Forbidden pairs with a common graph generating almost the same sets

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Submitted: Jun 3, 2016; Accepted: Apr 4, 2017; Published: Apr 13, 2017
Mathematics Subject Classifications: 05C75

Abstract

Let $\mathcal{H}$ be a family of connected graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain any members of $\mathcal{H}$ as an induced subgraph. Let $\mathcal{F}(\mathcal{H})$ be the family of connected $\mathcal{H}$-free graphs. In this context, the members of $\mathcal{H}$ are called forbidden subgraphs.

In this paper, we focus on two pairs of forbidden subgraphs containing a common graph, and compare the classes of graphs satisfying each of the two forbidden subgraph conditions. Our main result is the following: Let H_1, H_2, H_3 be connected graphs of order at least three, and suppose that H_1 is twin-less. If the symmetric difference of $\mathcal{F}(\{H_1, H_2\})$ and $\mathcal{F}(\{H_1, H_3\})$ is finite and the tuple $(H_1; H_2, H_3)$ is non-trivial in a sense, then H_2 and H_3 are obtained from the same vertex-transitive graph by successively replacing a vertex with a clique and joining the neighbors of the original vertex and the clique. Furthermore, we refine a result in [Combin. Probab. Comput. **22** (2013) 733–748] concerning forbidden pairs.

Keywords: forbidden subgraph; star-free graph; vertex-transitive graph.

*Partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (B) 26800083

†Partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (B) 24340021 and Grant-in-Aid for Young Scientists (B) 26800085

‡Partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (B) 26800086

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. If there is no fear of confusion, we often identify G with its vertex set $V(G)$. For $X, Y \subseteq V(G)$, by *joining* X and Y we mean adding all the missing edges between vertices of X and vertices of Y. For $x \in V(G)$, we let $N_G(x)$ and $N_G[x]$ denote the *open neighborhood* and the *closed neighborhood* of x, respectively; thus $N_G[x] = N_G(x) \cup \{x\}$. For $X \subseteq V(G)$, we let $N_G[X] = \bigcup_{x \in X} N_G[x]$. For $x \in V(G)$, we let $d_G(x)$ denote the *degree* of x; thus $d_G(x) = |N_G(x)|$. We let $\delta(G)$ and $\Delta(G)$ denote the *minimum degree* and the *maximum degree* of G, respectively. We let K_n denote the *complete graph* of order n, and let $K_{1,n}$ denote the *star* of order $n+1$. The graph K_3 is called the *triangle*, and the graph $K_{1,3}$ is called the *claw*.

For a connected graph H, G is said to be *H-free* if G does not contain H as an induced subgraph. For a family $\mathcal{H}$ of connected graphs, G is said to be *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$, and we let $\mathcal{F}(\mathcal{H})$ denote the family of all connected $\mathcal{H}$-free graphs. When $\mathcal{H}$ is finite and the members of $\mathcal{H}$ are specified, we use a sequence of members of $\mathcal{H}$ instead of $\mathcal{H}$ for $\mathcal{F}(\mathcal{H})$; thus if $\mathcal{H} = \{H_1, \dots, H_m\}$, we write $\mathcal{F}(H_1, \dots, H_m)$ instead of $\mathcal{F}(\mathcal{H})$. In this context, members of $\mathcal{H}$ are often referred to as *forbidden subgraphs*. For a graph H, we write $H \prec G$ if G contains H as an induced subgraph (i.e., G is not H-free). For terms and symbols not defined here, we refer the reader to [2].

In graph theory, many researchers have studied forbidden subgraph conditions for graphs to satisfy a given property. For example, there are complete characterizations of perfect graphs, intersection graphs and line graphs using forbidden subgraphs. However, in general, for many properties P, it seems difficult to characterize completely the forbidden subgraph conditions concerning P (in fact, if P is not hereditary, then there is no forbidden subgraphs which yield P). Thus we usually look for forbidden subgraph conditions which imply P. Further we frequently restrict the number of forbidden subgraphs, and in many cases, we focus on pairs of forbidden subgraphs. For example, Duffus, Gould and Jacobson [3] proved that every 2-connected $\{K_{1,3}, N\}$-free graph has a Hamiltonian cycle, where N is the graph obtained from a triangle by adding a pendant edge to each vertex, and Broersma and Veldman [1] proved that every 2-connected $\{K_{1,3}, P_6\}$-free graph has a Hamiltonian cycle, where P_6 is the path of order 6. Similar problems have widely been studied (see [5]).

Here we assume that for a property P, there exist two families $\mathcal{H}_1, \mathcal{H}_2$ of forbidden subgraphs such that every graph in $\mathcal{F}(\mathcal{H}_i)$ of sufficiently large order satisfies P. If $\mathcal{F}(\mathcal{H}_1)$ is essentially different from $\mathcal{F}(\mathcal{H}_2)$, then it might be important to study $\mathcal{H}_1$-freeness and $\mathcal{H}_2$-freeness. On the other hand, if $\mathcal{F}(\mathcal{H}_1)$ relates to $\mathcal{F}(\mathcal{H}_2)$, then it seems redundant to consider both $\mathcal{H}_1$-freeness and $\mathcal{H}_2$-freeness. Thus it is important to judge whether $\mathcal{F}(\mathcal{H}_1)$ relates to $\mathcal{F}(\mathcal{H}_2)$ or not. For such a reason, Fujita, Furuya and Ozeki [7] studied two families $\mathcal{H}_1, \mathcal{H}_2$ of forbidden subgraphs such that $\mathcal{F}(\mathcal{H}_1) \triangle \mathcal{F}(\mathcal{H}_2)$ is finite, where $\mathcal{F}(\mathcal{H}_1) \triangle \mathcal{F}(\mathcal{H}_2)$ is the symmetric difference of $\mathcal{F}(\mathcal{H}_1)$ and $\mathcal{F}(\mathcal{H}_2)$. (For detailed historical background and related results, we refer the reader to [7]. For example, they proved that for two connected graphs H_1 and H_2, if $|\mathcal{F}(H_1) \triangle \mathcal{F}(H_2)| < \infty$, then $H_1 \simeq H_2$.) They gave some results concerning the above problem and, in particular, proved a theorem on forbidden pairs. Before we introduce their theorem, we give a fundamental definition. A tuple $(H_1; H_2, H_3)$ of connected graphs of order at least three is *trivial* if either

