On spherical designs of some harmonic indices

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Abstract

A finite subset Y on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ is called a spherical design of harmonic index t, if the following condition is satisfied: $\sum_{\mathbf{x}\in Y} f(\mathbf{x}) = 0$ for all real homogeneous harmonic polynomials $f(x_1, \ldots, x_n)$ of degree t. Also, for a subset T of $\mathbb{N} = \{1, 2, \cdots\}$, a finite subset $Y \subseteq S^{n-1}$ is called a spherical design of harmonic index T, if $\sum_{\mathbf{x}\in Y} f(\mathbf{x}) = 0$ is satisfied for all real homogeneous harmonic polynomials $f(x_1, \ldots, x_n)$ of degree k with $k \in T$.

In the present paper we first study Fisher type lower bounds for the sizes of spherical designs of harmonic index t (or for harmonic index T). We also study 'tight' spherical designs of harmonic index t or index T. Here 'tight' means that the size of Y attains the lower bound for this Fisher type inequality. The classification problem of tight spherical designs of harmonic index t was started by Bannai-Okuda-Tagami (2015), and the case $t = 4$ was completed by Okuda-Yu (2016). In this paper we show the classification (non-existence) of tight spherical designs of harmonic index 6 and 8, as well as the asymptotic non-existence of tight spherical designs of harmonic index $2e$ for general $e \geqslant 3$. We also study the existence problem for tight spherical designs of harmonic index T for some T, in particular, including index $T = \{8, 4\}$.

Keywords: spherical designs of harmonic index; Gegenbauer polynomial; Fisher type lower bound; tight design; Larman-Rogers-Seidel's theorem; Delsarte's method; semidefinite programming; elliptic diophantine equation.

1 Introduction and spherical designs of harmonic index t (or T)

Throughout this paper Y is assumed to be a finite non-empty set, and we denote the set of positive (resp. non-negative) integers by $\mathbb{N}$ (resp. $\mathbb{N}_0$).

Let $S^{n-1} = \{\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ be the unit sphere in the Euclidean space $\mathbb{R}^n$. Delsarte-Goethals-Seidel [6, Definition 5.1] (1977) gave the following definition of spherical designs.

Definition 1 (Spherical t-designs)**.** Let $t \in \mathbb{N}_0$. A subset $Y \subseteq S^{n-1}$ is called a *spherical t-design* on S^{n-1}, if

$$\frac{1}{|S^{n-1}|} \int_{\mathbf{x} \in S^{n-1}} f(\mathbf{x}) d\sigma(\mathbf{x}) = \frac{1}{|Y|} \sum_{\mathbf{y} \in Y} f(\mathbf{y}) \tag{1}$$

for any real polynomial $f(x_1, \ldots, x_n)$ of degree at most t, where $|S^{n-1}|$ denotes the volume (or the surface area) of the sphere S^{n-1}, and the integral is the surface integral on S^{n-1}.

The condition (1) is known to be equivalent to the condition:

$$\sum_{\mathbf{y} \in Y} f(\mathbf{y}) = 0 \text{ for all } f(x_1, \ldots, x_n) \in \mathrm{Harm}_k^n,\ 1 \leqslant k \leqslant t\,,$$

where Harm_k^n is the space of real homogeneous harmonic polynomials of degree k in n indeterminates.

In connection with the latter equivalent defining condition for spherical t-designs, we define a weaker concept which we call designs of harmonic index t as follows.

Definition 2 (Spherical designs of harmonic index t)**.** A subset $Y \subseteq S^{n-1}$ is called a *spherical design of harmonic index t* on S^{n-1}, if

$$\sum_{\mathbf{y} \in Y} f(\mathbf{y}) = 0 \tag{$*$}$$

for all real homogeneous harmonic polynomial $f(x_1, \ldots, x_n)$ of degree exactly t.

More generally we have the following definition.

Definition 3 (Spherical designs of harmonic index T)**.** Let T be a subset of $\mathbb{N}$. A subset $Y \subseteq S^{n-1}$ is called a *spherical design of harmonic index T* on S^{n-1} if

$$\sum_{\mathbf{y} \in Y} f(\mathbf{y}) = 0 \text{ for all } f(x_1, \ldots, x_n) \in \mathrm{Harm}_k^n \text{ with } k \in T.$$

A spherical design of harmonic index T with $T = \{1, 2, \ldots, t\}$ corresponds to a usual spherical t-design, and the case $T = \{t\}$ corresponds to a spherical design of harmonic index t.

We should remark that the concept of spherical designs of harmonic index t (or T) was already introduced by Delsarte-Seidel [7, Definition 4.1] (1989). However the study of this topic is started in Bannai-Okuda-Tagami [2] (2015).

The purpose of this paper is to study spherical designs of harmonic index t as well as harmonic index T for some T, and to convince the reader that these are interesting mathematical objects. Our main concerns are Fisher type lower bounds for spherical designs of harmonic index t and T, as well as the classification problems of so-called 'tight' designs. Here 'tight' means those that attain the lower bound in a Fisher type inequality. In Section 2 we provide a linear programming bound for spherical designs of harmonic index T. We also formulate Fisher type inequalities and tight spherical designs of harmonic index t or T. In Section 3, we discuss our philosophy how to find our test functions. In the subsequent sections we will study some specific problems. In Section 4 the complete non-existence results for tight spherical designs of harmonic index 6 and 8 are proved. Note that the case of $t = 4$ was already settled by Okuda-Yu [17] in a beautiful way by applying the SDP (semidefinite programming) to the existence problem of equiangular lines. Also note that our proofs for $t = 6$ and $t = 8$ are obtained in an elementary level without recourse to such deeper consideration as SDP. In Section 5 we show the asymptotic non-existence of harmonic index $2e$ case for general $e \geqslant 3$. Then we turn our attention to the case of $T = \{t_1, t_2\}$. The central model is the case $T = \{8, 4\}$ in Section 6. In Section 7 we study the cases $T = \{8, 2\}$, $\{8, 6\}$, $\{6, 2\}$, $\{6, 4\}$, as well as $\{10, 6, 2\}$, and $\{12, 8, 4\}$. We conclude the paper in Section 8 with some remarks.

