Maximal partial spreads of polar spaces

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Abstract

Some constructions of maximal partial spreads of finite classical polar spaces are provided. In particular we show that, for $n \geqslant 1$, $\mathcal{H}(4n-1, q^2)$ has a maximal partial spread of size $q^{2n}+1$, $\mathcal{H}(4n+1, q^2)$ has a maximal partial spread of size $q^{2n+1}+1$ and, for $n \geqslant 2$, $\mathcal{Q}^+(4n-1, q)$, $\mathcal{Q}(4n-2, q)$, $\mathcal{W}(4n-1, q)$, q even, $\mathcal{W}(4n-3, q)$, q even, have a maximal partial spread of size $q^n + 1$.

Keywords: finite classical polar space, maximal partial spread, Singer cycle, Segre variety

1 Introduction

Let $\mathcal{P}$ be a finite classical polar space, i.e., $\mathcal{P}$ arises from a vector space of finite dimension over a finite field equipped with a non–degenerate reflexive sesquilinear form. Hence $\mathcal{P}$ is a member of one of the following classes: a symplectic space $\mathcal{W}(2n+1, q)$, a quadric $\mathcal{Q}(2n, q)$, $\mathcal{Q}^+(2n+1, q)$, $\mathcal{Q}^-(2n+1, q)$ or a Hermitian variety $\mathcal{H}(n, q^2)$. A projective subspace of maximal dimension contained in $\mathcal{P}$ is called a *generator* of $\mathcal{P}$.

For further details on finite classical polar spaces we refer to [9],[14].

Definition 1. A **partial spread** $\mathcal{S}$ of $\mathcal{P}$ is a set of pairwise disjoint generators. A partial spread is said to be **maximal** if it is maximal with respect to set–theoretic inclusion. A partial spread $\mathcal{S}$ is called a **spread** if $\mathcal{S}$ partitions the point set of $\mathcal{P}$.

If a polar space does not admit spreads, the question on the size of a maximal partial spread in such a space naturally arises. In general constructing maximal partial spreads and obtaining reasonable upper and lower bounds for the size of such partial spreads is an interesting problem. Recently maximal partial spreads of symplectic polar spaces received particular attention due to their applications in quantum information theory. In fact they correspond to so–called weakly unextendible mutually unbiased bases [13],[18]. In this paper we are interested in maximal partial spreads of $\mathcal{H}(2n+1, q^2)$, $\mathcal{Q}^+(4n-1, q)$, $\mathcal{Q}(4n-2, q)$, $n \geqslant 2$, for any q and of $\mathcal{W}(2n+1, q)$, $n \geqslant 2$, when q is even. In particular we show that, for $n \geqslant 1$, $\mathcal{H}(4n-1, q^2)$ has a maximal partial spread of size $q^{2n}+1$, $\mathcal{H}(4n+1, q^2)$ has a maximal partial spread of size $q^{2n+1}+1$ and, for $n \geqslant 2$, $\mathcal{Q}^+(4n-1, q)$, $\mathcal{Q}(4n-2, q)$, $\mathcal{W}(4n-1, q)$, q even, $\mathcal{W}(4n-3, q)$, q even, have a maximal partial spread of size q^n+1. These results are obtained by investigating particular Segre varieties $\mathcal{S}_{1,n}$ "embedded" in polar spaces of Hermitian or hyperbolic type.

2 The geometric setting

We will use the term n-space to denote an n–dimensional projective subspace of the ambient projective space. Also in the sequel we will use the following notation $\theta_{n,q} := \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q = q^n + \cdots + q + 1$.

2.1 Linear representations and spreads of projective spaces

Let (V, k) be a non–degenerate formed space with associated polar space $\mathcal{P}$ where V is a $(d+1)$–dimensional vector space over $\mathrm{GF}(q^e)$ and k is a sesquilinear (quadratic) form. The vector space V can be considered as an $(e(d+1))$–dimensional vector space V' over $\mathrm{GF}(q)$ via the inclusion $\mathrm{GF}(q) \subset \mathrm{GF}(q^e)$. Composition of k with the trace map $Tr : z \in \mathrm{GF}(q^e) \mapsto \sum_{i=1}^{e} z^{q^i} \in \mathrm{GF}(q)$ provides a new form k' on V' and so we obtain a new formed space (V', k'). If our new formed space (V', k') is non–degenerate, then it has an associated polar space $\mathcal{P}'$. The isomorphism types and various conditions are presented in [10], [7]. Now each point in $\mathrm{PG}(d, q^e)$ corresponds to a 1–dimensional vector subspace in V, which in turn corresponds to an e–dimensional vector subspace in V', that is an $(e-1)$–space of $\mathrm{PG}(e(d+1)-1, q)$. Extending this map from points of $\mathrm{PG}(d, q^e)$ to subspaces of $\mathrm{PG}(d, q^e)$, we obtain an injective map from subspaces of $\mathrm{PG}(d, q^e)$ to certain subspaces of $\mathrm{PG}(e(d+1)-1, q)$:

$$\phi : \mathrm{PG}(d, q^e) \to \mathrm{PG}(e(d+1)-1, q).$$

The map ϕ is called the $\mathrm{GF}(q)$–*linear representation* of $\mathrm{PG}(d, q^e)$.

A t*–spread* of a projective space $\mathbf{P}$ is a collection $\mathcal{S}$ of mutually disjoint t–spaces of $\mathbf{P}$ such that each point of $\mathbf{P}$ is contained in an element of $\mathbf{P}$. The set $\mathcal{D} = \{\phi(P) \mid P \in \mathrm{PG}(d, q^e)\}$ is an example of $(e-1)$–spread of $\mathrm{PG}(e(d+1)-1, q)$, called a *Desarguesian spread* (see [15], Section 25). The incidence structure whose points are the elements of $\mathcal{D}$ and whose lines are the $(2e-1)$–spaces of $\mathrm{PG}(e(d+1)-1, q)$ joining two distinct elements of $\mathcal{D}$, is isomorphic to $\mathrm{PG}(d, q^e)$. One immediate consequence of the definitions is that the image of the pointset of the original polar space $\mathcal{P}$ is contained in the new polar space $\mathcal{P}'$ (but is not necessarily equal to it).

2.2 Segre varieties $\mathcal{S}_{1,n}$

Consider the map defined by

$$\sigma : \mathrm{PG}(1, q) \times \mathrm{PG}(n, q) \to \mathrm{PG}(2n + 1, q),$$

taking a pair of points $x = (x_1, x_2)$ of $\mathrm{PG}(1, q)$, $y = (y_1, \ldots, y_n, y_{n+1})$ of $\mathrm{PG}(n, q)$ to their product $(x_1y_1, \ldots, x_2y_{n+1})$. This is a special case of a wider class of maps called *Segre maps* [9]. The image of σ is an algebraic variety called the *Segre variety* and denoted by $\mathcal{S}_{1,n}$. The Segre variety $\mathcal{S}_{1,n}$ has two rulings, say $\mathcal{R}_1$ and $\mathcal{R}_2$, where $\mathcal{R}_1$ contains $\theta_{n,q}$ lines and $\mathcal{R}_2$ consists of $q+1$ n–spaces such that two subspaces in the same ruling are disjoint, and each point of $\mathcal{S}_{1,n}$ is contained in exactly one member of each ruling. Also, a member of $\mathcal{R}_1$ meets an element of $\mathcal{R}_2$ in exactly one point.

