On the smallest non-trivial tight sets in Hermitian polar spaces

Jan De Beule*

Department of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2
B–1050 Brussel, Belgium

jan@debeule.eu

Klaus Metsch

Justus-Liebig-Universität
Mathematisches Institut
Arndtstraße 2
D-35392 Gießen

Klaus.Metsch@math.uni-giessen.de

Submitted: Sep 16, 2016; Accepted: Mar 16, 2017; Published: Mar 31, 2017
Mathematics Subject Classifications: 05B25, 51A50, 05E30

Abstract

We show that an x-tight set of the Hermitian polar spaces $\mathrm{H}(4, q^2)$ and $\mathrm{H}(6, q^2)$ respectively, is the union of x disjoint generators of the polar space provided that x is small compared to q. For $\mathrm{H}(4, q^2)$ we need the bound $x < q + 1$ and we can show that this bound is sharp.

Keywords: polar space, tight set

1 Introduction

Let q be a prime power and let $\mathrm{GF}(q)$ be the finite field of order q. The vector space of dimension d over $\mathrm{GF}(q)$ will be written as $V(d, q)$, and $\mathrm{PG}(n, q)$ will denote the projective space with underlying vector space $V(n+1, q)$. Let f be a non-degenerate (reflexive) sesquilinear or non-singular quadratic form on $V(n + 1, q)$. The elements of the finite classical polar space $\mathcal{P}$ associated with f are the totally singular or totally isotropic subspaces of $\mathrm{PG}(n, q)$ with relation to f, according to whether f is a quadratic or sesquilinear form. The Witt index of the form f determines the dimension of the subspaces of maximal dimension contained in $\mathcal{P}$; the *rank* of $\mathcal{P}$ equals the Witt index of its form, and the (projective) dimension of generators will be one less than the Witt index. Hence, a finite classical polar space of rank r embedded in $\mathrm{PG}(n, q)$ has an underlying form of Witt index r, and contains points, lines, ..., $(r - 1)$-dimensional subspaces. The elements of maximal dimension are called its *generators*. A finite polar space of rank 2 is a point-line geometry, and is also called a finite *generalized quadrangle*.

*This author has been a postdoctoral fellow of the Research Foundation Flanders – FWO (Belgium)

There exist the following finite classical polar spaces of rank $r \geqslant 1$, which are, up to transformation of the coordinate system, described as follows inside their ambient projective space $\mathrm{PG}(d,q)$ or $\mathrm{PG}(d,q^2)$ with $d \in \{2r-1, 2r, 2r+1\}$.

1. The elliptic quadric $\mathrm{Q}^-(2r+1,q)$ formed by all points of $\mathrm{PG}(2r+1,q)$ which satisfy the standard equation $x_0x_1 + \cdots + x_{2r-2}x_{2r-1} + f(x_{2r}, x_{2r+1}) = 0$, where f is a homogeneous irreducible polynomial of degree 2 over $\mathrm{GF}(q)$.

2. The parabolic quadric $\mathrm{Q}(2r,q)$ formed by all points of $\mathrm{PG}(2r,q)$ which satisfy the standard equation $x_0x_1 + \cdots + x_{2r-2}x_{2r-1} + x_{2r}^2 = 0$.

3. The hyperbolic quadric $\mathrm{Q}^+(2r-1,q)$ formed by all points of $\mathrm{PG}(2r-1,q)$ which satisfy the standard equation $x_0x_1 + \cdots + x_{2r-2}x_{2r-1} = 0$.

4. The symplectic polar space $\mathrm{W}(2r-1,q)$, which consists of all points of $\mathrm{PG}(2n-1,q)$ together with the totally isotropic subspaces with respect to the standard symplectic form $\theta(x,y) = x_0y_1 - x_1y_0 + \cdots + x_{2r-2}y_{2r-1} - x_{2r-1}y_{2r-2}$.

5. The Hermitian variety $\mathrm{H}(2r,q^2)$ (also called a Hermitian polar space) formed by all points of $\mathrm{PG}(2r,q^2)$ which satisfy the standard equation $x_0^{q+1} + \cdots + x_{2r}^{q+1} = 0$.

6. The Hermitian variety $\mathrm{H}(2r-1,q^2)$ (also called a Hermitian polar space) formed by all points of $\mathrm{PG}(2r-1,q^2)$ which satisfy the standard equation $x_0^{q+1} + \cdots + x_{2r-1}^{q+1} = 0$.

Remark. For q even, the polar spaces $\mathrm{W}(2r-1,q)$ and $\mathrm{Q}(2r,q)$ are isomorphic, but there are no other isomorphisms between the polar spaces in the above list.

Tight sets in generalized quadrangles were introduced by Payne [6] and his definition has been generalized to polar spaces by Drudge [3]. Tight sets are extremal point sets of a finite classical polar space in the sense that the number of pairs of collinear points of such a set reaches a theoretical upper bound. Tight sets or, more generally, weighted tight sets have the important property that they meet every ovoid of a polar space in the same number of points, which depends only on the size of the tight set. Tight sets of the hyperbolic quadric $Q^+(5,q)$ (a polar space of rank three) appeared in a different setting in a paper by Cameron and Liebler [2]; in fact, in [2] Cameron-Liebler line classes were introduced, and it was noticed by Drudge that these correspond to tight sets via the Klein-correspondence. For this reason, tight sets of the hyperbolic quadrics of rank three have been intensively studied for more than two decades.