The techniques which we used in the present paper are: (i) the linear programming method by Delsarte, (ii) the detailed information on the locations of the zeros as well as the local minimum values of Gegenbauer polynomials, (iii) the generalization by Nozaki of the Larman-Rogers-Seidel theorem on 2-distance sets to s-distance sets, (iv) the theory of elliptic diophantine equations, and (v) the semidefinite programming method of eliminating some 2-angular line systems for small dimensions.

2 Linear programming method for spherical designs of harmonic index T

In this section we consider a linear programming bound for spherical designs of harmonic index T. We also introduce some terminology and notation which will be used in the subsequent sections.

Let $Q_{n,k}(x)$ be the *Gegenbauer polynomial* of degree k in one variable x as introduced in [6, Definition 2.1]. Recall how the polynomials $Q_{n,k}(x)$ are normalized [6, Theorem 2.4, Theorem 3.2]:

$$Q_{n,k}(1) = \dim \mathrm{Harm}_k^n = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1} =: h_{n,k} \,.$$

The Gegenbauer polynomials $Q_{n,k}(x)$ are orthogonal polynomials on the closed interval $[-1, 1]$ with respect to the weight function $(1 - x^2)^{(n-3)/2}$, i.e.,

$$\int_{-1}^{1} Q_{n,k}(x)\, Q_{n,\ell}(x)\, (1 - x^2)^{\frac{n-3}{2}}\, dx = a_{n,k}\delta_{k,\ell} \,,$$

where $a_{n,k}$ is some (normalization) constant depending on n and k, and $\delta_{k,\ell}$ is the Kronecker delta. From this orthogonality it is well established that to any real polynomial $F(x)$ of degree r we can associate its *Gegenbauer expansion*

$$F(x) = \sum_{k=0}^{r} f_k \, Q_{n,k}(x) \,, \tag{2}$$

where the *Gegenbauer coefficients* f_k can be computed as follows:

$$f_k = \frac{1}{a_{n,k}} \int_{-1}^{1} F(x) \, Q_{n,k}(x) \, (1-x^2)^{\frac{n-3}{2}} \, dx.$$

We denote by $\mathbf{x} \cdot \mathbf{y}$ the standard inner product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. For a subset $Y \subseteq \mathbb{R}^n$ we set $I(Y) := \{\mathbf{x} \cdot \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in Y, \ \mathbf{x} \neq \mathbf{y}\}$. If $\{e_{k,1}, \ldots, e_{k,h_{n,k}}\}$ is any orthonormal basis for Harm_k^n with respect to the inner product $\langle f, g \rangle = \frac{1}{|S^{n-1}|} \int_{\mathbf{x} \in S^{n-1}} f(\mathbf{x}) g(\mathbf{x}) d\sigma(\mathbf{x})$ in Harm_k^n, then the well-known *addition formula* says that, for every $\mathbf{x}, \mathbf{y} \in S^{n-1}$,

$$Q_{n,k}(\mathbf{x} \cdot \mathbf{y}) = \sum_{i=1}^{h_{n,k}} e_{k,i}(\mathbf{x}) e_{k,i}(\mathbf{y}).$$

From the addition formula we have, for any $Y \subseteq S^{n-1}$,

$$\begin{aligned} M_k(Y) \ &:= \ \sum_{\mathbf{x},\mathbf{y} \in Y} Q_{n,k}(\mathbf{x} \cdot \mathbf{y}) = \sum_{\mathbf{x},\mathbf{y} \in Y} \sum_{i=1}^{h_{n,k}} e_{k,i}(\mathbf{x}) e_{k,i}(\mathbf{y}) \\ &= \ \sum_{i=1}^{h_{n,k}} \sum_{\mathbf{x},\mathbf{y} \in Y} e_{k,i}(\mathbf{x}) e_{k,i}(\mathbf{y}) = \sum_{i=1}^{h_{n,k}} \left(\sum_{\mathbf{x} \in Y} e_{k,i}(\mathbf{x}) \right)^2. \end{aligned}$$

Thus we obtain (see Definition 3 for (M2)) the following two simple observations:

(M1) The quantity $M_k(Y)$ is always non-negative;

(M2) Moreover, $M_k(Y) = 0$ for any $k \in T$ if and only if $Y \subseteq S^{n-1}$ is a spherical design of harmonic index T.

We introduce our main identity (see (3) below). (See [6] for the original discussion about so-called '*linear programming bounds*' for spherical designs.) Suppose $F(x)$ is a non-constant real polynomial of degree r which is of the form (2). For any $Y \subseteq S^{n-1}$, if we calculate $\sum_{\mathbf{x},\mathbf{y} \in Y} F(\mathbf{x} \cdot \mathbf{y})$ in two different ways, we find

$$|Y| \, F(1) + \sum_{\substack{\mathbf{x},\mathbf{y} \in Y, \\ \mathbf{x} \neq \mathbf{y}}} F(\mathbf{x} \cdot \mathbf{y}) = |Y|^2 \, f_0 + \sum_{k=1}^{r} f_k \, M_k(Y). \tag{3}$$

Now suppose $Y \subseteq S^{n-1}$ is a spherical design of harmonic index T. We are interested in finding a good lower bound for $|Y|$. (If $Y_1, Y_2 \subseteq S^{n-1}$ are spherical designs of harmonic index T with $Y_1 \cap Y_2 = \emptyset$, then so is $Y_1 \cup Y_2$.) If $F(x)$ satisfies

(LP1) $F(u) \geqslant 0$ for each $u \in [-1, 1]$;

(LP2) $f_k \leqslant 0$ for each non-negative integer k not in T.

