Transitive Avoidance Games

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Abstract

Positional games are a well-studied class of combinatorial games. In their usual form, two players take turns to play moves in a set ('the board'), and certain subsets are designated as 'winning': the first person to occupy such a set wins the game. For these games, it is well known that (with correct play) the game cannot be a second-player win.

In the avoidance (or misère) form, the first person to occupy such a set *loses* the game. Here it would be natural to expect that the game cannot be a first-player win, at least if the game is transitive, meaning that all points of the board look the same. Our main result is that, contrary to this expectation, there are transitive games that are first-player wins, for all board sizes which are not prime or a power of 2.

Further, we show that such games can have additional properties such as stronger transitivity conditions, fast winning times, and 'small' winning sets.

1 Introduction

Many natural combinatorial games may be viewed as 'achievement games' as follows. We have a finite set (the *board*) and some of its subsets are designated as special and called *lines*. Two players take it in turn to claim a (previously unclaimed) point of the board; the first player to complete a line is declared the winner. If all points have been claimed but neither player has completed a line then the game is considered a draw.

In this paper we consider the avoidance (or misère) variant – the game is played exactly as above except the first player to complete a line *loses*. The first example of such a game that we are aware of was introduced by Simmons [13] (the game of Sim). Subsequently Harary [7] introduced the more general mathematical framework for such games. Since then these games have been considered by many authors – see, e.g., Beck [3, 2] and Slany [14]. For more general misère games see, e.g., Conway [4], and Albert and Nowakowski [1]. There has also been a substantial amount of work on Avoider/Enforcer games, which are the 'Maker/Breaker' analogue for avoidance games – see, e.g., Lu [12], Beck [3], and Hefetz, Krivelevich and Szabó [9].

For achievement games, a simple strategy stealing argument shows that the game is either a draw or a first player win (with perfect play). However, for avoidance games the situation is not clear. Indeed Beck [2] states "The general open problem is to find the avoidance version of the strategy stealing argument."

At first glance it looks as though the first player has a disadvantage since he has 'more' points than the second player. Of course in general this is not true: indeed take any game that is a second player win and add a single point not in any line. The first player picks this new point on his first turn and thus reduces the new game to the old game with Player II (the second player) playing first.

However, this is a rather trivial example: we have artificially given Player I an advantage by giving him a special 'safe' point he can pick. Thus, it is natural to insist that the game is transitive: that is, that the automorphism group of the game acts transitively on the board's points. (The automorphism group is the group of all permutations of the board X that preserve the family $\mathcal{L}$ of lines.) Informally, 'all points look the same.'

One might guess that a transitive avoidance game must be a second player win: what advantage can there be to going first when all points are the same? Indeed, this intuition is correct when all the lines have size 2. In general, however, it turns out that there may be advantages to playing first.

A priori, there are two reasons why it seems to be harder to find a game that is a Player I win when n (the board size) is odd: first, Player I has an extra move and, secondly, Player II has the final choice of move (as Player I's last move is forced). However, whilst we shall give examples of first player wins for both n even and n odd, it turns out that the simplest examples of such a game occur in the case n odd (see Section 3).

Having found examples of Player I wins we turn to the central question we address in this paper: namely for which n does there exist a transitive avoidance game that is a Player I win. Our main result shows that, for most n, such games do exist.

Theorem 1. *Suppose that n is neither a power of 2 nor a prime. Then there is a transitive avoidance game on n points that is a Player I win.*

The case when n is divisible by a large power of 2 is much harder than the case when n is odd, or n is equal to $2 \mod 4$. It relies on a careful study of subsets of $\mathbb{Z}_{2^m}$ (the integers modulo 2^m) under rotation. This may be of independent interest.

There are two cases not covered by Theorem 1: powers of 2, and primes. For powers of 2 we have the following result.

Theorem 2. *Suppose that n is a power of 2. Then no transitive avoidance game on n points is a Player I win.*

It turns out that this is a fairly simple consequence of a well known group theoretic result.

For the remaining case of n prime there are examples for which there are Player I wins (e.g., 11 and 13), but we do not know what happens for any prime larger than 13.

In many of our constructions Player II does not lose until his last move. It is easy to modify these examples so that Player I can force Player II to lose before time $(1-\varepsilon)n$. However, we also give examples of games where Player II loses in time $o(n)$. Interestingly these even include examples where the losing lines are all of bounded size (size 3 in fact).

We also consider avoidance games where on each turn each player is allowed to pick more than one point (this is usually called the *plus* version of the game; see Slany [14]). One might think that picking more than one point would never help a player, but it turns out that in any transitive plus game Player II can use this extra freedom to guarantee that he does not lose. A related phenomenon was proved by Hefetz, Krivelevich, Stojaković and Szabó [8] for a biased version of the Avoider/Enforcer game where moving to the plus version of the game simplified the behaviour substantially.

Our argument applies to some previously studied games. For example, it gives a simple strategy stealing proof of the previously unknown fact that the plus version of the Ramsey Avoidance Game (see Section 2) is never a Player I win.

In Section 2 we give two situations where strategy stealing arguments do work: when there is a line of size 2, and for the plus version of the games. Then in Section 3 we give a simple construction of Player I wins for all composite odd n.

Section 4 is the heart of the paper. In it we show that there are Player I wins for all even sizes, except powers of 2.

In Section 5 we discuss some natural variants of the game such as lines with bounded size, games which are 'more' transitive, and games where Player I can win quickly. We show that our constructions so far (or simple modifications of them) are able to provide first player wins in these cases too.

Then, in Section 6 we give a natural game which satisfies all of these stronger properties simultaneously – it is a fast Player I win, the lines have bounded size, and it is more transitive (more precisely it is edge/line transitive – see the section for the formal definition).

We conclude with a discussion of some open problems.

We will use the following notation for games throughout the paper. We will denote the board by X, the board size $|X|$ by n, and the set of losing lines by $\mathcal{L}$. We abbreviate Player I and Player II to PI and PII respectively.

2 Games for which strategy stealing works

2.1 A line of size 2

If the family of losing lines $\mathcal{L}$ has any set of size 2 then there is a strategy stealing argument. Note that this includes the case when all lines have size 2: i.e., where $\mathcal{L}$ corresponds to the edge set of some vertex transitive graph.

Theorem 3. *A transitive avoidance game with any line of size* 2 *is not a PI win.*

Proof. Suppose (for contradiction) that the game is a PI win.

Suppose that PI's first move is x. Since the game is transitive, every point is in a line of size 2 so we can choose y such that $\{x, y\} \in \mathcal{L}$.

Using the transitivity of the game again we see that there is a winning strategy, Φ say, for PI with first move y.

PII now plays strategy Φ ignoring the fact that PI has already claimed x. This could only go wrong if Φ tells PII to play x. However, since $\{x, y\}$ is a line, and thus PII would lose anyway if he played x, this cannot happen. □

Note that we made crucial use of the existence of a line of size 2 in the 'PII would lose anyway' statement at the end of the proof.

We remark that a similar argument shows that in any transitive avoidance game any strategy of PI that promises never to play a particular point is not winning. Thus, winning strategies for PI must be global; i.e., they must examine the entire board.

2.2 The plus version

As described in the introduction, the plus variant of an avoidance game is the same as the avoidance game except that, on each move, a player may choose to pick as many points as he likes rather than just one; in other words, on each move, each player chooses a (non-empty) set of points.

