A construction of uniquely n-colorable digraphs with arbitrarily large digirth

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Submitted: Nov 10, 2014; Accepted: Feb 22, 2017; Published: Apr 13, 2017
Mathematics Subject Classifications: 05C20, 05C15, 68R10

Abstract

A natural digraph analogue of the graph-theoretic concept of an 'independent set' is that of an 'acyclic set', namely a set of vertices not spanning a directed cycle. Hence a digraph analogue of a graph coloring is a decomposition of the vertex set into acyclic sets and we say a digraph is uniquely n-colorable when this decomposition is unique up to relabeling. It was shown probabilistically in [A. Harutyunyan et al., Uniquely D-colorable digraphs with large girth, Canad. J. Math., **64**(6): 1310–1328, 2012] that there exist uniquely n-colorable digraphs with arbitrarily large girth. Here we give a construction of such digraphs and prove that they have circular chromatic number n. The graph-theoretic notion of 'homomorphism' also gives rise to a digraph analogue. An acyclic homomorphism from a digraph D to a digraph H is a mapping $\varphi : V(D) \to V(H)$ such that $uv \in A(D)$ implies that either $\varphi(u)\varphi(v) \in A(H)$ or $\varphi(u) = \varphi(v)$, and all the 'fibers' $\varphi^{-1}(v)$, for $v \in V(H)$, of φ are acyclic. In this language, a core is a digraph D for which there does not exist an acyclic homomorphism from D to a proper subdigraph of itself. Here we prove some basic results about digraph cores and construct highly chromatic cores without short cycles.

Keywords: digraph; chromatic number; acyclic homomorphism; girth

1 Introduction

The author previously *constructed* digraphs with arbitrarily large digirth and chromatic number in [13]. In fact, the construction strengthens the probabilistic result in [2] because

*Supported in part by a grant from the Simons Foundation (#279367 to Mark Kayll) and a University of Montana Graduate Student Summer Research Award funded by the George and Dorothy Bryan Endowment.

it produces a digraph with digirth k and chromatic number n for each pair k, n of integers exceeding one. It is also of interest that unlike the analogous graph constructions in [7],[8], and [11], the construction is primitively recursive in n. The main result appearing here constructs uniquely n-colorable digraphs with digirth k for every reasonable pair n, k of integers. In Section 3 we show that the digraphs constructed in [13] are 'cores'. Finally, we give a simple construction of highly chromatic digraphs without two cycles in Section 4.

Although basic terminology can be found in [1], we include our main definitions for completeness. The *digirth* of a digraph D, denoted $\vec{g}(D)$, is the length of its shortest directed cycle. Following [2] we define the *chromatic number* $\chi(D)$ of D to be the minimum number of parts in a partition of $V(D)$ into acyclic sets, and we say that D is *n-chromatic* if $\chi(D) = n$. It should be noted that this chromatic number of a digraph was first defined by Neumann-Lara in [12], where it was called the 'dichromatic number'. We would now like to relate digraph colorings to homomorphisms. An *acyclic homomorphism* from a digraph D to a digraph H is a mapping $\varphi : V(D) \to V(H)$ such that $uv \in A(D)$, where $A(D)$ is the arc set of D, implies that either $\varphi(u)\varphi(v) \in A(H)$ or $\varphi(u) = \varphi(v)$, and all the fibers of φ are acyclic. If φ is an acyclic homomorphism such that all $uv \in A(D)$ satisfy $\varphi(u) \neq \varphi(v)$ then φ is a *non-contracting homomorphism*. As with graphs, if there exists an acyclic homomorphism from a digraph D to a digraph H we say that D is *homomorphic* to H, denoted by $D \to H$, and D is *H-colorable*. Since we deal almost exclusively with acyclic homomorphisms when considering digraphs, we often write 'homomorphism' when it is clear from the context that we mean 'acyclic homomorphism'. As in the case of the graph coloring analogue, an equivalent definition of the chromatic number is $\chi(D) = \min\{n \mid D \to \overset{\leftrightarrow}{K}_n\}$ (where $\overset{\leftrightarrow}{K}_n$ denotes the complete biorientation, see [1], of K_n). One nice property of the chromatic number of a digraph is that it generalizes the chromatic number of a graph because $\chi(G) = \chi(\overset{\leftrightarrow}{G})$ for every finite simple graph G.

We define a digraph D to be *uniquely n-colorable* if D is n-chromatic and any two n-colorings of D induce the same partition of $V(D)$. A digraph D is *uniquely H-colorable* if it is surjectively H-colorable, and for any two H-colorings ϕ, ψ of D, the functions ϕ and ψ differ by an automorphism of H, and a digraph D is a *core* if it is uniquely D-colorable. It is well known that a digraph D is uniquely n-colorable if and only if it is uniquely $\overset{\leftrightarrow}{K}_n$-colorable. In order to confirm the correctness of our theorems, we will also need the fact that $D \to H$ implies that $\vec{g}(D) \geqslant \vec{g}(H)$, which is a direct consequence of Propositions 1.2 and 1.3 in [2]. It is worth noticing the subtle difference between the last statement and its graph analogue which is true only for odd girth.

2 Uniquely n-colorable digraphs without short cycles

The proof of Theorem 8 constructs uniquely n-colorable digraphs with digirth k for any pair n, k of suitable integers. This result is a constructive version of (an important case of) the probabilistic proof appearing in [5] and is analogous to the undirected construction appearing in [14]. For the proof of Theorem 8 we first need to prove a few lemmas and to construct a few digraphs. The first of which, denoted D_n, was constructed in [13] in order to prove the following theorem.

Theorem 1. *For any given integers k and n exceeding one, there exists an n-chromatic digraph D with $\vec{g}(D) = k$.*

We will provide the reader with the construction but we refer the reader to [13] for the proof of Theorem 1. For $n = 2$, the directed k-cycle will suffice. For $n \geqslant 2$, we proceed by induction on n and suppose that we have already constructed a digraph D_n with chromatic number n, digirth k, and $V(D_n) = \{d_1, d_2, \dots, d_m\}$. We now define D_{n+1}.

For each $i \in [m]$, let D_n^i be a digraph with vertex set

$$V(D_n^i) = \{(d_1, i), (d_2, i), \dots, (d_m, i)\}$$

which is isomorphic to D_n in the natural way. Next construct m directed paths P_{d_i}, for $1 \leqslant i \leqslant m$, each of length $k-2$, with vertex sets $\{(d_i, p_1), (d_i, p_2), \dots (d_i, p_{k-1})\}$ and arc sets $A(P_{d_i}) := \{\overrightarrow{(d_i, p_j)(d_i, p_{j+1})} \mid j \in [k-2]\}$. Now define m digraphs $H(n, i)$, for $1 \leqslant i \leqslant m$, in the following manner. The vertex sets are $V(H(n, i)) := V(D_n^i) \cup V(P_{d_i})$, and the arc sets are

$$A(H(n,i)) := A(D_n^i) \cup A(P_{d_i}) \cup \left\{\overrightarrow{(d,i)(d_i,p_1)} \mid d \in V(D_n)\right\} \cup \left\{\overrightarrow{(d_i,p_{k-1})(d,i)} \mid d \in V(D_n)\right\}.$$

Finally, we define D_{n+1} to be the digraph with

$$V(D_{n+1}) := \bigcup_{i=1}^{m} V(H(n,i))$$

and

$$A(D_{n+1}) := \bigcup_{i=1}^{m} A(H(n,i)) \cup \left\{\overrightarrow{(d_i,p_\ell)(d_j,p_h)} \mid d_i d_j \in A(D_n) \text{ and } \ell, h \in [k-1]\right\}.$$

In order to illustrate this construction, we include in Figure 1 a diagram of D_3 with $k = 3$. All double-tailed arrows represent numerous arcs in the diagram. The double-tailed arrows running horizontally indicate an arc from every vertex at the tail to every vertex at the head. The double-tailed arrows running up and down indicate an arc from every vertex at the tail to one vertex at the head and from one vertex at the tail to every vertex at the head respectively. The three remaining diagrams in this section follow similar schematics. It is also worth pointing out that the subdigraph Σ of D_{n+1} induced by the vertices of the P_{d_i}'s is isomorphic to the lexicographic product of D_n with the directed path of length $k-2$. The *lexicographic product* $D \circ H$ of two digraphs D, H is defined to be the digraph with $V(D) \times V(H)$ as its vertex set and with an arc from (d_1, h_1) to (d_2, h_2) if $d_1 d_2$ is an arc in D, or $d_1 = d_2$ and $h_1 h_2$ is an arc in H.

The following lemma provides us with an important result about the arcs of D_n that will be useful both in this section and Section 3.