then we obtain (recall (M1) and (M2)) that

$$|Y|\, F(1) \leqslant \text{LHS of (3)} = \text{RHS of (3)} \leqslant |Y|^2\, f_0\,, \tag{4}$$

where the first and second inequalities are due to (LP1) and (LP2), respectively. Moreover we necessarily have $f_0 > 0$, since by (LP1) the integrand of the following integral

$$f_0 = \frac{1}{a_{n,0}} \int_{-1}^{1} F(x)\,(1-x^2)^{\frac{n-3}{2}}\, dx$$

is non-negative and $a_{n,0} = \int_{-1}^{1} (1-x^2)^{\frac{n-3}{2}}\, dx > 0$. By (4) we have

$$|Y| \geqslant \frac{F(1)}{f_0}. \tag{5}$$

Therefore we want to find the following quantity:

$$S := \sup \left\{ \frac{F(1)}{f_0} \mid F(x) \text{ satisfies (LP1) and (LP2)} \right\}. \tag{6}$$

Definition 4. We call a polynomial $F(x)$ a *test function* provided that both (LP1) and (LP2) hold.

Note that the multiplication of $F(x)$ by any *positive* real number does not affect the consistency of (LP1) and (LP2), and does not change the value $\frac{F(1)}{f_0}$. Thus we might adopt an *equivalence relation* on the set of test functions defined using the multiplication by a positive real number.

It is also harmless to restrict the range of the above supremum by only considering $F(x)$ with (any) fixed constant term $f_0 = c$. If we introduce new variables $g_k := \frac{f_k}{f_0}$ ($k \geqslant 0$), then the computation of S in (6) becomes the following *linear programming* of infinite variables $(g_1, g_2, \ldots)$. (Notice that (6) itself is not a linear programming of the variables $(f_0, f_1, \ldots)$ because f_0 appears in the denominator of $\frac{F(1)}{f_0}$ so that $\frac{F(1)}{f_0}$ is not actually linear in f_0.)

$$S = \sup \left\{ 1 + \sum_{k=1}^{\infty} g_k\, Q_{n,k}(1) \right\} \quad \text{subject to} \tag{7}$$

(i) All but finite number of $g_1, g_2, \ldots \in \mathbb{R}$ are zero;

(ii) A polynomial $G(x) = 1 + \sum_{k=1}^{\infty} g_k\, Q_{n,k}(x)$ satisfies (LP1);

(iii) All $g_1, g_2, \ldots$ satisfy (LP2).

3 How do we find a good test function?

In this paper we consider the case where T consists of positive even integers $t_1 > t_2 > \cdots > t_\ell \geqslant 2$. Given n and T, what is the *best* choice of a test function $F(x)$ for (6)? The answer is not easy: First of all, what is the definition of "*goodness*" for $F(x)$? Of course, the most obvious one is that a "better" $F(x)$ provides a larger $\frac{F(1)}{f_0}$. However, this approach seems hopeless because the exact determination of S in (6) is too difficult. (It is not even apparent that there exists an optimal test function which attains the supremum in (6).) This means that there can be many approaches for choosing a test function with different definitions of "goodness".

In this section we show our way for choosing a test function, and explain its philosophy. We should emphasize here that our goal is not to maximize $\frac{F(1)}{f_0}$. In this paper, for our purpose, we only deal with a test function $F(x)$ of the following form:

$$F(x) = f_0 + \sum_{k \in T} f_k \, Q_{n,k}(x). \tag{8}$$

One reason why we only concern test functions of the form (8) is that it is the easiest one we can handle. More precisely, Eq. (8) is the easiest form for a polynomial $F(x)$ to satisfy (LP2). Another reason is that if $F(x)$ has the form (8), then we are able to guess when the equality in (5) is attained:

Proposition 5. *Let $Y \subseteq S^{n-1}$ be a spherical design of harmonic index T. Suppose a polynomial $F(x)$ has the form* (8)*, and satisfies* (LP1)*.* (*Thus $F(x)$ becomes a test function.*) *Then, $|Y| = \frac{F(1)}{f_0}$ if and only if $F(\alpha) = 0$ for all $\alpha \in I(Y)$.*

Proof. Recall $|Y| \geqslant \frac{F(1)}{f_0}$. Thus, $|Y| = \frac{F(1)}{f_0}$ if and only if equality in (4) holds if and only if $\sum_{\substack{\mathbf{x},\mathbf{y} \in Y, \\ \mathbf{x} \neq \mathbf{y}}} F(\mathbf{x} \cdot \mathbf{y}) = 0 = \sum_{k \in T} f_k \, M_k(Y)$. Suppose $|Y| = \frac{F(1)}{f_0}$. Since $F(x)$ has the form (8) and $M_k(Y) = 0$ for all $k \in T$ (Definition 3), the term $\sum_{k \in T} f_k M_k(Y)$ vanishes. We see from (LP1) that the equation $\sum_{\substack{\mathbf{x},\mathbf{y} \in Y, \\ \mathbf{x} \neq \mathbf{y}}} F(\mathbf{x} \cdot \mathbf{y}) = 0$ is equivalent to that $F(\alpha) = 0$ for all $\alpha \in I(Y)$. $\square$

Thus, if $Y \subseteq S^{n-1}$ is a spherical design of harmonic index $T = \{t_1 > \cdots > t_\ell\}$ (t_i : all even) and $F(x)$ is a test function of the form (8), then the previous proposition says that $|Y| = \frac{F(1)}{f_0}$ if and only if Y is a *distance set* where the distances between two distinct points in Y should occur in the zero set of $F(x)$.

Now it is the time when we should explain our definition of "goodness" for $F(x)$. Our philosophy for choosing $F(x)$ (of the form (8)) is that we want a distance set Y with large $|I(Y)|$ whenever $|Y|$ attains $\frac{F(1)}{f_0}$, i.e., we hope an *interesting* (*=large*) "tight" object. Therefore we require that

$$\text{a test function } F(x) \text{ has exactly } \ell \text{ non-negative zeros} \tag{9}$$

because we think that ℓ is the naturally expected largest number of non-negative zeros for $F(x)$ of the form (8). (Of course, it is still a difficult problem to determine the precise maximum number of minima for $F(x)$ in (8).)