Notice that the set $\mathcal{R}_1$ consists of all the lines of $\mathrm{PG}(2n + 1, q)$ incident with three distinct members of $\mathcal{R}_2$ and, from [9, Theorem 25.6.1], three mutually disjoint n–spaces of $\mathrm{PG}(2n + 1, q)$ define a unique Segre variety $\mathcal{S}_{1,n}$. A line of $\mathrm{PG}(2n + 1, q)$ shares with $\mathcal{S}_{1,n}$ $0, 1, 2$ or $q + 1$ points. Also, the automorphism group of $\mathcal{S}_{1,n}$ in $\mathrm{PGL}(2n + 2, q)$ is a group isomorphic to $\mathrm{PGL}(2, q) \times \mathrm{PGL}(n+1, q)$ [9, Theorem 25.5.13]. For more details on Segre varieties, see [9].

2.3 $\mathcal{S}_{1,n}$ and Hermitian varieties

Let $\mathcal{H}(2n + 1, q^2)$ be a Hermitian variety of $\mathrm{PG}(2n + 1, q^2)$ and let $G \simeq \mathrm{PGU}(2n + 2, q^2)$ be the stabilizer of $\mathcal{H}(2n+1, q^2)$ in $\mathrm{PGL}(2n+2, q^2)$. Let g_1, g_2, g_3 be three mutually skew generators of $\mathcal{H}(2n + 1, q^2)$. We recall the following lemma due to J.A. Thas [17, p. 538].

Lemma 2. *The points of g_1, that lie on a line of $\mathcal{H}(2n + 1, q^2)$ intersecting g_2 and g_3, form a Hermitian variety $\mathcal{H}(n, q^2)$ in g_1.*

Remark 3. If $\perp$ denotes the unitary polarity of $\mathrm{PG}(2n + 1, q^2)$ induced by $\mathcal{H}(2n + 1, q^2)$ and $\perp'$ the unitary polarity of g_1 induced by $\mathcal{H}(n, q^2)$, then $\perp'$ is defined as follows. Let P be a point of g_1 and let ℓ be the unique line of $\mathrm{PG}(2n + 1, q^2)$ through P intersecting g_2 and g_3, too. Then $P^{\perp'} = g_1 \cap \ell^{\perp}$.

From [10, Table 4.2.A], the stabilizer of g_1 and g_2 in G contains a subgroup K isomorphic to $\mathrm{PGL}(n + 1, q^2)$ inducing a group isomorphic to $\mathrm{PGL}(n + 1, q^2)$ on both g_1 and g_2. If $\mathcal{S}_{1,n}$ denotes the unique Segre variety containing g_1, g_2, g_3, then it follows that, among the θ_{n,q^2} lines of $\mathcal{R}_1$, there are $|\mathcal{H}(n, q^2)|$ lines that are contained in $\mathcal{H}(2n + 1, q^2)$ and $\theta_{n,q^2} - |\mathcal{H}(n, q^2)|$ that are $(q + 1)$–secant to $\mathcal{H}(2n + 1, q^2)$. Hence, we have that $q + 1$ members of $\mathcal{R}_2$ are generators of $\mathcal{H}(2n + 1, q^2)$ and the remaining $q^2 - q$ elements of $\mathcal{R}_2$ meet $\mathcal{H}(2n + 1, q^2)$ in a Hermitian variety $\mathcal{H}(n, q^2)$.

If $\mathcal{H}_{1,n} := \mathcal{S}_{1,n} \cap \mathcal{H}(2n + 1, q^2)$, it turns out that

$$|\mathcal{H}_{1,n}| = (q + 1)\theta_{n,q^2} + (q^2 - q)|\mathcal{H}(n, q^2)|.$$

We will refer to $\mathcal{H}_{1,n}$ as *Hermitian Segre variety* of $\mathcal{H}(2n+1,q^2)$.

Let H be the subgroup of K fixing $\mathcal{H}_{1,n}$. Notice that $|H| \leqslant |\mathrm{PGU}(n+1,q^2)|$. From [11, Corollary 12] the number of generators of $\mathcal{H}(2n+1,q^2)$ disjoint from g_1 and g_2 equals

$$q^{\frac{n(n+1)}{2}} \prod_{i=1}^{n+1} (q^i + (-1)^i).$$

On the other hand, the number of Hermitian varieties $\mathcal{H}(n,q^2)$ in g_1 equals the index of $\mathrm{PGU}(n+1,q^2)$ in $\mathrm{PGL}(n+1,q^2)$, that is

$$|\mathrm{PGL}(n+1,q^2) : \mathrm{PGU}(n+1,q^2)| = q^{\frac{n(n+1)}{2}} \prod_{i=2}^{n+1} (q^i + (-1)^i).$$

Therefore we may conclude that $H \simeq \mathrm{PGU}(n+1,q^2)$. We have proved the following proposition which shows that the converse of Thas's lemma holds true.

Proposition 4. *There exists a one to one correspondence between the set of Hermitian Segre varieties $\mathcal{H}_{1,n}$ of $\mathcal{H}(2n+1,q^2)$ containing g_1 and g_2 and the set of Hermitian varieties $\mathcal{H}(n,q^2)$ of g_1.*

Remark 5. From [10, Table 4.4.A] the stabilizer of $\mathcal{H}_{1,n}$ in G is isomorphic to $\mathrm{PGU}(2,q^2) \times \mathrm{PGU}(n+1,q^2)$.

We will need the following lemma.

Lemma 6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two distinct Hermitian Segre varieties of $\mathcal{H}(2n+1,q^2)$ containing g_1 and g_2. Let $\mathcal{H}_i$ be the Hermitian variety of g_1 determined by $\mathcal{S}_i$ and let $\perp_i$ be the unitary polarity of g_1 induced by $\mathcal{H}_i$, $i = 1,2$. Then $|\mathcal{S}_1 \cap \mathcal{S}_2| = 2\theta_{n,q^2} + x_1(q^2-1) + x_2(q-1)$, where $x_1 = |\{P \in \mathcal{H}_1 \cap \mathcal{H}_2 \mid P^{\perp_1} = P^{\perp_2}\}|$ and $x_2 = |\{P \in g_1 \setminus (\mathcal{H}_1 \cap \mathcal{H}_2) \mid P^{\perp_1} = P^{\perp_2}\}|$.*