Lemma 2. *For $k = \vec{g}(D_n)$, every arc of D_n is in a directed k-cycle.*

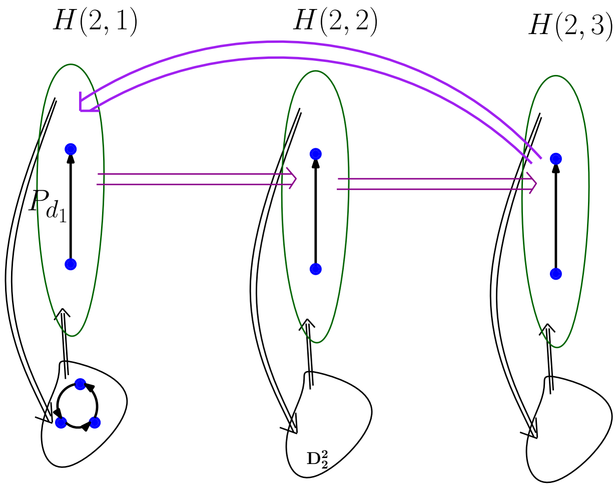

Figure 1: D_3 with $k = 3$

Proof. We notice that the assertion is true for $n = 2$ because $D_2 \cong \vec{C}_k$ and proceed by induction. Next assume its truth for D_n and let uv be an arc in D_{n+1}. If, for an $i \in [m]$, uv is an arc in D_n^i, which is isomorphic to D_n, then we may use the inductive hypothesis to see that uv is in a k-cycle. Another easy case is when uv is an arc of some P_{d_i} since uv is in a k-cycle for all $i \in [m]$ by our construction. Similarly, for all $i \in [m]$, our construction implies that uv is in a k-cycle when either $u \in V(D_n^i)$ and $v = (d_i, p_1)$ or $v \in V(D_n^i)$ and $u = (d_i, p_{k-1})$. The last case to inspect is when $u \in V(P_{d_i})$ and $v \in V(P_{d_j})$ for $i, j \in [m]$ with $i \neq j$. In this case, by our construction, uv is an arc in D_{n+1} if and only if $d_i d_j$ is an arc in D_n. Thus $d_i d_j$ is in a k-cycle of D_n by the induction hypothesis. Finally this in turn implies that uv is in a k-cycle of D_{n+1} and the proof is complete. $\square$

Another of these digraphs, denoted B_n, is a spanning subdigraph of D_n. We will define B_n inductively and start by setting B_2 to be the path of length $k-1$. We now define B_{n+1} from B_n. Suppose that $V(B_n) = \{d_1, d_2, \dots, d_m\} = V(D_n)$ and set $V(B_{n+1}) = V(D_{n+1})$. For $i \in [m]$ let B_n^i be B_n tagged with an i. Now define m digraphs $F(n, i)$, for $1 \leqslant i \leqslant m$, in the following manner. The vertex sets are $V(F(n, i)) \coloneqq V(B_n^i) \cup V(P_{d_i})$, and the arc sets are

$$A(F(n,i)) \coloneqq A(B_n^i) \cup A(P_{d_i}) \cup \left\{ \overrightarrow{(d,i)(d_i,p_1)} \mid d \in V(D_n) = V(B_n) \right\}.$$

Finally, we define B_{n+1} to be the digraph with

$$V(B_{n+1}) \coloneqq \bigcup_{i=1}^{m} V(F(n,i))$$

and

$$A(B_{n+1}) \coloneqq \bigcup_{i=1}^{m} A(F(n,i)) \cup \left\{ \overrightarrow{(d_i,p_\ell)(d_j,p_h)} \mid d_i d_j \in A(B_n) \text{ and } \ell, h \in [k-1] \right\}.$$

It may be helpful for the reader to view Figure 2 for an example of this construction.

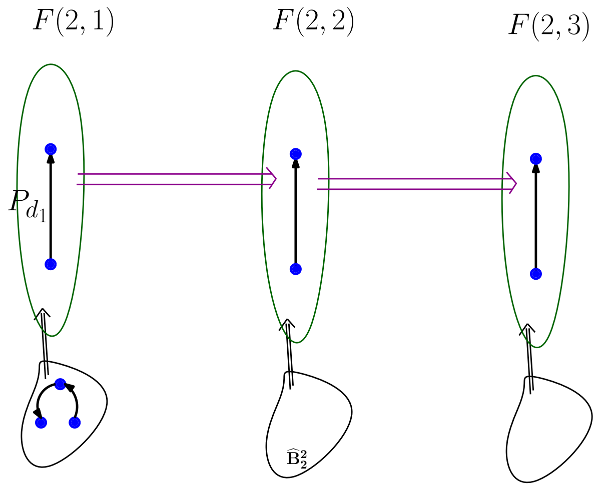

Figure 2: B_3 with $k = 3$

Lemma 3. *B_n is acyclic for all n.*

Proof. We proceed by induction and notice first that B_2 is acyclic as it is just a directed path. Now, assuming that B_n is acyclic, we see that each subdigraph B_n^i of B_{n+1} is acyclic by our induction hypothesis. Thus there does not exist a cycle in B_{n+1} containing a vertex from any B_n^i since there are no arcs from any vertex of the P_{d_i}'s to any vertex of the B_n^i's. Since the subdigraph $\widehat{\Sigma}$ of B_{n+1} induced by the vertices of the P_{d_i}'s is homomorphic to B_n via projection onto the first coordinate, we see that $\vec{g}(\widehat{\Sigma}) \geqslant \vec{g}(B_n) = \infty$. Therefore B_{n+1} is also acyclic. □

Lemma 4. *If $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ is a shortest cycle in D_n, then there exists a unique $\ell \in \{0, 1, \ldots, k-1\}$ such that the arc $\alpha_\ell \alpha_{\ell+1}$ is in $A(D_n) \smallsetminus A(B_n)$.*

Proof. Proceeding by induction, again we see that the statement is true for B_2. Thus by our induction hypotheses the statement is true for any shortest cycle contained in any D_n^i. Since all shortest cycles containing a vertex (d, i) from a D_n^i and a vertex from a P_{d_j} have the form $\big((d, i), (d_i, p_1), .., (d_i, p_{k-1})\big)$, the unique arc in $A(D_{n+1}) \smallsetminus A(B_{n+1})$ for such cycles is $(d_i, p_{k-1})(d, i)$. Thus it remains to show that the statement is true for shortest cycles contained in Σ (defined on p. 3). So let us now suppose that the cycle $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ is contained in Σ; referencing the proof of Theorem 1, [13], we may assume that $\alpha_j = (d_j, p_{r_j})$, where $(d_0, d_1, \ldots, d_{k-1})$ is a shortest cycle in D_{n-1}. Thus the induction hypothesis yields that there exists a unique ℓ such that $d_\ell d_{\ell+1}$ is in $A(D_{n-1}) \smallsetminus A(B_{n-1})$. Therefore $\alpha_\ell \alpha_{\ell+1}$ is the unique arc of $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ in $A(D_n) \smallsetminus A(B_n)$, and the proof is complete. □

Next we construct digraphs D'_n with digirth k. For $n = 2$, the directed k-cycle will suffice. For $n \geqslant 2$, we proceed by induction on n and suppose that we have already constructed a digraph D'_n with digirth k and $V(D'_n) = \{d_1, d_2, \ldots, d_m\}$. We now define D'_{n+1}.

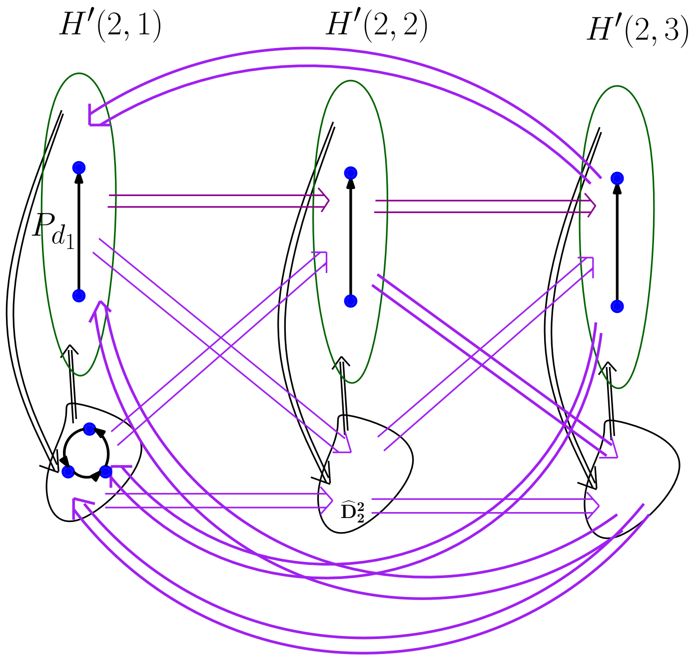

Figure 3: D_3' with $k = 3$

For each $i \in [m]$, let $\widehat{D}_n^i$ be a digraph with vertex set

$$V(\widehat{D}_n^i) = \{(d_1, i), (d_2, i), \ldots, (d_m, i)\}$$

which is isomorphic to D_n' in the natural way. Next construct m directed paths P_{d_i}, for $1 \leqslant i \leqslant m$, each of length $k-2$, with vertex sets $\{(d_i, p_1), (d_i, p_2), \ldots (d_i, p_{k-1})\}$ and arc sets $A(P_{d_i}) = \{\overrightarrow{(d_i, p_j)(d_i, p_{j+1})} \,|\, j \in [k-2]\}$. Now define m digraphs $H'(n, i)$, for $1 \leqslant i \leqslant m$, in the following manner. The vertex sets are $V(H'(n, i)) \coloneqq V(\widehat{D}_n^i) \cup V(P_{d_i})$, and the arc sets are

$$A(H'(n, i)) \coloneqq A(\widehat{D}_n^i) \cup A(P_{d_i}) \cup \left\{\overrightarrow{(d, i)(d_i, p_1)} \,|\, d \in V(D_n')\right\} \cup \left\{\overrightarrow{(d_i, p_{k-1})(d, i)} \,|\, d \in V(D_n')\right\}.$$

Finally, we define D_{n+1}' to be the digraph with

$$V(D_{n+1}') \coloneqq \bigcup_{i=1}^{m} V(H'(n, i))$$

and

$$A(D_{n+1}') \coloneqq \bigcup_{i=1}^{m} A(H'(n, i)) \cup \{\alpha\beta \,|\, \alpha \in V(H'(n, i)), \beta \in V(H'(n, j)) \text{ and } d_i d_j \in A(D_n')\}.$$

Figure 3 is included in order to clarify the construction of D_n'.

