tableaux of the same shape from a standard Young tableau T of shape $\langle (k+1)^n \rangle$ with entries $[(k+1)n]$ such that running the RSK algorithm backwards will send (P,Q) to a permutation $\sigma \in \mathcal{L}(n,k,I)$.

Given a standard Young tableau T of shape $\langle (k+1)^n \rangle$ with entries $[(k+1)n]$. Let P be the standard Young tableau obtained by removing the boxes with $n - |I|$ largest entries from T, and denote by λ the shape of P. The other standard Young tableau Q of shape λ will be constructed by filling the Young diagram λ step by step as i runs from 1 to n. Let $c_0 = 0$ and $c_i = j_1 + j_2 + \cdots + j_i$ for $1 \leqslant i \leqslant n$. At the i-th step, the fillings are as follows. If $i \in I$, for those boxes of λ that remain empty (i.e., not filled with entries) at the $(i-1)$-th step, we fill the top empty box of each column with entries $c_{i-1}+1, c_{i-1}+2, \ldots, c_i$ successively from left to right. If $i = i_t \notin I$ and the entry $(k+1)n + 1 - t$ is located on the v_t-th column of T, we fill the top empty box of each column, except for column v_t, with entries $c_{i-1}+1, c_{i-1}+2, \ldots, c_i$ successively from left to right. Performing this algorithm finally yields a standard Young tableau Q of shape λ, which together with P determines a permutation σ by running the procedure $\sigma \xrightarrow{RSK} (P,Q)$ backwards. It is clear by Lemma 2.2 that σ avoids $12 \cdots (k+2)$ since P has at most $(k+1)$ columns. From the construction of Q, it is clear by Lemma 2.1 that σ consists of n increasing subsequences of length either k or $k+1$ consecutively. More precisely, $\sigma = \sigma_{11} \cdots \sigma_{1j_1} \cdots \sigma_{n1} \cdots \sigma_{nj_n}$ with $\sigma_{i1} < \cdots < \sigma_{ij_i}$, and $j_i = k+1$ if $i \in I$ and $j_i = k$ otherwise. Therefore, the above map sends a standard Young tableau T of shape $\langle (k+1)^n$ to a permutation $\sigma \in \mathcal{L}(n,k,I)$. $\square$

Example 2.4. *Let $I = \{1,3\}$ and $\sigma = 7\ 10\ 12\ 6\ 9\ 1\ 5\ 11\ 3\ 8\ 2\ 4 \in \mathcal{L}(5,2,I)$. Using the algorithm mentioned in the above proof, we will construct a standard Young tableau T of shape $\langle 3^5 \rangle$. Denote $a = 10$, $b = 11$, $c = 12$, $d = 13$, $e = 14$, $f = 15$ for convenience. Performing the row insertions $T_i = (T_{i-1} \leftarrow \sigma_{i1} \leftarrow \sigma_{i2} \leftarrow \cdots \leftarrow \sigma_{ij_i})$ for $1 \leqslant i \leqslant 5$, we have*

$$T_0 \to T_1 \to T_2 \to T_3 \to T_4 \to T_5$$

$$\varnothing \qquad \begin{array}{|c|c|c|}\hline 7 & a & c \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 6 & 9 & c \\ \hline 7 & a & \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 5 & b \\ \hline 6 & 9 & c \\ \hline 7 & a & \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 8 \\ \hline 5 & 9 & b \\ \hline 6 & a & c \\ \hline 7 & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 8 & b \\ \hline 5 & 9 & c \\ \hline 6 & a & \\ \cline{1-2} 7 & & \\ \cline{1-1} \end{array}.$$

The corresponding fillings of $\langle 3^5 \rangle$ at each step is

$$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & & \\ \hline \ & & \\ \hline \ & & \\ \hline \ & & \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline 7 & a & c \\ \hline \ & & \\ \hline \ & & \\ \hline \ & & \\ \hline \ & & \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline 6 & 9 & c \\ \hline 7 & a & \\ \hline \ & & \\ \hline \ & & \\ \hline \ & & f \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline 1 & 5 & b \\ \hline 6 & 9 & c \\ \hline 7 & a & \\ \hline \ & & \\ \hline \ & & f \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline 1 & 3 & 8 \\ \hline 5 & 9 & b \\ \hline 6 & a & c \\ \hline 7 & & \\ \hline \ & e & f \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 8 & b \\ \hline 5 & 9 & c \\ \hline 6 & a & d \\ \hline 7 & e & f \\ \hline \end{array}$$