Proof. Let $\mathcal{S}_i'$ be the Segre variety of the ambient projective space such that $\mathcal{S}_i = \mathcal{S}_i' \cap \mathcal{H}(2n+1,q^2)$, $i=1,2$. Notice that if $P \notin g_1 \cup g_2$, then there exists a unique line ℓ through P meeting both g_1 and g_2. Therefore if $P \in \mathcal{S}_1' \cap \mathcal{S}_2'$, then $\ell \subset \mathcal{S}_1' \cap \mathcal{S}_2'$. Let $P_i = \ell \cap g_i$, $i=1,2$ and assume that $P \in \mathcal{S}_1 \cap \mathcal{S}_2$. Then $P_1^{\perp_1} = \ell^{\perp} \cap g_1 = P^{\perp} \cap g_1 = P_1^{\perp_2}$. On the other hand, assume that there exists a point $P_1 \in g_1$ such that $P_1^{\perp_1} = P_1^{\perp_2}$. Let ℓ_i' be the unique line of $\mathcal{S}_i'$, $i=1,2$, passing through P_1 and having a point in common with g_2. Then $(P_1^{\perp_i})^{\perp} \cap g_2$ coincides with the point $g_2 \cap \ell_i'$, $i=1,2$. Therefore we have $g_2 \cap \ell_1' = g_2 \cap \ell_2'$ and hence $\ell_1' = \ell_2'$. The result now follows from the fact that if $\ell \subset \mathcal{S}_1' \cap \mathcal{S}_2'$ is a line meeting both g_1 and g_2, then $|\ell \cap \mathcal{H}(2n+1,q^2)| = q^2+1$ if and only if $\ell \cap g_1 \in \mathcal{H}_1 \cap \mathcal{H}_2$. $\square$

2.4 $\mathcal{S}_{1,n}$ and hyperbolic quadrics

Let $\mathcal{Q}^+(4n-1,q)$ be a hyperbolic quadric of $\mathrm{PG}(4n-1,q)$ and let $G \simeq \mathrm{PGO}^+(4n,q)$ be the stabilizer of $\mathcal{Q}^+(4n-1,q)$ in $\mathrm{PGL}(4n,q)$. Let g_1,g_2,g_3 be three mutually skew generators of $\mathcal{Q}^+(4n-1,q)$. We recall the following lemma due to A. Klein, K. Metsch and L. Storme [11, Theorem 6].

Lemma 7. *The lines of g_1 lying in a solid of $\mathcal{Q}^+(4n-1,q)$ that intersects g_2, g_3 in a line, are those of a symplectic space $\mathcal{W}(2n-1,q)$ in g_1.*

From [10, Table 4.2.A], the stabilizer of g_1 and g_2 in G contains a subgroup K isomorphic to $\mathrm{PGL}(2n,q)$ fixing both g_1 and g_2. If $\mathcal{S}_{1,2n-1}$ denotes the unique Segre variety containing g_1, g_2, g_3, then it follows that the $\theta_{2n-1,q}$ lines of $\mathcal{R}_1$ are contained in $\mathcal{Q}^+(4n-1,q)$. Hence, we have that the $q+1$ members of $\mathcal{R}_2$ are generators of $\mathcal{Q}^+(4n-1,q)$. In particular $\mathcal{S}_{1,2n-1} \subset \mathcal{Q}^+(4n-1,q)$ and, in this case, we will refer to $\mathcal{S}_{1,2n-1}$ as *hyperbolic Segre variety* of $\mathcal{Q}^+(4n-1,q)$.

Let H be the subgroup of K fixing a hyperbolic Segre variety of $\mathcal{Q}^+(4n-1,q)$. Notice that $|H| \leqslant |\mathrm{PSp}(2n,q)|$. From [11, Corollary 5] the number of generators of $\mathcal{Q}^+(4n-1,q)$ disjoint from g_1 and g_2 equals

$$q^{n(n-1)}\prod_{i=1}^{n}(q^{2i-1}-1).$$

On the other hand, the number of symplectic polar space $\mathcal{W}(2n-1,q)$ of g_1 is equal to

$$|\mathrm{PGL}(2n,q) : \mathrm{PSp}(2n,q)| = q^{n(n-1)}\prod_{i=2}^{n}(q^{2i-1}-1).$$

Therefore we may conclude that $H \simeq \mathrm{PSp}(2n,q)$. We have proved the following proposition which shows that the converse of Lemma 7 holds true.

Proposition 8. *There exists a one to one correspondence between the set of hyperbolic Segre varieties of $\mathcal{Q}^+(4n-1,q)$ containing g_1 and g_2 and the set of symplectic spaces $\mathcal{W}(2n-1,q)$ of g_1.*

Remark 9. From [10, Table 4.4.A] the stabilizer of a hyperbolic Segre variety in G is isomorphic to $\mathrm{PSp}(2,q) \times \mathrm{PSp}(2n,q)$.

Arguing as in Lemma 6, we obtain the following.

Lemma 10. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two distinct hyperbolic Segre varieties of $\mathcal{Q}^+(4n-1,q)$ containing g_1 and g_2. Let $\mathcal{W}_i$ be the symplectic polar space of g_1 determined by $\mathcal{S}_i$ and let $\perp_i$ be the symplectic polarity of g_1 induced by $\mathcal{W}_i$, $i=1,2$. Then $|\mathcal{S}_1 \cap \mathcal{S}_2| = 2\theta_{2n-1,q} + x(q-1)$, where $x = |\{P \in g_1 \mid P^{\perp_1} = P^{\perp_2}\}|$.*

3 Hermitian polar spaces

3.1 $\mathcal{H}(4n-1,q^2)$, $n \geqslant 1$

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the two distinct Hermitian varieties of $\mathrm{PG}(4n-1,q^2)$ having the following homogeneous equations

$$f_1 : X_1X_{2n+1}^q + \cdots + X_{2n}X_{4n}^q + X_1^qX_{2n+1} + \cdots + X_{2n}^qX_{4n} = 0,$$

$$f_2 : X_1X_{2n+1}^q + \cdots + X_{2n}X_{4n}^q + \omega^{q-1}(X_1^qX_{2n+1} + \cdots + X_{2n}^qX_{4n}) = 0,$$

respectively, where ω is a primitive element of $\mathrm{GF}(q^2)$. Then the Hermitian pencil $\mathcal{F}$ defined by $\mathcal{H}_1$ and $\mathcal{H}_2$ is the set of all Hermitian varieties with equations $af_1 + bf_2 = 0$, as a and b vary over the subfield $\mathrm{GF}(q)$, not both zero. Note that there are $q+1$ distinct Hermitian varieties in the pencil $\mathcal{F}$, none of which is degenerate. The set $\mathcal{X} = \mathcal{H}_1 \cap \mathcal{H}_2$ is called the base locus of $\mathcal{F}$. Since the Hermitian varieties of a pencil cover all the points of $\mathrm{PG}(4n-1, q^2)$, a counting argument shows that

$$|\mathcal{X}| = \frac{(q^{4n-2}+1)(q^{4n}-1)}{q^2-1}$$

and any two distinct varieties in $\mathcal{F}$ intersect precisely in $\mathcal{X}$. In particular $\mathcal{X}$ is a variety defined by the following equation:

$$X_1X_{2n+1}^q + \cdots + X_{2n}X_{4n}^q = 0.$$

Straightforward computations show that $\mathcal{X}$ contains the following two $(2n-1)$–dimensional projective spaces:

$$\Sigma : X_1 = \cdots = X_{2n} = 0, \Sigma' : X_{2n+1} = \cdots = X_{4n} = 0.$$

Also, through a point P of Σ (resp. Σ') there pass θ_{2n-2,q^2} lines entirely contained in $\mathcal{X}$ and these lines are contained in a generator of $\mathcal{H}(4n-1, q^2)$ meeting Σ (resp. Σ') exactly in P.