A tight set of a finite classical polar space can be defined in several ways. Combinatorially, it behaves as if it was a disjoint union of generators of the polar space. A tight set can be defined in an exclusively combinatorial way, as we will do formally below. Considering the point graph of the polar space, which is a strongly regular graph, it turns out that the characteristic vector of a tight set is a vector orthogonal to one of the eigenspaces of the adjacency matrix of the graph. This connection with algebraic graph theory has been studied as well, and was important to derive further geometric properties of tight sets and other substructures such as m-ovoids of finite classical polar spaces, see e.g. [1].

polar space	rank	number of points
$\mathrm{Q}(2n,q)$, $n \geqslant 1$	n	$\theta_{2n-1}(q) = \frac{q^{2n}-1}{q-1}$
$\mathrm{W}(2n+1,q)$, $n \geqslant 1$	$n+1$	$\theta_{2n+1}(q) = \frac{q^{2n+2}-1}{q-1}$
$\mathrm{H}(2n,q^2)$, $n \geqslant 1$	n	$(q^{2n+1}+1)\theta_{n-1}(q^2) = (q^{2n+1}+1)\frac{q^{2n}-1}{q^2-1}$
$\mathrm{H}(2n+1,q^2)$, $n \geqslant 1$	$n+1$	$(q^{2n+1}+1)\theta_{n}(q^2) = (q^{2n+1}+1)\frac{q^{2n+2}-1}{q^2-1}$

Table 1: Number of points in some polar spaces

As such, geometric characterizations of tight sets have applications when studying related structures. In this paper, we derive a geometric characterization of small tight sets of the Hermitian polar spaces of rank 2 and 3 in respectively projective dimension 4 and 6.

For integers $s \geqslant -1$, we use the notation

$$\theta_s(q) = \frac{q^{s+1}-1}{q-1}.$$

Note that this is the number of points in an s-dimensional projective space over the field $\mathrm{GF}(q)$. In particular, $\theta_{-1}(q) = 0$, $\theta_0(q) = 1$ and $\theta_1(q) = q+1$.

Collinearity in polar spaces is described completely using the underlying form. For a polar space defined by a quadratic form Q, one defines the associated bilinear form f as $f(u,v) := Q(u+v) - Q(u) - Q(v)$. Two points U and V of the polar space are collinear if and only if $f(U,V) = 0$. We will denote the set of points collinear with a given point P as $P^\perp$, and similarly, for a set A of points, $A^\perp := \cap_{P \in A} P^\perp$. Note that $P \in P^\perp$.

A fundamental property of polar spaces is the following. Let $\mathcal{P}_r$ be a polar space of rank r, then the set $P^\perp$ of points collinear with P is the intersection of a hyperplane of the ambient space and $\mathcal{P}_r$, and this intersection is a cone with vertex P and base a polar space $\mathcal{P}_{r-1}$ of the same type as $\mathcal{P}_r$ but of rank $r-1$. By same type we mean that they correspond to the same number in the above list of polar spaces.

We give a definition of tight sets that can also be found in the literature, e.g. in [1].

Definition 1.1. Let $\mathcal{P}$ be a polar space of rank r over the field $\mathrm{GF}(q)$. A *tight* set of $\mathcal{P}$ is a subset T of the point set of $\mathcal{P}$ such that for some integer $x > 0$,

$$|P^\perp \cap T| = \begin{cases} q^{r-1} + x\theta_{r-2}(q) & \text{when } P \in T, \\ x\theta_{r-2}(q) & \text{when } P \notin T. \end{cases}$$

The integer x is called the *parameter* of the tight set; a tight set with parameter x is called an *x-tight set.*

In the next section, we deal with tight sets with *small* parameter x. The notion "small" is derived from the following example, which is based on the natural embedding of the the polar space $\mathrm{W}(2n-1,q)$ in $\mathrm{H}(2n-1,q^2)$. Clearly, $\mathrm{H}(2n-1,q^2)$ can be embedded as a hyperplane intersection in $\mathrm{H}(2n,q^2)$, which yields an embedding of $\mathrm{W}(2n-1,q)$ in

$H(2n,q^2)$. It is well known that $W(2n-1,q)$ is a $(q+1)$-tight of $H(2n,q^2)$, see [1]. In particular, $W(3,q)$ can be embedded as a $(q+1)$-tight set in $H(4,q^2)$. Of course, $W(3,q)$ does not contain a line of $H(4,q^2)$ and the main result of this paper is that there is no smaller tight set with this property.

Theorem 1.2. *Every tight set of* $H(4,q^2)$ *with parameter* $x < q+1$ *is the union of* x *lines.*

We strongly believe that every $(q+1)$-tight set of $H(4,q^2)$ is either the union of $q+1$ lines or consists of the points set of a generalized subquadrangle of $H(4,q^2)$, such as $W(3,q)$ as explained above. More generally, we conjecture that every x-tight set of $H(2n,q^2)$, $n \geqslant 2$, with $x < q+1$ is the union of x generators of the polar space. However, a possible proof for $n > 2$ will be technically more difficult than in the case $n = 2$. We tried the case $n = 3$, where we can show the following.

Theorem 1.3. *Every tight set of* $H(6,q^2)$ *with parameter* $x \leqslant q+1-\sqrt{2q}$ *is the union of* x *disjoint generators.*

Remarks 1.

1. For odd q it is known that $H(4,q^2)$ has besides the subquadrangles that are isomorphic to $W(3,q)$ also subquadrangles of order (q,q) that are isomorphic to the parabolic generalized quadrangle $Q(4,q)$. More generally, $Q(2n,q)$ can naturally be embedded in $H(2n,q^2)$. We show in Section 4 that $Q(2n,q)$ in its natural embedding is an $(x+1)$-tight set of $H(2n,q^2)$. Up to our knowledge, this was not noticed in the literature.

2. Theorems 1.2 and 1.3 significantly improve results of [5].

2 Small tight sets in the polar space $H(4,q^2)$

In this section we show that a tight set of the Hermitian quadrangle $H(4,q^2)$ with parameter $x < q+1$ is trivial in the sense that it is the union of x disjoint lines. Recall from the definition of tight sets that every point of T is collinear with q^2+x points of T, and that every point of the polar space that is not in T is collinear with x points of T. We will frequently use these properties. The first lemma is well-known, we provide a short proof.