Then we have

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 8 & b \\ \hline 5 & 9 & c \\ \hline 6 & a & d \\ \hline 7 & e & f \\ \hline \end{array}$$

Conversely, using the second algorithm in the proof, we can easily construct a pair (P, Q) of standard Young tableaux of the same shape from T, which is

$$P = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 8 & b \\ \hline 5 & 9 & c \\ \hline 6 & a & \\ \cline{1-2} 7 & & \\ \cline{1-1} \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 8 \\ \hline 6 & 7 & a \\ \hline 9 & c & \\ \cline{1-2} b & & \\ \cline{1-1} \end{array}.$$

Taking $I = \varnothing$ and $[n]$ respectively, Proposition 3.1 and Theorem 4.1 of [13] are easy consequences of Theorem 2.3.

Corollary 2.5. [13, Proposition 3.1] *There is a bijection between $\mathcal{L}(n, k, [n])$ and the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$.*

Corollary 2.6. [13, Theorem 4.1] *There is a bijection between $\mathcal{L}(n, k, \varnothing)$ and the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$.*

3 Catalan Numbers

The inclusion-exclusion principle states that for finite sets $A_1, \ldots, A_n$, one has

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

In this section, we give a class of combinatorial realizations for Catalan numbers which are new to our knowledge, and apply the inclusion-exclusion principle to obtain several recursive formula of Catalan numbers.

As mentioned in Section 1, the classical pattern-avoiding interpretation of Catalan numbers C_n is $S_n(\pi)$ for any $\pi \in S_3$, i.e., the collection of all π-avoiding permutations on the set $[n]$. Another easy combinatorial realization of Catalan numbers is the set of standard Young tableaux of shape $\langle 2^n \rangle$, namely, $f^{\langle 2^n \rangle} = C_n = \frac{1}{n+1}\binom{2n}{n}$. More combinatorial models of Catalan numbers can be found in [18]. Applying Theorem 2.3 at $k = 1$, below we obtain a new class of combinatorial realizations of Catalan numbers by permutations avoiding patterns.

Theorem 3.1. *For any $I \subseteq [n]$, we have $|\mathcal{L}(n, 1, I)| = C_n$.*

For the index set $I = \{i_1, \ldots, i_m\} \subseteq [n]$, we define

$$I^* = \{i_j + j - 1 \mid j = 1, \ldots, m\} \subseteq [n + m - 1].$$

Note that I^* contains no consecutive numbers of $[n]$ since $(i_{j+1} + j) - (i_j + j - 1) = i_{j+1} - i_j + 1 \geqslant 2$. If $\sigma = \sigma_1 \ldots \sigma_{n+m} \in \mathcal{L}(n, 1, I)$, it is easy from the definition of $\mathcal{L}(n, 1, I)$ that $\sigma_{i_j+j-1} < \sigma_{i_j+j}$ for each $j \in [m]$, i.e., $I^* \subseteq \mathrm{Asc}(\sigma)$, where the set $\mathrm{Asc}(\sigma)$ of all ascents of σ is defined by

$$\mathrm{Asc}(\sigma) = \{i \in [n + m - 1] \mid \sigma_i < \sigma_{i+1}\}.$$

Conversely, if $\sigma \in S_{n+m}(123)$, it is clear that $I^* \subseteq \mathrm{Asc}(\sigma)$ implies $\sigma \in \mathcal{L}(n, 1, I)$. Thus

$$\mathcal{L}(n, 1, I) = \{\sigma \in S_{n+m}(123) \mid I^* \subseteq \mathrm{Asc}(\sigma)\}.$$

We say that a permutation σ contains the ascent set A if $A \subseteq \mathrm{Asc}(\sigma)$. It is clear that the ascent set $\mathrm{Asc}(\sigma)$ contains no consecutive numbers if σ avoids 123. Let

$$\mathcal{A}_n = \{A \subseteq [n - 1] \mid A \,\text{contains no consecutive numbers}\}.$$

For $A \subseteq [n - 1]$, let

$$\begin{aligned} S_n(123, A) &= \{\sigma \in S_n(123) \mid A \subseteq \mathrm{Asc}(\sigma)\}, \\ S_n^\circ(123, A) &= \{\sigma \in S_n(123) \mid A = \mathrm{Asc}(\sigma)\}. \end{aligned}$$

Applying Theorem 3.1, the following result is obvious.