It is clear from the construction of D_n' that D_n is a spanning subdigraph of D_n' for all n which implies that $\chi(D_n') \geqslant \chi(D_n) = n$ and $\vec{g}(D_n') \leqslant \vec{g}(D_n)$.

Lemma 5. *For each integer $n \geqslant 2$, the digraph D'_n has digirth $\vec{g}(D'_n) = \vec{g}(D_n) = k$.*

Proof. We observe that D'_2 has digirth k and proceed by induction. Using the induction hypothesis, we see that $\vec{g}(\widehat{D}^i_n) = k$ for all $i \in [m]$, which combined with the construction of D'_{n+1} implies that $\vec{g}(H'(n,i)) = k$ for all $i \in [m]$. Thus we need only consider cycles which contain vertices α and β where $\alpha \in V(H'(n,i))$, $\beta \in V(H'(n,j))$ and $i \neq j$. But this implies, from the construction of D'_{n+1}, that there exists a path in D'_n from d_i to d_j and from d_j to d_i. Thus the induction hypothesis also implies that the cycle containing α and β has length at least k. Combining these observations, we see that $\vec{g}(D'_{n+1}) = k$, and induction gives the lemma. $\square$

Finally we define B'_n inductively and start by letting B'_2 be the path of length $k-1$. We now define B'_{n+1} from B'_n. Suppose that $V(B'_n) = \{d_1, d_2, \ldots, d_m\} = V(D'_n)$ and set $V(B'_{n+1}) = V(D'_{n+1})$. For $i \in [m]$ let $\widehat{B}^i_n$ be B'_n tagged with an i. Now define m digraphs $F'(n,i)$, for $1 \leqslant i \leqslant m$, in the following manner. The vertex sets are $V(F'(n,i)) \coloneqq V(\widehat{B}^i_n) \cup V(P_{d_i})$, and the arc sets are

$$A(F'(n,i)) \coloneqq A(\widehat{B}^i_n) \cup A(P_{d_i}) \cup \left\{ \overrightarrow{(d,i)(d_i,p_1)} \mid d \in V(D_n) = V(B'_n) \right\}.$$

Finally, we define B'_{n+1} to be the digraph with

$$V(B'_{n+1}) \coloneqq \bigcup_{i=1}^{m} V(F'(n,i))$$

and, for all $s, t \in [m]$,

$$A(B'_{n+1}) \coloneqq$$

$$\begin{array}{cccc} & \bigcup_{i=1}^m A(F'(n,i)) & \bigcup & \left\{ \overrightarrow{(d_i,p_\ell)(d_j,p_h)} \mid d_id_j \in A(B'_n) \text{ and } \ell, h \in [k-1] \right\} \\ \bigcup & \left\{ \overrightarrow{(d_s,i)(d_t,j)} \mid d_id_j \in A(B'_n) \right\} & \bigcup & \left\{ \overrightarrow{(d_s,i)(d_j,p_h)} \mid d_id_j \in A(D'_n) \text{ and } h \in [k-1] \right\}. \end{array}$$

It is clear from the construction that B_n is a spanning subdigraph of B'_n. Figure 4 may help bring to light some of the nuances of this construction.

Lemma 6. *B'_n is acyclic for all n.*

Proof. It is easy to see that B'_2 is acyclic and thus we continue by induction. Once again the induction hypothesis and the construction of B'_{n+1} imply that $F'(n,i)$ is acyclic for every $i \in [m]$. Since there is no arc from any vertex of a P_{d_i} to any vertex of a $\widehat{B}^j_n$, it suffices to consider the subdigraph of B'_{n+1} induced by the vertices of the P_{d_i}'s and the subdigraph induced by the $\widehat{B}^i_n$'s. Both of these subdigraphs are homomorphic to B'_n and thus are acyclic by induction. Therefore B'_{n+1} is acyclic. $\square$

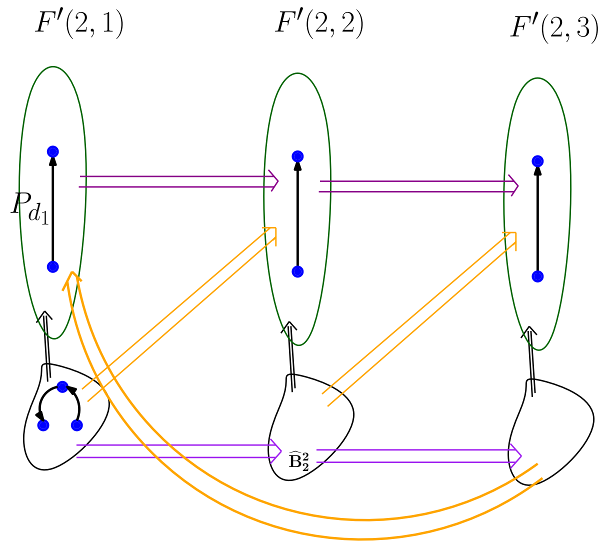

Figure 4: B'_3 with $k = 3$

We may now define directed graphs $D'_m * \overset{\leftrightarrow}{K}_n$ which will be shown to be uniquely n-colorable with digirth equal to $\vec{g}(D'_m)$. The vertex set of $D'_m * \overset{\leftrightarrow}{K}_n$ is $V(D'_m * \overset{\leftrightarrow}{K}_n) = V(D'_m) \times V(\overset{\leftrightarrow}{K}_n)$ and there is an arc from (d_1, h_1) to (d_2, h_2) if $d_1 d_2 \in A(D'_m)$ and $h_1 h_2 \in A(\overset{\leftrightarrow}{K}_n)$, or $d_1 d_2 \in A(B'_m)$ and $h_1 = h_2$. It is worth noting that the direct product $D'_m \times \vec{K}_n$ is a spanning subdigraph of $D'_m * \overset{\leftrightarrow}{K}_n$. The first two properties to notice about $D'_m * \overset{\leftrightarrow}{K}_n$ are that it has digirth at least $\vec{g}(D'_m)$ and is n-colorable because the projections are homomorphisms. (The projection onto $\overset{\leftrightarrow}{K}_n$ being a homomorphism relies on the fact that B'_m is acyclic.) We now introduce some notation for future use. Let the vertices of $\overset{\leftrightarrow}{K}_n$ be $0, 1, \ldots, n-1$ and, for $t \in V(\overset{\leftrightarrow}{K}_n)$, let $H^t(m-1, i)$ be the set of vertices $V(H'(m-1, i)) \times \{t\}$. Similarly define $P^t_{d_i}$ and $D^t(m-1, i)$ to be $V(P_{d_i}) \times \{t\}$ and $V(\widehat{D}^i_{m-1}) \times \{t\}$ respectively. Lastly define $\Omega^n(m-1, j)$ to be the subdigraph of $D'_m * \overset{\leftrightarrow}{K}_n$ induced by $\bigcup_{i=0}^{n-1} D^i(m-1, j)$. The next lemma provides the linchpin to proving that $D'_m * \overset{\leftrightarrow}{K}_n$ is uniquely n-colorable when $m > n$.

Lemma 7. *If $n \leqslant m - 1$ and $j \in \{1, 2, \ldots, |V(D'_{m-1})|\}$, then the chromatic number of $\Omega^n(m-1, j)$ is n.*

Proof. First we show that there exists an m-coloring ϕ_m of D_m such that $\alpha\beta \in A(D_m) \smallsetminus A(B_m)$ implies that $\phi_m(\alpha) \neq \phi_m(\beta)$. It is easy to see that ϕ_2 exists and thus we may proceed by induction. Define a mapping $\phi_m : V(D_m) \to V(\overset{\leftrightarrow}{K}_m)$ as follows. For vertices $(d_j, i) \in V(D^i_{m-1})$, let

$$\phi_m((d_j, i)) = \begin{cases} \phi_{m-1}(d_j) & \text{if } \phi_{m-1}(d_j) \neq \phi_{m-1}(d_i), \\ m-1 & \text{if } \phi_{m-1}(d_j) = \phi_{m-1}(d_i). \end{cases}$$

For vertices $(d_i, p) \in V(P_{d_i})$, define $\phi_m((d_i, p)) = \phi_{m-1}(d_i)$.