(T1) $H_2 \simeq H_3$, or

(T2) $H_1 \prec H_2$ and $H_1 \prec H_3$.

If (T1) holds, then we clearly obtain $\mathcal{F}(H_1, H_2) = \mathcal{F}(H_1, H_3)$; while if (T2) holds, then $\mathcal{F}(H_1, H_2) = \mathcal{F}(H_1) = \mathcal{F}(H_1, H_3)$. Hence if $(H_1; H_2, H_3)$ is a trivial tuple, then $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$.

Fujita, Furuya and Ozeki [7] proved the following theorem.

Theorem A (Fujita, Furuya and Ozeki [7]). *Let H_1, H_2, H_3 be connected graphs of order at least three, and suppose that $\Delta(H_1) \leqslant |V(H_1)| - 2$ and $\delta(H_1) \geqslant 2$. Then $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$ if and only if $(H_1; H_2, H_3)$ is a trivial tuple.*

Now we focus on claw-freeness, or star-freeness in general. It has been known that the claw or the stars are important forbidden subgraphs (see [4]). For example, Fujisawa, Fujita, Plummer, Saito and Schiermeyer [6] proved that stars appear in all of the forbidden pairs assuring us the existence of a perfect matching in highly-connected graphs of sufficiently large order. Furthermore, Faudree and Gould [5] proved that for each one of the following three properties, the claw appears in all of the forbidden pairs assuring us the property:

- The existence of a Hamiltonian path in connected graphs of sufficiently large order.
- The existence of a Hamiltonian cycle in 2-connected graphs of sufficiently large order.
- The Hamiltonian-connectedness in 3-connected graphs of sufficiently large order.

Thus it is worthwhile to compare two forbidden pairs having the same star. However, no star can be H_1 in Theorem A. Our first purpose in this paper is to give a necessary condition for $|\mathcal{F}(K_{1,n}, H_2) \triangle \mathcal{F}(K_{1,n}, H_3)| < \infty$.

Let H be a graph. We define the relation $\equiv_H$ on $V(H)$ by letting $u \equiv_H v$ if and only if $N_H[u] = N_H[v]$. Then we can verify that $\equiv_H$ is an equivalence relation on $V(H)$. We let $\mathcal{C}(H)$ denote the quotient set with respect to $\equiv_H$. Then $\mathcal{C}(H)$ is a partition of $V(H)$ such that

(C1) every $C \in \mathcal{C}(H)$ is a clique of H, and

(C2) for two elements C, C' of $\mathcal{C}(H)$ with $C \neq C'$, either all vertices in C are joined to all vertices in C' in H or there is no edge of H between C and C'.

Let $B(H)$ be the graph on $\mathcal{C}(H)$ such that $CC' \in E(B(H))$ if and only if all vertices in C are joined to all vertices in C' in H. Note that $B(H)$ is isomorphic to the graph obtained from H by contracting each $C \in \mathcal{C}(H)$ to a vertex and replacing resulting parallel edges with a single edge. On the other hand, H is isomorphic to a graph obtained from $B(H)$ by successively replacing a vertex with a new clique (and joining the neighbors of the original vertex and the clique).

A *twin* is a pair u, v of vertices of a graph H such that $N_H[u] = N_H[v]$. A graph is *twin-less* if the graph has no twin. A graph H is *vertex-transitive* if for any $u, v \in V(H)$, there exists an automorphism ϕ of H such that $\phi(u) = v$.

We prove the following theorem. (Considering that stars of order at least 3 are twin-less, the assumption on H_1 in Theorem 1.1 is appropriate for our purpose.)

Theorem 1.1. *Let H_1, H_2, H_3 be connected graphs of order at least three, and suppose that H_1 is twin-less. If $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$, then one of the following holds:*

(i) *$(H_1; H_2, H_3)$ is a trivial tuple, or*

(ii) (a) *$\delta(H_1) = 1$ or $\Delta(H_1) = |V(H_1)| - 1$, and*

(b) *for some $i \in \{2, 3\}$, H_{5-i} is obtained from H_i by replacing a vertex with a clique and $B(H_i)$ is vertex-transitive.*

Our second purpose is to show that the condition $\Delta(H_1) \leqslant |V(H_1)| - 2$ in Theorem A can be dropped as follows.

Theorem 1.2. *Let H_1, H_2, H_3 be connected graphs of order at least three, and suppose that $\delta(H_1) \geqslant 2$. Then $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$ if and only if $(H_1; H_2, H_3)$ is a trivial tuple.*

In Section 2, we prove Theorem 1.1 and show that the conclusion (ii) of Theorem 1.1 cannot be dropped. In Section 3, we prove Theorem 1.2.

The following lemma will be used in our proof.