Lemma 2.1. *An* x*-tight set of* $H(4,q^2)$ *has* $x(q^2+1)$ *points.*

Proof. Count ordered pairs of perpendicular points $(P,Q) \in T \times (H(4,q^2) \setminus T)$ in two ways. This gives $|T|(1+q^2(q^3+1)-(q^2+x)) = (|H(4,q^2)|-|T|)x$, since each point of T is perpendicular to $1+q^2(q^3+1)$ points of $H(4,q^2)$ of which q^2+x lie in T, and since each point of $H(4,q^2) \setminus T$ is perpendicular to x points of T. □

The following crucial lemma uses a counting argument that also played a prominent role in a result by Gavrilyuk and Mogilnykh [4].

Lemma 2.2. *Let T be a tight set of* $\mathrm{H}(4,q^2)$ *with parameter* $x \leqslant q+1$. *Suppose that* $|P^\perp \cap R^\perp \cap T| \leqslant x$ *for any two different points P and R of T. Then $x = q+1$, and T is a subquadrangle of the generalized quadrangle* $\mathrm{H}(4,q^2)$ *of order q.*

Proof. We consider a point $P \in T$ and count pairs of perpendicular points $(Q,R) \in (T \cap P^\perp) \times (T \setminus P^\perp)$. From the definition of tight sets, we see that there are q^2+x choices for $Q \in P^\perp \cap T$. One of these points is P but P is not contained in any of the pairs we are counting. For each of the remaining q^2+x-1 points Q, the hypothesis of the lemma requires that $|P^\perp \cap Q^\perp \cap T| \leqslant x$. As $|Q^\perp \cap T| = q^2+x$, this implies that are at least $|Q^\perp \cap T| - x = q^2$ choices for a point R that is perpendicular to Q and lies in $T \setminus P^\perp$. Therefore the number of pairs (Q,R) under consideration is at least $(q^2+x-1)q^2$.

There are $|T| - |T \cap P^\perp| = (x-1)q^2$ choices for R. The hypothesis of the lemma shows that each is contained in at most x pairs (Q,R). Therefore the number of pairs (Q,R) is at most $(x-1)q^2x$.

It follows that $(q^2+x-1)q^2 \leqslant (x-1)q^2x$. Hence $q^2 \leqslant (x-1)^2$. As $x \leqslant q+1$ by hypothesis in the lemma, we find that $x = q+1$ and we have equality in all estimations used above. This shows that every two different points P and Q of T satisfy $|P^\perp \cap Q^\perp \cap T| = x = q+1$. We show that this implies that T is a subquadrangle of $\mathrm{H}(4,q^2)$.

If P and Q are different perpendicular points of T, then $|P^\perp \cap Q^\perp \cap T| = q+1$ implies that the line on P and Q meets T in exactly $q+1$ points. Hence every line of $\mathrm{H}(4,q^2)$ meets T in at most one or in exactly $q+1$ points. Consider the incidence structure I consisting of the points of T and the lines of $\mathrm{H}(4,q^2)$ that meet T in $q+1$ points where incidence is the natural one. Then every line of I has $q+1$ points of I. For $P \in T$ we have $|P^\perp \cap T| = q^2+x = q^2+q+1$ and hence P lies on exactly $q+1$ lines of I. Our argument shows that two points of I are collinear in I if and only if they are perpendicular in $\mathrm{H}(4,q^2)$. In order to show that I satisfies the one-or-all axiom for generalized quadrangles, consider a line ℓ of I and a point P of I that is not on ℓ. As $\ell = h \cap T$ for some line h of $\mathrm{H}(4,q^2)$, then P is perpendicular to at most one point of ℓ. To see that P is perpendicular to exactly one point of ℓ, consider a point Q of I on ℓ that is not perpendicular to P. Then Q lies on $q+1$ lines of I and we have seen that P is perpendicular to at most one point of each of these lines. As $|P^\perp \cap Q^\perp \cap T| = q+1$, it follows that P is perpendicular to exactly one point of each line of I on Q. Hence, P is perpendicular to one point of ℓ. This means that P is in I collinear to one point of ℓ. We have shown that I is a generalized quadrangle of order q. □

Lemma 2.3. *Consider* $\mathrm{H}(4,q^2)$ *naturally embedded in* $\mathrm{PG}(4,q^2)$ *and suppose that T is an x-tight set of* $\mathrm{H}(4,q^2)$ *with $x \leqslant q+1$. Let h be a line of* $\mathrm{PG}(4,q^2)$ *and suppose that* $h \cap \mathrm{H}(4,q^2)$ *contains a point that is not in T. Then $|h \cap T| \leqslant x$.*

Proof. Put $y := |h \cap T|$ and $u := |h^\perp \cap T|$ and count pairs of perpendicular points $(P,Q) \in (h \cap T) \times T$. The definition of tight set implies that each $P \in h \cap T$ occurs in q^2+x pairs (P,Q). The points Q of $h^\perp \cap T$ occur in y pairs (P,Q), but the points Q of $T \setminus h^\perp$ occur in at most one pair (P,Q). Hence $y(q^2+x) \leqslant u \cdot y + (|T| - u) \cdot 1$ and therefore $y(q^2+x) \leqslant |T| + u \cdot (y-1)$.

If $h \cap \mathrm{H}(4,q^2)$ contains a point R that is not in T, then $|R^\perp \cap T| = x$, since T is a tight set. Since $h^\perp \subseteq R^\perp$, then $u \leqslant x$ and hence $x(q^2+1)+(y-1)x \geqslant y(q^2+x)$. It follows that $x \geqslant y$. □

Remark. If the x-tight set T contains a line ℓ of $\mathrm{H}(4,q^2)$, then $T \setminus \ell$ is again a tight set of $\mathrm{H}(4,q^2)$ but with parameter $x-1$. This is well-known and follows immediately from the definition of tight sets.