Let Π_{r-1} be an $(r-1)$–dimensional projective space of Σ, $1 \leqslant r \leqslant 2n-1$, and let $\Pi_{r-1}^{\perp}$ be the polar space of Π_{r-1} with respect to the unitary polarity $\perp$ of $\mathcal{H}_1$ (or, equivalently, $\mathcal{H}_2$). The intersection of $\Pi_{r-1}^{\perp}$ and Σ' is a $(2n-r-1)$–dimensional projective space, say Π'_{2n-r-1}. Note that $\langle \Pi_{r-1}, \Pi'_{2n-r-1} \rangle$ is a generator of $\mathcal{H}_1$ contained in $\mathcal{X}$. In particular, an $(n-1)$–dimensional subspace of Σ corresponds to a uniquely determined $(n-1)$–dimensional subspace of Σ'.

Let $\mathcal{D}$ be an $(n-1)$–spread of Σ. From the above discussion, $\mathcal{D}$ defines an $(n-1)$–spread, say $\mathcal{D}'$ of Σ'. Corresponding elements of $\mathcal{D}$ and $\mathcal{D}'$ determine a generator of any member of $\mathcal{F}$ and hence $\mathcal{D}$ ($\mathcal{D}'$) produces a partial spread $\mathcal{S}$ of $\mathcal{H}_1$ of size $|\mathcal{D}| = q^{2n}+1$.

In the next result we prove that the partial spread $\mathcal{S}$ is maximal.

Theorem 11. *Any $(n-1)$–spread $\mathcal{D}$ of $\mathrm{PG}(2n-1, q^2)$ gives rise to a maximal partial spread $\mathcal{S}$ of $\mathcal{H}(4n-1, q^2)$ of size $q^{2n}+1$.*

Proof. Let $\bar{\mathcal{S}}$ denote the pointset of $\mathcal{H}_1$ covered by $\mathcal{S}$. Let $\mathcal{T}$ be a generator of $\mathcal{H}_1$ disjoint from Σ and Σ' and let $\mathcal{H}_{1,2n-1}$ be the Hermitian Segre variety obtained by intersecting the unique Segre variety $\mathcal{S}_{1,2n-1}$ defined by Σ, Σ' and $\mathcal{T}$ with $\mathcal{H}_1$. Let $\mathcal{H}_\Sigma$ and $\mathcal{H}_{\Sigma'}$ denote the Hermitian varieties of Σ and Σ', respectively, arising from Thas's lemma. Let d be a member of $\mathcal{D}$. Then $d^\perp \cap \Sigma'$ defines a unique member d' of $\mathcal{D}'$. Here $\perp$ denotes the unitary polarity induced by $\mathcal{H}_1$. Let $L_d \subset \mathcal{R}_1$ be the set of lines of $\mathcal{S}_{1,2n-1}$ meeting d. The set L_d determines an $(n-1)$–space of Σ', say d''. From Remark (3), it turns out that d' and d'' are conjugate with respect to the polarity of Σ' induced by $\mathcal{H}_{\Sigma'}$.

Since members of $\mathcal{D}'$ are pairwise disjoint and $|\mathcal{H}_{\Sigma'}| = (q^{2n+2}-1)(q^{2n+1}+1)/(q^2-1)$, it follows that there exists at least one member X of $\mathcal{D}'$ meeting $\mathcal{H}_{\Sigma'}$ in a degenerate Hermitian variety. With respect to the unitary polarity of Σ' induced by $\mathcal{H}_{\Sigma'}$ the conjugate of X meets X in at least a point, say P, of $\mathcal{H}_{\Sigma'}$. Let ℓ be the unique line of $\mathcal{S}_{1,2n-1}$ through P. Then, on one hand ℓ meets $\mathcal{T}$ in a point Q. On the other hand, ℓ is contained $\langle d, d' \rangle$. It follows that $\mathcal{T}$ has a non–empty intersection with $\bar{\mathcal{S}}$. □

Remark 12. The automorphism group of the maximal partial spread $\mathcal{S}$ in $\mathrm{PGU}(4n, q^2)$ contains the automorphism group of $\mathcal{D}$ as a subgroup of $\mathrm{PGL}(2n, q^2)$.

Remark 13. In [1] the authors observed that the $q^{m+1}+1$ elements of a spread of $\mathcal{W}(2m+1, q)$ embedded in $\mathcal{H}(2m+1, q^2)$ extend to pairwise disjoint generators of $\mathcal{H}(2m+1, q^2)$. Maximality of partial spreads of $\mathcal{H}(2m+1, q^2)$ constructed in this way was shown for $m = 1, 2$ in [1] and for all even m in [12]. When $m = 2n-1$, $m > 1$, it is not immediately clear whether this partial spread is maximal or not. From Theorem 11 it can be deduced that if the spread of $\mathcal{W}(4n-1, q)$ arises from a spread of $\mathcal{W}(2n-1, q^2)$ in the $\mathrm{GF}(q)$–linear representation, then the "extended" partial spread of $\mathcal{H}(4n-1, q^2)$ is maximal. Indeed, if $\mathcal{L}$ is a Desarguesian line–spread of a $\mathcal{W}(4n-1, q)$ embedded in $\mathcal{H}(4n-1, q^2)$, then we may assume that Σ and Σ' play the role of director spaces of $\mathcal{L}$. Extending the elements of a spread $\mathcal{S}'$ of $\mathcal{W}(4n-1, q)$, we get an $(n-1)$–spread of Σ (Σ') if and only if $\mathcal{L}$ induces a line–spread on each of the members of $\mathcal{S}'$. Under this assumption, the partial spread of $\mathcal{H}(4n-1, q^2)$, obtained by extending the elements of $\mathcal{S}'$, coincides with the construction given above.

3.2 $\mathcal{H}(4n+1, q^2)$, $n \geqslant 1$

As already observed in Remark 13, the $q^{2n+1}+1$ elements of a spread of $\mathcal{W}(4n+1, q)$ embedded in $\mathcal{H}(4n+1, q^2)$ extend to pairwise disjoint generators of $\mathcal{H}(4n+1, q^2)$ forming a maximal partial spread of $\mathcal{H}(4n+1, q^2)$ as shown in [1] for $n = 1$ and in [12] for $n \geqslant 1$. On the other hand, it is known that the size of a partial spread in $\mathcal{H}(4n+1, q^2)$ is at most $q^{2n+1}+1$, see [6], [19], [20]. In what follows we prove that $\mathcal{H}(4n+1, q^2)$ has a maximal partial spread attaining the upper bound. We do not know if such a partial spread arises from a spread of a $\mathcal{W}(4n+1, q)$ embedded in $\mathcal{H}(4n+1, q^2)$.