Lemma 1.3 (Fujita, Furuya and Ozeki [7])**.** *Let H_1, H_2 be connected graphs of order at least three. If $|\mathcal{F}(H_1) - \mathcal{F}(H_2)| < \infty$ and $H_1 \not\prec H_2$, then $\delta(H_1) = 1$ and H_1 has a twin.*

2 Proof of Theorem 1.1

Let H be a graph. For an integer $n \geqslant 0$ and an element C of $\mathcal{C}(H)$, let $G_1^n(H; C)$ be the graph obtained from H by adding a clique of size n and joining the clique and $N_H[C]$. By the definition of $\mathcal{C}(H)$ and $G_1^n(H; C)$, we have $B(G_1^n(H; C)) \simeq B(H)$. Set $c(H) = \max\{|C| : C \in \mathcal{C}(H)\}$. Note that $c(G_1^n(H; C)) > c(H)$ if $|C| = c(H)$ and $n \geqslant 1$.

Proof of Theorem 1.1. Let H_i $(1 \leqslant i \leqslant 3)$ be as in Theorem 1.1. We first suppose that $H_1 \prec H_i$ for some $i \in \{2, 3\}$. Then $\mathcal{F}(H_1, H_i) \triangle \mathcal{F}(H_1, H_{5-i}) = \mathcal{F}(H_1) \triangle \mathcal{F}(H_1, H_{5-i}) = \mathcal{F}(H_1) - \mathcal{F}(H_{5-i})$, and hence $|\mathcal{F}(H_1) - \mathcal{F}(H_{5-i})| < \infty$. Since H_1 is twin-less, it follows from Lemma 1.3 that $H_1 \prec H_{5-i}$, which implies that $(H_1; H_2, H_3)$ satisfies (T2). In particular, the tuple $(H_1; H_2, H_3)$ is trivial, as desired. Thus we may assume that $H_1 \not\prec H_i$ for each $i \in \{2, 3\}$.

Claim 2.1. *Let $i \in \{2,3\}$, and suppose that $H_i \not\prec H_{5-i}$. Let $C \in \mathcal{C}(H_{5-i})$. Then the following hold.*

(a) *There exists an integer $n_C \geqslant 1$ such that $G_1^{n_C}(H_{5-i}; C)$ contains H_i as an induced subgraph.*

(b) $c(H_i) > c(H_{5-i})$.

(c) $H_{5-i} \prec H_i$.

(d) *If $|C| = c(H_{5-i})$, then there exists an integer $n_0 \geqslant 1$ such that $H_i \simeq G_1^{n_0}(H_{5-i}; C)$.*

Proof. We first show that $H_1 \not\prec G_1^n(H_{5-i}; C)$ for any $n \geqslant 0$. Suppose that $H_1 \prec G_1^n(H_{5-i}; C)$ for some $n \geqslant 0$, and let H_1^0 be an induced subgraph of $G_1^n(H_{5-i}; C)$ isomorphic to H_1. Note that for any $x \in C$ and any $y \in V(G_1^n(H_{5-i}; C)) - V(H_{5-i})$, the subgraph of $G_1^n(H_{5-i}; C)$ induced by $(V(H_{5-i}) - \{x\}) \cup \{y\}$ is isomorphic to H_{5-i}. Since $H_1 \not\prec H_{5-i}$, it follows that H_1^0 contains at least $|C|+1$ ($\geqslant 2$) vertices of $C \cup (V(G_1^n(H_{5-i}; C)) - V(H_{5-i}))$. Then H_1^0 contains a twin, which is a contradiction. Thus $H_1 \not\prec G_1^n(H_{5-i}; C)$ for any $n \geqslant 0$. In particular, $\{G_1^n(H_{5-i}; C) : n \geqslant 0\} \subseteq \mathcal{F}(H_1) - \mathcal{F}(H_{5-i})$. Since $|\mathcal{F}(H_1, H_i) - \mathcal{F}(H_{5-i})| < \infty$, it follows that there exists an integer $n_C \geqslant 0$ such that $G_1^{n_C}(H_{5-i}; C)$ contains H_i as an induced subgraph. Since $H_i \not\prec H_{5-i}$, $n_C \geqslant 1$, and hence we obtain (a).

In the rest of the proof of the claim, we let C be a member of $\mathcal{C}(H_{5-i})$ such that $|C| = c(H_{5-i})$, and take n_C as small as possible. Recall that $B(G_1^{n_C}(H_{5-i}; C)) \simeq B(H_{5-i})$. Let C_0 be the element of $\mathcal{C}(G_1^{n_C}(H_{5-i}; C))$ such that $C \subseteq C_0$. Then $|C_0| = |C| + n_C = c(H_{5-i}) + n_C$. Let H_i^0 be an induced subgraph of $G_1^{n_C}(H_{5-i}; C)$ isomorphic to H_i. If $C_0 \not\subseteq V(H_i^0)$, then $G_1^{n_C-1}(H_{5-i}; C)$ contains H_i^0 as an induced subgraph, which contradicts the choice of n_C. Thus $C_0 \subseteq V(H_i^0)$. In particular, an element of $\mathcal{C}(H_i^0)$ contains C_0, and hence $c(H_i) \geqslant |C_0| = c(H_{5-i}) + n_C > c(H_{5-i})$. Consequently we obtain (b).

If $H_{5-i} \not\prec H_i$, then applying (b) with roles of H_i and H_{5-i} interchanged, we get $c(H_{5-i}) > c(H_i)$, which contradicts (b). Thus we obtain (c).