Lemma 2.4. *Let T be a tight set of $\mathrm{H}(4,q^2)$ with parameter $x \leqslant q+1$. Suppose that $|P^\perp \cap R^\perp \cap T| \leqslant x$ for any two non-collinear points P and R of T. Then either T is a union of x lines or $x = q+1$ and T is a subquadrangle of order q of the generalized quadrangle $\mathrm{H}(4,q^2)$.*

Proof. We proceed by induction on x, the case $x = 0$ being trivial. Suppose now that $0 < x \leqslant q+1$. If T contains a line ℓ of $\mathrm{H}(4,q^2)$, then $T \setminus \ell$ is an $(x-1)$-tight set of $\mathrm{H}(4,q^2)$. In this case the induction hypothesis implies that T is a union of x lines. If T contains no line, then Lemma 2.3 shows that every line of $\mathrm{H}(4,q^2)$ meets T in at most x points. Hence $|P^\perp \cap R^\perp \cap T| \leqslant x$ for any two different collinear points P and Q of T. Together with the hypothesis of the present lemma, we see that $|P^\perp \cap R^\perp \cap T| \leqslant x$ for any two different points of T. Now Lemma 2.2 can be applied. □

Theorem 2.5. *Every tight set of $\mathrm{H}(4,q^2)$ with parameter $x < q+1$ is the union of x lines.*

Proof. Let T be a tight set of $\mathrm{H}(4,q^2)$ with parameter $x < q+1$. We have to show that T is the union of x lines. This follows from Lemma 2.4 provided that we can show that $|P^\perp \cap Q^\perp \cap T| \leqslant x$ for any two non-perpendicular points P and Q of T.

Assume on the contrary that T contains non-perpendicular points P and Q with $|P^\perp \cap Q^\perp \cap T| > x$. Let $\mathrm{PG}(4,q^2)$ be the ambient space, and consider the line $s = PQ$ of $\mathrm{PG}(4,q^2)$. Since P and Q are not perpendicular, then s is a secant line of $\mathrm{H}(4,q^2)$. We have $s^\perp = P^\perp \cap Q^\perp$. Therefore $|s^\perp \cap T| > x$. It follows that $|R^\perp \cap T| > x$ for all points R of $s \cap \mathrm{H}(4,q^2)$. As T is a tight set with parameter x, this implies that all $q+1$ points of $s \cap \mathrm{H}(4,q^2)$ belong to T. The plane $s^\perp$ meets $\mathrm{H}(4,q^2)$ in a Hermitian curve $\mathrm{H}(2,q^2)$. Each point of the Hermitian curve is perpendicular to all points of s and hence to the $q+1$ points of $s \cap T$. Since T is a tight set with parameter $x < q+1$, it follows that all q^3+1 points of the Hermitian curve belong to T.

Then $|P^\perp \cap T| \geqslant |\mathrm{H}(2,q^2)| = q^3+1$. But T is a tight set containing P, so $|P^\perp \cap T| = q^2+x$, which is the desired contradiction. □

Based on Lemma 2.4 and Theorem 1.2, we formulate the following conjecture. Note that the only subquadrangles of order q of $\mathrm{H}(4,q^2)$ are precisely the generalized quadrangles $\mathrm{W}(3,q)$ and $\mathrm{Q}(4,q)$ embedded in $\mathrm{H}(4,q^2)$, which are exactly the embedded subquadrangles yielding the examples of tight sets with parameter $x = q+1$.

Conjecture 2.6. *Every $(q+1)$-tight set of $\mathrm{H}(4,q^2)$ that is not the union of $q+1$ lines is the set of points of an embedded $\mathrm{W}(3,q)$, or, when q is odd, the set of points of an embedded $\mathrm{W}(3,q)$ or $\mathrm{Q}(4,q)$.*

3 Small tight sets in the polar space $\mathrm{H}(6, q^2)$

In this section we prove Theorem 1.3, which is comparable to Theorem 1.2, however we need a stronger bound on x. Recall that a tight set T with parameter x of $\mathrm{H}(6, q^2)$ has by definition the property that $|P^\perp \cap T| = q^4 + x(q^2+1)$ when $P \in T$ and $|P^\perp \cap T| = x(q^2+1)$ when $P \in \mathrm{H}(6, q^2) \setminus T$. As in the case of tights sets in $\mathrm{H}(4, q^2)$, a double counting argument shows that T has precisely $x\theta_2(q^2) = x(q^4 + q^2 + 1)$ points.

For the next series of lemmas, we consider $\mathrm{H}(6, q^2)$ naturally embedded in $\mathrm{PG}(6, q^2)$, and denote by T a tight set of $\mathrm{H}(6, q^2)$ with parameter x.

Lemma 3.1. *Let ℓ be a line of* $\mathrm{PG}(6, q^2)$*.*

(a) *If $|\ell^\perp \cap T| > x(q^2 + 1)$, then $\ell \cap \mathrm{H}(6, q^2) \subseteq T$.*

(b) *If ℓ is a line of* $\mathrm{H}(6, q^2)$ *with $\ell \subseteq T$, then*

$$|\ell^\perp \cap T| = q^4 + q^2 + x.$$

(c) *If ℓ is a secant line of* $\mathrm{H}(6, q^2)$ *with $\ell \cap \mathrm{H}(6, q^2) \subseteq T$, then*

$$|\ell^\perp \cap T| \geqslant (q + 1 - x)q^3 + x(q^2 + 1).$$

Proof. (a) For every point P of $\ell \cap \mathrm{H}(6, q^2)$ we have $|P^\perp \cap T| \geqslant |\ell^\perp \cap T| > x(q^2 + 1)$. As T is a tight set with parameter x, it follows that all points of $\ell \cap \mathrm{H}(6, q^2)$ belong to T.