Corollary 3.2. *For $A \in \mathcal{A}_n$, we have $|S_n(123, A)| = C_{n-|A|}$.*

Also, we have the following facts,

- $S_n(123, \varnothing) = S_n(123)$.
- $S_n(123, A) = \varnothing$ if $A \notin \mathcal{A}_n$.
- $S_n(123, A) \subseteq S_n(123, B)$ if $A, B \in \mathcal{A}_n$ and $B \subseteq A$.
- For any $A, B \in \mathcal{A}_n$, we have

$$S_n(123, A \cup B) = \bigcap_{i \in B} S_n(123, A \cup \{i\}).$$

  In particular, $S_n(123, A) = \bigcap_{i \in A} S_n(123, \{i\})$.

- For any $A \in \mathcal{A}_n$, we have

$$S_n^\circ(123, A) = S_n(123, A) - \bigcup_{A \subsetneq B} S_n(123, B) = S_n(123, A) - \bigcup_{i \notin A} S_n(123, A \cup \{i\}).$$

  In particular, $\{\sigma = n(n-1)\cdots 21\} = S_n^\circ(123, \varnothing) = S_n(123) - \bigcup_{i=1}^{n-1} S_n(123, \{i\})$.

Since the map

$$\{a_1, a_2, \ldots, a_s\} \mapsto \{a_1, a_2 - 1, \ldots, a_s - s + 1\}$$

is a bijection between the set $\{A \in \mathcal{A}_n : |A| = s\}$ and $\binom{[n-s]}{s}$, we have

$$\#\{A \in \mathcal{A}_n : |A| = s\} = \binom{n-s}{s}.$$

Let $A = \{a_1, \ldots, a_s\} \in \mathcal{A}_n$ and for $k = 1, \ldots, n-s$,

$$\alpha_k(A) = \#\{B \in \mathcal{A}_n \mid A \subseteq B, |B - A| = k\}.$$

We have $\alpha_k(\varnothing) = \binom{n-k}{k}$ and

$$\alpha_k(A) = \sum_{\substack{b_i \geqslant 0 \\ b_0 + \cdots + b_s = k}} \prod_{i=0}^{s} \binom{a_{i+1} - a_i - 2 - b_i}{b_i},$$

where $a_0 = -1$ and $a_{s+1} = n+1$. Indeed, we have $\binom{a_{i+1} - a_i - 2 - b_i}{b_i}$ choices to insert b_i integers between a_i and a_{i+1} so that the sequence contains no consecutive numbers.

Theorem 3.3. *For any $A \in \mathcal{A}_n$, we have*

$$|S_n^\circ(123, A)| = \sum_{k \geqslant 0} (-1)^k \alpha_k(A) C_{n-k-|A|}.$$

Proof. The above facts imply

$$\begin{aligned} |S_n^\circ(123, A)| &= \left| S_n(123, A) - \bigcup_{i \notin A} S_n(123, A \cup \{i\}) \right| \\ &= |S_n(123, A)| - \left| \bigcup_{i \notin A} S_n(123, A \cup \{i\}) \right|. \end{aligned}$$

Using the inclusion-exclusion principle, we have

$$\begin{aligned} |S_n^\circ(123, A)| = C_{n-|A|} + \sum_{k=1}^{n-s} \sum_{\substack{A \subseteq B \in \mathcal{A}_n \\ |B - A| = k}} (-1)^k |S_n(123, B)| \\ = \sum_{k \geqslant 0} (-1)^k \alpha_k(A) C_{n-k-|A|}. \end{aligned}$$

□

Taking $A = \varnothing$, the following result is immediate.

Corollary 3.4. *For any nonnegative integer n, we have*

$$\sum_{k \geqslant 0} (-1)^k \binom{n-k}{k} C_{n-k} = 1.$$

Taking $A = \{i\}$, each $\sigma \in S_n^\circ(123, \{i\})$ has exactly one ascent i. Writing $\sigma = \sigma_1 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_n \in S_n^\circ(123, \{i\})$, both subsequences $\sigma_1 \cdots \sigma_i$ and $\sigma_{i+1} \cdots \sigma_n$ are decreasing and $\sigma_i < \sigma_{i+1}$. It is clear that there are $\binom{n}{i} - 1$ such permutations σ, i.e., $|S_n^\circ(123, \{i\})| = \binom{n}{i} - 1$. On the other hand, setting $\binom{x}{k} = 0$ for $x < 0$, we have

$$\begin{aligned} \alpha_k(\{i\}) &= \sum_{\substack{b_0, b_1 \geqslant 0 \\ b_0 + b_1 = k}} \binom{i-1-b_0}{b_0} \binom{n-i-1-b_1}{b_1} \\ &= \sum_{j=0}^{k} \binom{i-j-1}{j} \binom{n-k-i+j-1}{k-j}. \end{aligned}$$