Now suppose that $\alpha\beta \in A(D_m) \smallsetminus A(B_m)$ which implies that either α, β are vertices of some D^i_{m-1}, or α is a vertex of some P_{d_i} and β is a vertex of D^i_{m-1}, or α is a vertex of some P_{d_i} and β is a vertex of some P_{d_j}. In the case where α, β are vertices of some D^i_{m-1}, we may assume that $\alpha = (d_j, i)$ and $\beta = (d_h, i)$, where $d_j d_h$ is in $A(D_{m-1}) \smallsetminus A(B_{m-1})$. Thus using the induction hypothesis we see that $\phi_{m-1}(d_j) \neq \phi_{m-1}(d_h)$ which implies that $\phi_m(\alpha) \neq \phi_m(\beta)$. In the second case, α is a vertex of some P_{d_i} and β is a vertex of D^i_{m-1}, i.e. $\alpha = (d_i, p_{k-2})$ and $\beta = (d_j, i)$. It is clear from the definition of $\phi_m$hh that $\phi_m(\alpha) \neq \phi_m(\beta)$. In the last case, in which α is a vertex of some P_{d_i} and β is a vertex of some P_{d_j}, we may assume that $d_i d_j$ is in $A(D_{m-1}) \smallsetminus A(B_{m-1})$. Thus the inductive hypotheses and the definition of ϕ_m imply that $\phi_m(\alpha) \neq \phi_m(\beta)$. Therefore ϕ_m is an m-coloring of D_m such that $\alpha\beta \in A(D_m) \smallsetminus A(B_m)$ implies that $\phi_m(\alpha) \neq \phi_m(\beta)$.

We now define Γ to be the subdigraph of $D'_{n+1} * \overset{\leftrightarrow}{K}_n$ induced by the set $\{((\alpha, 1), \phi_n(\alpha)) | (\alpha, 1) \in V(D^1_n)\}$ and notice that Γ is a subdigraph of $\Omega^n(n, 1)$ and the digraph induced by $D^i(m-1, 1)$ is isomorphic to the digraph induced by $D^i(m-1, j)$ for all $j \in \{1, 2, \ldots, |V(D'_{m-1})|\}$. Consider the mapping $\rho : V(D_n) \to V(\Gamma)$ defined by $\rho(\alpha) = ((\alpha, 1), \phi_n(\alpha))$. Since ϕ_n is an n-coloring of D_n (i.e. a homomorphism to $\overset{\leftrightarrow}{K}_n$) ρ is well-defined and bijective. We now suppose that $\alpha\beta \in A(D_n)$ in order to show that ρ is in fact a homomorphism. The first case is when $\alpha\beta$ is an arc in B_n which implies that $(\alpha, 1)(\beta, 1) \in A(B'_{n+1})$ and thus $((\alpha, 1), \phi_n(\alpha))((\beta, 1), \phi_n(\beta)) \in A(\Gamma)$ whether or not $\phi_n(\alpha) = \phi_n(\beta)$ (because of the definition of our $*$-product). The second case is when $\alpha\beta$ is an arc in D_n but not B_n. From the preceding paragraph we know that this implies that $\phi_n(\alpha) \neq \phi_n(\beta)$. Also the construction of D'_{n+1} implies that $(\alpha, 1)(\beta, 1)$ is an arc in D'_{n+1}. Hence $((\alpha, 1), \phi_n(\alpha))((\beta, 1), \phi_n(\beta))$ is an arc in Γ and ρ is a homomorphism. This now implies that $\chi(\Gamma) \geqslant n$ and in fact $\chi(\Gamma) = n$ since we saw above that $D'_{n+1} * \overset{\leftrightarrow}{K}_n$ is n-colorable. Recall that Γ is a subdigraph of $\Omega^n(n, 1)$. Since, for $m > n$, the digraph induced by $D^i(n, 1)$ is isomorphic to a subdigraph of the digraph induced by $D^i(m-1, 1)$ for all $i \in \{0, 1, \ldots, n-1\}$, the digraph Γ is isomorphic to a subdigraph of $\Omega^n(m-1, 1)$. Therefore the chromatic number of $\Omega^n(m-1, j)$ is n for all $j \in \{1, 2, \ldots, |V(D'_{m-1})|\}$. $\square$

Finally we have all the necessary tools to prove the deepest result of this paper.

Theorem 8. *For every integer $n \geqslant 2$, the digraph $D'_m * \overset{\leftrightarrow}{K}_n$ is uniquely n-colorable whenever $n \leqslant m-1$.*

Proof. We have seen that the canonical projection $\pi : V(D'_m * \overset{\leftrightarrow}{K}_n) \to V(\overset{\leftrightarrow}{K}_n)$ is a surjective homomorphism and thus $D'_m * \overset{\leftrightarrow}{K}_n$ is n-colorable. Now suppose that there exists another surjective homomorphism $\psi : V(D'_m * \overset{\leftrightarrow}{K}_n) \to V(\overset{\leftrightarrow}{K}_n)$ and we will show that ψ is a composition of π with an automorphism of $\overset{\leftrightarrow}{K}_n$. Notice that since the target digraph is $\overset{\leftrightarrow}{K}_n$ this amounts to showing that $\psi((\alpha, i)) = \psi((\beta, i))$ for all vertices α, β of D'_m and $i \in V(\overset{\leftrightarrow}{K}_n)$. In other words we need only show that the fibers of ψ are a relabeling of the fibers of π. The preceding lemma has the direct consequence that for all

$j \in \{1, 2, \ldots, |V(D'_{m-1})|\}$ and $s \in V(\overleftrightarrow{K}_n)$ there exists an $\alpha \in V(\Omega^n(m-1, j))$ such that $\psi(\alpha) = s$ because $\Omega^n(m-1, j)$ is n-chromatic. For each such j and s, let α_j^s be such that $\alpha_j^s \in V(\Omega^n(m-1, j))$ and $\psi(\alpha_j^s) = s$.

Consider two vertices d_0 and d_1 of D_{m-1} such that there is an arc from d_0 to d_1 in D_{m-1}. Lemmas 2 and 4 imply that there exists a cycle $(d_0, d_1, \ldots, d_{k-1})$ in D_{m-1} and there exists a unique $\ell \in [k]$ such that $d_\ell d_{\ell+1} \in A(D_{m-1}) \setminus A(B_{m-1})$. Thus for all $\beta_h \in V(\widehat{D}^h_{m-1})$, where $\widehat{D}^h_{m-1}$ is the copy corresponding to d_h, the sequence $\big((\beta_0, i_0), (\beta_1, i_1), \ldots, (\beta_{k-1}, i_{k-1})\big)$ is a cycle in $D'_m * \overleftrightarrow{K}_n$ whenever $i_\ell \neq i_{\ell+1}$. Hence for all $s \in [n]$ and some $i \in [n]$ (which depends on s) the vertices α_ℓ^s and $\alpha_{\ell+1}^s$ lie in $D^i(m-1, \ell)$ and $D^i(m-1, \ell+1)$ respectively, for otherwise $(\alpha_0^s, \alpha_1^s, \ldots, \alpha_{k-1}^s)$ would be a monochromatic cycle with respect to ψ. Similarly, supposing that $\alpha_\ell^s \in D^i(m-1, \ell)$,

$$\begin{aligned} &\text{there is no } \nu \in D^r(m-1, \ell+1) \text{ with } \psi(\nu) = s \\ &\qquad\qquad \text{nor a vertex } \mu \in H^r(m-1, \ell) \text{ with } \psi(\mu) = s \text{ when } r \neq i, \end{aligned} \tag{1}$$