Theorem 4. *The plus version of a transitive avoidance game is not a PI win.*

Proof. Since the players can pick an arbitrary number of points we cannot tell which player's move it is just by looking at the current position, and we will make use of this fact in our proof. Thus, we introduce some notation for the position. A position is a pair of disjoint sets (S, T) where S denotes the set of all points picked so far by the player whose turn it is, and T the set of all points picked by the other player. From this position we will name the player whose move it is the *current* player.

Let G be the automorphism group of the game.

Suppose for the sake of a contradiction that PI has a winning strategy and let S be PI's first (set) move in this strategy. After this move the position is $(\emptyset, S)$. Since PI is playing a winning strategy we know that position (S, T) is a current player win for any non-empty set T. Thus, for any $g \in G$ and any non-empty set T disjoint from $g(S)$, the position $(g(S), T)$ is also a current player win.

To obtain a contradiction we now show that the game is, in fact, a PII win. Suppose that, on his first move, PI plays some set U. If U is the whole board then the game is over and PII has definitely not lost. Thus, we may assume that PI does not pick all the points.

We have to give a strategy for PII. Pick $x \notin U$ and $g \in G$ which maps some $s \in S$ to x. Let $S' = g(S)$. PII plays $S' \setminus U$, which contains x so is non-empty.

Now, suppose that PI plays W. Then, it is PII's turn and the position is $(S' \setminus U, U \cup W)$. Since $S' \setminus U \subset S'$ and

$$(S' \setminus U) \cup (U \cup W) = S' \cup U \cup W = S' \cup ((U \setminus S') \cup W),$$

we see that $(S' \setminus U, U \cup W)$ is no worse for the current player than $(S', (U \setminus S') \cup W)$. As noted above, this latter position is a current player win so $(S' \setminus U, U \cup W)$ is also a current player win; i.e., PII wins as claimed. □

Some plus version avoidance games have been studied previously: in particular, the Ramsey Avoidance Game $RAG(N, s)$ (see, e.g., [14, 2]). This game is played with board the edge set $E = E(K_N)$ and lines are the $\binom{s}{2}$-subsets of E corresponding to K_s subgraphs for some fixed s.

Corollary 5. *The plus version of Ramsey Avoidance Game $RAG(N, s)$ is not a PI win. In particular, if $N \geqslant R(s, s)$ (where $R(s, s)$ denotes the Ramsey number) then it is a PII win.*

Proof. The game is obviously transitive so Theorem 4 implies that it is not a PI win. In the case $N \geqslant R(s, s)$ the game cannot be a draw – when the board is full (at least) one player must have a K_s – and so must be a PII win. □

3 First player wins for odd board sizes

In this section we show that there are first player wins for all odd composite board sizes. This is in contrast to the special cases discussed in the previous section.

Theorem 6. *Suppose that $n = pq$ is odd. Then there is a transitive avoidance game on $[n]$ that is a PI win.*

Proof. First we define the game. View the board, $[n]$, as q sets $A_1, A_2, \dots, A_q$ of p points. Let $p' = (p + 1)/2$ and $q' = (q + 1)/2$. We call each A_i a *bucket*.

Let $\mathcal{W}$ be all subsets of $[n]$ of size $p'q'$ consisting of p' from each of q' buckets. Define the lines $\mathcal{L} = [n]^{(p'q')} \setminus \mathcal{W}$.

We claim that the avoidance game on $(n, \mathcal{L})$ is transitive and a PI win. The transitivity is trivial since we can permute the bins, and permute the points in any bin. Thus we just need to prove that it is a PI win.

Since, every pair of sets in $\mathcal{W}$ meet, we see that if PI can form a set in $\mathcal{W}$ with his first $p'q'$ points then PII's first $p'q'$ points must form a set not in $\mathcal{W}$, i.e., these points must form a line in $\mathcal{L}$. Thus, it suffices to show that PI can guarantee to form a set in $\mathcal{W}$ with his first $p'q'$ points.

We call a bucket *active* if PI has played at least one point in it but not yet p' points in it. We say it is *full* if he has played p' points in it. PI's strategy is to play according to first of the following rules that applies.

1. If PII has just played in an active bucket then PI plays in the same bucket.

2. If less than q' buckets are either full or active then PI plays in an empty bucket.

3. PI plays in any active bucket.

These rules imply that, after his turn, PI always has strictly more points than PII in any active bucket, and strictly more than half the non-empty buckets are active or full. Thus after $p'q'$ moves PI has exactly p' points in exactly q' buckets. □

4 First player wins for even board sizes

4.1 Isbell Families

We saw in the introduction that one advantage for the first player is that he goes first and can pick a 'special' point, but obviously this is not possible for a transitive game. A second possible advantage for the first player occurs if the board has an even number of points: the second player has no choice on his last turn, whereas the first player does always have a choice. We use this 'forced move' to construct examples of games with even size boards that are first player wins.

It turns out that our avoidance game is closely related to an existing game idea, namely that of a *fair game*, from a very different context (see Isbell [10] and [11]). To avoid confusion with other notions of fair game we will call the families involved *Isbell families*.

Definition. *An* Isbell family *on a set* $[n]$ *is a family* $\mathcal{F}$ *of subsets of* $[n]$ *such that* $\mathcal{F}$ *is an up-set containing exactly one of* X *and* $[n] \setminus X$ *for each set* X*, and having a transitive automorphism group.*

We remark that an Isbell family must be *intersecting*, that is any two sets in the family meet.

Proposition 7. *Suppose that* n *is even and that there exists an Isbell family on* $[n]$*. Then there is a transitive avoidance game with board* $[n]$ *that is a first player win.*

Proof. Let $\mathcal{F}$ be the sets in the Isbell family. We define the lines $\mathcal{L} = \mathcal{F} \cap [n]^{(n/2)}$. We show that this avoidance game is a PI win. Obviously, this game is transitive.

Consider the achievement game on board $[n]$ with winning lines $\mathcal{W} = [n]^{(n/2)} \setminus \mathcal{L}$. By the definition of an Isbell family every $n/2$ sized set is either in $\mathcal{W}$ or its complement is in $\mathcal{W}$. Hence, a draw is impossible in this achievement game. Thus, by the standard strategy stealing result, this achievement game must be a PI win.

PI follows exactly the same strategy in the avoidance game. At the end his $n/2$ points form a set in $\mathcal{W}$ so not in $\mathcal{L}$ (so he has not lost) and PII's $n/2$ points are the complement of PI's set, so must form a set in $\mathcal{L}$ and PII has lost. □

It is not known for exactly which n Isbell families exist but most cases are known. In particular, Isbell [10] showed that they do exist for $n = 2b$ with $b > 1$ odd; and Cameron, Frankl and Kantor [5] showed they do exist for $n = 4b$ with $b > 3$ and odd. Thus, for all these cases we have transitive avoidance games which are first player wins. However, Cameron, Frankl and Kantor [5] also showed that Isbell families do not exist for $n = 2^a$, or for $n = 3 \times 2^a$ for $a \geqslant 2$, so we cannot use them to prove all the remaining cases of Theorem 1.

For concreteness let us describe the Isbell family, and thus the PI win avoidance game, on 6 points. We think of the six points as being arranged in a grid of two rows and three columns. The Isbell family $\mathcal{F}$ is the up-set generated by the family of all 3-sets that either contain one point from each column and an even number of points in the top row, or contain both points in one column and one point in the next column cyclicly. Since we can cycle the columns or swap the elements in each of two columns we see that this family is transitive. Also it is easy to see that every set is either in $\mathcal{F}$ or its complement is in $\mathcal{F}$. Thus, $\mathcal{F}$ is indeed an Isbell family, and the avoidance game with lines $[6]^{(3)} \cap \mathcal{F}$ (i.e., the lines are exactly the generating sets described above) is a PI win. Indeed, this is easy to verify by hand.