We only deal with the case where there are only finitely many test functions satisfying our conditions (9), although the choices of test functions are possibly infinite for some index sets. Suppose that there are only m test functions $F_1(x), F_2(x), \ldots, F_m(x)$ (up to equivalence) which satisfy our condition (9) for a given index set T. For each test function $F_i(x)$, put $b_{n,T,F_i} := \frac{F_i(1)}{f_0^{(i)}}$, where $f_0^{(i)}$ is the constant term in the Gegenbauer expansion of $F_i(x)$.

Definition 6. With the above conditions and notation, we define $b_{n,T} := \max_{1 \leqslant i \leqslant m} \{b_{n,T,F_i}\}$. If the size of a spherical design Y of harmonic index T attains the lower bound $b_{n,T}$, then Y is said to be *tight*.

If there exists a unique test function $F(x)$ (up to equivalence), then Definition 6 implies that $b_{n,T} = \frac{F(1)}{f_0}$.

Remark 7. We should note that it is a very delicate problem to define "tight" designs for general T in a rigorous way. In the above argument, we required that our test function should have as many zeros as possible so that $b_{n,T}$ could be as large as possible. The reason to get as many zeros is that the "tight" set can be an s-distance set for larger s, otherwise the size cannot be large.

In the first section we defined the concept of spherical design of harmonic index t or more general T. This notion was already essentially defined in the literature, as "a spherical design which admits indices T". (See Delsarte-Seidel [7], say.) On the other hand, the terminology of spherical design of harmonic index t is already defined as "a spherical design for which equality in Definition 2 $(*)$ holds for any *homogeneous polynomials of degree* t", say in [7, 14], etc. In order to avoid the confusion with these terminologies, we use the term 'spherical designs of harmonic index t (or T)'. It seems that no systematic study of spherical designs of harmonic index T has been made, before Bannai-Okuda-Tagami [2]. They used the test function $F(x) = c_{n,t} + Q_{n,t}(x)$, with $c_{n,t} = -\min_{-1 \leqslant x \leqslant 1} Q_{n,t}(x)$ for $T = \{t\}$, and obtained the following theorem. (Note that this theorem is a special case of Proposition 5.)

Theorem 8 ([2, Theorem 1.2]). *Let Y be a design of harmonic index t on S^{n-1}. Then the following inequality holds:*

$$|Y| \geqslant 1 + \frac{Q_{n,t}(1)}{c_{n,t}}, \tag{10}$$

where $c_{n,t} = -\min_{-1 \leqslant x \leqslant 1} Q_{n,t}(x)$. Moreover equality in (10) *holds if and only if $Q_{n,t}(\alpha) = -c_{n,t}$ holds for any $\alpha \in I(Y)$.*

Remark 9. For $T = \{t\}$, the choice of test function is unique (up to equivalence). The reason is that our test function is of form (8), i.e. $F(x) = f_0 + Q_{n,t}(x)$. Moreover,

$f_0 = -\min_{-1 \leqslant x \leqslant 1} Q_{n,t}(x)$ is uniquely determined, from the monotonicity of the local minima of the Gegenbauer polynomial. Hence $b_{n,\{t\}} = 1 + \frac{Q_{n,t}(1)}{c_{n,t}}$ and we write $b_{n,t} = b_{n,\{t\}}$ for simplicity.

In [2, 17] they discussed the cases when $t = 2$ and 4. In the following section we discuss the existence problem of tight spherical designs of harmonic index t for $t = 6$ and 8. As is seen from Remark 9, if we consider $T = \{2e\}$, then there is only one positive zero of $F(x)$, equivalently, only one positive minima of $Q_{n,2e}(x)$ exists.

If we take $T = \{8, 4\}$, say, there are only at most two positive zeros. With property (9) and some calculation, it is shown that our test function $F(x)$ is determined uniquely. For $T = \{t_1, t_2, \ldots, t_\ell\}$ with t_i even, it seems that we can expect that there are at most ℓ non-negative zeros although we do not know the exact answer for general T. Among the possible test functions with this property, finding the best one, namely with the largest $b_{n,T}$, is not easy for general T. For example, in the discussion below, in the case of $T = \{12, 8, 4\}$ the candidates of test function $F(x)$ are not necessarily unique. Also, in some cases, no good test function exists. Thus, here we are compromising in taking the test function which seems to be the most natural one. We cannot eliminate the possibility of the existence of a better test function, in general case. For specific T, which are discussed in subsequent sections, we believe the choices of our test functions are natural and meaningful, although we do not show that rigorously at this stage. This situation may look to be an embarrassing situation, but this is even true for the definition of tight spherical t-designs, originally defined by Delsarte-Goethals-Seidel [6]. In the case of ordinary tight spherical $2e$-designs, the specifically chosen test function $(R_e(x))^2 = (Q_{n,0}(x) + \cdots + Q_{n,e}(x))^2$ satisfies the requirement that there are possible maximum e positive zeros. Moreover, it is expected to give the maximum $b_{n,T}$ (with $T = \{1, 2, \ldots, 2e\}$), so it satisfies our criterion of 'good' test function. On the other hand, there is no easy proof that it is the best test function. Still, the concept of tight spherical t-design in this particular choice of the test function was very meaningful. Our definition of tight spherical designs of harmonic index T has the same feature, and we have to compromise that the definitions of tight designs are not completely rigorously defined in the general case of T. (To show the test function is a best one is not easy and in many cases it is still undecided. However, such a problem is an unavoidable fact in this kind of theories.)

4 The non-existence of tight spherical designs of harmonic index 6 and 8

In this section we will prove the non-existence of tight spherical designs of harmonic index $t = 6$ and $t = 8$. The lower bound in Theorem 8 is obtained by the inequality (5) using following test function $F(x)$.

$$F(x) = c_{n,t} + Q_{n,t}(x) \quad \text{with} \quad c_{n,t} := -\min Q_{n,t}(x). \tag{11}$$

Throughout this section, $F(x) = c_{n,2e} + Q_{n,2e}(x)$ and we say Y is a tight spherical design of harmonic index $2e$ if $|Y|$ attains the lower bound $b_{n,2e}$ in (10).

