A Note on the Weak Dirac Conjecture

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Abstract

We show that every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil \frac{n}{3} \rceil + 1$ of the lines determined by $\mathcal{P}$.

Keywords: Configurations of points; Incident-line-numbers; Weak Dirac Conjecture, Hirzebruch-type inequalities

In this note we denote by $\mathcal{P}$ a set of non-collinear points in the plane, and by $\mathcal{L}(\mathcal{P})$ the set of lines determined by $\mathcal{P}$, where a line that passes through at least two points of $\mathcal{P}$ is said to be *determined* by $\mathcal{P}$. For a point $P \in \mathcal{P}$, we denote by $d(P)$ the number of lines of $\mathcal{L}(\mathcal{P})$ that are incident to P, called the *incident-line-number* or *multiplicity* of P; see [4] and [14]. Finally, we denote by l_r the number of lines that pass through precisely r points of $\mathcal{P}$.

Dirac's conjecture is a well-known problem in combinatorial geometry. In 1951, Dirac [5] showed that:

Theorem 1. *Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil \sqrt{n+1} \rceil$ lines of $\mathcal{L}(\mathcal{P})$.*

Dirac [5] made (and verified for $n \leqslant 14$) the following conjecture.

Conjecture 2 (Dirac Conjecture). Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lfloor \frac{n}{2} \rfloor$ lines of $\mathcal{L}(\mathcal{P})$.

The conjectured bound is tight, for instance, Dirac [5] constructed a set $\mathcal{P}$ of n non-collinear points with $(l_2, l_3, l_{\frac{n}{2}}) = (\frac{n^2}{4} - \frac{3n}{2} + 3, \frac{n}{2} - 1, 2)$ for every even-integer $n \geqslant 6$. In 2011, Akiyama, Ito, Kobayashi, and Nakamura [1] proved there exists a set $\mathcal{P}$ of n non-collinear points for every integer $n \geqslant 8$ except $n = 12k + 11 (k \geqslant 4)$, satisfying $d(P) \leqslant \lfloor \frac{n}{2} \rfloor$ for every point $P \in \mathcal{P}$. However, Dirac's conjecture is false, some counter-examples were found in [1, 7–11].

The following natural conjecture arises [4].

Conjecture 3 (**Strong Dirac Conjecture**)**.** Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lfloor \frac{n}{2} \rfloor - c_0$ lines of $\mathcal{L}(\mathcal{P})$ with $c_0 > 0$.

In 1961, Erdős [6] proposed the following weakened conjecture.

Conjecture 4 (**Weak Dirac Conjecture**)**.** Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil \frac{n}{c_1} \rceil$ lines of $\mathcal{L}(\mathcal{P})$ with $c_1 > 0$.

In 1983, the Weak Dirac Conjecture was proved independently by Beck [2] and Szemerédi and Trotter [20] with c_1 unspecified or very large.

In 2012, based on Crossing Lemma, Szemerédi-Trotter Theorem, and Hirzebruch's inequality, Payne and Wood [17] proved the following theorem,

Theorem 5. *Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil \frac{n}{37} \rceil$ lines of $\mathcal{L}(\mathcal{P})$.*

In 2016, Pham and Phi [18] refined the result of Payne and Wood to give:

Theorem 6. *Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil \frac{n}{26} \rceil + 2$ lines of $\mathcal{L}(\mathcal{P})$.*

There are some results in algebraic geometry providing constraints on line arrangements in the projective plane. In [12, 13], Hirzebruch studied algebraic surfaces constructed as abelian covers of the projective plane branched along line arrangements in the context of the so-called ball-quotients. It turned out that he obtained, as a by-product, the following result which is known as Hirzebruch's inequality.

Theorem 7 (**Hirzebruch's Inequality**)**.** *Let $\mathcal{P}$ be a set of n points in the plane with at most $n-3$ collinear. Then*

$$l_2 + \frac{3}{4} l_3 \geqslant n + \sum_{r \geqslant 5} (2r-9) l_r.$$

In 2003, Langer [15] provided a variation on the classical Bogomolov-Miyaoka-Yau inequality [16] using the so-called orbifold Euler numbers.

Theorem 8 (**Orbifold Langer-Miyaoka-Yau Inequality**)**.** *Let (X, D) be a normal projective surface with a $\mathbb{Q}$-divisor $D = \sum_i a_i D_i$ with $0 \leqslant a_i \leqslant 1$. Assume that the pair (X, D) is log canonical and $K_X + D$ is $\mathbb{Q}$-effective. Then*

$$(K_X + D)^2 \leqslant 3 e_{\mathrm{orb}}(X, D),$$

where $e_{\mathrm{orb}}(X, D)$ denotes the global orbifold number for $(X, \sum_i a_i D_i)$. Moreover, if equality holds, then $K_X + D$ is nef.

Bojanowski in [3] provided the following Hirzebruch-type inequality for line arrangements in the projective plane, which is also a special case of a much stronger result from the same thesis [3, Theorem 2.3]. It is worth pointing out that following Langer's ideas, Pokora [19] provided some Hirzebruch-type inequalities for curve configurations in the projective plane with transversal intersection points where Bojanowski's result is a special case.

Theorem 9 (**Bojanowski-Pokora Inequality**). *Let $\mathcal{P}$ be a set of n points in the plane with at most $\lfloor \frac{2n}{3} \rfloor$ collinear. Then*

$$l_2 + \frac{3}{4} l_3 \geqslant n + \frac{1}{4} \sum_{r \geqslant 5} r(r-4) l_r.$$

Based on the Bojanowski-Pokora inequality, we show the following result.