In contrast, there do not exist first player wins when n is a power of 2 and the board is transitive. In order to prove this we start with a simple lemma.

Lemma 8. *Let $(n, \mathcal{L})$ be a transitive avoidance game and suppose that its automorphism group contains a fixed-point-free involution. Then the game is not a PI win.*

Proof. Let G be the game's automorphism group and let $g \in G$ be a fixed-point-free involution. Then g partitions the board into pairs. PII's strategy is to play the point paired with PI's previous move; i.e., if PI plays x, then PII plays $g(x)$.

Suppose that, after PII's move, PII has played all the points in T. Then PI must have played all the points in $g(T)$. Thus, if PII's move was losing – that is, if T contains a losing set – then PI must have already lost. □

The following theorem is an immediate consequence.

Theorem 9. *A transitive avoidance game on a set of size 2^a for some a is not a PI win.*

Proof. Let G be the automorphism group of the game. Then G acts transitively on the board X which has size 2^a. By a standard result (see e.g., [5]) from group theory, it follows that g has a fixed-point-free involution. Thus, by Lemma 8 the game is not a PI win. □

We would like to extend Proposition 7 to show that there is a transitive avoidance game that is a PI win for all even sizes except powers of 2. In the proof of Proposition 7 we relied on the existence of Isbell families. These were useful because they gave a transitive intersecting family consisting of half of the $\frac{n}{2}$-subsets. Since there do not exist Isbell families for sizes 3×2^a, we cannot use the same technique. Indeed, since the union of any such intersecting family and the family of all subsets of size strictly greater than $n/2$ is an Isbell family, such intersecting families only exist when Isbell families exist.

Instead, we look for a smaller intersecting family $\mathcal{F}$ for which PI can win the achievement game. Since $\mathcal{F}$ is not a maximal intersecting family it is possible that neither player forms a set from $\mathcal{F}$ (i.e., a draw is possible in the achievement game), so we cannot use a strategy stealing argument to show that PI has a winning strategy for the achievement game. Thus, we need to define a winning strategy explicitly.

4.2 Special case: $n = 2b$ with b odd

As our proof in the general case is rather involved, we start by illustrating it in the simple case of $n = 2b$ with b odd; this case is covered by Isbell's results on the existence of Isbell Families, but we will give an example with a much smaller intersecting family for all $n \geqslant 10$. (For $n = 6$ the family is exactly the Isbell family on six points described earlier.) We remark that we give a proof that generalises to the full case, rather than the simplest proof for this special case.

Proposition 10. *For any odd $b \geqslant 3$ there is a transitive avoidance game of size $2b$.*

Proof. Let $n = 2b$ and $b' = (b-1)/2$. We think of $[n]$ as $\mathbb{Z}_b \times \mathbb{Z}_2$: i.e., as b pairs. We start by defining some *winning* sets $\mathcal{W}$. They are all of size b, and are

1. all sets which contain exactly one point from each pair and an odd number of which have a 1 in the $\mathbb{Z}_2$-coordinate,
2. all sets with both elements of exactly one pair, that have the (unique) pair with neither element at most b' later cyclicly.

Obviously the complement of any set in this family is not in the family, and thus we see that this family is intersecting. We define the board of our avoidance game to be $[n]$ and the lines $\mathcal{L}$ to be the complements of the sets in $\mathcal{W}$. We claim that this avoidance game is transitive and a PI win.

To see that the game is transitive, it is enough to observe that its automorphism group contains the following elements

- cycle the b pairs,
- in two pairs swap the elements (i.e., swap $(x, 0)$ and $(x, 1)$, and swap $(y, 0)$ and $(y, 1)$ for some x and y).

Thus, to complete the proof, we just need to show that the game is a PI win. Indeed, if we can show that PI can guarantee to make a set in $\mathcal{W}$, then we know that PII will finish with a set in $\mathcal{L}$, and so will lose.

We will denote positions in the game as ordered pairs (A, B) of subsets of the board where this means PI has played the points in A and PII has played the points in B. For a set A containing at most one of each pair, we write $\bar{A}$ for the 'opposite' points: that is the points in the other half of each pair that meets A.

First, define a position to be a *direct win* if it is of the form $(A \cup \{(x, y)\}, \bar{A} \cup \{(z, w)\})$ for some A and $1 \leqslant z - x \leqslant b'$, and observe that PI has a simple winning strategy from any such position. Indeed, he plays $(x, y + 1)$ and for the rest of the game makes sure he gets one of every other pair set except he never plays the point $(z, w + 1)$ opposite to (z, w). This means that after PI's last turn he has one of every pair, except he has both of the x^{th} pair and neither of the z^{th} pair. The constraint on z means that this is a winning set under Condition 2 above.

Our strategy for PI is as follows. Unless the position is a direct win, he will ensure that the position after his turn is of the form $(A \cup \{(x, y)\}, \bar{A})$ for some set A and point (x, y).

Now from the position $(A \cup \{(x, y)\}, \bar{A})$ suppose that PII plays a point (z, w). We split into several cases.

1. If $1 \leqslant z - x \leqslant b'$ then the position is a direct win and he wins by following the simple strategy given above.

2. If (z, w) is any point other than $(x, y+1)$ (i.e., unless PII plays the opposite point to (x, y)) then PI plays the point $(z, w+1)$ (i.e., opposite where PII played). Obviously this is possible and keeps the position being of the form above, so PI follows the same strategy from the new position.

3. The last case is if PII plays the point $(x, y + 1)$, i.e., the opposite point to (x, y). Thus, before PI plays, the position is $(B, \bar{B})$ where $B = A \cup \{(x, y)\}$: i.e., every pair is either full or empty. Let x' be the first empty pair. If $x' < b'$ then PI just plays $(x', 0)$ making the position of the required form and continues as above.

   (Note this case implies that PI plays $(0, 0)$ on his first go.)

   There is one remaining case: $x' \geqslant b'$. We deal with this below.

To have reached this position all pairs $[0, b')$ must already be filled. There may be some points in pairs in the interval (b', b) but the b'^{th} pair must be empty as PI did not play in it, and if PII had played in it then he would have lost under Case 1 above (as for the whole game so far the 'extra point' in the position was in a pair in the interval $[0, b')$).

For the remainder of the game PI is going to play in a certain fashion filling up the empty pairs $[b', b)$ in turn, except if the position is a direct win, in which case PI follows that strategy. Since, when PI plays in the empty pair x all pairs $[0, x)$ have already been filled, unless PII plays the other point in pair x then the position will be a direct win and PI wins. Thus, we may assume that PII plays the other point in pair x.

Hence, PI gets to pick one point from each of the remaining empty pairs, and PII has to pick the other from each of them. In particular, when PI picks his point from the last empty pair he can choose the correct element to ensure the correct parity under Condition 1. Thus, PI finishes with a set in $\mathcal{W}$ as required. □

4.3 General even case

Having seen this special case, we extend these ideas to all board sizes of the form $n = 2^a b$ with b odd and greater than 1 (i.e., all even board sizes except powers of 2). As above, we view this as b copies of 2^a; we call each copy of 2^a a *bin*. In the previous example there were very few possibilities for what happened in one copy of 2^a (i.e. in a pair) but in the general case there will be a lot more. This makes the proof substantially more difficult.

We need some definitions. Let $m = 2^a$ and let $m' = m/4$.