Finally we show (d). By (c), there exists a set $X \subseteq V(H_i)$ such that $H_i - X \simeq H_{5-i}$. Since $c(H_i) \geqslant c(H_{5-i}) + n_C$, there exists a subset of X which is a clique with size at least n_C. This together with the fact that $H_i \prec G_1^{n_C}(H_{5-i}; C)$ leads to

$$|V(H_{5-i})| = |V(H_i)| - |X| \leqslant |V(H_i)| - n_C \leqslant |V(G_1^{n_C}(H_{5-i}; C))| - n_C = |V(H_{5-i})|.$$

This forces $H_i \simeq G_1^{n_C}(H_{5-i}; C)$. In particular, we obtain (d). $\square$

If $H_2 \simeq H_3$, then $(H_1; H_2, H_3)$ is a trivial tuple, as desired. Thus we may assume that $H_2 \not\simeq H_3$. Then either $H_2 \not\prec H_3$ or $H_3 \not\prec H_2$. We may assume that $H_2 \not\prec H_3$. Then by Claim 2.1(c), we have $H_3 \prec H_2$.

Fix an element C_3^* of $\mathcal{C}(H_3)$ such that $|C_3^*| = c(H_3)$. By Claim 2.1(d), there exists an integer $n_0 \geqslant 1$ such that $H_2 \simeq G_1^{n_0}(H_3; C_3^*)$ (i.e., H_2 is obtained from H_3 by replacing a vertex with a clique). Hence $B(H_2) \simeq B(H_3)$ and $\mathcal{C}(H_2)$ has exactly one element C_2^* such that $|C_2^*| = c(H_2)$ $(= c(H_3) + n_0)$.

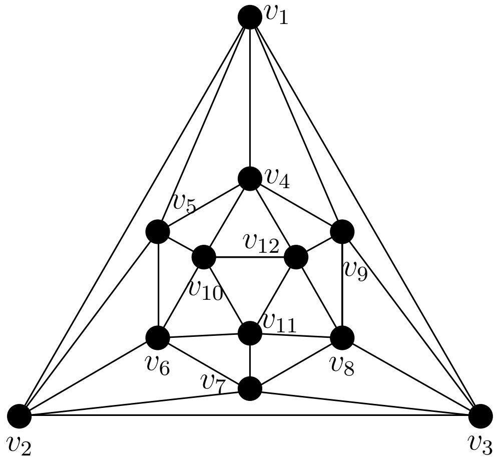

Figure 1: Icosahedron A

Claim 2.2. *For $C \in \mathcal{C}(H_3)$, there exists an isomorphic mapping ϕ_C from $B(H_3)$ to $B(H_2)$ such that $\phi_C(C) = C_2^*$.*

Proof. Set $D = C \cup (V(G_1^{n_C}(H_3; C)) - V(H_3))$, where n_C is an integer assured in Claim 2.1(a). By the definition of $G_1^{n_C}(H_3; C)$, we have $D \in \mathcal{C}(G_1^{n_C}(H_3; C))$ and every $C' \in \mathcal{C}(G_1^{n_C}(H_3; C)) - \{D\}$ satisfies $|C'| \leqslant c(H_3)$. Since $H_2 \prec G_1^{n_C}(H_3; C)$ and $c(H_2) > c(H_3)$ by Claim 2.1, we have $|D| = c(G_1^{n_C}(H_3; C)) \geqslant c(H_2) > c(H_3)$. Furthermore, there exists an isomorphic mapping ϕ_1 from $B(H_3)$ to $B(G_1^{n_C}(H_3; C))$ such that $\phi_1(C) = D$.

Let H_2^1 be an induced subgraph of $G_1^{n_C}(H_3; C)$ isomorphic to H_2. Since $B(H_2^1) \simeq B(H_2) \simeq B(H_3) \simeq B(G_1^{n_C}(H_3; C))$, it follows that for each $\tilde{C} \in \mathcal{C}(G_1^{n_C}(H_3; C))$, there exists exactly one element $D_{\tilde{C}}$ of $\mathcal{C}(H_2^1)$ such that $D_{\tilde{C}} \subseteq \tilde{C}$. Let $\phi_2 : \mathcal{C}(G_1^{n_C}(H_3; C)) \to \mathcal{C}(H_2^1)$ be the mapping such that $\phi_2(\tilde{C}) = D_{\tilde{C}}$ for $\tilde{C} \in \mathcal{C}(G_1^{n_C}(H_3; C))$. Then we can verify that ϕ_2 is an isomorphic mapping from $B(G_1^{n_C}(H_3; C))$ to $B(H_2^1)$. Recall that $\mathcal{C}(H_2)$ has exactly one element of size $c(H_2)$. Let C_2^{**} be the unique element of $\mathcal{C}(H_2^1)$ such that $|C_2^{**}| = c(H_2^1)$. Since $|C_2^{**}| = c(H_2^1) > c(H_3)$ and D is the unique element of $\mathcal{C}(G_1^{n_C}(H_3; C))$ such that $|D| > c(H_3)$, we have $\phi_2(D) = C_2^{**}$. Furthermore, there clearly exists an isomorphic mapping ϕ_3 from $B(H_2^1)$ to $B(H_2)$ such that $\phi_3(C_2^{**}) = C_2^*$. Therefore the mapping $\phi_C = \phi_3 \circ \phi_2 \circ \phi_1$ is an isomorphic mapping from $B(H_3)$ to $B(H_2)$ such that $\phi_C(C) = C_2^*$. □

Let $C, C' \in \mathcal{C}(H_3)$. Then $\phi = \phi_{C'}^{-1} \circ \phi_C$ is an automorphism of $B(H_3)$ such that $\phi(C) = C'$. Consequently $B(H_3)$ is vertex-transitive.