Theorem 10. *Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil \frac{n}{3} \rceil + 1$ lines of $\mathcal{L}(\mathcal{P})$.*

Proof. Suppose some line L passes through $\lceil \frac{n}{3} \rceil + 1$ or more points of $\mathcal{P}$. Since $\mathcal{P}$ is non-collinear, there exists a point $P \in \mathcal{P}$ such that $P \notin L$. Consider the (distinct) lines determined by P and $\mathcal{P} \cap L$. Then P is incident to at least $\lceil \frac{n}{3} \rceil + 1$ lines of $\mathcal{L}(\mathcal{P})$, and the theorem holds. Now assume that $\mathcal{P}$ does not contain $\lceil \frac{n}{3} \rceil + 1$ collinear points.

According to Theorem 9,

$$l_2 + \frac{3}{4} l_3 \geqslant n + \frac{1}{4} \sum_{r \geqslant 5} r(r-4) l_r = n + \frac{1}{2} \sum_{r \geqslant 5} \binom{r}{2} l_r - \frac{3}{4} \sum_{r \geqslant 5} r l_r.$$

Since $\sum_{r \geqslant 2} \binom{r}{2} l_r = \binom{n}{2}$,

$$l_2 + \frac{3}{4} l_3 \geqslant n + \frac{1}{2} \left(\binom{n}{2} - \sum_{r=2}^{4} \binom{r}{2} l_r \right) - \frac{3}{4} \sum_{r \geqslant 5} r l_r.$$

That is,

$$\sum_{r \geqslant 2} r l_r \geqslant \frac{n(n+3)}{3}.$$

Since $\sum_{P \in \mathcal{P}} d(P) = \sum_{r \geqslant 2} r l_r$,

$$\sum_{P \in \mathcal{P}} d(P) \geqslant \frac{n(n+3)}{3}.$$

By the pigeonhole principle, $\mathcal{P}$ contains a point incident to at least $\lceil \frac{n}{3} \rceil + 1$ lines of $\mathcal{L}(\mathcal{P})$. □

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References

[1] Jin Akiyama, Hiro Ito, Midori Kobayashi, and Gisaku Nakamura. Arrangements of n points whose incident-line-numbers are at most $n/2$. *Graphs Combin.*, 27(3):321-326, 2011.

[2] József Beck. On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. *Combinatorica*, 3(3-4):281-297, 1983.

[3] R. Bojanowski. Zastosowania uogólnionej nierówności Bogomolova-Miyaoka-Yau. *Master Thesis (in Polish)*, 2003. http://www.mimuw.edu.pl/~alan/postscript/bojanowski.ps.

[4] P. Brass, W. Moser, J. Pach. *Research Problems in Discrete Geometry*. Springer, 2005.

[5] Gabriel A. Dirac. Collinearity properties of sets of points. *Quart. J. Math., Oxford Ser. (2)*, 2:221-227, 1951.

[6] Paul Erdős. Some unsolved problems. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 6:221-254, 1961.

[7] S. Felsner. *Geometric Graphs and Arrangements*. Vieweg, Wiesbaden, 2004.

[8] B. Grünbaum. A catalogue of simplicial arrangements in the real projective plane. *Ars Mathematica Contemporanea*, 2(1):1-25, 2009.

[9] B. Grünbaum. *Arrangements and Spreads*. CBMS Regional Conference Series in Mathematics, No. 10. AMS, Providence, RI, 1972.

[10] B. Grünbaum. *Arrangements of hyperplanes*. Proc. Second Louisiana Conf. on Combinatorics, Graph Theory and Computing, R. C. Mullin et al., eds., Louisiana State University, Baton Rouge, 1971.

[11] B. Grünbaum. Dirac's conjecture concerning high-incidence elements in aggregates. *Geombinatorics*, 2:48-55, 2010.

[12] F. Hirzebruch. Arrangements of lines and algebraic surfaces. *Progress in Mathematics 36, Birkhäuser Boston*, 113-140, 1983.

[13] F. Hirzebruch. Singularities of algebraic surfaces and characteristic numbers. *Proc. Lefschetz Centennial Conf. (Mexico City, 1984), Part I, Contemporary Mathematics 58, AMS*, 141-155, 1986.

[14] V. Klee and S. Wagon. *Old and New Unsolved Problems in Plane Geometry and Number Theory*. The Mathematical Association of America, 1991.

[15] A. Langer. Logarithmic orbifold Euler numbers of surfaces with applications. *Proceedings of the London Mathematical Society*, 86(2), 358-396, 2003.

[16] Y. Miyaoka. On the Chern numbers of surfaces of general type. *Invent. Math.* 42(1): 225-237, 1977.

[17] Michael S. Payne, David R. Wood. Progress on Dirac's Conjecture. *The Electronic Journal of Combinatorics*, 21(2), #P2.12, 2014.

[18] H. H. Pham, and T. C. Phi. A new progress on Weak Dirac conjecture. http://arxiv.org/abs/1607.08398.

[19] P. Pokora. The orbifold Langer-Miyaoka-Yau inequality and Hirzebruch-type inequalities. http://arxiv.org/abs/1612.05141.

[20] Endre Szemerédi and William T. Trotter, Jr.. Extremal problems in discrete geometry. *Combinatorica*, 3(3-4):381-392, 1983.