Definition. *For $x \in [m]$ we define the* opposite point *to x to be the point $x + m/2$, where addition is modulo m. We call the pair $\{x, x + m/2\}$ an* opposite pair*. We say a set $A \subset [m]$ is a* cyclic pair set *if it contains exactly one of each opposite pair. We say it is a* partial cyclic pair set *if it contains at most one of each opposite pair and is not empty. In a partial pair set we call any point where neither it nor its opposite point are in the set a* free *point.*

The proof relies on a careful examination of the lexicographic order on a set and its rotations and, in particular, its 'maximum' rotate defined as follows.

Definition. *The lexicographic order on subsets of $[m]$ is defined as follows: $A \leqslant B$ if the first point in the symmetric difference $A \Delta B$ is in A.*

*For any $r > 1$, an r-*maximal point *of a partial cyclic pair set A is a point x where the intersection of A with the interval $\{x, x+1, x+2, \ldots, x+r-1\}$ is maximal in lexicographic order over all the sets formed by intersecting A with an interval of length r of $[m]$. We say x is* maximal *if it is an m-maximal point. We define r-*minimal *and* minimal *similarly.*

Lemma 11. *Any partial cyclic pair set in $[m]$ has a unique maximal point.*

Proof. Since the set contains exactly one of some opposite pair the set is not fixed by cycling by $m/2$, the order of the stabiliser is not divisible by 2. Since the set has size $m = 2^a$ this means the stabiliser has order 1; in particular, no two cyclic shifts of A are the same so there must be a unique maximal point. □

In the case when $a = 1$ discussed above, i.e., for board sizes $2b$, it was important that when PI came to decide what happened in the final empty pair he already knew what would happen in all the remaining pairs. In that case that was trivial: they were already completely filled. In the general case, when PI decides what happens in the final empty bin the remaining bins are non-empty but this does not mean they are full. However, the following key lemma shows that PI has some control over what happens in these later bins.

Lemma 12. *Suppose that m and m' are as above, and that A is a partial cyclic pair set in $\mathbb{Z}_m$. Then there exist values s, t and z_1, z_2 such that*

- *if all free points in $[z_1, z_1 + m')$ are placed in A then regardless of which of the remaining free points are placed in A the maximum lies in $[t - s, t]$*
- *if all free points in $[z_2, z_2 + m')$ are placed in A then regardless of which of the remaining free points are placed in A the maximum lies in $[t, t + 2m' - s)$.*

We postpone the proof of this technical lemma to later. First, we show that we can deduce the existence of transitive avoidance games of all even sizes (except powers of 2) that are PI wins from it. The deduction is similar to the proof of Proposition 10.

Theorem 13. *Suppose that $n = 2^a b$ for some odd $b > 1$. Then there is a transitive avoidance game on n points which is a Player 1 win.*

Proof. Let $m = 2^a$, $m' = m/4$ and $b' = (b-1)/2$. First, we define the winning sets $\mathcal{W}$. These will all have size $n/2$ and the lines in our avoidance game will be the family of their complements. We view n as being $\mathbb{Z}_b \times Z_m$. For any point (x, y) we define its opposite point to be the opposite point in the same bin: i.e., $(x, y + 2m')$.

The winning sets $\mathcal{W}$ are all the sets

1. that contain exactly one of each opposite pair and the sum over all bins of the maximal points mod m lies in the interval $[0, m/2)$.
2. that contain both elements of exactly one opposite pair, say the pair $\{y, y + 2m'\}$ in bin j, and the unique empty opposite pair is either
   (a) in one of the bins between $j + 1$ and $j + b'$,
   (b) in bin j and of the form $\{z, z + 2m'\}$ for some $y + 1 \leqslant z \leqslant y + m' - 1$ (i.e., the empty pair is between 1 and $m' - 1$ after the full pair in the same bin)

This is an intersecting family; indeed, since all the sets have size $n/2$, we just need to check that the complement of any set in the family is not in the family. In the first case, this follows since the maximal point of the complement of a set containing exactly one of each pair is the maximal point of the set plus $m/2$, so the sum of the maximal points changes by $bm/2 \equiv m/2$ modulo m. In the second case it is trivial.

Also this family is transitive: indeed, the automorphism group contains the elements

- cycle the b bins
- rotate each bin i by an amount r_i with $\sum_i r_i = 0$.

As in Proposition 10 we will denote a position in the game by an ordered pair (A, B) of subsets of the board, where A denotes the points played by PI, and B the points played by PII. For any set A containing at most one of each opposite pair we write $\bar{A}$ for the set of points opposite to A.

Also as in Proposition 10, some positions have simple direct wins. There are two types. A position of the form $(A\cup\{(x,y)\},\bar{A}\cup\{(z,w)\})$ is a *direct win of type-1* if $1 \leqslant z-x \leqslant b'$. It is a *direct win of type-2* if $z=x$ and $0 < w-y < m'$ or $2n' < w-y < 3m'$. Let us see that both of these are indeed winning positions for PI.

If the position is a direct win of type-1 then PI plays $(x, y+2m')$ and for the rest of the game makes sure he gets one of every other pair set except he never plays the point $(z, w+2m')$ opposite to (z,w). This means that after PI's last turn he has one of every pair except he has both of a pair in the x^{th} bin and neither of a pair in the z^{th} bin. The constraint on z means that this is a winning set under Condition 2a above.

If the position is a direct win of type-2 then PI plays $x, y+2m'$ and again makes sure he gets one of each opposite pair apart from he never plays $(z, w+2m')$. This means that Maker finishes with a set that is winning under Condition 2b above.

PI's strategy is as follows. Unless the position is a direct win (of either type), he will make sure that the position after his turn is of the form $(A\cup\{(x,y)\},\bar{A})$. Now suppose that, from this position, PII plays any point (z,w). We split into several cases

1. If $1 \leqslant z-x \leqslant b'$ then the position is a direct win of type-1 and PI wins.
2. If $z=x$ and $0 < w-y < m'$ or $2n' < w-y < 3m'$ then the position is a direct win of type-2 and PI wins.
3. If (z,w) is any point other than $(x, y+2m')$ (i.e., unless PII plays the opposite point to (x,y)) then PI plays the point $(z, w+2m')$ (i.e., opposite where PII played). Obviously, this is possible and it keeps the position being of the form above, so PI follows the same strategy from the new position.
4. If PII plays $(x, y+2m')$, the point opposite to (x,y). Then, before PI plays, the position is $(B,\bar{B})$ where $B = A\cup\{(x,y)\}$. Let (x',y') be the empty point where x' is smallest and y' is smallest amongst the empty points for that value of x'. If $x' < b'$ then PI just plays (x',y') making the position of the required form, and continues as above.

   (Note this case implies that PI plays $(0,0)$ on his first go.)

   There is one remaining case: $x' \geqslant b'$. We deal with this below.

There is one remaining case. To have reached this position all points in bins $[0,b')$ must already be filled. There may be some points in bins (b',b) but the bin b' must be empty as PI did not play in it and if PII had played in it then he would have lost under Case 1 above (as for the whole game so far the 'extra point' in the position was in bin $[0,b')$.

For the remainder of the game PI is going to play in a certain fashion filling up the bins $[b',b)$ in turn, except if a direct win of either type occurs, in which case PI follows the appropriate winning strategy as described above. Since, when PI plays in a bin x all pairs $[0,x)$ have already been filled, unless PII also plays in the bin x the position will be a direct win and PI wins. Thus, we may assume that PII plays in the same bin as PI for the rest of the game.