This completes the proof of Theorem 1.1. □

In the rest of this section, we show that the conclusion (ii) of Theorem 1.1 cannot be dropped. We first prove the following proposition.

Proposition 2.1. *Let H_2 be the icosahedron, and let $H_3 = G_1^1(H_2; \{v\})$, where v is a vertex of H_2. Let G be a connected $K_{1,3}$-free graph of order at least 13. If $H_2 \prec G$, then $H_3 \prec G$.*

Proof. Let A be an induced subgraph of G isomorphic to H_2, and label each vertex of A as in Figure 1. Since $|V(G)| > |V(A)|$ and G is connected, there exists a vertex $x \in V(G) - V(A)$ such that $N_G(x) \cap V(A) \neq \emptyset$. Note that for each $v \in V(A)$, $N_A(v)$ induces a cycle of order 5.

Claim 2.3. *Let $v \in V(A)$ be a vertex with $vx \in E(G)$, and let C be the cycle of A induced by $N_A(v)$. Then there exist three consecutive vertices of C adjacent to x.*

Proof. If no three consecutive vertices of C are adjacent to x, then there exist two non-adjacent vertices $w, w' \in V(C)$ such that $xw, xw' \notin E(G)$, and hence $\{v, x, w, w'\}$ induces $K_{1,3}$ in G, which is a contradiction. □

Let $v \in V(A)$ be a vertex such that

(I1) $vx \in E(G)$, and

(I2) subject to (I1), $|N_A(v) \cap N_G(x)|$ is as large as possible.

We may assume that $v = v_1$. Applying Claim 2.3, we may further assume that $v_2x, v_5x, v_4x \in E(G)$.

We first show that $N_A(v_1) \subseteq N_G(x)$. Suppose $N_A(v_1) \not\subseteq N_G(x)$. We may assume that $v_9x \notin E(G)$. Applying Claim 2.3 to $v = v_4$, we have $v_{10}x \in E(G)$. Then $|N_A(v_5) \cap N_G(x)| \geqslant 4$. It follows from (I2) that $|N_A(v_1) \cap N_G(x)| \geqslant 4$, and hence $v_3x \in E(G)$. Applying Claim 2.3 to $v = v_3$, we have $v_7x \in E(G)$. Then $\{x, v_1, v_7, v_{10}\}$ induces $K_{1,3}$ in G, which is a contradiction. Consequently $N_A(v_1) \subseteq N_G(x)$.

Suppose that $N_G(x) \cap V(A) \neq N_A[v_1]$, and let $v' \in (N_G(x) \cap V(A)) - N_A[v_1]$. Then v' is adjacent to at most two vertices in $N_A(v_1)$. Hence there exist two non-adjacent vertices $w, w' \in N_A(v_1) - N_G(v')$. Then $\{x, v', w, w'\}$ induces $K_{1,3}$ in G, which is a contradiction. Thus $N_G(x) \cap V(A) = N_A[v_1]$. Therefore $V(A) \cup \{x\}$ induces H_3 in G. □

Here we consider the conclusion (ii) of Theorem 1.1. Let $H_1 = K_{1,3}$, and let H_2 and H_3 be as in Proposition 2.1. Then H_1 is twin-less and $H_2 \prec H_3$. In particular, $\mathcal{F}(H_1, H_2) - \mathcal{F}(H_3) = \emptyset$. Furthermore, it follows from Proposition 2.1 that $\mathcal{F}(H_1, H_3) - \mathcal{F}(H_2) = \{H_2\}$. Consequently $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$. On the other hand, it is clear that $(H_1; H_2, H_3)$ is not a trivial tuple. This shows that there exists an example requiring the conclusion (ii) of Theorem 1.1. In fact, we can construct infinitely many such examples. Let $H_1 = K_{1,3}$. Let H_2 be the graph obtained from the icosahedron by replacing each vertex with a clique of size m, and let $H_3 = G_1^1(H_2; C)$, where C is an element of $\mathcal{C}(H_2)$. Then by arguments similar to the ones used in the proof of Proposition 2.1, we can verify that the tuple $(H_1; H_2, H_3)$ is not trivial and satisfies $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$.

3 Proof of Theorem 1.2

Let H be a graph. For an integer $n \geqslant 0$ and a subset X of $V(H)$,

- let $G_2^n(H; X)$ be the graph obtained from H by adding an independent set of size n and joining the independent set and X, and

- let $G_3^n(H;X)$ be the graph obtained from H by adding a path of order n and joining an end-vertex of the path and X.

If X consists of a vertex, say x, we write $G_2^n(H;x)$ and $G_3^n(H;x)$ instead of $G_2^n(H;X)$ and $G_3^n(H;X)$, respectively.

A *leaf* of a graph is a vertex of degree 1. For $x \in V(H)$, let $L_H(x)$ be the set of leaves of H adjacent to x. Set $l_H(x) = |L_H(x)|$ for $x \in V(G)$ and $l(H) = \max\{l_H(x) : x \in V(H)\}$. For $x \in V(H)$, a path $P = x_1x_2\cdots x_t$ of H is *x-good* if

(P1) $x_1 = x$, and

(P2) either $t = 1$, or $t \geqslant 2$, $d_H(x_t) = 1$ and $d_H(x_i) = 2$ for all $2 \leqslant i \leqslant t-1$.

Set $p_H(x) = \max\{|V(P)| : P \text{ is an } x\text{-good path}\}$ for $x \in V(H)$ and $p(H) = \max\{p_H(x) : x \in V(H)\}$.