(b) Let $P_0, \dots, P_{q^2}$ be the points of ℓ. Each point of T is either perpendicular to all points P_i or to exactly one. Therefore a double counting argument shows that

$$|\ell^\perp \cap T|(q^2 + 1) + (|T| - |\ell^\perp \cap T|) \cdot 1 = \sum_{i=0}^{q^2} |P_i^\perp \cap T|.$$

As $P_i \in T$, then $|P_i^\perp \cap T| = q^4 + x(q^2 + 1)$. Using also that $|T| = x(q^4 + q^2 + 1)$, it follows that $|\ell^\perp \cap T| = q^4 + q^2 + x$.

(c) By the hypothesis the $q + 1$ points $P_0, \dots, P_q$ of $\ell \cap \mathrm{H}(6, q^2)$ lie in T. Each point of T is either perpendicular to all points P_i or to at most one. Therefore a double counting argument shows that

$$|\ell^\perp \cap T|(q + 1) + (|T| - |\ell^\perp \cap T|) \cdot 1 \geqslant \sum_{i=0}^{q} |P_i^\perp \cap T|.$$

Using $|P_i^\perp \cap T| = q^4 + x(q^2 + 1)$ and $|T| = x(q^4 + q^2 + 1)$ as in (b), it follows that $|\ell^\perp \cap T| \geqslant (q + 1 - x)q^3 + x(q^2 + 1)$. □

Lemma 3.2. *Let $x \leqslant q$. If T contains two intersecting lines, then T contains a plane.*

Proof. Suppose that h_1 and h_2 are lines that are contained in T and meet in a point P. These two lines span a plane π of $\mathrm{PG}(6,q^2)$. As π contains two lines of $\mathrm{H}(6,q^2)$, then either π is contained in $\mathrm{H}(6,q^2)$ or π meets $\mathrm{H}(6,q^2)$ in the union of $q+1$ lines on P. Lemma 3.1 implies that

$$\begin{aligned} q^4 + x(q^2+1) &= |P^\perp \cap T| \\ &\geqslant |h_1^\perp \cap T| + |h_2^\perp \cap T| - |\pi^\perp \cap T| \\ &= 2(q^4+q^2+x) - |\pi^\perp \cap T|\,. \end{aligned}$$

Thus $|\pi^\perp \cap T| \geqslant q^4 + 2q^2 + x - xq^2$. Since $x \leqslant q$, it follows that $|\pi^\perp \cap T| > x(q^2+1)$. Since T is a tight set, this implies that all points of $\pi \cap \mathrm{H}(6,q^2)$ are contained in T. It remains to show that π is a plane of $\mathrm{H}(6,q^2)$.

Assume on the contrary that $\pi \cap \mathrm{H}(6,q^2)$ is the union of $q+1$ lines on P. Thus π has $1+(q+1)q^2$ points in $\mathrm{H}(6,q^2)$ and we have already shown that all these belong to T. Since $1+(q+1)q^2 > x(q^2+1)$ and since T is a tight set, it follows that all points of $\pi^\perp \cap \mathrm{H}(6,q^2)$ lie in T. Since π meets $\mathrm{H}(6,q^2)$ in the union of $q+1$ lines on P, then $\pi^\perp$ meets $\mathrm{H}(6,q^2)$ in a cone with vertex P over a Hermitian curve $\mathrm{H}(2,q^2)$. Hence $\pi^\perp$ has $1+q^2(q^3+1)$ points in $\mathrm{H}(6,q^2)$ and all these belong T. Since $\pi \cap \pi^\perp = \{P\}$, it follows that

$$\begin{aligned} |T| &\geqslant |\pi^\perp \cap \mathrm{H}(2n,q^2)| + |\pi \cap \mathrm{H}(6,q^2)| - 1 \\ &= 1 + (q^3+1)q^2 + (q+1)q^2. \end{aligned}$$

But $|T| = x(q^4+q^2+1)$ and $x \leqslant q$, a contradiction. □

Lemma 3.3. *If $1 \leqslant x \leqslant q$ and if T does not contain a plane, then there exists a point of T that does not lie on a line that is completely contained in T.*

Proof. Since $x \leqslant q$ and T contains no plane, the preceding lemma shows that every point of T lies on at most one line that is contained in T. Thus, the number of points of T on such a line is divisible by the number q^2+1 of points on a line. Since $|T| = x(q^4+q^2+1)$ with $1 \leqslant x \leqslant q$, then $|T|$ is not divisible by q^2+1. Hence, not all points of T can be on a line completely contained in $\mathrm{H}(6,q^2)$. □

Lemma 3.4. *Suppose that $1 \leqslant x \leqslant q$ and that T does not contain a plane. Then there exists a point $P \in T$ that lies on at least $2q$ lines h being secant to $\mathrm{H}(6,q^2)$ and satisfying $|h^\perp \cap T| > x(q^2+1)$.*

Proof. The previous lemma proves the existence of a point P such that no line of $\mathrm{H}(6,q^2)$ on P is completely contained in T. We count the number n of pairs of perpendicular points $(Q,R) \in (T \cap P^\perp) \times (T \setminus P^\perp)$ in two ways.