Observe that, when PI first plays in any of the bins $[b', b)$, he can pick a point u and guarantee to play all the empty points in $[u, u + m')$. Indeed, he starts by playing the first empty point after u. Then he follows his normal 'pick the opposite point' strategy (i.e., Case 3 of the strategy above) but whenever he gets a free choice (i.e., Case 4) he picks the next free point in $[u, u + m')$. By doing this PI does get all the free points in $[u, u + m')$ since, if PII ever takes any of the points in $[u, u + m')$ or $[u + 2m', u + 3m')$, then PI wins directly under Case 2 above.

Let r be the final empty bin. PI plays arbitrarily in the bins $[b', r)$, just ensuring that he gets one of each pair. (For example he could guarantee to play all the points in $[0, m')$ as described above.)

Now PI is about to play in bin r. We define a key quantity which PI will try and control for the rest of the game; this quantity will be defined for all positions where, for some $j > r$, bins $[0, j)$ are full but bin j is not. In such a position, the maximal points u_i for bins $[0, j)$ have all been determined and, moreover, all the remaining bins are non-empty. Thus, for $j \leqslant i < b$, there exist s_i, t_i as given by by Lemma 12. We will think of the number

$$G(j) = \sum_{i<j} u_i + \sum_{i \geqslant j} t_i$$

as being the current 'guess' at the sum of the maximum points. PI's strategy is to ensure that, for all $j > r$, this stays in the region $[0, 2m')$ as the bins fill up. We show that PI can achieve this by induction.

First, we show that he can play in bin r to ensure that $G(r + 1) \in [0, 2m')$. This is trivial: PI picks a point u and, as above, guarantees to pick all the points in $[u, u + m')$. It is easy to see that, however PII plays, the maximum lies in $[u - m', u]$. Thus, regardless of the game so far, by choosing u correctly, PI can ensure that $G(r + 1) \in [0, 2m')$.

Now suppose that $j > r$ and that $x = G(j)$. Inductively, we know that $x \in [0, 2m')$, so one of $[x - s_j, x]$ and $[x, x + 2m' - s_j)$ is a subset of $[0, 2m')$. Therefore, by the above observation, PI can play all the free points in $[z_1, z_1 + m')$ or $[z_2, z_2 + m')$ respectively, where z_1, z_2 are as in Lemma 12. Then, however PII plays, $u_j \in [t_j - s_j, t_j]$ or $u_j \in [t_j, t_j + 2m' - s_j)$ and

$$G(j + 1) = G(j) + u_j - t_j \in [0, 2m').$$

Thus, $G(j + 1) \in [0, 2m')$ and the induction is complete.

When the process finishes the 'guess' $G(b)$ equals the actual sum of the maximum points and so this sum is in $[0, 2m')$ and PI wins. □

4.4 Proof of Lemma 12

In this section fix $m = 2^a$ and let $m' = m/4$. We start with some some notation.

Suppose that A is a partial cyclic pair set in $\mathbb{Z}_m$. Then $A|_{[x,x+r)}$ denotes the restriction of A to the interval $[x, x + r)$, rotated to be a subset of $[0, r)$.

We write $A + \mathbb{1}|_{x,x+r}$ for the union of A and the set of all free points in $[x, x + r)$. Note that this is not just $A \cup [x, x + r)$ as there may be points of $[x, x + r)$ that are opposite to points in A and these are not added; in particular, $A + \mathbb{1}|_{x,x+r}$ is a partial cyclic pair set.

Finally we write $A_{\max}$ for the set A with all the free points added. Note that $A_{\max}$ is not a pair set (unless A was already a full pair set).

Next we need some simple lemmas about maximal points. We start with a trivial observation that we use repeatedly.

Lemma 14. *Suppose that A is a full pair set and that x is r-maximal. Then $x + 2m'$ is r-minimal.*

Proof. Immediate from the definition of a cyclic pair set. □

Lemma 15. *Let A be any subset of $[m]$. Then any r-maximal point is r'-maximal for all $r' \leqslant r$.*

Proof. This is trivial from properties of the lexicographic order. □

Lemma 16. *Let A be any subset of $[m]$ and suppose that x is an r-maximal point and $x + r$ is an r'-maximal point. Then x is an $(r + r')$-maximal point.*

Proof. This is trivial from properties of the lexicographic order. □

Corollary 17. *Suppose that A is a partial cyclic pair set and x is a maximal point. Then the point $x - r$ is not an r-maximal point.*

Proof. By the previous lemma $x - r$ would be an $(r + m)$-maximal point so an m-maximal point (i.e., a maximal point) contradicting the uniqueness of the maximum point (Lemma 11). □

The same holds for $A_{\max}$ even though it is not a cyclic pair set.

Corollary 18. *Suppose that A is a partial cyclic pair set and x is a maximal point of $A_{\max}$. Then the point $x - r$ is not an r-maximal point of $A_{\max}$.*

Proof. Again, by Lemma 16, $x - r$ would be an $(r + m)$-maximal point of $A_{\max}$ so an m-maximal point of $A_{\max}$. However, by a similar argument to Lemma 11, $A_{\max}$ has a unique maximal point (the proof of that lemma just uses the fact that the set has exactly one point of some pair). □

Lemma 19. *Suppose that A is a (full) pair set, that 0 is maximal, that x is r-minimal for some x and r, and that y is any x-maximal point. Then $y + x$ is r-minimal.*

Proof. Since 0 and y are both x-maximal we have that $A|_{[0,x)} = A|_{[y,y+x)}$ and, since 0 is maximal, we have $A|_{[0,x+r)} \geqslant A|_{[y,y+x+r)}$. Combining these we get $A|_{[x,x+r)} \geqslant A|_{[y+x,y+x+r)}$. But since x is r-minimal, we have $A|_{[x,x+r)} \leqslant A|_{[y+x,y+x+r)}$ and hence $A|_{[x,x+r)} = A|_{[y+x,y+x+r)}$. Thus, $y + x$ is r-minimal. □

Lemma 20. *Suppose that A is a (full) pair set and that 0 is maximal. Then, for all $r < 2m'$, no point in $[0, r]$ is r-minimal. In particular, this holds for $r = m'$.*

Proof. Suppose that $x \in [0, r]$ is r-minimal. Since $r \geqslant x$ Lemma 19 implies that, for any y that is x-maximal, $y + x$ is x-minimal, and this is the form we shall use.

Now x is r-minimal, so $x + 2m'$ is r-maximal and thus, since $x \leqslant r$, also x-maximal. By Lemma 19, $2x + 2m'$ is x-minimal, and hence $2x$ is x-maximal. By Lemma 19 again, we have that $3x$ is x-minimal, so $3x + 2m'$ is x-maximal. Repeating we see that $4x + 2m'$ is x-minimal so $4x$ is x-maximal etc. In particular kx is x-maximal for even k and x-minimal for odd k. Thus the pair set A is periodic with period $2x \leqslant 2r < m$. Since a cyclic pair set cannot be periodic this is a contradiction. ☐

Lemma 21. *Suppose that A is (full) pair set. Suppose that x is an m'-maximal point. Then the maximal point lies in $(x - 2m', x]$.*

Proof. Let y be the maximal point.

First, suppose that $y \in [x + 1, x + m']$. Then, by Corollary 17, x is not m'-maximal, which is a contradiction.