Lemma 10. *If Y is a tight spherical design of harmonic index $2e$, then $|Y| \leqslant \frac{n(n+1)}{2}$.*

Proof. Recall that if $Y \subseteq S^{n-1}$ is a tight spherical design of harmonic index t, then $I(Y) \subseteq \{x \mid F(x) = 0\}$. It is known that $F(x)$ have exactly two roots $\{\alpha, -\alpha\}$, which means that Y is bounded above by the cardinality of spherical 2-distance set. For any spherical 2-distance set $X \subseteq S^{n-1}$ with $I(X) = \{\alpha, \beta\}$ and $\alpha + \beta \geqslant 0$, Musin [15] proved $|X| \leqslant \frac{n(n+1)}{2}$. Therefore $|Y| \leqslant \frac{n(n+1)}{2}$. □

4.1 The non-existence of tight spherical designs of harmonic index 6

In this subsection Y denotes a tight spherical design of harmonic index 6. The Gegenbauer polynomial $Q_{n,6}(x)$ (with our normalization $Q_{n,6}(1) = \dim \mathrm{Harm}_6^n$) is given by

$$Q_{n,6}(x) \;=\; \tfrac{n(n+2)(n+10)}{6!}\left\{(n+4)(n+6)(n+8)x^6 - 15(n+4)(n+6)x^4 + 45(n+4)x^2 - 15\right\}.$$

By taking the largest root for $Q'_{n,6}(x) = 0$, we get (see e.g. the proof of [2, Corollary 4.1]) the point α at which $Q_{n,6}(x)$ takes the minimum value, i.e., $Q_{n,6}(\alpha) = -c_{n,6}$. The lower bound $b_{n,6}$ of $|Y|$ defined in (10) can be obtained as well. The following are our results:

$$\begin{aligned}
\alpha^2 &= \tfrac{5(n+6)+\sqrt{10(n+3)(n+6)}}{(n+6)(n+8)},\\
c_{n,6} &= -\tfrac{n(n+2)(n+10)\left(2(n-2)(n+3)(n+6)+(n+3)(n+4)\sqrt{10(n+3)(n+6)}\right)}{36(n+6)(n+8)^2},\\
b_{n,6} &= \tfrac{(n+4)\left(20\sqrt{10(n+3)(n+6)}+(n+3)(n+6)(n^2+9n-12)\right)}{20\left(2(n-2)(n+6)+(n+4)\sqrt{10(n+3)(n+6)}\right)}.
\end{aligned}$$

It is not difficult to check that $|Y| = b_{n,6} > \frac{n(n+1)}{2}$ if $n \geqslant 37$. Moreover, $b_{2,6} = 2$ and $b_{24,6} = 231$ are the only two cases for which $b_{n,6} \in \mathbb{Z}$ when $n \leqslant 36$.

Remark 11. We should remark that not all the roots of $F(x)$ in (11) will necessarily appear in $I(Y)$ when n is small. Consider the case $n = 2$. Recall that $b_{2,2e} = 2$ is proved for general e in [2, p. 6]. Let y_1, y_2 be two unit vectors in $\mathbb{R}^2$ with angle $\theta = j\pi/2e$ for odd j. Then, by the argument in [2, p. 2], $Y = \{y_1, y_2\}$ is a tight spherical design of harmonic index $2e$ on S^1.

Larman-Rogers- Seidel (1977) proved the following fact.

Theorem 12 ([12, Theorem 2])**.** *Let X be a 2-distance set in $\mathbb{R}^n$ with Euclidean distances c and d $(c < d)$. If $|X| > 2n + 3$, then we have*

$$\frac{c^2}{d^2} = \frac{(k-1)}{k}$$

for some integer k with $2 \leqslant k \leqslant \frac{1+\sqrt{2n}}{2}$.

Suppose $n = 24$. Then Y is an at most 2-distance set in S^{23} with $I(Y) \subseteq \{\pm\alpha\}$. Assume X is a spherical 2-distance set with $I(X) = \{\pm\alpha\}$ such that $|Y| \leqslant |X|$. (There exists such X, otherwise $|Y|$ is strictly larger than the cardinality of any 2-distance set.) If we put $c = \sqrt{2 - 2\alpha}$ and $d = \sqrt{2 + 2\alpha}$, then c and d become the Euclidean distances between two distinct vectors in X. Note that $|X| \geqslant |Y| = b_{24,6} = 231 > 2 \times 24 + 3 = 51$. However, in this case, we obtain $c^2/d^2 = 1/3$ from easy calculation, contrary to Theorem 12. Hence there exists no tight spherical design of harmonic index 6 when $n = 24$.

4.2 The non-existence of tight spherical designs of harmonic index 8

In this subsection $Y \subseteq S^{n-1}$ denotes a tight spherical design of harmonic index 8. The Gegenbauer polynomial $Q_{n,8}(x)$ is

$$\begin{aligned} Q_{n,8}(x) \;=\; & \tfrac{n(n+2)(n+4)(n+14)}{8!}\Big\{(n+6)(n+8)(n+10)(n+12)x^8 - 28(n+6)(n+8)(n+10)x^6 \\ & +210(n+6)(n+8)x^4 - 420(n+6)x^2 + 105\Big\}. \end{aligned}$$

As in the preceding subsection we can obtain α, $c_{n,8}$, and also

$$b_{n,8} = \frac{1}{252 \times 12.03144913\cdots} \times n^4(1 + o(1)). \tag{12}$$

It can be checked that $|Y| = b_{n,8} > \frac{n(n+1)}{2}$ if $n \geqslant 20$ and, if $n \leqslant 19$, the only integral value is $b_{2,8} = 2$. By a similar argument as in Remark 11 one trivial example exists when $n = 2$.