There are $q^4 + x(q^2+1) - 1$ choices for $Q \in T \cap P^\perp$ with $Q \neq P$. For such a point Q, Lemma 3.1 (a) implies that $|P^\perp \cap Q^\perp \cap T| \leqslant x(q^2+1)$. Hence $|(Q^\perp \setminus P^\perp) \cap T| \geqslant q^4$. This implies that Q occurs in at least q^4 pairs (Q,R). Therefore

$$(q^4 + x(q^2+1) - 1)q^4 \;\leqslant\; n. \tag{1}$$

Now we count n in a second way by first choosing R from the set $T \setminus P^\perp$. For each point $R \in T \setminus P^\perp$, the line PR is a secant line and thus has $q+1$ points in $\mathrm{H}(6, q^2)$. Let s be the number of secant lines h on P that satisfy $|h^\perp \cap T| > x(q^2+1)$. Since T is a tight set, all $q+1$ points of such a secant line belong to T. Hence these s secant lines contain P and sq further points of T. For each of these sq points R we use

$$|P^\perp \cap R^\perp \cap T| \leqslant |R^\perp \cap T| - 1 = q^4 - 1 + x(q^2+1).$$

Since $|T \setminus P^\perp| = |T| - q^4 - x(q^2+1) = (x-1)q^4$, there are $(x-1)q^4 - sq$ points R in $T \setminus P^\perp$ that are not in one of the s secant lines. Consider such a point R. Then the secant line PR contains a point U of $\mathrm{H}(6, q^2)$ that does not belong to T. Hence $|P^\perp \cap R^\perp \cap T| \leqslant |U^\perp \cap T| = x(q^2+1)$. It follows that

$$\begin{aligned} n &\leqslant sq\left(q^4 - 1 + x(q^2+1)\right) + \left((x-1)q^4 - sq\right) \cdot x(q^2+1) \\ &= (x-1)q^4 x(q^2+1) + sq(q^4-1). \end{aligned}$$

Combining this with the lower bound for n in (1), we obtain

$$q^4(q^2+1)(q+1-x)(q-1+x) \leqslant sq(q^4-1).$$

As $x \leqslant q$, then $(q+1-x)(q+1-x) = q^2 - (x-1)^2 \geqslant 2q - 1$. Hence $s \geqslant q^3(2q-1)/(q^2-1)$ and thus $s > 2q - 1$. □

Lemma 3.5. *Suppose that π is a plane of $\mathrm{PG}(6, q^2)$ that meets $\mathrm{H}(6, q^2)$ either in a Hermitian curve $\mathrm{H}(2, q^2)$ or in $q+1$ concurrent lines. Then $x \geqslant q$ or $|\pi^\perp \cap T| \leqslant x(q^2+1)$.*

Proof. Assume on the contrary that $x \leqslant q - 1$ and $|\pi^\perp \cap T| > x(q^2+1)$. We shall derive a contradiction.

As T is an x-tight set and $|\pi^\perp \cap T| > x(q^2+1)$, all points of $\pi \cap \mathrm{H}(6, q^2)$ belong to T. There are the following two possibilities for the structure of $\pi \cap \mathrm{H}(6, q^2)$ and $\pi^\perp \cap \mathrm{H}(6, q^2)$.

- $\pi \cap \mathrm{H}(6, q^2)$ is the union of $q+1$ concurrent lines and $\pi^\perp \cap \mathrm{H}(6, q^2)$ is a cone with point vertex over a Hermitian curve. In this case $|\pi \cap \mathrm{H}(6, q^2)| = q^3 + q^2 + 1$ and $|\pi^\perp \cap \mathrm{H}(6, q^2)| = 1 + q^2(q^3+1)$.
- $\pi \cap \mathrm{H}(6, q^2)$ is a Hermitian curve $\mathrm{H}(2, q^2)$ and $\pi^\perp \cap \mathrm{H}(6, q^2)$ is a $\mathrm{H}(3, q^2)$. Then $|\pi \cap \mathrm{H}(6, q^2)| = q^3 + 1$ and $|\pi^\perp \cap \mathrm{H}(6, q^2)| = (q^2+1)(q^3+1)$.

Thus π has at least $q^3 + 1$ points in $\mathrm{H}(6, q^2)$ and these belong to T. Since $x \leqslant q - 1$, then all points $Q \in \pi^\perp \cap \mathrm{H}(6, q^2)$ satisfy

$$|Q^\perp \cap T| \geqslant |\pi \cap \mathrm{H}(6, q^2)| \geqslant q^3 + 1 > x(q^2+1).$$

As T is an x-tight set, it follows that all points of $\pi^\perp \cap \mathrm{H}(6, q^2)$ belong to T. Hence

$$|T \cap \pi^\perp| = |\pi^\perp \cap \mathrm{H}(6, q^2)|.$$

In the first case above for the structure of π we have $\pi \cap \pi^\perp = \{P\}$ and since all points of $\pi \cap \mathrm{H}(6, q^2)$ and of $\pi^\perp \cap \mathrm{H}(6, q^2)$ belong to T, it follows that

$$|T| \geqslant q^3 + q^2 + 1 + q^2(q^3 + 1).$$

Similarly, in the second case $\pi \cap \pi^\perp = \emptyset$ and

$$|T| \geqslant q^3 + 1 + (q^2 + 1)(q^3 + 1).$$

As $|T| = x(q^4 + q^2 + 1)$ and $x \leqslant q - 1$, this is a contradiction. □

Lemma 3.6. *Suppose that T contains a point P that lies on more than $\sqrt{2q}$ secant lines h of $\mathrm{H}(6, q^2)$ that satisfy $|h^\perp \cap T| > x(q^2 + 1)$. Then $x \geqslant q - \sqrt{2q} + 2$.*