Now suppose that $y \in [x + m', x + 2m']$. Then $x + 2m'$ is m'-minimal but, since $x + 2m' \in [y, y + m')$, this contradicts Lemma 20. ☐

Corollary 22. *Suppose that A is a full pair set and that 0 is m'-maximal. Then the actual maximal point is the first point in the set $(2m', 4m']$ that is m'-maximal.*

Proof. Let x be the first point in $(2m', 4m']$ that is m'-maximal. By Lemma 21 applied to 0, we know that the actual maximum is in $(2m', 4m']$ and, by applying it to x, that the actual maximum is not in $(x - 2m', x]$. Combining these we see that the actual maximum must be in $(2m', x]$. Since the actual maximum must be m'-maximal this implies that x is the actual maximum. ☐

Lemma 23. *Suppose that A is a partial cyclic pair set and that x is m'-maximal in $A_{\max}$ and that y is such that there are no free points in the interval $[x, y)$. Let $A' = A + \mathbb{1}|_{[y,y+m')} + \mathbb{1}|_{[y-m',y)}$ and let x' be the maximum point of A'. Then*

(a) $x' \in (x - 2m', x]$,

(b) *x' is m'-maximal in $A_{\max}$,*

(c) *if $x' \notin (y - m', x]$ then A has no free points in the interval $[x', y - m')$,*

(d) *for any (full) pair set B extending $A + \mathbb{1}|_{[y,y+m')}$, the maximum point of B lies in the interval $[x', x]$.*

Proof. First observe that, since there are no free points in $[x, y)$ we have $A'|_{[x,x+m')} = A_{\max}|_{[x,x+m')}$. This observation, together with x' being m'-maximal in A', shows that

$$A_{\max}|_{[x',x'+m')} \geqslant A'|_{[x',x'+m')} \geqslant A'|_{[x,x+m')} = A_{\max}|_{[x,x+m')}.$$

Since x is m'-maximal in $A_{\max}$ this shows that x' is also m'-maximal in $A_{\max}$ and that each of the inequalities must actually be an equality.

This shows that x is m'-maximal in A', so by Corollary 22, $x' \in (x-2m', x]$ which is part (a). It also shows that x' is m'-maximal in $A_{\max}$ which is part (b).

Moreover, it shows that $A_{\max}|_{[x',x'+m')} = A'|_{[x',x'+m')}$ so, in particular, if $x' \notin (y-m', x]$ then, since any free points in the interval $[x', y-m')$ would be present in $A_{\max}$ but absent in A' there cannot be any such free points.

To prove part (d) observe that, since there are no free points in $[x, y)$, $B|_{[x,y)} = A'|_{[x,y)}$, and by definition $B|_{[y,y+m')} = A'|_{[y,y+m')}$. Thus,

$$B|_{[x,y+m')} = A'|_{[x,y+m')}. \tag{$*$}$$

In particular $B|_{[x,x+m')} = A'|_{[x,x+m')} = A_{\max}|_{[x,x+m')}$, so x is m'-maximal in B. Let z be the maximum point of B. By Corollary 22, we see that $z \in (x-2m', x]$.

Also, since A' and B are full pair sets, $(*)$ shows that $B|_{[x-2m',y-m')} = A'|_{[x-2m',y-m')}$. By definition $A'|_{[y-m',y)} \geqslant B|_{[y-m',y)}$ (i.e. restricted to this interval, A' contains B). Combining these, we see that $A'|_{[x-2m',y)} \geqslant B|_{x-2m',y)}$. In particular, any m'-maximal point of B in $[x-2m', x]$ is also m'-maximal in A'. Thus by Corollary 22 we see that $x' \leqslant z$. □

We are now in a position to prove Lemma 12.

Proof of Lemma 12.

We may assume that 0 is the maximum point of $A_{\max}$. Write A^k for the set $A + \mathbb{1}|_{[km',(k+1)m')}$ (so $A^4 = A^0$ etc – we use whichever expression is convenient). Similarly, write $A^{k,k+1}$ for the set $A + \mathbb{1}|_{[km',(k+1)m')} + \mathbb{1}|_{[(k+1)m',(k+2)m')}$ and note that $A^{k,k+1}$ is a cyclic pair set.

Let x_k be the maximum point of $A^{k,k+1}$. Observe that $A^{4,5}|_{[0,2m')} = A_{\max}|_{[0,2m')}$ so 0 is maximal in $A^{4,5}$; i.e., $x_4 = 0$.

By Lemma 23(b) applied with $x = y = 0$, the maximum point x_3 of $A^{3,4}$ is m'-maximal in $A_{\max}$. By Corollary 18 either $x_3 = x_4$ or $x_3 < 3m'$. If $x_3 = x_4$ then, by Lemma 23(d), we see that the maximum point of any set B extending A^0 is 0, and the claimed result is trivially true with $z_1 = z_2 = 0$.

Thus, we may assume $x_3 < 3m'$. Then Lemma 23(c) shows that there are no free points in $[x_3, 3m')$. Now we can apply Lemma 23(b) again but this time with $x = x_3$ and $y = 3m'$. This shows that x_2 is m'-maximal in $A_{\max}$.

If $x_2 > 2m'$ then, by Lemma 23(d), any set extending A^3 has maximum point in $[x_2, x_3]$ and any set extending A^4 has maximum point in $[x_3, 0]$ (recall $x_4 = 0$). Thus, in this case we are done with $t = x_3$ and $s = x_3 - x_2$.

Thus we may assume $x_2 \leqslant 2m'$ and thus, by Lemma 23(c), that that there are no free points in $[x_2, 2m')$. We apply Lemma 23(b) again, this time with $x = x_2$ and $y = 2m'$. This shows that x_1 is m'-maximal in $A_{\max}$.

If $x_3 - x_1 < 2m'$ then, as above, we are done: any set extending A^2 has maximum point in $[x_1, x_2]$ and any set extending A^3 has maximum point in $[x_2, x_3]$.

Thus we may assume $x_3 - x_1 \geqslant 2m'$ and, in particular, that $x_1 \leqslant m'$. Hence, Lemma 23(c) implies that A has no free points in $[x_1, m')$.

To summarise, we have that, for each k, x_k is m'-maximal in $A_{\max}$, $x_k \leqslant km'$, and A does not have any free points in the interval $[x_k, km')$.

If, for any k we have $x_{k+2} - x_k < 2m'$ then as above we are done: any set extending A^{k+1} has maximum point in $[x_k, x_{k+1}]$ and any set extending A^{k+2} has maximum point in $[x_{k+1}, x_{k+2}]$.

Hence the only remaining case is that both $x_2 = 2m'$ (i.e $x_2 = x_0 + 2m'$) and $x_3 = x_1 + 2m'$. Then both 0 and $2m'$ are m'-maximal in $A_{\max}$ and, in particular, $A|_{[0,m')} = A|_{[2m',3m')}$. This means that there can be no fixed point (i.e., non-free point) in these intervals as such a point would be fixed differently in the two sets. Hence all points in these intervals are free.

But, if this is the case, then 0 is the maximal point in any set B extending A^0. Indeed, since 0 is maximal in $A_{\max}$ we must have that the preceding element (element '-1') is fixed not in A. Thus, in the interval $[2m', 4m')$, the set B consists of m' zeros, $m' - 1$ points we don't know about followed by another zero. Thus it is trivial to see that, whatever B is, no point in $[2m', 4m')$ is m'-maximal (it would have to have m' ones following it) and the result follows. □

5 Some Simple Observations

Before proceeding further we collect some simple observations. Note that, although some of the proofs are a little long to write out, all the results in this section are essentially trivial consequences of our work so far.