for otherwise $(\alpha_0^s, \alpha_1^s, \ldots, \alpha_\ell^s, \nu, \alpha_{\ell+2}^s, \ldots, \alpha_{k-1}^s)$ and $(\alpha_0^s, \alpha_1^s, \ldots, \alpha_{\ell-1}^s, \mu, \alpha_{\ell+1}^s, \alpha_{\ell+2}^s, \ldots, \alpha_{k-1}^s)$ would be monochromatic cycles with respect to ψ. We claim that for all $i \in [n]$ there exists an $\alpha_\ell^s \in D^i(m-1, \ell)$ for some $s \in [n]$. To this end we consider the set $\{i \in [n] |$ there exists an s such that $\alpha_\ell^s \in D^i(m-1, \ell)\}$ and suppose that the size of this set is less than n in order to reach a contradiction. This assumption implies that there exists a $j \in [n]$ such that $D^j(m-1, \ell)$ does not contain any α_ℓ^s. This implies further that no $\alpha_{\ell+1}^s$, for $s \in [n]$, lies in $D^j(m-1, \ell+1)$ as we've established that for each fixed $s \in [n]$, the vertices α_ℓ^s and $\alpha_{\ell+1}^s$ lie in $D^i(m-1, \ell)$ and $D^i(m-1, \ell+1)$ respectively (i.e. share the same superscript i here). However there exists some $s_1 \in [n]$ and $\beta \in D^j(m-1, \ell)$ such that $\psi(\beta) = s_1$ because every vertex is sent to some color, and we just concluded that $\alpha_{\ell+1}^{s_1}$ cannot be in $D^j(m-1, \ell+1)$. Hence we reach a contradiction because this leads to the cycle $(\alpha_0^{s_1}, \alpha_1^{s_1}, \ldots, \alpha_{\ell-1}^{s_1}, \beta, \alpha_{\ell+1}^{s_1}, \ldots, \alpha_{k-1}^{s_1})$ being monochromatic with respect to ψ. Thus the claim is true and we may conclude that for all $i \in [n]$ there exists an $\alpha_\ell^s \in D^i(m-1, \ell)$ for some $s \in [n]$. Suppose that $\alpha_\ell^{s_j} \in D^i(m-1, \ell)$. Appealing to (1), we now see that for every $i \in [n]$, when $r \neq i$, there does not exist a vertex in $D^r(m-1, \ell+1)$ nor a vertex in $H^r(m-1, \ell)$ either of which is colored s_j. Therefore for each $s \in [n]$ there exists a unique $i \in [n]$ such that $H^i(m-1, \ell)$ and $D^i(m-1, \ell+1)$ are both monochromatic of the same color with respect to ψ.

We now aim to prove that the sets $D^i(m-1, \ell+1)$ and $D^i(m-1, \ell+2)$ are monochromatic of the same color with respect to ψ for all $i \in [n]$. In order to reach a contradiction suppose that for some color s_1 there exists a vertex $\mu \in D^{t_1}(m-1, \ell+2)$ with $\psi(\mu) = s_1$, where $D^{t_2}(m-1, \ell+1)$ is colored s_1 and $t_1 \neq t_2$. This implies that there does not exist a vertex $\beta \in V(P^{t_2}_{\ell+1})$ such that $\psi(\beta) = s_1$, for otherwise the cycle $(\alpha_0^{s_1}, \alpha_1^{s_1}, \ldots, \alpha_\ell^{s_1}, \beta, \mu, \alpha_{\ell+3}^{s_1} \ldots, \alpha_{k-1}^{s_1})$ would be monochromatic with respect to ψ. Consider if $P_{\ell+1}$ is monochromatic of the color s, $s \neq s_1$. We have established the there exists an $i \in [n]$ with $i \neq t_2$ such that $D^i(m-1, \ell+1)$ is monochromatic of color s. However, every vertex in $D^i(m-1, \ell+1)$ is in a cycle with the vertices of $P^{t_2}_{\ell+1}$ according to the construction of $D'_m * \overleftrightarrow{K}_n$. Thus we would have a monochromatic cycle with respect to ψ. Hence there exist vertices $\beta_2, \beta_3 \in V(P_{\ell+1}^{t_2})$ such that $\psi(\beta_2) = s_2$, $\psi(\beta_3) = s_3$, and s_1, s_2 and s_3 are distinct. This implies that there does not exist a vertex $\nu \in V(\Omega^n(m-1, \ell+2)) \smallsetminus D^{t_2}(m-1, \ell+2)$ such that $\psi(\nu) = s_i$, for $i = 2$ or $i = 3$, for otherwise the cycle $(\alpha_0^{s_i}, \alpha_1^{s_i}, \ldots, \alpha_\ell^{s_i}, \beta_i, \nu, \alpha_{\ell+3}^{s_i}, \ldots, \alpha_{k-1}^{s_i})$ would be monochromatic with respect to ψ. However this implies that $\Omega^n(m-1, \ell+2) \smallsetminus D^{t_2}(m-1, \ell+2)$, which is isomorphic to $\Omega^{n-1}(m-1, \ell+2)$, is $(n-2)$-colored, contradicting Lemma 7. Thus for all $i \in [n]$, the sets $D^i(m-1, \ell+1)$ and $D^i(m-1, \ell+2)$ are monochromatic with respect to ψ.

Assume now that there exists a vertex $\gamma \in P_{\ell+1}^i$ such that $\psi(\gamma) = s_4$ and the vertices in $D^i(m-1, \ell+1)$ and $D^i(m-1, \ell+2)$ are colored s_5 where $s_4 \neq s_5$. We again reach a contradiction since $(\alpha_0^{s_4}, \alpha_1^{s_4}, \ldots, \alpha_\ell^{s_4}, \gamma, \alpha_{\ell+2}^{s_4}, \alpha_{\ell+3}^{s_4} \ldots, \alpha_{k-1}^{s_4})$ would be a monochromatic cycle with respect to ψ. Thus $P_{\ell+1}^i$ and $D^i(m-1, \ell+1)$ are monochromatic for all $i \in n$. This fact along with the previous two paragraphs we see that $H^i(m-1, \ell)$, $H^i(m-1, \ell+1)$, and $D^i(m-1, \ell+2)$ are monochromatic with respect to ψ for all $i \in [n]$. Therefore we may inductively argue that all $H^i(m-1, j)$, for $j \in [k]$, are monochromatically colored the same. As D_m is strongly connected, this implies that $\psi((\alpha, i)) = \psi((\beta, i))$ for all vertices α, β of D'_m and $i \in V(\overleftrightarrow{K}_n)$, and as we noted in the first paragraph of this proof, this is enough to show that $D'_m * \overleftrightarrow{K}_n$ is uniquely n-colorable. $\square$

Recall from p. 8 and Lemma 5 that $\vec{g}(D'_m * \overleftrightarrow{K}_n) \geqslant \vec{g}(D'_m) = k$. In the preceding proof we encountered a number of directed k-cycles in $D'_m * \overleftrightarrow{K}_n$ implying that $\vec{g}(D'_m * \overleftrightarrow{K}_n) = \vec{g}(D'_m)$. We also note that we were able to construct D'_m with $\vec{g}(D'_m) = k$ for any pair m, k of integers exceeding one. Therefore our proof of Theorem 8 constructs a uniquely n-colorable digraph with digirth k for every pair n, k of integers both exceeding one. Thus we now have a constructive and more precise version of (an important case of) the main theorem appearing in [5].

We remark here on the number of vertices in the D_n's (and consequently the number of vertices in $D'_m * \overleftrightarrow{K}_n$). We see from the construction that we have the recurrence relation:

$$|V(D_n)| = |V(D_{n-1})|^2 + (k-1) \cdot |V(D_{n-1})| \text{ with } |V(D_2)| = k.$$

It is easy to confirm that $|V(D_n)| = 2^{2^{n-1}} - 1$ for $k = 3$. For a general k we do not have a closed form. However we may inductively argue that $|V(D_n)|$ is $O\left(2^{2^{n-3}} \cdot k^{2^{n-2}}\right)$ after observing that $|V(D_2)| = k$ and $|V(D_3)| = 2k^2 - k$. Therefore $|V(D'_m * \overleftrightarrow{K}_n)|$ is $O\left(n \cdot 2^{2^{m-3}} \cdot k^{2^{m-2}}\right)$ since D_m is a spanning subdigraph of D'_m. In order to put the size of these digraphs into context we point to a construction of uniquely 3-colorable graphs with girth 4 by Nešetřil [10] in which the smallest example has over 500 million vertices. The digraph constructed in the proof of Theorem 8 with the least vertices while still being uniquely 3-colorable with digirth 4 is $D'_4 * \overleftrightarrow{K}_3$ (where $\vec{g}(D'_4) = 4$) and $|V(D'_4 * \overleftrightarrow{K}_3)| = 2604$.

In [2] the authors define the circular chromatic number χ_c of a digraph This digraph invariant can be seen to generalize the graph version of this invariant. For $p \geqslant q$, we define the *directed complete rational graph* $\overrightarrow{K}_{p/q}$ to be the digraph with vertex set $\{0, 1, \ldots, p-1\}$ and an arc from i to j if $j - i \in \{q, q+1, \ldots, p-1\}$ (with arithmetic modulo p). It is easy to check that $\overrightarrow{K}_{p/1} \cong \overleftrightarrow{K}_p$ for every positive integer p. It was shown in [2] that we may define

$$\chi_c(D) = \min\{p/q \mid D \to \overrightarrow{K}_{p/q}\}.$$

Next we would like to show that $\chi_c(D'_m * \overleftrightarrow{K}_n) = n$ in order to construct digraphs with digirth k, chromatic number n, and circular chromatic number n for every pair k, n of integers exceeding one. To this end we will prove the following proposition.