Proof of Theorem 1.2. Let H_i $(1 \leqslant i \leqslant 3)$ be as in Theorem 1.2. It suffices to show that if $|\mathcal{F}(H_1,H_2)\triangle\mathcal{F}(H_1,H_3)| < \infty$, then $(H_1;H_2,H_3)$ is a trivial tuple. If $\Delta(H_1) \leqslant |V(H_1)|-2$, then Theorem A leads to the desired conclusion. Thus we may assume that

$$\Delta(H_1) = |V(H_1)| - 1. \tag{3.1}$$

Since $\delta(H_1) \geqslant 2$, it follows from (3.1) that

$$H_1 \text{ contains a triangle.} \tag{3.2}$$

We first suppose that $H_1 \prec H_i$ for some $i \in \{2,3\}$. Then $\mathcal{F}(H_1,H_i) \triangle \mathcal{F}(H_1,H_{5-i}) = \mathcal{F}(H_1) \triangle \mathcal{F}(H_1,H_{5-i}) = \mathcal{F}(H_1) - \mathcal{F}(H_{5-i})$, and hence $|\mathcal{F}(H_1) - \mathcal{F}(H_{5-i})| < \infty$. Since $\delta(H_1) \geqslant 2$, it follows from Lemma 1.3 that $H_1 \prec H_{5-i}$, which implies that $(H_1;H_2,H_3)$ satisfies (T2). In particular, $(H_1;H_2,H_3)$ is a trivial tuple, as desired. Thus we may assume that $H_1 \not\prec H_i$ for each $i \in \{2,3\}$.

Claim 3.1. *Let $i \in \{2,3\}$, and suppose that $H_i \not\prec H_{5-i}$. Then the following hold.*

(a) $l(H_i) > l(H_{5-i})$.

(b) $H_{5-i} \prec H_i$.

(c) *There exists an integer $n_1 \geqslant 1$ such that $H_i \simeq G_2^{n_1}(H_{5-i};v)$ for any vertex $v \in V(H_{5-i})$ with $l_{H_{5-i}}(v) = l(H_{5-i})$.*

Proof. Let v be a vertex of H_{5-i} such that $l_{H_{5-i}}(v) = l(H_{5-i})$. We first show that $H_1 \not\prec G_2^n(H_{5-i};v)$ for any $n \geqslant 0$. Suppose that $H_1 \prec G_2^n(H_{5-i};v)$ for some $n \geqslant 0$, and let H_1^0 be an induced subgraph of $G_2^n(H_{5-i};v)$ isomorphic to H_1. Note that for any $x \in L_{H_{5-i}}(v)$ and any $y \in V(G_2^n(H_{5-i};v)) - V(H_{5-i})$, the subgraph of $G_2^n(H_{5-i};v)$ induced by $(V(H_{5-i}) - \{x\}) \cup \{y\}$ is isomorphic to H_{5-i}. Since $H_1 \not\prec H_{5-i}$, it follows that H_1^0 contains at least $l(H_{5-i})+1$ vertices in $L_{H_{5-i}}(v) \cup (V(G_2^n(H_{5-i};v)) - V(H_{5-i}))$. Then H_1^0 has a leaf, which is a contradiction. Thus $H_1 \not\prec G_2^n(H_{5-i};v)$ for any $n \geqslant 0$. In particular,

$\{G_2^n(H_{5-i};v) : n \geqslant 0\} \subseteq \mathcal{F}(H_1) - \mathcal{F}(H_{5-i})$. Since $|\mathcal{F}(H_1, H_i) - \mathcal{F}(H_{5-i})| < \infty$, it follows that there exists an integer $n_1 \geqslant 0$ such that $G_2^{n_1}(H_{5-i};v)$ contains H_i as an induced subgraph. Since $H_i \not\prec H_{5-i}$, we have $n_1 \geqslant 1$. We take n_1 as small as possible. Let H_i^0 be an induced subgraph of $G_2^{n_1}(H_{5-i};v)$ isomorphic to H_i. If $L_{G_2^{n_1}(H_{5-i};v)}(v) \not\subseteq V(H_i^0)$, then $G_2^{n_1-1}(H_{5-i};v)$ contains H_i^0 as an induced subgraph, which contradicts the choice of n_1. Thus $L_{G_2^{n_1}(H_{5-i};v)}(v) \subseteq V(H_i^0)$. This implies that $l(H_i) \geqslant |L_{G_2^{n_1}(H_{5-i};v)}(v)| = l(H_{5-i}) + n_1 > l(H_{5-i})$. Consequently we obtain (a).

If $H_{5-i} \not\prec H_i$, then applying (a) with roles of H_i and H_{5-i} interchanged, we get $l(H_{5-i}) > l(H_i)$, which contradicts (a). Thus we obtain (b).

Next we show (c). By (b), there exists a set $X \subseteq V(H_i)$ such that $H_i - X \simeq H_{5-i}$. Since $l(H_i) \geqslant l(H_{5-i}) + n_1$, there exists a subset of X consisting of n_1 leaves of H_i. This together with the fact that $H_i \prec G_2^{n_1}(H_{5-i};v)$ leads to

$$|V(H_{5-i})| = |V(H_i)| - |X| \leqslant |V(H_i)| - n_1 \leqslant |V(G_2^{n_1}(H_{5-i};v))| - n_1 = |V(H_{5-i})|.$$

This forces $H_i \simeq G_2^{n_1}(H_{5-i};v)$. We also get $n_1 = |V(H_i)| - |V(H_{5-i})|$, which shows that n_1 does not depend on the choice of v such that $l_{H_{5-i}}(v) = l(H_{5-i})$. In particular, we obtain (c). □

Suppose that $H_i \not\prec H_{5-i}$ for some $i \in \{2, 3\}$. We may assume that $H_2 \not\prec H_3$. By Claim 3.1(b), we have $H_3 \prec H_2$. By Claim 3.1(c), $H_2 \simeq G_2^{n_1}(H_3;v)$ for any vertex $v \in V(H_3)$ with $l_{H_3}(v) = l(H_3)$.