5.1 Bounded size lines

In Theorem 3 we showed that no transitive game with a line of size 2 is a PI win. However, this is best possible since our first example of an even sized PI win was a game on 6 points with all lines of size 3. It is easy to extend this to find arbitrarily large examples of PI win games with all lines of size 3.

Theorem 24. *For all $n = 12k + 6$ there is a transitive avoidance game on n points, with all lines of size 3, that is a first player win.*

Proof. Let $\mathcal{H}_0$ be the transitive avoidance game on 6 points that is a PI win given by Proposition 10 and let Φ be a winning strategy for $\mathcal{H}_0$. Let $\mathcal{H}$ be the disjoint union of $2k + 1$ copies of $\mathcal{H}_0$ which we write as $[2k + 1] \times \mathcal{H}_0$.

Obviously $\mathcal{H}$ has $12k + 6$ points, all lines have size 3, and it is transitive. Thus we just need to show that this game is a PI win. The idea is that, if PII plays in $\mathcal{H}_0$ then PI also plays in $\mathcal{H}_0$ following the winning strategy for $\mathcal{H}$, and otherwise PI 'mirrors' PII's move by playing in the same place in the 'opposite' copy of $\mathcal{H}$.

Let f be an involution from $[2k + 1] \to [2k + 1]$ fixing 1 and having no other fixed point. PI starts by playing according to the winning strategy in the first copy of $\mathcal{H}_0$. For all subsequent moves he plays as follows. Suppose that PII has just played a point $(x, y) \in [2k+1] \times \mathcal{H}_0$. If $x \neq 1$ then PI plays $(f(x), y)$; if $x = 1$ then PI plays according to

Φ in the first copy of $\mathcal{H}_0$. It is easy to see that PI can follow this strategy (i.e., he never has to play a point that has already been played), and that this strategy is a PI win. $\square$

One might wonder whether size 3 is special in the above theorem, but it is easy to show that there are arbitrarily large games that are PI wins and have all lines of any fixed size greater than 2.

Corollary 25. *For any $r \geqslant 3$ and n_0 there is a transitive game on $n > n_0$ points with lines all of size r that is a first player win.*

Proof. Take the game $\mathcal{H}$ with lines $\mathcal{L}$ given by Theorem 24 with $k = \max(n_0, 2r)$. Define the new game $\mathcal{H}'$ to have the same board and to have lines

$$\mathcal{L}' = \{L' \in [n]^{(r)} : L' \supset L \text{ for some } L \in \mathcal{L}\}.$$

Let PI play as above. Then at some point PII forms a set $L \in \mathcal{L}$. If PII has played at least r points then he has formed a set $L' \in \mathcal{L}'$ (any superset of a set L which has size r is in $\mathcal{L}'$). If PII has not played this many points then PI just has to continue playing until PII has played r points without forming a set in $\mathcal{L}'$ himself. This is trivial: on each turn PI plays in a copy of $\mathcal{H}_0$ that has not been played in previously (where $\mathcal{H}_0$ is as in the construction for $\mathcal{H}$). This is possible as $k \geqslant 2r$. $\square$

5.2 Fast Player I wins

In the examples of PI wins we have given so far PII can avoid losing for a long time: indeed, in the even case he need not lose until the last turn of the game, and in the odd case he can play for at least time $n/4$ (in the construction in Theorem 6 for board size pq all lines have size $p'q' \geqslant pq/4$).

However, an iterated variant of the odd size 'majority of majorities' construction in Theorem 6 shows that there are games where PII must lose in time $o(n)$.

Theorem 26. *For any $\varepsilon > 0$ there is an $n \in \mathbb{N}$ and a game $\mathcal{H}$ on n points such that PII must lose before time εn.*

The proof of this Theorem, whilst simple, is a little tedious to write out. Since we prove a substantially stronger result in the next section we only sketch the construction.

Sketch proof. The board will be the set $X_k = [a_1] \times [a_2] \times \cdots \times [a_k]$ where $a_1, a_2, \ldots, a_k$ are odd numbers. We define some winning sets $\mathcal{W}_k$ inductively based on the winning sets in the game X_{k-1}. In X_1 the winning sets are the subsets of $[a_1]$ which contain exactly $(a_1 + 1)/2$ points. In X_k the winning sets are the sets which are the union of $(a_k + 1)/2$ disjoint sets each of which is entirely contained in $X_{k-1} \times \{i\}$ for some i and, when viewed as a subset of X_{k-1} is a winning set (i.e., it is in W_{k-1}). Note that the winning sets all have size $\prod_{i=1}^{k} \frac{a_i+1}{2}$. The losing sets $\mathcal{L}$ are all other sets of this size.

The proof that this game is a PI win is very similar to the proof of Theorem 6. We omit the details but remark that the key fact is that $\mathcal{W}_k$ is intersecting, and that PI can guarantee to complete a set in $\mathcal{W}_k$ in exactly $\prod_{i=1}^{k} \frac{a_i+1}{2}$ moves (i.e., as quickly as possible). These, together, imply that PII will lose on his following move. $\square$

5.3 Transitivity

So far we have required that the game be transitive on the points. However, many natural games are more transitive than this. For example the Ramsey Avoidance Game is also line transitive (recall that edges in that game correspond to our points, and complete graphs there are our lines). In fact, the Ramsey Avoidance Game is point/line transitive in that any line, and any point in that line can be mapped, by an element of the automorphism group, to any other line and point in that line.

All our PI wining games so far are not transitive on the lines so it is natural to ask whether this extra transitivity is enough to rule out the possibility of a PI win. We answer this in the next section.

We remark that our 'majority of majorities' game may appear transitive on the lines, and it is true that it is transitive on the set $\mathcal{W}$ of 'allowed subsets'. However, it is *not* transitive on the set $\mathcal{L}$ of lines.

6 The Torus Game

In this section we introduce a very natural game that simultaneously satisfies all of the properties we discussed in the previous section: PI can force a win in time $o(n)$, all lines have size 3, and it is point/line transitive (indeed it is even more: it is also possible to map any point and line not containing it to any other point and line not containing it).

Definition. *The torus game $\mathcal{T}_q(d)$ is the game on board $\mathbb{Z}_q^d$, with lines $\mathcal{L}$ defined to be all subsets of the form $\{x, x+y, x+2y, \dots x+(q-1)y\}$ for $x, y \in \mathbb{Z}_q^d$ and $y \neq 0$.*

We show that the game $\mathcal{T}_3(d)$ is an avoidance game satisfying all the conditions mentioned at the start of this section. Obviously, all lines have size 3, and it is transitive (translation by any element of $\mathbb{Z}_3^d$ is in its automorphism group). The other stronger transitive properties mentioned above are also simple to verify.

Theorem 27. *The game $\mathcal{T}_3(d)$ is not a PII win.*

Proof. The map $g : \mathbb{Z}_3^d \to \mathbb{Z}_3^d$ defined by $g(x) = -x$ is an involution of the board X with a single fixed point 0. PI plays 0 on his first go. Then on each subsequent turn he plays $g(y)$ where y is the point PII just played (informally, PI just mirrors PII's move).

Observe that this is a valid strategy: PI never plays a point that has already been played.