Proposition 9. *If a digraph D is uniquely n-colorable, then $\chi_c(D) = \chi(D) = n$.*

Proof. By way of contradiction we assume D is uniquely n-colorable, $\chi_c(D) = p/q < n$, and $\zeta : V(D) \to \overrightarrow{V}(K_{p/q})$ is an acyclic homomorphism. It has been proved that every digraph D satisfies $\chi(D) - 1 < \chi_c(D) \leqslant \chi(D)$, see e.g. [2], allowing us to assume that $n - 1 < p/q$. Now consider the mapping $\varphi : V(\overrightarrow{K}_{p/q}) \to V(\overleftrightarrow{K}_n)$ defined by $\varphi(i) = \lfloor i/q \rfloor$ (taking $V(\overleftrightarrow{K}_n) = \{0, 1, \ldots, n-1\}$). In order to show that φ is an acyclic homomorphism it suffices to show that the fibers of φ are acyclic because the target digraph is complete. It is easy to check that for $j \in \{0, 1, \ldots, n-2\}$ $\varphi^{-1}(j) = \{jq, jq+1, \ldots, (j+1)q - 1\}$ and $\varphi^{-1}(n-1) = \{(n-1)q, (n-1)q+1, \ldots, p-1\}$. Thus for every $j \in V(K_n)$ and pair of vertices $s, t \in \varphi^{-1}(j)$ we have $0 < t - s < q$ whenever $s < t$ implying that st is not an arc in $K_{p/q}$. Therefore the fibers of φ are acyclic and φ is an acyclic homomorphism. Next consider the mapping $\sigma : V(\overrightarrow{K}_{p/q}) \to V(\overleftrightarrow{K}_n)$ defined by $\sigma(i) = \lfloor (i+1)/q \rfloor$ for $i = 0, 1, \ldots, p-2$ and $\sigma(p-1) = 0$. A similar argument to that for φ yields that σ is an acyclic homomorphism. Thus we see that $\varphi \circ \zeta$ and $\sigma \circ \zeta$ are two homomorphisms from D to K_n that induce different partitions of $V(D)$ contradicting the assumption that D is uniquely n-colorable. □

Because any subdigraph H of D satisfies $\chi_c(H) \leqslant \chi_c(D)$ we have the following immediate corollary to Proposition 9.

Corollary 10. *If a digraph D is n-chromatic and contains a subdigraph that is uniquely n-colorable, then $\chi_c(D) = \chi(D) = n$.*

Thus we see that Proposition 9 implies that $\chi_c(D'_m * \overleftrightarrow{K}_n) = n$ and hence we have a construction of digraphs with digirth k, chromatic number n, and circular chromatic number n for every pair k, n of integers exceeding one.

To conclude this section, we leave the reader with a conjecture. Noting that $\overleftrightarrow{K}_n$ is a core, and with an eye to the nonconstructive results of [5], we would like to construct digraphs with arbitrarily large girth which are uniquely H-colorable for any core H. In fact we feel confident that the construction is done, but the proof still eludes us. Consider the following conjecture concerning the digraphs D'_m constructed for Theorem 8, and notice that $\vec{g}(D'_m * H) \geqslant \vec{g}(D'_m)$.

Conjecture 11. *For all cores H and some constant c, the digraph $D'_m * H$ is uniquely H-colorable for $m > c \cdot \chi(H)$.*

3 D_n are cores

We now turn our attention to saying something substantially stronger about the digraphs constructed in the proof of Theorem 1: they are cores. This suggests that there is some sort of minimality to this construction. In order to proceed we need a few lemmas about cores. For the sake of completeness we have also decided to include a few basic lemmas about cores of digraphs that confirm that they behave similarly to their graph analogues. Recall that a digraph D is a *core* if it is uniquely D-colorable. The following useful lemma was proved in [5].

Lemma 12. *A digraph D is a core if and only if every acyclic homomorphism $V(D) \to V(D)$ is a bijection.*

The condition that every acyclic homomorphism $V(D) \to V(D)$ is a bijection is equivalent to saying that D is not homomorphic to a proper subdigraph of itself. An *(acyclic) retraction* of a digraph D is an acyclic homomorphism ϕ from D to a subdigraph H of D such that the restriction $\phi|_H$ is the identity map on H. Now we can state an equivalent definition of a core that will be used to prove Theorem 20.

Lemma 13. *A digraph D is a core if and only if it does not retract to a proper subdigraph of itself.*

Proof. The necessity is clear since an acyclic retraction is an acyclic homomorphism. Now suppose that D is not a core, and let H be a proper subdigraph of D such that ϕ is an acyclic homomorphism from D to H and D is not homomorphic to any proper subdigraph of H. The existence of such an H is ensured by Lemma 12. We claim that H is a core, for suppose it is not and let f be an acyclic homomorphism from H to a proper subdigraph of H. Then $f \circ \phi$ is an acyclic homomorphism from D to a proper subdigraph of H, contradicting our choice of H. Because of the claim, any homomorphism from H to itself is an automorphism of H. Let $\varphi : V(D) \to V(H)$ be an acyclic homomorphism. Since the restriction of a homomorphism is a homomorphism, $\psi \coloneqq \varphi|_H$ is an automorphism of H. Hence ψ^{-1} exists and $\psi^{-1} \circ \varphi : V(D) \to V(H)$ is an acyclic retraction to a proper subdigraph of D. □

We now define a subdigraph H of a digraph D to be a *core in D* if there exists an acyclic retraction from D to H and H is a core.

Lemma 14. *An acyclic retract of a digraph D is an induced subdigraph of D.*

Proof. Let ϕ be an acyclic retraction from D to a subdigraph H of D. Suppose that $x, y \in V(H)$ and $xy \in A(D)$. Since H is a retract, both $\phi(x) = x$ and $\phi(y) = y$ and, as ϕ is a homomorphism, xy is an arc in H. $\square$

We say that two digraphs D and H are *homomorphically equivalent* if H is homomorphic to D and D is homomorphic to H.

Lemma 15. *If H and K are cores then they are homomorphically equivalent if and only if they are isomorphic.*

Proof. Let $\phi : H \to K$ and $\psi : K \to H$ be acyclic homomorphisms. This implies that $\psi \circ \phi$ and $\phi \circ \psi$ are bijections since H and K are cores. Thus ϕ and ψ are both bijective and hence $H \cong K$ since, e.g., ϕ is a bijective homomorphism. $\square$

Lemma 16. *Every finite digraph D has a core, which is an induced subdigraph and is unique up to isomorphism.*

Proof. Since D is finite and the identity mapping is an acyclic retraction, the family of subdigraphs of D to which D has an acyclic retraction is finite and nonempty and thus has a minimal element $D^\bullet$ with respect to inclusion. From the definition of 'core in D' and Lemma 13, we see that $D^\bullet$ is a core in D. Since $D^\bullet$ is an acyclic retract, it is an induced subdigraph by Lemma 14. Now let H_1 and H_2 be cores of D, and, for $i = 1, 2$, let ϕ_i be an acyclic retraction from D to H_i. Then $\phi_1|_{H_2}$ is an acyclic homomorphism from H_2 to H_1 and similarly there exists an acyclic homomorphism from H_1 to H_2. Therefore, by the preceding lemma, $H_1 \cong H_2$. $\square$

In the remainder of this paper we will always use $D^\bullet$ for 'core of D' as is done for the graph-theoretic analogue in [4].

Lemma 17. *Cores of connected digraphs are connected.*

Proof. Let D be a connected digraph and φ a retraction to $D^\bullet$. Suppose that $x, y \in V(D^\bullet)$. Then x, y are vertices of D because φ is a retraction. Since D is connected there exists a sequence of vertices $x = u_1, u_2, \ldots, u_n = y$ in D such that for all $i \in [n-1]$ we have $u_i u_{i+1} \in A(D)$ or $u_{i+1} u_i \in A(D)$ (possibly both). For $i \in [n]$, define $v_i \coloneqq \varphi(u_i)$. The fact that φ is a retraction implies that $v_1 = x$, $v_n = y$, and the sequence of vertices $v_1, v_2, \ldots, v_n$ has the property that all $i \in [n-1]$ satisfy $v_i v_{i+1} \in A(D)$, $v_{i+1} v_i \in A(D)$, or $v_i = v_{i+1}$. Therefore $D^\bullet$ is connected. $\square$

The following result displays one use of cores for testing homomorphic equivalence.

Lemma 18. *Two digraphs are homomorphically equivalent if and only if their cores are isomorphic.*

Proof. Clearly, a digraph and its core are homomorphically equivalent. The sufficiency of the condition follows. For necessity, let $D^\bullet$ and $H^\bullet$ be cores of the digraphs D and H respectively. Assuming D and H are homomorphically equivalent, we have that $D^\bullet$ is homomorphic to D, D is homomorphic to H, and H is homomorphic to $H^\bullet$. Thus $D^\bullet$ is homorphic to $H^\bullet$ using the fact that the composition of acyclic homomorphisms is an acyclic homomorphism. Similarly $H^\bullet$ is homomorphic to $D^\bullet$. Hence by Lemma 15, $H^\bullet$ and $D^\bullet$ are isomorphic. $\square$

Earlier we defined a digraph H to be a core if it is uniquely H-colorable. In fact we will see that there is a looser condition governing whether H is a core. The next result shows that if we find any digraph which is uniquely H-colorable, then H is a core.