Remark 13. We do not give the formulas of α, $b_{n,8}$ and $c_{n,8}$ explicitly, since they are extremely complicated. Here, $b_{n,8} > \frac{n(n+1)}{2}$ is checked from the formula of $b_{n,8}$ rather than from the asymptotic form (12).

5 The asymptotic non-existence of tight spherical designs of harmonic index $2e$ for general e

In this section we consider the existence of tight spherical designs of harmonic index $2e$ for $e \geqslant 5$, since the cases $e = 2, 3, 4$ were already treated. Our main result in this section is the following theorem.

Theorem 14. *Let $e \geqslant 2$ be fixed. Then there exist positive constants A_{2e} and B_{2e} such that*

$$\lim_{n\to\infty} \frac{c_{n,2e}}{n^e} = A_{2e} \quad \text{and} \quad \lim_{n\to\infty} \frac{b_{n,2e}}{n^e} = B_{2e},$$

where A_{2e} and B_{2e} depend only on e. Therefore,

$$c_{n,2e} = A_{2e} n^e (1 + o(1)) \quad \text{and} \quad b_{n,2e} = B_{2e} n^e (1 + o(1)).$$

Corollary 15. *Let $e \geqslant 3$ be fixed. If n is sufficiently large, then there exist no tight spherical designs of harmonic index $2e$.*

Proof. If Y is a tight spherical design of harmonic index $2e$, then $I(Y) \subseteq \{\pm\alpha\}$ for some $\alpha > 0$, and it follows from Lemma 10 that $|Y| \leqslant \frac{n(n+1)}{2}$. On the other hand, if n is sufficiently large, then Theorem 14 implies

$$|Y| = b_{n,2e} = B_{2e}n^e(1 + o(1)),$$

a contradiction. $\square$

Proof of Theorem 14. Szegő [19, p. 107] gives the asymptotic property of Gengebauer polynomial $C_t^\lambda(x)$:

$$\lim_{\lambda\to\infty} \lambda^{-\frac{t}{2}} C_t^\lambda(\lambda^{-\frac{1}{2}}x) = \frac{H_t(x)}{t!},$$

where $H_t(x)$ is the Hermite polynomial of degree t.

Recall that if $n \geqslant 3$ then $Q_{n,t}(x) = \frac{n+2t-2}{n-2} C_t^{(n-2)/2}(x)$. (See e.g. [6, p. 365].) Putting $\lambda = \frac{n-2}{2}$ and $t = 2e$, we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \lambda^{-\frac{t}{2}} C_t^\lambda(\lambda^{-\frac{1}{2}}x) &= \lim_{n\to\infty} \left(\frac{n-2}{2}\right)^{-e} \frac{n-2}{n+4e-2} Q_{n,2e}(\sqrt{\frac{2}{n-2}}\,x) \\
&= 2^e \lim_{n\to\infty} n^{-e} Q_{n,2e}(\sqrt{\frac{2}{n-2}}\,x).
\end{aligned}$$

Set $P_{n,e}(x) = n^{-e}Q_{n,2e}(\sqrt{\frac{2}{n-2}}\,x)$ for simplicity. Then we have

$$2^e \lim_{n\to\infty} P_{n,e}(x) = \frac{H_{2e}(x)}{(2e)!}. \tag{13}$$

Take the derivative with respect to x on both sides of (13). For fixed e, since $P'_{n,e}(x)$ uniformly converges to $\frac{2^e}{(2e-1)!}x^{2e-1}$ as n tends to be infinity, we get the following result:

$$2^e \lim_{n\to\infty} \frac{d}{dx} P_{n,e}(x) = 2^e \frac{d}{dx}\left(\lim_{n\to\infty} P_{n,e}(x)\right) = \frac{H'_{2e}(x)}{(2e)!} = \frac{4e}{(2e)!} H_{2e-1}(x),$$

where the last equality is due to the property $H'_t(x) = 2tH_{t-1}(x)$. Let x_1 be the largest zero of $H_{2e-1}(x)$. Then

$$2^e \lim_{n\to\infty} \frac{d}{dx}\left(n^{-e}Q_{n,2e}(\sqrt{\frac{2}{n-2}}\,x)\right)\Big|_{x=x_1} = \frac{1}{(2e)!} H'_{2e}(x_1) = 0.$$

Thus the following equality can be obtained.

$$A_{2e} = -\lim_{n\to\infty} \frac{\min Q_{n,2e}(x)}{n^e} = -\lim_{n\to\infty} \frac{Q_{n,2e}(\sqrt{\frac{2}{n-2}}\,x_1)}{n^e} = -\frac{H_{2e}(x_1)}{2^e(2e)!}.$$

Recall that $b_{n,t} = 1 + \frac{Q_{n,t}(1)}{c_{n,t}}$ and $Q_{n,t}(1) = \binom{n+t-1}{n-1} - \binom{n+t-3}{n-1}$. This implies

$$B_{2e} = \lim_{n\to\infty} \frac{b_{n,2e}}{n^e} = \frac{1}{(2e)!A_{2e}} = -\frac{2^e}{H_{2e}(x_1)}.$$

$\square$

Remark 16. In Theorem 14 we did not give explicit evaluation of B_{2e}, but it is possible to give it, since the locations of the zeros of Hermite polynomials and the (local) minimum values of $H_{2e}(x)$ are well studied. Also, if we want to evaluate $b_{n,2e}$ explicitly from below, rather than evaluating B_{2e}, it is also possible, although we will not discuss it in this paper. For this purpose, the following papers [4], [8], [9], [11] may be useful to do that. It seems that there are many literature on this.