Corollary 3.5. *For any $i \in [n-1]$, we have*

$$\binom{n}{i} - 1 = \sum_{k \geqslant 0} (-1)^k C_{n-k-1} \sum_{j=0}^{k} \binom{i-j-1}{j} \binom{n-k-i+j-1}{k-j}.$$

If $i = 1$ and 2, then

$$\begin{aligned} \sum_{k \geqslant 0} (-1)^k \binom{n-k-2}{k} C_{n-k-1} &= n - 1, \\ \sum_{k \geqslant 0} (-1)^k \binom{n-k-3}{k} C_{n-k-1} &= \binom{n}{2} - 1. \end{aligned}$$

Suppose $A = \{a_1, \ldots, a_s\}$ and $a_{i+1} - a_i \leqslant 3$ for all $i = 0, \ldots, s$, where $a_0 = -1$ and $a_{s+1} = n+1$. From Corollary 3.2, one has

$$S_n^{\circ}(123, A) = S_n(123, A) = C_{n-s}.$$

In particular, for any alternating permutation $\sigma \in S_n$, we have either $\mathrm{Asc}(\sigma) = \{1, 3, 5, \ldots\}$ or $\mathrm{Asc}(\sigma) = \{2, 4, 6, \ldots\}$. Hence, we can easily obtain the following result. (We can't find references, although we believe it is a known result)

Corollary 3.6. *The number of alternating permutations in $S_n(123)$ is $C_k + C_{k+1}$ if $n = 2k$, and $2C_k$ if $n = 2k+1$.*

4 Further Discussions

Let I and I' be distinct subsets of [n]. Recall that the proof of Theorem 2.3 gives a bijection sending each permutation $\sigma \in \mathcal{L}(n, k, I)$ to a standard Young tableau T of shape $\langle (k+1)^n \rangle$. An inverse of the bijection will send each standard Young tableau T of shape $\langle (k+1)^n \rangle$ to a permutation $\sigma' \in \mathcal{L}(n, k, I')$. Composing both will yields a bijection between $\mathcal{L}(n, k, I)$ and $\mathcal{L}(n, k, I')$ for distinct I and I'. However, since the RSK correspondence and its inverse are esoteric, the bijection obtained in this way is extremely not intuitive. Hence, it would be exciting if there is a direct bijection, without RSK correspondence involved, between $\mathcal{L}(n, k, I)$ and $\mathcal{L}(n, k, I')$. Next we give an easy example.

Proposition 4.1. *There is a bijection between $\mathcal{L}(n, 1, \{1\})$ or $\mathcal{L}(n, 1, \{n\})$ and $\mathcal{L}(n, 1, \varnothing)$.*

Proof. First note that $\sigma_1 < \sigma_2$ for any $\sigma = \sigma_1\sigma_2\sigma_3 \cdots \sigma_n\sigma_{n+1} \in \mathcal{L}(n, 1, \{1\})$. Since σ avoids the pattern 123, one has $\sigma_i < \sigma_2$ for all $3 \leqslant i \leqslant n+1$, otherwise $\sigma_1\sigma_2\sigma_i$ forms an increasing subsequence of σ. So $\sigma_2 = n+1$. Let $\sigma' = \sigma_1\sigma_3\sigma_4 \cdots \sigma_{n+1}$ be the permutation on $[n]$ obtained by removing σ_2 from a permutation $\sigma \in \mathcal{L}(n, 1, \{1\})$. Obviously, σ' avoids 123, i.e, $\sigma \in \mathcal{S}_n(123) = \mathcal{L}(n, 1, \varnothing)$. Thus we obtain a map $\sigma \mapsto \sigma'$ from $\mathcal{L}(n, 1, \{1\})$ to $\mathcal{L}(n, 1, \varnothing)$. Its inverse can be obtained by running the above process backwards. Hence the map $\sigma \mapsto \sigma'$ is a bijection from $\mathcal{L}(n, 1, \{1\})$ to $\mathcal{L}(n, 1, \varnothing)$. The other bijection between $\mathcal{L}(n, 1, \{n\})$ and $\mathcal{L}(n, 1, \varnothing)$ can be constructed similarly. $\square$

Obvioulsy, finding a bijection between $\mathcal{L}(n, k, I)$ and $\mathcal{L}(n, k, I')$ can be reduced to a bijection between $\mathcal{L}(n, k, I)$ and $\mathcal{L}(n, k, \varnothing) = S_{nk}(123)$, which is raised below as an open problem.

Problem 4.2. *Find a bijection between $\mathcal{L}(n, k, I)$ and $S_{nk}(123)$ for any $I \subseteq [n]$.*

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