Lemma 19. *If there exists a uniquely H-colorable digraph, then H must be a core.*

Proof. Let D be uniquely H-colorable and $\phi : V(D) \to V(H)$ a surjective acyclic homomorphism. Now suppose that $\psi : V(H) \to V(H^\bullet)$ is an acyclic retraction and hence $\psi \circ \phi : V(D) \to V(H)$ is an acyclic homomorphism. Thus $\psi \circ \phi = \pi \circ \phi$ for some $\pi \in \operatorname{Aut}(H)$, since D is uniquely H-colorable. Now since ϕ is surjective, $\operatorname{Im}(\pi \circ \phi) = V(H)$. This implies that $\operatorname{Im}(\psi \circ \phi) = V(H)$. But $\operatorname{Im}(\psi) = V(H^\bullet)$, so we've shown that $V(H) \subseteq V(H^\bullet)$. Since the reverse containment is always true $V(H) = V(H^\bullet)$ and thus we conclude that H is a core. $\square$

In the proof of the following theorem we use the notion of directed distance in a digraph. For a digraph D and $u, v \in V$ the directed distance from u to v, denoted $\overrightarrow{\operatorname{dist}(u,v)}$, is the length of the shortest directed path from u to v in D.

Theorem 20. *For any given integers $n > 1$ and $k > 2$ the digraph D_n constructed in Theorem 1 is a core with $\vec{g}(D_n) = k$.*

Proof. It is clear that D_2 is a core since $D_2 \cong \vec{C}_k$ so that we may proceed by induction. Assume that D_n is a core with $n \geqslant 2$ and let $\varphi : V(D_{n+1}) \to V(D_{n+1})$ be a retraction. Define Γ to be the image of φ and thus our goal is to show that $\Gamma = V(D_{n+1})$. Since φ is a retraction and Γ induces a subdigraph of D_{n+1}, the function φ must map k-cycles (i.e. shortest cycles) to k-cycles. Thus

$$\text{if two vertices } u, v \text{ are in the same } k\text{-cycle then } \varphi(u) \neq \varphi(v). \tag{2}$$

Thus Lemma 2 implies that all arcs uv of D_{n+1} satisfy $\varphi(u) \neq \varphi(v)$. In other words φ is a non-contracting homomorphism (defined on p. 2).

We first proceed to show that for the subdigraph $\Sigma = D_{n+1}\left[\bigcup_{i=1}^{m} V(P_{d_i})\right]$ of D_{n+1}, the image of φ restricted to $V(\Sigma)$ is contained in $V(\Sigma)$. Let $u \in V(P_{d_i})$ and $v \in V(D_n^j)$, for some i, j, and we will show that $\varphi(u) \neq v$. This and the fact that D_n is strongly connected will suffice to show that $\varphi(V(\Sigma)) \subseteq V(\Sigma)$, for we can repeat our argument below as necessary to force every such $\varphi(u)$ into $V(\Sigma)$. If $i = j$, then the construction of D_{n+1} puts u and v together in a k-cycle and hence they cannot be in the same fiber of φ. This proves that $\varphi(u) \neq v$ for otherwise, with φ being the identity on Γ, we'd have $\varphi(u) = \varphi \circ \varphi(u) = \varphi(v)$, contradicting (2). Notice that there exists a directed path from

d_i to d_j for all $i, j \in [m]$ because D_n is strongly connected. For the case $i \neq j$, we proceed by induction on the distance, s, from d_i to d_j in D_n. Assume that this distance is $s+1$ and that for every $r \in [m]$ with $0 \leqslant \overrightarrow{\text{dist}(d_i, d_r)} \leqslant s$ we have $\varphi(u) \neq z$ for all $z \in V(D_n^r)$. By assumption there is a path $P = (d_i, d_{i+1}, \ldots, d_{i+s}, d_j)$ in D_n which by our construction implies that $u(d_{i+1}, p_\ell)$ is an arc in D_{n+1} for all $\ell \in [k-1]$. Hence $\varphi(u) \neq \varphi((d_{i+1}, p_\ell))$ for all $\ell \in [k-1]$ because φ is a non-contracting homomorphism. Similarly if $\varphi(u) = v$ then $v\varphi(d_{i+1}, p_\ell) \in A(D_{n+1})$ for all $\ell \in [k-1]$. However the induction hypothesis implies that $\varphi((d_{i+1}, p_\ell)) \notin V(D_n^j)$ which forces $\varphi((d_{i+1}, p_\ell))$ to be (d_j, p_1) for every $\ell \in [k-1]$. This cannot happen because $k > 2$ and (d_{i+1}, p_ℓ) and (d_{i+1}, p_t) are in a k-cycle together for all $\ell, t \in [k-1]$ with $\ell \neq t$. Therefore the restriction of φ to $V(\Sigma)$ is indeed contained in $V(\Sigma)$.

The next step is to show that $D_{n+1}[\Gamma]$ cannot be a subdigraph of $D_{n+1} - H(n,i)$ for any $i \in [m]$. By way of contradiction we assume that there is an i such that $D_{n+1}[\Gamma]$ is a subdigraph of $D_{n+1} - H(n,i)$. Choose exactly one vertex v_j from each P_{d_j} and define Λ to be the subdigraph of D_{n+1} induced by the v_j's which by our construction is isomorphic to D_n. Define $\psi : V(\Sigma) \to V(\Lambda)$ by $\psi((d_\ell, p_s)) = v_\ell$ for all $\ell \in [m]$ and $s \in [k-1]$. It is easy to check that ψ is an acyclic homomorphism. Consider the homomorphism $\psi|_\zeta \circ \varphi|_{V(\Lambda)} : V(\Lambda) \to V(\Lambda)$ where $\zeta := \text{Im}(\varphi|_{V(\Lambda)})$. (Note that these restrictions compose because ψ is defined on $\zeta \subset \text{Im}(\varphi|_{V(\Sigma)}) \subseteq V(\Sigma)$.) Since we're under the assumption that the image of φ is contained in $V(D_{n+1} - H(n,i))$, there exists an $\alpha \in [m]$ such that $\varphi(v_\alpha) \neq v_\alpha$. Thus v_α is not in the image of $\psi|_\zeta \circ \varphi|_{V(\Lambda)}$ which implies that $\psi|_\zeta \circ \varphi|_{V(\Lambda)}$ is not a bijection. Thus by Lemma 12 the digraph Λ is not a core. This contradicts the induction hypothesis that D_n is a core because Λ is isomorphic to D_n. Therefore $D_{n+1}[\Gamma]$ cannot be a subdigraph of $D_{n+1} - H(n,i)$ for any $i \in [m]$.

We now show that

$$\text{for all } i \in [m] \text{ there exists a } j \in [k-1] \text{ such that } \varphi((d_i, p_j)) = (d_i, p_j) \in \Gamma. \tag{3}$$

By way of contradiction assume that Γ is contained in $V(D_{n+1} - P_{d_i})$. This follows from the negation of (3) because $\text{Im}(\varphi|_{V(\Sigma)}) \subseteq V(\Sigma)$ and the definition of retraction implies that

$$\text{for any vertices } u, v \in V(D_{n+1}), \text{ if } \varphi(u) = v \text{ then } \varphi(v) = v.$$

Notice that for all $\alpha \in V(D_n^i)$ the arc $\varphi(\alpha)\varphi((d_i, p_1))$, $(d_i, p_1) \in V(D_{n+1})$, is in $D_{n+1}[\Gamma]$ because $\alpha(d_i, p_1)$ is an arc in D_{n+1}. However, in the construction of D_{n+1}, the only arcs from D_n^i to Σ are those from D_n^i to (d_i, p_1). Thus $\varphi(\alpha) \in V(D_{n+1} - H(n,i))$ for every $\alpha \in V(D_n^i)$. However this and the fact that φ is a retraction imply that Γ is contained in $V(D_{n+1} - H(n,i))$ which contradicts the preceding paragraph. Thus we've established (3). This fact will now allow us to show that for every $i \in [m]$ the vertices (d_i, p_1) and (d_i, p_{k-1}) are sent to the same P_d by φ. We again proceed by way of contradiction and assume that for two distinct vertices u_1, u_{k-1} of D_n, we have $\varphi((d_i, p_1)) \in V(P_{u_1})$ and $\varphi((d_i, p_{k-1})) \in V(P_{u_{k-1}})$. Considering an $\alpha \in V(D_n^i)$ we see that both $\varphi((d_i, p_{k-1}))\varphi(\alpha)$ and $\varphi(\alpha)\varphi((d_i, p_1))$ must be in the arc set of $D_{n+1}[\Gamma]$ since φ is a non-contracting homomorphism. The preceding sentence and our construction of D_{n+1} thus imply that

$\varphi(\alpha) \in V(\Sigma)$, say $\varphi(\alpha) \in V(P_{u_0})$, for some $u_0 \in V(D_n)$, because $u_1 \neq u_{k-1}$. Similarly $\big(\varphi(\alpha), \varphi((d_i, p_1)), \varphi((d_i, p_2)), \ldots, \varphi((d_i, p_{k-1}))\big)$ is a $k-$cycle in $D_{n+1}[\Gamma]$. This implies that $\big(u_0, u_1, \ldots, u_{k-1}\big)$, where $\varphi((d_i, p_j)) \in V(P_{u_j})$, is a cycle in D_n. We may thus assume that the u_j's are distinct for otherwise $\big(u_0, u_1, \ldots, u_{k-1}\big)$ would contain a cycle of length less than k. Now (3) shows that there exists $j \in [k-1]$ such that $u_j = d_i$. If we let $\ell = j - 1 \pmod{k}$, then the preceding sentence implies that $u_\ell d_i \in A(D_n)$. Thus for all $r, s \in [k-1]$ the arc $(u_\ell, p_r)(d_i, p_s)$ is in D_{n+1} and (3) implies that $\varphi((d_i, p_\ell)) = (u_\ell, p_t)$ for some $t \in [k-1]$. Hence (u_ℓ, p_t) is in the image of φ implying that $\varphi((u_\ell, p_t)) = (u_\ell, p_t)$ because φ is a retraction (φ is the identity on its image). This contradicts the fact that φ is a non-contracting homomorphism. Thus for every $i \in [m]$ the vertices (d_i, p_1) and (d_i, p_{k-1}) are indeed sent to the same P_d by φ.