6 Tight spherical designs of harmonic index $\{8, 4\}$

In what follows, we assume $T = \{t_1, t_2, \ldots, t_\ell\}$ with $t_1 = 2e > t_2 > \cdots > t_\ell$ and t_i $(1 \leqslant i \leqslant \ell)$ even. And we investigate the case where our test function is of the form

$$F(x) = f_0 + Q_{n,2e}(x) + f_{t_2}Q_{n,t_2}(x) + \cdots + f_{t_\ell}Q_{n,t_\ell}(x).$$

Let $L(x) = Q_{n,2e}(x) + f_{t_2}Q_{n,t_2}(x) + \cdots + f_{t_\ell}Q_{n,t_\ell}(x)$. Then as we mentioned in Section 3, it is better that $F(x)$ have as many zeros in $[-1, 1]$. For this purpose we are interested in the case where $L(x)$ attains minimum value at ℓ non-negative points $\alpha_1, \ldots, \alpha_\ell \in [0, 1]$.

Delsarte-Goethals-Seidel (1977) gave an upper bound for a spherical s-distance set $X \subseteq S^{n-1}$.

Theorem 17 ([6, Theorem 4.8])**.** *If X is a spherical s-distance set in S^{n-1}, then* $|X| \leqslant \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$.

Using the above theorem, we obtain the following lemma which gives the relation between the size of tight spherical design of harmonic index T and the cardinality of spherical s-distance set.

Lemma 18. *If Y is a tight spherical design of harmonic index T, and $L(x) = Q_{n,2e}(x) + f_{t_2}Q_{n,t_2}(x) + \cdots + f_{t_\ell}Q_{n,t_\ell}(x)$ takes minimum value at ℓ non-negative points, then*

$$|Y| \leqslant \begin{cases} \binom{n+2\ell-2}{n-1} + \binom{n+2\ell-3}{n-1} & \text{if } 2 \nmid e \\ \binom{n+2\ell-1}{n-1} + \binom{n+2\ell-2}{n-1} & \text{if } 2 \mid e \end{cases}$$

Proof. Let $c_{n,T,L} = -\min L(x)$ and $F(x) = L(x) + c_{n,T,L}$. If e is even (resp. odd), then $F(x)$ has 2ℓ (resp. $2\ell - 1$) zeros. By the assumption, $F(x)$ has ℓ non-negative roots. Namely, $|I(Y)| \leqslant 2\ell$ if e is even, then $Y \in S^{n-1}$ is an at most 2ℓ-distance set. It follows from Theorem 17 that $|Y| \leqslant \binom{n+2\ell-1}{n-1} + \binom{n+2\ell-2}{n-1}$. Similarly, we can get the conclusion if e is odd. □

Consider the case of $T = \{t_1, t_2\}$ with $t_2 =$ even and $t_1 = 2e > t_2$. In this case we use the test function $F(x) = f_0 + Q_{n,2e}(x) + f_{t_2}Q_{n,t_2}(x)$. Let $L(x) = Q_{n,2e}(x) + f_{t_2}Q_{n,t_2}(x)$. In this case we are interested in the cases when $L(x)$ has the minimum value $-c_{n,T}$ at exactly two non-negative points α, β (with $\alpha > \beta$) and choose $f_0 = c_{n,T}$. For this purpose we take

$$L(x) = Q_{n,2e}(x) + f_{t_2}Q_{n,t_2}(x),$$

and we want to determine f_{t_2} such that

$$Q_{n,2e}(x) + f_{t_2} Q_{n,t_2}(x) = a(x^2-\alpha^2)^2(x^2-\beta^2)^2 - c_{n,T}$$

for some α, β and $c_{n,T}$.

Suppose $t_1 = 8$ and $t_2 = 4$. Then the problem is to find f_4 such that

$$Q_{n,8}(x) + f_4 Q_{n,4}(x) = a(x^2-\alpha^2)^2(x^2-\beta^2)^2 - c_{n,T}. \tag{14}$$

The Gegenbauer polynomial $Q_{n,4}(x)$ is

$$Q_{n,4}(x) = \frac{n(n+6)}{4!}\left\{(n+2)(n+4)x^4 - 6(n+2)x^2 + 3\right\}.$$

By comparing the coefficients in (14), we obtain the following equations:

$$a = \tfrac{n(n+2)(n+4)(n+6)(n+8)(n+10)(n+12)(n+14)}{8!},$$

$$2a(\alpha^2+\beta^2) = \tfrac{28n(n+2)(n+4)(n+6)(n+8)(n+10)(n+14)}{8!},$$

$$a(\alpha^4+\beta^4+4\alpha^2\beta^2) = \tfrac{210n(n+2)(n+4)(n+6)(n+8)(n+14)}{8!} + \tfrac{n(n+2)(n+4)(n+6)}{4!} f_4,$$

$$2a\alpha^2\beta^2(\alpha^2+\beta^2) = \tfrac{420n(n+2)(n+4)(n+6)(n+14)}{8!} + \tfrac{6n(n+2)(n+6)}{4!} f_4,$$

$$a\alpha^4\beta^4 - c_{n,T} = \tfrac{105n(n+2)(n+4)(n+14)}{8!} + \tfrac{3n(n+6)}{4!} f_4.$$

Therefore,

$$\alpha^2+\beta^2 = \frac{14}{n+12},$$

$$(\alpha^2+\beta^2)^2 + 2\alpha^2\beta^2 = \tfrac{210}{(n+10)(n+12)} + \tfrac{1680}{(n+8)(n+10)(n+12)(n+14)} f_4,$$

$$\alpha^2\beta^2(\alpha^2+\beta^2) = \tfrac{210}{(n+8)(n+10)(n+12)} + \tfrac{5040}{(n+4)(n+8)(n+10)(n+12)(n+14)} f_4.$$

We have

$$f_4 = \frac{(n+4)(n+5)(n+14)}{60(n+12)}.$$

Hence α^2 and β^2 are the solutions of the following quadratic equation in the variable u:

$$u^2 - \frac{14}{n+12}u + \frac{21}{(n+8)(n+12)} = 0.$$

Finally, we obtain

$$\alpha^2, \beta^2 = \frac{7(n+8) \pm 2\sqrt{7(n+5)(n+8)}}{(n+8)(n+12)},$$

$$b_{n,T} = \frac{1}{252}(n+1)(n+2)(n+5)(n+6),$$

where $b_{n,T}$ is the lower bound in (5).