Proof. We choose $s := \lfloor \sqrt{2q} + 1 \rfloor$ secant lines $h_1, \ldots, h_s$ on P with $|h_i^\perp \cap T| > x(q^2 + 1)$. We have $|h_i^\perp \cap T| \geqslant (q + 1 - x)q^3 + x(q^2 + 1)$ from Lemma 3.1 (a) and (c). For different indices i and j, the plane generated by h_i and h_j meets $\mathrm{H}(6, q^2)$ either in a Hermitian curve or in $q + 1$ concurrent lines. If for some indices $i \neq j$, the plane π_{ij} generated by h_i and h_j satisfies $|\pi_{ij}^\perp \cap T| > x(q^2 + 1)$, then Lemma 3.5 gives $x \geqslant q$. In this case we are done. We may therefore assume that $|\pi_{ij}^\perp \cap T| \leqslant x(q^2 + 1)$ for all indices $i \neq j$. This means that $|h_i^\perp \cap h_j^\perp \cap T| \leqslant x(q^2 + 1)$ for all $i \neq j$. As $h_i^\perp \cap T \subseteq (P^\perp \setminus \{P\}) \cap T$ for all i, the Inclusion-Exclusion Principle implies that

$$\begin{aligned} q^4 + x(q^2 + 1) - 1 &= |P^\perp \cap T| - 1 \\ &\geqslant \sum_i |h_i^\perp \cap T| - \sum_{i<j} |h_i^\perp \cap h_j^\perp \cap T| \\ &\geqslant s(q + 1 - x)q^3 + sx(q^2 + 1) - \binom{s}{2} x(q^2 + 1). \end{aligned}$$

Since $s \leqslant \sqrt{2q} + 1$, it follows that

$$q^4 + x(q^2 + 1) \geqslant s(q + 1 - x)q^3 + sx(q^2 + 1) - \frac{1}{2} s\sqrt{2q}x(q^2 + 1).$$

Since $s \geqslant \sqrt{2q}$, this remains true, if we replace s in the inequality by $\sqrt{2q}$ (whether the coefficient of s on the right hand side is positive or not is irrelevant for this). Hence

$$q^4 + x(q^2 + 1) \geqslant \sqrt{2q}(q + 1 - x)q^3 + \sqrt{2q}x(q^2 + 1) - qx(q^2 + 1).$$

Solving for x, this implies that $x \geqslant q - \sqrt{2q} + 2$. □

Theorem 3.7. *Every tight set of $\mathrm{H}(6, q^2)$ with parameter $x \leqslant q + 1 - \sqrt{2q}$ is the union of x disjoint generators.*

Proof. Let T be a tight set of $\mathrm{H}(6, q^2)$ with parameter $x \leqslant q + 1 - \sqrt{2q}$. We have to show that T is the union of x disjoint generators, that is of x planes that are contained in $\mathrm{H}(6, q^2)$.

We use induction on x. If $x = 0$, then $T = \emptyset$ and there is nothing to show. Now suppose that $x \geqslant 1$. It suffices to show that T contains a plane. In fact, if T contains the plane π, then $\pi \subseteq \mathrm{H}(6, q^2)$ and the definition of a tight set implies that $T \setminus \pi$ is a tight set with parameter $x - 1$. The induction hypothesis implies then that T is the disjoint union of $x - 1$ planes, so T is the union of x planes. It remains to show that T contains a plane.

Assume on the contrary that T does not contain a plane. Then by Lemma 3.4, there exists a point $P \in T$ that lies on more than $\sqrt{2q}$ lines h being secant to $\mathrm{H}(6, q^2)$ and satisfying $|h^\perp \cap T| > x(q^2 + 1)$. Therefore Lemma 3.6 shows that $x \geqslant q - \sqrt{2q} + 2$, a contradiction to the assumed upper bound on x. □

4 A new tight set

In this section, we show that $\mathrm{Q}(2n, q)$, q odd, in its natural embedding in $\mathrm{H}(2n, q^2)$ provides a $(q + 1)$-tight set of $\mathrm{H}(2n, q^2)$. We mention that the embedding of $\mathrm{Q}(2n, q)$ in $\mathrm{H}(2n, q^2)$ is seen by restricting the canonical Hermitian form $f(x, y) = x_0 y_0^q + \cdots + x_{2n} y_{2n}^q$ to the subfield of order q. Thus $\mathrm{Q}(2n, q)$ is represented by the quadratic form $x_0^2 + \cdots + x_{2n}^2$ over the field $\mathrm{GF}(q)$. Notice that the bilinear form b associated to this quadratic form is given by $b(x, y) := 2(x_0 y_0 + \cdots + x_{2n} y_{2n})$. Hence b is proportional to f restricted to the subfield. Therefore two points of the embedded polar space $\mathrm{Q}(2n, q)$ are perpendicular in $\mathrm{Q}(2n, q)$ if they are in $\mathrm{H}(2n, q^2)$, and two different points of $\mathrm{Q}(2n, q)$ therefore lie on a line of $\mathrm{Q}(2n, q)$ if and only if they lie on a line of $\mathrm{H}(2n, q^2)$. This is the crucial tool in the following proof.

Theorem 4.1. *Let q be odd and $n \geqslant 2$. The set of points of the polar space $\mathrm{Q}(2n, q)$, q odd, embedded in $\mathrm{H}(2n, q^2)$, is a $(q + 1)$-tight set of $\mathrm{H}(2n, q^2)$.*

Proof. For a point P in $\mathrm{Q}(2n, q)$ the hyperplane $P^\perp$ of $\mathrm{PG}(2n, q^2)$ intersects $\mathrm{PG}(2n, q)$ in the tangent hyperplane of $\mathrm{Q}(2n, q)$ at P. Therefore $P^\perp \cap \mathrm{Q}(2n, q)$ is a cone with vertex P over a quadric $\mathrm{Q}(2n-2, q)$ (for $n = 2$, this is just a line on P), and hence $|P^\perp \cap \mathrm{Q}(2n, q)| = 1 + q|\mathrm{Q}(2n - 2, q)| = q\theta_{2n-3}(q) + 1$. Since $q\theta_{2n-3}(q) + 1 = (q^2)^{n-1} + (q + 1)\theta_{n-2}(q^2)$, the first condition for the set of points of $\mathrm{Q}(2n, q)$ to be a $(q + 1)$-tight set is satisfied. For the second condition, we have to show that $|P^\perp \cap \mathrm{Q}(2n, q)| = (q + 1)\theta_{n-2}(q^2)$ for points P of $\mathrm{H}(2n, q^2)$ that are not contained in $\mathrm{Q}(2n, q)$.