Claim 3.2. (a) *We have* $p(H_2) > p(H_3)$.

(b) *There exists an integer* $n_2 \geqslant 1$ *such that* $H_2 \simeq G_3^{n_2}(H_3;v)$ *for any vertex* $v \in V(H_3)$ *for which there exists* $x \in V(H_3)$ *such that there is an* x*-good path of order* $p(H_3)$ *ending at* v.

Proof. Let v and x be as in (b). Then $p_{H_3}(x) = p(H_3)$, and there exists an x-good path $P = x_1x_2\cdots x_t$ with $x_1 = x$ and $x_t = v$ such that $|V(P)| = p(H_3)$. We first show that $H_1 \not\prec G_3^n(H_3;v)$ for any $n \geqslant 0$. Suppose that $H_1 \prec G_3^n(H_3;v)$ for some $n \geqslant 0$, and let H_1^0 be an induced subgraph of $G_3^n(H_3;v)$ isomorphic to H_1. Since $H_1 \not\prec H_3$, H_1^0 contains a vertex in $V(G_3^n(H_3;v)) - V(H_3)$. Then H_1^0 has a leaf, which is a contradiction. Thus $H_1 \not\prec G_3^n(H_3;v)$ for any $n \geqslant 0$. In particular, $\{G_3^n(H_3;v) : n \geqslant 0\} \subseteq \mathcal{F}(H_1) - \mathcal{F}(H_3)$. Since $|\mathcal{F}(H_1, H_2) - \mathcal{F}(H_3)| < \infty$, it follows that there exists an integer $n_2 \geqslant 0$ such that $G_3^{n_2}(H_3;v)$ contains H_2 as an induced subgraph. Since $H_2 \not\prec H_3$, we have $n_2 \geqslant 1$. We take n_2 as small as possible. Let H_2^0 be an induced subgraph of $G_3^{n_2}(H_3;v)$ isomorphic to H_2. Set $S = V(P) \cup (V(G_3^{n_2}(H_3;v)) - V(H_3))$. If $S \not\subseteq V(H_2^0)$, then we can verify that $G_3^{n_2-1}(H_3;v)$ contains H_2^0 as an induced subgraph, which contradicts the choice of n_2. Thus $S \subseteq V(H_2^0)$. This implies that $p(H_2) \geqslant p_{H_2^0}(x) \geqslant |S| = p(H_3) + n_2 > p(H_3)$, and hence we obtain (a).

Since $H_3 \prec H_2$, there exists a set $X \subseteq V(H_2)$ such that $H_2 - X \simeq H_3$. Since $p(H_2) \geqslant p(H_3) + n_2$, there exists a subset of X inducing a path of order n_2 in H_2. This together with the fact that $H_2 \prec G_3^{n_2}(H_3;v)$ leads to

$$|V(H_3)| = |V(H_2)| - |X| \leqslant |V(H_2)| - n_2 \leqslant |V(G_3^{n_2}(H_3;v))| - n_2 = |V(H_3)|.$$

This forces $H_2 \simeq G_3^{n_2}(H_3; v)$. We also get $n_2 = |V(H_2)| - |V(H_3)|$, which shows that n_2 does not depend on the choice of a vertex v satisfying the condition stated in (b). In particular, we obtain (b). $\square$

Suppose that $\delta(H_3) = 1$, and let v be a vertex of H_3 with $l_{H_3}(v) = l(H_3)$. Then by Claim 3.1(c), we have $H_2 \simeq G_2^{n_1}(H_3; v)$. Since $l_{H_3}(v) = l(H_3)$, v is not a leaf of H_3, and hence $p(H_3) = p(G_2^{n_1}(H_3; v)) = p(H_2)$, which contradicts Claim 3.2(a). Thus $\delta(H_3) \geqslant 2$. In particular, $l(H_3) = 0$ and $p(H_3) = 1$, and every $v \in V(H_3)$ satisfies the condition in Claim 3.1(c) as well as that in Claim 3.2(b). Take $v \in V(H_3)$. By Claim 3.1(c) and Claim 3.2(b), $H_2 \simeq G_2^{n_1}(H_3; v)$ for some $n_1 \geqslant 1$, and $H_2 \simeq G_3^{n_2}(H_3; v)$ for some $n_2 \geqslant 1$. Hence H_2 is isomorphic to the graph obtained from H_3 by adding one pendant edge to v. Since $v \in V(H_3)$ is arbitrary, this implies that

- for any $v \in V(H_3)$, H_2 is isomorphic to the graph obtained from H_3 by adding one pendant edge to v, and
- H_3 is vertex-transitive.

In particular, H_3 is an r-regular graph for some $r \geqslant 2$.

Case 1: $r = |V(H_3)| - 1$ (i.e., H_3 is complete).

Note that if H_1 is complete, then $|V(H_1)| > |V(H_3)| \geqslant 3$ because $H_1 \not\prec H_3$. If H_1 is complete, then for two vertices $x, y \in V(H_3)$ with $x \neq y$ and an integer $n \geqslant 1$, we can verify that $H_1 \not\prec G_2^n(H_3; \{x, y\})$, $H_2 \not\prec G_2^n(H_3; \{x, y\})$ and $H_3 \prec G_2^n(H_3; \{x, y\})$; if H_1 is not complete, then for an integer $n \geqslant |V(H_3)|$, we have $H_1 \not\prec K_n$, $H_2 \not\prec K_n$ and $H_3 \prec K_n$. In either case, it follows that $\mathcal{F}(H_1, H_2) - \mathcal{F}(H_3)$ is an infinite family, which contradicts the assumption $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$.