Now suppose that at some point PI forms a set $L \in \mathcal{L}$. This set cannot contain 0 since all lines containing 0 are of the form $\{-y, 0, y\}$, and if PI has played y then PII has played $-y$. Thus, if PI has all the points of L then PII has already played all points of $-L$ which is also a line in $\mathcal{L}$. Thus PII has already lost. $\square$

Corollary 28. *For all sufficiently large d the game $\mathcal{T}_3(d)$ is a PI win. Moreover, if t_d is the longest PII can avoid losing then $t_d/3^d \to 0$ as $d \to \infty$.*

Proof. View the torus $\mathbb{Z}_3^d$ as a Hales-Jewett cube $[3]^d$. Observe that any combinatorial line in the Hales-Jewett cube is a line in the game sense (i.e. is in $\mathcal{L}$). Now for any d greater than the Hales-Jewett number $HJ(q,2)$ (i.e., the smallest d such that any two colouring of the cube $[3]^d$ contains a monochromatic combinatorial line) the game cannot be a draw, and thus must be a PI win.

To prove the bound on the time observe that, by the density version of the Hales-Jewett Theorem [6] (or, indeed, standard cap-set results), there exists a sequence ε_d tending to zero such that any set of size $\varepsilon_d 3^d$ in $[3]^d$ contains a combinatorial line. Thus, by time $\varepsilon_d 3^d$ one player must have lost, so PII must have lost. Hence $t_d/3^d < \varepsilon_d$ so $t_d/3^d \to 0$ as claimed. □

We conclude with an example showing that there are also even-sized boards where PI can win quickly. Moreover, in this example all lines have size 3.

Theorem 29. *There are games $\mathcal{H}_d$ with board size $n_d = 6 \cdot 3^d$ which are PI wins and, moreover, PII loses game $\mathcal{H}_d$ in time $o(n_d)$.*

Proof. The construction is an extension of the above. Let $\mathcal{H}_0$ be the game on six points that is a PI win given by Proposition 10. We define $\mathcal{H}_d$ to be $\mathcal{T}_3(d) \times \mathcal{H}_0$ where the lines in this product are a line in one of the component directions and constant in the other. More precisely, the board is the set $\mathbb{Z}_q^d \times \mathcal{H}_0$ and the lines are the set

$$\left(\bigcup_{\substack{x \in \mathcal{T}_3(d) \\ L \in \mathcal{L}(\mathcal{H}_0)}} \{x\} \times L \right) \cup \left(\bigcup_{\substack{L \in \mathcal{L}(\mathcal{T}_3(d)) \\ y \in \mathcal{H}_0}} L \times \{y\} \right).$$

We have to show that PI wins this game and that the game ends quickly.

First, we give a winning PI strategy. Let Φ denote the winning strategy for PI in $\mathcal{H}_0$. As usual we view the board of $\mathcal{T}_3(d)$ as $\{-1,0,1\}^d$. PI starts by playing in the $(0,0,\dots,0)$ copy of $\mathcal{H}_0$ and plays according to Φ. We call this copy of $\mathcal{H}_0$ the zero copy.

Now for all subsequent moves PI does the following. If PII just played in the zero copy then PI also plays in the zero copy and follows the strategy Φ. If PII played in any other copy $(x_1, x_2, \dots, x_d)$ of $\mathcal{H}_0$ then PI plays the same point in the antipodal copy $(-x_1, -x_2, \dots, -x_d)$ of $\mathcal{H}_0$.

Observe that PI cannot lose by forming any triple not including a point in the zero copy of $\mathcal{H}_0$ as PII would have formed the antipodal set and would have already lost. Furthermore PI cannot lose by forming a triple not wholly contained in the zero copy as PII would have the antipodal point. Finally, PI does not form a losing triple in $\mathcal{H}_0$ as he is following the winning strategy Φ there.

To show that the game ends quickly consider the points in $\mathcal{T}_3(d) \times \{y\}$ for some $y \in \mathcal{H}_0$. If a player has more than $\varepsilon \|\mathcal{T}^d\|$ points in this set then, by the density Hales-Jewett theorem, provided d is sufficiently large, he must have a combinatorial line which is a losing set in our game. Thus, if PII has not lost he has played a total of at most $6\varepsilon|\mathcal{T}^d| = \varepsilon|\mathcal{H}_d|$ points. □

7 Open Problems

Our first open question concerns the case when n is prime.

Question 1. *For which primes n does there exist a transitive avoidance game on n points that is a first player win?*

We know almost nothing in this case. When $n = 3$, 5 or 7 there is no transitive avoidance game that is a PI win. Indeed, 3 is trivial; 5 follows immediately from Theorem 3. The case $|V| = 7$ is slightly trickier but, by Theorem 3, we only need to consider the case where all lines have size three and, since 7 is prime, we may assume that the cyclic group C_7 acts on the board. This reduces the problem to a manageable number of cases.

However, for 11 and 13 there *are* transitive avoidance games that are first player wins. These were found by computer search. The games we find are of the following form: there is a transitive intersecting family $\mathcal{W}$ of $\frac{n-1}{2}$-sets that PI can guarantee to make one of in his first $\frac{n-1}{2}$ moves. Thus setting $\mathcal{L}$ to be all other $\frac{n-1}{2}$-subsets we get the required game.

For $n = 11$, $\mathcal{W}$ is the set of all affine copies of the set $\{0, 1, 2, 4, 5\}$. For $n = 13$, $\mathcal{W}$ is the set of all affine copies of the any of the sets $\{0, 1, 2, 4, 5, 6\}$, $\{0, 1, 2, 4, 5, 7\}$ or $\{0, 1, 3, 4, 5, 7\}$. However, we do not have a 'nice' strategy for either game.

We remark that we searched for games that are transitive under the affine group, as this reduced the number of orbits to a level manageable by computer search. There are other PI wins which are less symmetric, and there are some which are not of this 'maker' form.

We do not know the anything about the answer to this question for any primes greater than 13. Indeed, we do not even know if there are infinitely many such primes. Moreover, the number of possible transitive games for 17 is extremely large (even if we restrict to affinely transitive games) which makes a computer search, even for the next open case, impractical.

Theorem 24 and Corollary 25 answer the question of for which n and r there exist transitive avoidance games on n points with lines all of size r that are PI wins, for infinitely many values of n and r. However, the full characterisation remains open.

Question 2. *For which n and r does there exist a transitive avoidance game on n points with all lines of size r that is a PI win.*

In particular we do not know the full characterisation even for $r = 3$.

A family of games of particular interest is the class of *sim-like* games, which is defined as follows. These games have board the edge set of the complete graph K_n. The lines are sets of edges that are isomorphic to some forbidden graph (or family of graphs). For example, the Ramsey Avoidance Game $RAG(n, k)$ is of this form: the board is the edge set of K_n and the lines are the subgraphs isomorphic to K_k. (They have been called *sim-like* as the first non-trivial Ramsey Avoidance Game, $RAG(6, 3)$, is commonly called Sim.)

We do not know if there are any sim-like games that are PI wins.

Question 3. *Does there exist a sim-like game that is a PI win?*

One particular property of sim-like games is that they have a large automorphism group. Indeed, the automorphism group of a sim-like game played on K_n trivially contains S_n. We do not know whether this itself is enough to force a PII win. More generally it would be interesting to characterise for which automorphism groups there exist PI wins.

Question 4. *For which groups G does there exist a transitive avoidance game with automorphism group G that is a PI win?*

Finally, there is a natural definition of an infinite avoidance game. In this case the board has infinite size (with all lines finite) and a player loses if he forms a line. If the play continues forever with neither player losing then the game is deemed a draw.

Question 5. *Is there an infinite transitive avoidance game that is a PI win?*

We have no intuition as to the correct answer to this question; indeed, we do not even know the answer when all losing lines have size three. We remark that any such game must have 'many' losing lines, to avoid PII having an easy draw – for example, by picking a faraway point that does not complete a losing line.

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