Our argument to show this uses the polarity $P \mapsto P^\perp$, which maps points P of $\mathrm{PG}(2n, q^2)$ to hyperplanes of $\mathrm{PG}(2n, q^2)$. This polarity induces the polarity of $\mathrm{PG}(2n, q)$ related to the embedded $\mathrm{Q}(2n, q)$ in the following sense. For points X of $\mathrm{PG}(2n, q)$, the hyperplane $X^\perp$ of $\mathrm{PG}(2n, q^2)$ meets $\mathrm{PG}(2n, q)$ in a hyperplane. Moreover every hyperplane of $\mathrm{PG}(2n, q)$ is equal to $X^\perp$ for exactly one point X of $\mathrm{PG}(2n, q)$. For points X of $\mathrm{PG}(2n, q^2)$ that are not in $\mathrm{PG}(2n, q)$, it follows that $X^\perp$ does not contain a hyperplane of $\mathrm{PG}(2n, q)$. Therefore the Grassmann formula for vector space dimensions implies for points X that are not in $\mathrm{PG}(2n, q)$ that $X^\perp \cap \mathrm{PG}(2n, q)$ is a subspace of $\mathrm{PG}(2n, q)$ of dimension $2n - 2$.

Now consider a point P of $\mathrm{H}(2n, q^2)$ that is not in $\mathrm{Q}(2n, q)$. Then $S := P^\perp \cap \mathrm{PG}(2n, q)$ is a subspace of $\mathrm{PG}(2n, q)$ of dimension $2n-2$. It follows that $P^\perp \cap \mathrm{Q}(2n, q) = S \cap \mathrm{Q}(2n, q)$

is either a parabolic quadric $\mathrm{Q}(2n-2,q)$, a cone with a line vertex over a parabolic quadric $\mathrm{Q}(2n-4,q)$, or a cone with a point vertex R over a hyperbolic or elliptic quadric $Q^+(2n-3,q)$ or $Q^-(2n-3,q)$.

Assume $S \cap \mathrm{Q}(2n,q)$ is a cone with point vertex R over $Q^+(2n-3,q)$ or $Q^-(2n-3,q)$. Then the polarity of $\mathrm{PG}(2n,q)$ related to $\mathrm{Q}(2n,q)$ maps S to a line ℓ of $\mathrm{PG}(2n,q)$ that meets $\mathrm{Q}(2n,q)$ only in R. As a set of point of $\mathrm{PG}(2n,q^2)$, the line ℓ spans a line $\bar{\ell}$ of $\mathrm{PG}(2n,q^2)$, and the subspace S spans a subspace $\bar{S}$ of $\mathrm{PG}(2n,q^2)$ of dimension $2n-2$. Furthermore $\bar{\ell} = \bar{S}^{\perp}$. As $S \subseteq P^{\perp}$, it follows that $P \in \bar{\ell}$. Thus ℓ contains the points P and R of $\mathrm{H}(2n,q^2)$, and since these are perpendicular, it follows that $\ell \subseteq \mathrm{H}(2n,q^2)$. As $\bar{\ell} \cap \mathrm{PG}(2n,q) = \ell$, it follows that $\ell \subseteq \mathrm{Q}(2n,q)$. But we have also seen that ℓ meets $\mathrm{Q}(2n,q)$ only in R.

This contradiction shows that $P^{\perp}$ meets $\mathrm{Q}(2n,q)$ either in a quadric $\mathrm{Q}(2n-2,q)$ or in a a cone with a line vertex over a quadric $\mathrm{Q}(2n-4,q)$. In both cases it follows that $|P^{\perp} \cap \mathrm{Q}(2n,q)| = \theta_{2n-3}(q) = (q+1)\theta_{n-2}(q^2)$. □

5 Open problems

Based on these *small* examples of $(q+1)$-tight sets in general dimension, and the results in dimensions 4 and 6, we formulate the following conjecture already mentioned in the introduction.

Conjecture 5.1. *The smallest parameter x for which there exists an x-tight set of $\mathrm{H}(2n,q^2)$ that is not the union of x generators is $x = q+1$, in which case the tight set is the set of points of an embedded $\mathrm{W}(2n-1,q)$, or, when q is odd, the set of points of an embedded $\mathrm{W}(2n-1,q)$ or an embedded $\mathrm{Q}(2n,q)$.*

However, expectedly in ascending order of difficulty, we formulate the following open problems.

1. Classify the tight sets with parameter $x = q+1$ in $\mathrm{H}(4,q^2)$.
2. Improve the bound on x of Theorem 1.3 to $x < q+1$.
3. Prove Conjecture 5.1.

References

[1] John Bamberg, Shane Kelly, Maska Law, and Tim Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A*, 114(7):1293–1314, 2007.

[2] Peter. J. Cameron and Robert A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.*, 46:91–102, 1982.

[3] Keldon W. Drudge. *Extremal sets in projective and polar spaces.* ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–The University of Western Ontario (Canada).

[4] Alexander L. Gavrilyuk and Ivan Yu. Mogilnykh. Cameron-Liebler line classes in $PG(n, 4)$. *Des. Codes Cryptogr.*, 73(3):969–982, 2014.

[5] Klaus Metsch. Small tight sets in finite elliptic, parabolic and hermitian polar spaces. *Combinatorica*, 36(6):725–744, 2016.

[6] Stanley E. Payne. Tight pointsets in finite generalized quadrangles. *Congr. Numer.*, 60:243–260, 1987. Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fla., 1987).