Case 2: $3 \leqslant r \leqslant |V(H_3)| - 2$.

Since H_3 is not complete, H_3 has non-adjacent vertices x and y. Let $n \geqslant 1$ be an integer. We show that $G_3^n(H_3; \{x, y\})$ contains neither H_1 nor H_2 as an induced subgraph. Set $U = V(G_3^n(H_3; \{x, y\})) - V(H_3)$, and let z be the unique vertex in U adjacent to x and y.

Suppose that $G_3^n(H_3; \{x, y\})$ contains H_1 as an induced subgraph, and let H_1^1 be an induced subgraph of $G_3^n(H_3; \{x, y\})$ isomorphic to H_1. Since $H_1 \not\prec H_3$ and $\delta(H_1) \geqslant 2$, we have $V(H_1^1) \cap U = \{z\}$ and $x, y \in V(H_1)$. Since $xy \notin E(G)$, it follows from (3.1) that H_1 is a path of order three, which contradicts the assumption that $\delta(H_1) \geqslant 2$. Thus $G_3^n(H_3; \{x, y\})$ does not contain H_1 as an induced subgraph.

Suppose that $G_3^n(H_3; \{x, y\})$ contains H_2 as an induced subgraph, and let H_2^1 be an induced subgraph of $G_3^n(H_3; \{x, y\})$ isomorphic to H_2. Recall that

- H_2 has exactly one vertex of degree 1 and exactly one vertex of degree $r + 1$,
- such two vertices are adjacent in H_2, and
- other vertices of H_2 have degree r.

Since $r \geqslant 3$, only x and y can have degree $r+1$ in H_2^1. Thus we may assume that $x \in V(H_2^1)$ and all neighbors of x in $G_3^n(H_3; \{x, y\})$ belong to $V(H_2^1)$. In particular, $z \in V(H_2^1)$. If $V(H_2^1) \cap (U - \{z\}) \neq \emptyset$, then H_2^1 has a vertex of degree 1 which is not adjacent to x, which is a contradiction. Thus $V(H_2) \cap (U - \{z\}) = \emptyset$. Then $d_{H_2^1}(z) \leqslant 2 < r$, and hence $d_{H_2^1}(z) = 1$. In particular, $y \notin V(H_2^1)$. This implies that $|V(H_2)| \leqslant |V(H_3)|$, which is a contradiction. Thus $G_3^n(H_3; \{x, y\})$ does not contain H_2 as an induced subgraph.

Since n is arbitrary, $\mathcal{F}(H_1, H_2) - \mathcal{F}(H_3)$ is an infinite family, which is a contradiction.

Case 3: $2 = r \leqslant |V(H_3)| - 2$.

Since H_3 is a cycle of order at least 4, we can write $H_3 = u_1u_2 \cdots u_m u_1$ with $m \geqslant 4$. Note that $u_1u_3 \notin E(H_3)$. By (3.2), H_1 contains a triangle. Recall that H_2 is obtained from H_3 by adding a pendant edge. Hence for an integer $n \geqslant 1$, we can verify that $H_1 \not\prec G_2^n(H_3; \{u_1, u_3\})$, $H_2 \not\prec G_2^n(H_3; \{u_1, u_3\})$ and $H_3 \prec G_2^n(H_3; \{u_1, u_3\})$. Thus we see that $\mathcal{F}(H_1, H_2) - \mathcal{F}(H_3)$ is an infinite family, which is a contradiction.

The contradictions in Cases 1–3 imply that $H_i \prec H_{5-i}$ for each $i \in \{2, 3\}$. In particular, $H_2 \simeq H_3$, and hence $(H_1; H_2, H_3)$ is a trivial tuple. This completes the proof of Theorem 1.2. $\square$

4 Concluding Remarks

In this paper, we studied the difference between forbidden pairs having a common graph, and we gave necessary conditions for $|\mathcal{F}(H_1, H_2) \triangle \mathcal{F}(H_1, H_3)| < \infty$. Since vertex-transitive graphs rarely appear as forbidden subgraphs, it follows from Theorems 1.1 and 1.2 that in most cases, $\mathcal{F}(H_1, H_2)$ will be different from $\mathcal{F}(H_1, H_3)$ if $(H_1; H_2, H_3)$ is not trivial and either H_1 is twin-less or $\delta(H_1) \geqslant 2$. However, since the twin-less seems to be a technical condition, we expect that same situation occurs without the twin-less condition. On the other hand, we cannot judge from Theorems 1.1 and 1.2 whether $\mathcal{F}_k(H_1, H_2)$ and $\mathcal{F}_k(H_1, H_3)$ are essentially different or not, where $\mathcal{F}_k(\mathcal{H})$ denotes the family of k-connected $\mathcal{H}$-free graphs. Fujita, Furuya and Ozeki [7] proved that $|\mathcal{F}_k(H_1) \triangle \mathcal{F}_k(H_2)| < \infty$ if and only if $H_1 \simeq H_2$. Thus we expect that almost all tuples $(H_1; H_2, H_3)$ satisfying $|\mathcal{F}_k(H_1, H_2) \triangle \mathcal{F}_k(H_1, H_3)| < \infty$ are trivial. We leave such a problem for the readers.

Acknowledgment

The authors would like to thank Professor Yoshimi Egawa for his assistance in correction of this paper.

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