From [5] we recall the following facts. Consider the projective space $\mathrm{PG}(2n, q^2)$ in its field–model representation: the points of $\mathrm{PG}(2n, q^2)$ are the nonzero elements of $\mathrm{GF}(q^{4n+2})$ and two elements $x, y \in \mathrm{GF}(q^{4n+2})$ define the same point of $\mathrm{PG}(2n, q^2)$ if and only if $x/y \in \mathrm{GF}(q^2)$. The point of $\mathrm{PG}(2n, q^2)$ represented by $x \in \mathrm{GF}(q^{4n+2}) \setminus \{0\}$ will be denoted by (x). The hyperplanes of $\mathrm{PG}(2n, q^2)$ are the sets $[u] := \{(x) \in \mathrm{GF}(q^{4n+2}) \pmod{\mathrm{GF}(q^2)} \mid Tr(ux) = 0\}$, where Tr denotes the usual trace map from $\mathrm{GF}(q^{4n+2})$ to $\mathrm{GF}(q^2)$, $u \in \mathrm{GF}(q^{4n+2}) \setminus \{0\}$. In this setting the map $\sigma : (x) \mapsto (\omega x)$, where ω is a primitive element of $\mathrm{GF}(q^{4n+2})$, is a Singer cycle of $\mathrm{PG}(2n, q^2)$.

Let $a \in \mathrm{GF}(q^{4n+2}) \setminus \{0\}$ such that $a^{q^{2n+1}-1} \in \mathrm{GF}(q^2)$. Then

$$\Pi_a : \begin{cases} (x) \mapsto [a x^{q^{2n+1}}], & x \in \mathrm{GF}(q^{4n+2}) \setminus \{0\} \\ [u] \mapsto (u^{q^{2n+1}}/a), & u \in \mathrm{GF}(q^{4n+2}) \setminus \{0\} \end{cases}$$

is a unitary polarity of $\mathrm{PG}(2n, q^2)$. In particular, varying a, we get a set consisting of $\theta_{2n,q}$ Hermitian varieties, such that the intersection of any $2n$ of them is a $(q^{2n+1}+1)/(q+1)$–cap (i.e., a set of points such that no three of them are collinear) and there is a partition of $\mathrm{PG}(2n, q^2)$ into such caps which is invariant under a Singer cyclic subgroup of order $(q^{2n+1}+1)/(q+1)$.

Proposition 14. *Let $a_i \in \mathrm{GF}(q^{4n+2}) \setminus \{0\}$ such that $a_i^{q^{2n+1}-1} \in \mathrm{GF}(q^2)$, $i = 1, 2$ and $a_1 \neq a_2$. Then $\Pi_{a_1}\Pi_{a_2}$ is a fixed point free projectivity of $\mathrm{PG}(2n, q^2)$.*

Proof. Since $a_i = \omega^{k_i(q^{2n+1}+1)/(q+1)}$, $k_1 \neq k_2$, [5, Theorem 2.2], it follows that the projectivity $\Pi_{a_1}\Pi_{a_2}$ sends (x) to $(\omega^{(k_2 q^{2n+1}-k_1)(q^{2n+1}+1)/(q+1)}x)$. □

We are ready to give the aforementioned construction.

Theorem 15. *$\mathcal{H}(4n + 1, q^2)$ has a maximal partial spread of size $q^{2n+1} + 1$ admitting a cyclic group of order θ_{2n,q^2} as an automorphism group.*

Proof. From Proposition 4, Lemma 6 and Proposition 14, we find $\theta_{2n,q}$ Hermitian Segre varieties containing two fixed generators g_1, g_2 of $\mathcal{H}(4n+1, q^2)$ and pairwise sharing exactly $g_1 \cup g_2$. Hence we get a set $\mathcal{S}$ of $\theta_{2n,q}(q - 1) + 2 = q^{2n+1} + 1$ mutually disjoint generators. Of course $\mathcal{S}$ is maximal and by construction it is left invariant by a cyclic group of order θ_{2n,q^2} corresponding to a Singer cyclic group of g_1. □

Remark 16. If $\mathcal{D}$ is the set of extended elements of a Desarguesian spread of a $\mathcal{W}(4n+1, q)$ embedded in $\mathcal{H}(4n + 1, q^2)$, then it is easily seen that the $q + 1$ generators contained in the Hermitian Segre variety determined by any three distinct elements of $\mathcal{D}$ are contained in $\mathcal{D}$. It is therefore plausible that the maximal partial spread constructed in Theorem 15 is equivalent to $\mathcal{D}$.

4 Orthogonal and symplectic polar spaces

Let $\mathcal{Q}^+(2n + 1, q)$ be a hyperbolic quadric of $\mathrm{PG}(2n + 1, q)$. The set of all generators of $\mathcal{Q}^+(2n + 1, q)$, that are n–spaces, is divided in two distinct subsets of the same size, called *systems of generators* and denoted by $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let A and A' two distinct generators of $\mathcal{Q}^+(2n + 1, q)$. Then their possible intersections are projective spaces of dimension

$$\begin{cases} 0, \quad 2, \quad 4, \quad \ldots, \quad n-2 & \text{if} \quad A, A' \in \mathcal{M}_i, i = 1, 2 \\ -1, \quad 1, \quad 3, \quad \ldots, \quad n-1 & \text{if} \quad A \in \mathcal{M}_i, A' \in \mathcal{M}_j, i, j \in \{1, 2\}, i \neq j \end{cases}$$

if n is even or

$$\begin{cases} 0, \quad 2, \quad 4, \quad \ldots, \quad n-1 & \text{if} \quad A \in \mathcal{M}_i, A' \in \mathcal{M}_j, i, j \in \{1, 2\}, i \neq j \\ -1, \quad 1, \quad 3, \quad \ldots, \quad n-2 & \text{if} \quad A, A' \in \mathcal{M}_i, i = 1, 2 \end{cases}$$

if n is odd.

It follows that, if n is even, a maximal partial spread of $\mathcal{Q}^+(2n + 1, q)$ has size two. Therefore we will assume that $n = 2m - 1$ and in this case all the elements of a partial spread of $\mathcal{Q}^+(4m - 1, q)$ belong to the same system of generators.

Remark 17. Let $\mathcal{Q}(4m-2, q)$ be a parabolic quadric obtained by intersecting $\mathcal{Q}^+(4m-1, q)$ with a non–degenerate hyperplane section. Then a (maximal) partial spread of $\mathcal{Q}^+(4m-1, q)$ induces a (maximal) partial spread of $\mathcal{Q}(4m-2, q)$. Actually the converse also holds true.