We next show that in fact for every $i \in [m]$ all the vertices of P_{d_i} are sent to the same P_d by φ for some $d \in V(D_n)$. By way of contradiction assume that some $s \in [k-1]$ and $\ell \in \{2, 3, \ldots, k-2\}$ satisfy $\varphi((d_i, p_\ell)) = (v, p_s)$ while $\varphi((d_i, p_1)) \in V(P_u)$, where u and v are distinct vertices of D_n. The preceding paragraph implies that (d_i, p_{k-1}) is sent to P_u as well. As P_{d_i} is a directed path of length $k-2$, the image under φ of $V(P_{d_i})$ induces a directed path of length not exceeding $k-2$. However since $\varphi((d_i, p_1)), \varphi((d_i, p_{k-1})) \in V(P_u)$ and $u \neq v$, the preceding sentence implies that there is a cycle in D_n containing u and v which has length less than k. This contradiction lets us deduce that for every $i \in [m]$ all the vertices of P_{d_i} are sent to P_d by φ for some $d \in V(D_n)$. This implies that the restriction of φ to $V(\Sigma)$ is the identity because of (3). For $u \in V(D_n^i)$, the image $\varphi(u)$ lies in neither D_n^j, for $j \neq i$, nor P_{d_i} because we already know that φ fixes P_{d_i}. Also, $\varphi(u) \in P_{d_j}$ with $d_j \neq d_i$ implies that $d_j d_i$ and $d_i d_j$ are arcs in D_n because $u(d_i, p_1)$ and $(d_i, p_{k-1})u$ are arcs in D_{n+1}. Hence we see that for all $u \in V(D_n^i)$ the image $\varphi(u)$ lies in D_n^i. This implies that the restriction of φ to each $V(D_n^i)$ is a retraction of D_n^i to itself. Since each D_n^i is isomorphic to D_n, a core by induction, each D_n^i is also a core. Thus, the restriction of φ to each $V(D_n^i)$, for $i \in [m]$, is the identity. Therefore, we've finally established that $\Gamma = V(D_{n+1})$ and hence by induction reached the conclusion that the D_n's are cores. □

Hence we have in fact constructed highly chromatic cores without short cycles. It would be of interest to construct graphs with the same properties. In the next section we give a nod to the elegant construction of highly chromatic triangle-free graphs by Mycielski, see e.g. [9], by constructing analogous digraphs in a similar manner. We note that the Mycielski Graph with chromatic number 4, M_4, is an example of a graph core without triangles. The truth of this fact comes to light via the minimality of M_4 proved in [3].

4 Highly chromatic digraphs with digirth three

We conclude this paper with a construction of digraphs with arbitrarily large chromatic number and digirth at least three (i.e. without bi-cycles). Although it is not as general (it does not work for larger digirth) as the construction appearing in [13], we decided to include it for it's simplicity and the fact that it is a directed analogue of Mycielski's construction. We also ask the question: does there exist a construction of digraphs retaining the desired properties with less vertices or arcs? It should be noted that in the case where the chromatic number is 3, non-isomorphic examples with the same number of vertices and arcs as the construction which follows appear in [6].

The construction will be inductive and similar to the construction appearing in [13]. We will again start with the directed three cycle and call it A_2. We now suppose A_n has chromatic number n and digirth 3 and construct A_{n+1} from A_n. We will let P_1 be the path of length one with $V(P_1) = \{p_1, p_2\}$ and $A(P_1) = \{p_1p_2\}$. Let $V(A_{n+1}) = V(A_n \circ \overrightarrow{P}_1) \bigcup v$ and

$$A(A_{n+1}) = A(A_n \circ \overrightarrow{P}_1) \bigcup \{v(a, p_1) \,|\, a \in V(A_n)\} \bigcup \{(a, p_2)v \,|\, a \in V(A_n)\}.$$

We first show that the digirth of A_{n+1} is three. The following lemma will be useful in attaining that goal.

Lemma 21. *If M_2 is an acyclic digraph then the projection $\pi_{M_1} : V(H) \to V(M_1)$ is a homomorphism for every subdigraph H of $M_1 \circ M_2$.*

Proof. Let $(u, v)(x, y)$ be an arc of H. If $u \neq x$ then $ux = \pi_{M_1}((u, v))\pi_{M_1}((x, y))$ is an arc of M_1. Thus we assume that $u = x$ which implies that vy is an arc of M_2 and $\pi_{M_1}((u, v)) = \pi_{M_1}((x, y)) = u$. Since the subdigraph of $M_1 \circ M_2$ induced by the vertex set $F_u = \{(u, w)|w \in V(M_2)\}$ is isomorphic to M_2, the set $\pi^{-1}_{M_1}(u)$ is acyclic in H for all $u \in V(M_1)$. Therefore the projection π_{M_1} is a homomorphism. $\square$

We will show that $\chi(A_{n+1}) = n+1$ and A_{n+1} has no digons. We proceed by induction. We see from the construction that the vertex v is not a part of any bi-cycles. Thus it suffices to show that $\overrightarrow{g}(A_n \circ P_1) = 3$. Indeed Lemma 21 gives the homomorphism $\pi_{A_n} : V(A_n \circ P_1) \to V(A_n)$ and thus $\overrightarrow{g}(A_n \circ P_1) \geqslant 3$. Therefore $\overrightarrow{g}(A_n \circ P_1) = 3$ as it is easy to see from the construction that $A_n \circ P_1$ contains a number of directed three cycles because, by induction, $\overrightarrow{g}(A_n) = 3$.

It is clear that $\chi(H_{n+1}) \geqslant \chi(A_n) = n$ since A_n is isomorphic to a subdigraph of A_{n+1}. If A_{n+1} is n-chromatic, then there exists an acyclic homomorphism $\sigma : V(A_{n+1}) \to V(\overleftrightarrow{K}_n)$. To set up the contradiction we are about to derive, fix a σ 'color' $\alpha \in V(\overleftrightarrow{K}_n)$ and suppose $\sigma(v) = \alpha$. Notice that v is in a cycle with the vertices of the subdigraph induced by $\{a\} \times V(P_1)\ \forall a \in V(A_n)$, which implies that there exists a vertex $u_a \in \{a\} \times V(P_1)$ such that $\sigma(u_a) \neq \alpha$. The subdigraph Λ of A_{n+1} induced by $\{u_a \,|\, a \in V(A_n)\}$ is isomorphic to A_n. This contradicts the fact that A_n has chromatic number n since σ, now seen to avoid α on $V(\Lambda)$, effectively maps $V(\Lambda)$ to $V(\overleftrightarrow{K}_{n-1})$ acyclically. Thus $\chi(A_{n+1}) \geqslant n+1$. We now show that $\chi(A_{n+1}) = n+1$ by giving an acyclic homomorphism from A_{n+1} to $\overleftrightarrow{K}_{n+1}$. Let ζ be an acyclic homomorphism from A_n to $\overleftrightarrow{K}_n$. Define a mapping $\phi : V(A_{n+1}) \to V(\overleftrightarrow{K}_{n+1})$ by $\phi((a, p)) = \zeta(a)$ and $\phi(v) = n + 1$.

As the target digraph of ϕ is complete, to show that ϕ is an acyclic homomorphism, it will suffice to show that each fiber of ϕ is acyclic. Since $\phi^{-1}(n+1) = \{v\}$ it suffices to show that the restriction of ϕ to $A_n \circ P_1$ is an acyclic homomorphism. Notice that $\phi|_{A_n \circ P_1} = \zeta \circ \pi_{A_n}$. As ζ and π_{A_n} are acyclic homomorphisms, so too is their composition $\phi|_{A_n \circ P_1}$. Therefore ϕ is an acyclic homomorphism which finally implies that $\chi(A_{n+1}) = n+1$.

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