Let $\mathcal{Q}^+(3, q^m)$ be a hyperbolic quadric of $\mathrm{PG}(3, q^m)$, $m \geqslant 2$. In this special case $\mathcal{Q}^+(3, q^m)$ coincides with $\mathcal{S}_{1,1}$. Hence, both systems of generators, $\mathcal{R}_1$ and $\mathcal{R}_2$, consist of $q^m + 1$ mutually disjoint lines each, and are called *reguli*, see [8]. Under the $\mathrm{GF}(q)$–linear representation of $\mathrm{PG}(3, q^m)$ the points of $\mathcal{Q}^+(3, q^m)$ are mapped to $(m-1)$–spaces of a hyperbolic quadric $\mathcal{Q}^+(4m-1, q)$ and a line of $\mathcal{Q}^+(3, q^m)$ is mapped to a generator of $\mathcal{Q}^+(4m-1, q)$. It follows that $\phi(\mathcal{R}_1) = \{\phi(\ell) \mid \ell \in \mathcal{R}_1\}$ is a partial spread of $\mathcal{Q}^+(4m-1, q)$ of size $q^m + 1$. We denote by $\mathcal{Q}$ the points of $\mathcal{Q}^+(4m-1, q)$ covered by members of $\phi(\mathcal{Q}^+(3, q^m))$.

Theorem 18. *$\phi(\mathcal{R}_i)$ is a maximal partial spread of $\mathcal{Q}^+(4m-1, q)$.*

Proof. Let $\ell_i \in \mathcal{R}_i$, $i = 1, 2$. Then $\phi(\ell_1) \cap \phi(\ell_2)$ is an $(m-1)$–space.

$\boxed{m \text{ odd}}$

Since m is odd, we have that $\phi(\ell_1)$ and $\phi(\ell_2)$ belong to distinct systems of generators of $\mathcal{Q}^+(4m-1, q)$. Therefore, we may assume that $\phi(\ell_i) \in \mathcal{M}_i$. This means that $\phi(\mathcal{R}_i) \subset \mathcal{M}_i$. The result follows.

$\boxed{m \text{ even}}$

In this case we may assume that both $\phi(\mathcal{R}_1)$, $\phi(\mathcal{R}_2)$ are contained in $\mathcal{M}_1$. Let $g \in \mathcal{M}_2$. Then $|g \cap \mathcal{Q}| \geqslant q^m + 1$. If $g \cap \mathcal{Q}$ contains a line, then every $(2m-2)$–space contained in g has at least one point in common with $\mathcal{Q}$. If $g \cap \mathcal{Q}$ does not contain any line, then every element of $\phi(\mathcal{R}_1)$ (and $\phi(\mathcal{R}_2)$) has exactly one point in common with g. Let P_1, P_2, P_3 be three distinct points of $g \cap \mathcal{Q}$. Let $g_i \in \phi(\mathcal{R}_1)$ such that $g_i \cap g = \{P_i\}$, $i = 1, 2, 3$. Then, since g_1, g_2, g_3 are three distinct mutually disjoint generators of $\mathcal{Q}^+(4m-1, q)$, there exists a unique hyperbolic Segre variety $\mathcal{S}_{1,2m-1}$ of $\mathcal{Q}^+(4m-1, q)$ containing g_1, g_2, g_3. First of all notice that the $q+1$ $(2m-1)$–spaces contained in $\mathcal{S}_{1,2m-1}$ are also contained in $\phi(\mathcal{R}_1)$. Indeed, if $A_i = \ell_2 \cap \phi^{-1}(g_i)$, $i = 1, 2, 3$, then there exists a unique subline of order q, say s, of ℓ_2 containing A_1, A_2, A_3. Therefore $\{\phi(\ell) \mid \ell \in \mathcal{R}_1, \ell \cap \ell_2 \in s\}$ is the set of $q + 1$ $(2m-1)$–spaces contained in $\mathcal{S}_{1,2m-1}$. Let π be the plane $\langle P_1, P_2, P_3 \rangle$ and let r_i be a line of g_i passing through P_i, $i = 1, 2, 3$. Then r_1, r_2, r_3 are three mutually disjoint lines and the set of points covered by the lines incident to r_1, r_2, r_3 is a three–dimensional hyperbolic quadric contained in $\mathcal{S}_{1,2m-1}$. It follows that $\pi \cap \mathcal{Q} = \pi \cap \mathcal{S}_{1,2m-1}$ is a conic, say C. Let P_4 be a point of $(g \cap \mathcal{Q}) \setminus C$. By considering the planes $\langle P, Q, P_4 \rangle$, where P, Q are distinct points of C and using again the previous argument we obtain that the solid $\langle P_1, P_2, P_3, P_4 \rangle \subset g$ meets $\mathcal{Q}$ in a three–dimensional elliptic quadric. Hence, it turns out that every $(2m-2)$–space contained in g has at least a point in common with $\mathcal{Q}$. The result follows from the fact that through every $(2m-2)$–space contained in $\mathcal{Q}^+(4m-1, q)$ there passes exactly one generator of $\mathcal{M}_1$ and one of $\mathcal{M}_2$. $\square$

Remark 19. The idea of considering the image of a $\mathcal{Q}^+(3, q^m)$ under the $\mathrm{GF}(q)$–linear representation to construct maximal partial spread of polar spaces was first pointed out by

M. Grassl during the conference ALCOMA '15. In particular, using completely different techniques, he presented a proof of the result of Corollary 23.

Now, we focus on the even characteristic case.

Proposition 20. *Assume that q is even. If $\mathcal{Q}^+(4m-1,q)$ has a maximal partial spread of size x then*

- $\mathcal{W}(4m-1,q)$ *has a maximal partial spread of size x,*
- $\mathcal{W}(4m-3,q)$ *has a maximal partial spread of size x.*

Proof. Let $\mathcal{S} \subset \mathcal{M}_1$ be a maximal partial spread of $\mathcal{Q}^+(4m-1,q)$ and let $\bar{\mathcal{S}}$ be the set of points covered by the members $\mathcal{S}$. Let X be an element of $\mathcal{M}_1 \setminus \mathcal{S}$. Since $\mathcal{S}$ is maximal, there exists $Y \in \mathcal{S}$ such that $|X \cap Y| \neq 0$. It follows that X contains at least a line of $\bar{\mathcal{S}}$. Hence, as through a $(2m-2)$–space contained in $\mathcal{Q}^+(4m-1,q)$ there passes a generator of $\mathcal{M}_1$, we have that every $(2m-2)$–space contained in $\mathcal{Q}^+(4m-1,q)$ has at least one point of $\bar{\mathcal{S}}$.

As q is even, there exists a unique symplectic polar space $\mathcal{W}(4m-1,q)$ of $\mathrm{PG}(4m-1,q)$ such that all the lines tangent to or contained in $\mathcal{Q}^+(4m-1,q)$ are lines of $\mathcal{W}(4m-1,q)$. In other words, the orthogonal polarity defined by $\mathcal{Q}^+(4m-1,q)$ polarizes to the symplectic polarity induced by $\mathcal{W}(4m-1,q)$. In this setting, a generator of $\mathcal{W}(4m-1,q)$ that does not belong to $\mathcal{M}_1 \cup \mathcal{M}_2$, meets $\mathcal{Q}^+(4m-1,q)$ in a $(2m-2)$–space. Furthermore, through a $(2m-2)$–space contained in $\mathcal{Q}^+(4m-1,q)$ there pass $q-1$ generators of $\mathcal{W}(4m-1,q)$ not contained in $\mathcal{M}_1 \cup \mathcal{M}_2$. It turns out that every generator of $\mathcal{W}(4m-1,q)$ contains at least a point of $\bar{\mathcal{S}}$ and hence $\mathcal{S}$ is a maximal partial spread of $\mathcal{W}(4m-1,q)$.

The second assertion follows from Remark 17 and the fact that, for q even, $\mathcal{Q}(4m-2,q)$ is isomorphic to $\mathcal{W}(4m-3,q)$ [17, Theorem 4]. □

In the following result, with similar arguments to that used in the previous section for Hermitian polar spaces, we give a construction of maximal partial spreads of $\mathcal{Q}^+(4m-1,q)$, q even.

Theorem 21. *Any $(m-1)$–spread of $\mathrm{PG}(2m-1,q)$, q even, gives rise to a maximal partial spread of $\mathcal{Q}^+(4m-1,q)$ of size q^m+1.*

Proof. Let Σ and Σ' be two disjoint generators of $\mathcal{Q}^+(4m-1,q)$. Let Π_{r-1} be a $(r-1)$–dimensional projective space of Σ, $1 \leqslant r \leqslant 2m-1$, and let $\Pi_{r-1}^{\perp}$ be the polar space of Π_{r-1}. Here $\perp$ denotes the orthogonal polarity induced by $\mathcal{Q}^+(4m-1,q)$. The intersection of $\Pi_{r-1}^{\perp}$ and Σ' is a $(2m-r-1)$–dimensional projective space, say Π'_{2m-r-1}. Note that $\langle \Pi_{r-1}, \Pi'_{2n-r-1} \rangle$ is a generator of $\mathcal{Q}^+(4m-1,q)$. In particular, an $(m-1)$–dimensional subspace of Σ corresponds to a uniquely determined $(m-1)$–dimensional subspace of Σ'.

Let $\mathcal{D}$ be an $(m-1)$–spread of Σ. From the above discussion, $\mathcal{D}$ defines an $(m-1)$–spread, say $\mathcal{D}'$ of Σ'. Corresponding elements of $\mathcal{D}$ and $\mathcal{D}'$ determine a generator of $\mathcal{Q}^+(4m-1,q)$ and hence $\mathcal{D}$ ($\mathcal{D}'$) produces a partial spread $\mathcal{S}$ of $\mathcal{Q}^+(4m-1,q)$ of size $|\mathcal{D}| = q^m+1$. Let $\bar{\mathcal{S}}$ denote the pointset of $\mathcal{Q}^+(4m-1,q)$ covered by members of $\mathcal{S}$. Let $\mathcal{T}$

be a generator of $\mathcal{Q}^+(4m-1,q)$ disjoint from Σ and Σ' and let $\mathcal{S}_{1,2m-1}$ be the hyperbolic Segre variety of $\mathcal{Q}^+(4m-1,q)$ containing Σ, Σ' and $\mathcal{T}$. Let $\mathcal{W}_\Sigma$ and $\mathcal{W}_{\Sigma'}$ denote the symplectic space of Σ and Σ', respectively, arising from Lemma 7. Let d be a member of $\mathcal{D}$. Then $d^\perp \cap \Sigma'$ defines a unique member d' of $\mathcal{D}'$. Let $L_d \subset \mathcal{R}_1$ be the set of lines of $\mathcal{S}_{1,2m-1}$ meeting d. The set L_d determines an $(m-1)$–space of Σ', say d''. It turns out that d' and d'' are conjugate with respect to the polarity of Σ' induced by $\mathcal{W}_{\Sigma'}$.

We claim that there exists at least one member X of $\mathcal{D}'$ meeting $\mathcal{W}_{\Sigma'}$ in a degenerate symplectic space. If m is odd, the assertion is trivial. If m is even, assume, by way of contradiction, that each of the q^m+1 elements of $\mathcal{D}'$ meets $\mathcal{W}_{\Sigma'}$ in a $\mathcal{W}(m-1,q)$. Consider a hyperbolic quadric $\mathcal{Q}^+(2m-1,q)$ of Σ' such that the orthogonal polarity defined by $\mathcal{Q}^+(2m-1,q)$ polarizes to the symplectic polarity induced by $\mathcal{W}_{\Sigma'}$. Then every element of $\mathcal{D}'$ meets $\mathcal{Q}^+(2m-1,q)$ either in a $\mathcal{Q}^+(m-1,q)$ or in a $\mathcal{Q}^-(m-1,q)$. Let y_1 and y_2 be the number of elements of $\mathcal{D}'$ meeting $\mathcal{Q}^+(2m-1,q)$ in a $\mathcal{Q}^+(m-1,q)$ and in a $\mathcal{Q}^-(m-1,q)$, respectively. Then $y_1+y_2 = q^m+1$, $|\mathcal{Q}^-(m-1,q)|y_1 + |\mathcal{Q}^+(m-1,q)|y_2 = |\mathcal{Q}^+(2m-1,q)|$ imply that $y_2 = (q^{m/2}+1)^2/2$, a contradiction, since q is even. So there exists an element X of $\mathcal{D}'$ meeting $\mathcal{W}_{\Sigma'}$ in a degenerate symplectic space.

With respect to the symplectic polarity of Σ' induced by $\mathcal{W}_{\Sigma'}$ the conjugate of X meets X in at least a point of $\mathcal{W}_{\Sigma'}$. It follows that $\mathcal{T}$ has a non–empty intersection with $\bar{\mathcal{S}}$. $\square$

Corollary 22. *$\mathcal{Q}(4m-2)$ has a maximal partial spread of size q^m+1.*

Corollary 23. *$\mathcal{W}(4m-1,q)$, q even, has a maximal partial spread of size q^m+1.*

Corollary 24. *$\mathcal{W}(4m-3,q)$, q even, has a maximal partial spread of size q^m+1.*

Remark 25. The result of Theorem 21 cannot be extended to the odd characteristic case. Indeed, consider in $\mathrm{PG}(3,3)$ a Desarguesian line–spread $\mathcal{D}$ such that $\mathcal{D}$ contains 4 lines of a regulus $\mathcal{R}$ of a fixed symplectic space $\mathcal{W}(3,3)$. In particular $\mathcal{W}(3,3)$ can be chosen in such a way that none of the lines of the opposite regulus of $\mathcal{R}$, say $\mathcal{R}'$, is a line of $\mathcal{W}(3,3)$. If $\mathcal{D}'$ denotes the Hall spread $(\mathcal{D} \setminus \mathcal{R}) \cup \mathcal{R}'$, then each of the 10 elements of $\mathcal{D}'$ is non–degenerate with respect to $\mathcal{W}(3,3)$. Magma computations show that the partial spread $\mathcal{S}$ of $\mathcal{Q}^+(7,3)$ constructed from $\mathcal{D}'$ (as in Theorem 21) is not maximal; indeed there exist 34 generators disjoint from all the members of $\mathcal{S}$. Furthermore, among the 34 disjoint generators, it is possible to find a subset of 18 generators that, together with the 10 members of $\mathcal{S}$, give rise to the unique spread of $\mathcal{Q}^+(7,3)$, admitting $\mathrm{PSp}(6,2)$ as an automorphism group.

4.1 The triality quadric

In this section we specialize to the case of the triality quadric $\mathcal{Q}^+(7,q)$. We will denote by $\perp$ the polarity of $\mathrm{PG}(7,q)$ induced by $\mathcal{Q}^+(7,q)$.

Let us denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the two families of generators of $\mathcal{Q}^+(7,q)$, the set of lines of $\mathcal{Q}^+(7,q)$ by $\mathcal{L}$ and the set of points of $\mathcal{Q}^+(7,q)$ by $\mathcal{P}$; then the rank 4 incidence geometry $\Omega = (\mathcal{P}, \mathcal{L}, \mathcal{M}_1, \mathcal{M}_2)$ can be attached to $\mathcal{Q}^+(7,q)$ as follows. An element $G_1 \in \mathcal{M}_1$ is said to be incident with an element $G_2 \in \mathcal{M}_2$ if and only if the intersection $G_1 \cap G_2$ is a plane of $\mathcal{Q}^+(7,q)$. Incidence between other elements is symmetrized containment. Every permutation of the set $\{\mathcal{P}, \mathcal{M}_1, \mathcal{M}_2\}$ defines a geometry Ω' isomorphic to Ω and hence the automorphism groups of Ω and Ω' are isomorphic.

A *triality* of the geometry Ω is a map τ:

$$\tau : \mathcal{L} \to \mathcal{L}, \mathcal{P} \to \mathcal{M}_1, \mathcal{M}_1 \to \mathcal{M}_2, \mathcal{M}_2 \to \mathcal{P}$$

preserving the incidence in Ω and such that τ^3 is the identity.

The triality image of a maximal partial spread $\mathcal{S} \subset \mathcal{M}_2$ of $\mathcal{Q}^+(7,q)$ is a maximal partial ovoid of $\mathcal{Q}^+(7,q)$, i.e. a set of mutually non–collinear points of $\mathcal{Q}^+(7,q)$ that is maximal with respect to set–theoretic inclusion.

In the case of $\mathcal{Q}^+(7,q)$ we can extend Theorem 21 to the case q odd as follows.

Theorem 26. *Any symplectic spread of* $\mathrm{PG}(3,q)$ *gives rise to a maximal partial spread of* $\mathcal{Q}^+(7,q)$ *of size* q^2+1.

Proof. With the notation introduced in Theorem 21, assume that $\mathcal{D}$ is a symplectic spread of $\mathrm{PG}(3,q)$, i.e. every line of $\mathcal{D}$ is a line of a fixed symplectic polar space $\mathcal{W}(3,q)$, and that $\mathcal{S} \subset \mathcal{M}_2$. Since the members of $\mathcal{S}$ are incident with Σ and Σ', under the triality map we get a set E of q^2+1 points of $\mathcal{Q}^+(7,q)$ such that each of them is conjugate to two points of $\mathcal{Q}^+(7,q)$, say P, P', with $P = \Sigma^\tau$ and $P' = \Sigma'^\tau$. This means that E is a partial ovoid of a Klein quadric $\mathcal{Q}^+(5,q)$. Since $\mathcal{D}$ is symplectic it follows that actually E lies on a parabolic quadric $\mathcal{Q}(4,q)$ of $\mathcal{Q}^+(5,q)$. Then, $\mathcal{S}$ is maximal on $\mathcal{Q}^+(7,q)$ if and only if E is maximal on $\mathcal{Q}^+(7,q)$ (as a partial ovoid). Indeed, let P be a point of $\mathcal{Q}^+(7,q) \setminus \mathcal{Q}(4,q)$. Then $P^\perp$ intersects $\mathcal{Q}(4,q)$ in either a cone or a hyperbolic quadric or an elliptic quadric. In the former two cases the quadric section contains lines that certainly meet E. In the latter case, from [4] an elliptic quadric of $\mathcal{Q}(4,q)$ must meet E, and we are done. $\square$

Remark 27. Assuming that $\mathcal{D}$ is Desarguesian, it follows that the triality image of $\mathcal{S}$ is actually an elliptic quadric section $\mathcal{Q}^-(3,q)$ of $\mathcal{Q}^+(7,q)$. From [2, 15.1 8)] $\mathcal{S}$ corresponds to the $\mathrm{GF}(q)$–linear representation of a regulus of a hyperbolic quadric $\mathcal{Q}^+(3,q^2)$ and hence we are in the case of Theorem 18.

Remark 28. It is not difficult to prove that the smallest maximal partial spread of $\mathcal{Q}^+(7,2)$ has size 5 and corresponds under triality to an elliptic solid section of $\mathcal{Q}^+(7,2)$. In the spin representation of $\mathrm{PSp}(6,2)$ such an elliptic solid section corresponds to a maximal partial spread of $\mathcal{W}(5,2)$. Also, the smallest maximal partial spread of $\mathcal{W}(5,2)$ has size 5.

Taking into account Lemma 10, one can ask for the maximum number m of symplectic polar spaces $\mathcal{W}(3,q)$ in a solid such that the projectivity obtained as the product of any two associated symplectic polarities is fixed point free. This is equivalent to ask for the maximum number of $\mathcal{W}(3,q)$ pairwise intersecting in an elliptic congruence (i.e. a Desarguesian line–spread of a solid), which in turn, using the Klein correspondence, is equivalent to ask for the maximum size of an *exterior set* of $\mathcal{Q}^+(5,q)$, i.e., a set X of points such that each line joining two distinct elements of X has no point in common with $\mathcal{Q}^+(5,q)$. Since a spread of $\mathcal{Q}^+(7,q)$ has size q^3+1, we have that $m \leqslant q^2+q+1$. On the other hand it is known that $|X| \leqslant q^2+q+1$ and equality holds if and only if $q=2$, see [16]. Under triality the $q+1$ generators of a hyperbolic Segre variety are mapped to $q+1$ points forming a conic section of $\mathcal{Q}^+(7,q)$. Therefore there exists no ovoid of $\mathcal{Q}^+(7,q)$, $q>2$, that is union of conics sharing two points. However, this fact was already noted in [3].

The next result is group–theoretic and gives some information on the stabilizer of a hyperbolic Segre variety $\mathcal{S}_{1,3}$ of $\mathcal{Q}^+(7,q)$, q even.

Proposition 29. *The stabilizer of a hyperbolic Segre variety $\mathcal{S}_{1,3}$ of $\mathcal{Q}^+(7,q)$, q even, is a subgroup of $\Omega(7,q)$ in its spin representation.*

Proof. Assume without loss of generality that the hyperbolic Segre variety of $\mathcal{Q}^+(7,q)$ consists of $q+1$ mutually disjoint solids of $\mathcal{M}_2$. From Remark 9 such solids are permuted by $K \simeq \mathrm{PSp}(2,q) \times \mathrm{PSp}(4,q)$. Under triality they are mapped to $q+1$ points forming a conic section C of $\mathcal{Q}^+(7,q)$. Since q is even C has a nucleus $N \notin \mathcal{Q}^+(7,q)$ that is fixed by K^τ. From [2] the triality image of the stabilizer of a point not on $\mathcal{Q}^+(7,q)$ is the spin representation of $\Omega(7,q)$, and we are done. $\square$

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