In the following we shall prove the non-existence of tight spherical designs of harmonic index $\{8,4\}$ stated in Theorem 23 below. If Y is a tight spherical design of harmonic index $\{8,4\}$, then $I(Y) \subseteq \{\pm\alpha, \pm\beta\}$. We define

$$U(h) := \left\lfloor \frac{1}{2} + \sqrt{\frac{h^2}{2h-2} + \frac{1}{4}} \right\rfloor.$$

For a spherical s-distance set, Nozaki (2011) generalized Larman-Rogers-Seidel theorem [12, Theorem 2] as follows.

Theorem 19 ([16, Theorem 5.1])**.** *Let Y be an s-distance set on S^{n-1} with $s \geqslant 2$ and $I(Y) = \{\beta_1, \ldots, \beta_s\}$. Put $N := \binom{n+s-2}{s-1} + \binom{n+s-3}{s-2}$. If $|Y| \geqslant 2N$, then for each $i = 1, \ldots, s$,*

$$k_i := \prod_{\substack{j=1,\ldots,s,\\ j\neq i}} \frac{1-\beta_j}{\beta_i - \beta_j}$$

must be an integer with $|k_i| \leqslant U(N)$.

If $X = Y \cup (-Y)$ is an antipodal spherical s-distance set, then Y is a spherical $(s-1)$-distance set. Nozaki (2011) proved the following theorem. The conditions of $|X|$ in Theorem 20 are less restrictive than that in Theorem 19.

Theorem 20 ([16, Theorem 5.2])**.** *Let X be an antipodal s-distance set on S^{n-1} where s is an odd integer at least 5.*
Suppose $I(X) = \{-1, \pm\beta_1, \pm\beta_2, \ldots, \pm\beta_{\frac{s-1}{2}}\}$.

(1) *Let $N = \binom{n+s-4}{s-3}$. If $|X| \geqslant 4N$, then for each $i = 1, \ldots, (s-1)/2$,*

$$k_i := \prod_{\substack{j=1,\ldots,\frac{s-1}{2},\\ j\neq i}} \frac{1-\beta_j^2}{\beta_i^2 - \beta_j^2}$$

must be an integer with $|k_i| \leqslant U(N)$.

(2) *Let $N = \binom{n+s-3}{s-2}$. If $|X| \geqslant 4N+2$, then for each $i = 1, \ldots, (s-1)/2$,*

$$k_i := \frac{1}{\beta_i} \prod_{\substack{j=1,\ldots,\frac{s-1}{2},\\ j\neq i}} \frac{1-\beta_j^2}{\beta_i^2 - \beta_j^2}$$

must be an integer with $|k_i| \leqslant \lfloor\sqrt{2N^2/(N+1)}\rfloor$.

With the above theorem and [16, Theorem 5.3], Nozaki showed that inner products for an antipodal spherical s-distance set are in fact rational.

Theorem 21 ([16, Theorem 5.4])**.** *Suppose X is an antipodal s-distance set on S^{n-1} with $s \geqslant 4$. If $|X| \geqslant 4\binom{n+s-3}{s-2} + 2$, then β is rational for any $\beta \in I(X)$.*

A tight spherical design of harmonic index $\{8, 4\}$ is regarded as an at most 4-distance set $Y \subset S^{n-1}$ with $I(Y) \subseteq \{\pm\alpha, \pm\beta\}$. We construct an antipodal set $X' = Y \cup (-Y)$. Note that $I(X') \subseteq \{-1, \pm\alpha, \pm\beta\}$. Assume $X \subset S^{n-1}$ is a spherical 5-distance set with $I(X) = \{-1, \pm\alpha, \pm\beta\}$ such that $|X| \geqslant |X'| = 2|Y|$. By applying Theorem 20 to the set X for $s = 5$, we obtain the next lemma.

Lemma 22. *Suppose Y' is a spherical 4-distance set $\{\pm\alpha, \pm\beta\}$. Let $X = Y' \cup (-Y')$.*

(1) *If $|Y'| \geqslant 2\binom{n+1}{2}$, then the following two numbers are integers:*

$$k_1 = \frac{1-\alpha^2}{\beta^2 - \alpha^2}, \qquad k_2 = \frac{1-\beta^2}{\alpha^2 - \beta^2}.$$

(2) *If $|Y'| \geqslant 2\binom{n+2}{3} + 1$, then the following two numbers are integers:*

$$k_1 = \frac{1-\alpha^2}{\beta(\beta^2 - \alpha^2)}, \qquad k_2 = \frac{1-\beta^2}{\alpha(\alpha^2 - \beta^2)}.$$

Theorem 23. *There exists no tight spherical design of harmonic index $\{8, 4\}$ on S^{n-1} for all n.*

Proof. If Y is a tight spherical design of harmonic index $\{8, 4\}$, then

$$|Y| = \frac{(n+1)(n+2)(n+5)(n+6)}{252} \text{ with } I(Y) \subseteq \{\pm\alpha, \pm\beta\},$$

where

$$\alpha, \beta = \sqrt{\frac{7(n+8) \pm 2\sqrt{7(n+5)(n+8)}}{(n+8)(n+12)}}.$$

Assume X is a spherical 4-distance set with $I(X) = \{\pm\alpha, \pm\beta\}$ such that $|Y| \leqslant |X|$. We shall consider three cases: $n \geqslant 76$, $9 \leqslant n \leqslant 75$, and $2 \leqslant n \leqslant 8$.

Case (1): If $n \geqslant 76$, then

$$|X| \geqslant |Y| \geqslant 2\left(\binom{n+2}{3} + \binom{n+1}{2}\right) > 2\binom{n+2}{3} + 1.$$

By Theorem 19, k_1, k_2 are integers. We have

$$k_2 = \frac{1-\beta^2}{\alpha^2 - \beta^2} = \frac{2 + \sqrt{\frac{(n+5)(n+8)}{7}}}{4} = z \in \mathbb{Z}.$$

Hence $(n+5)(n+8) = 7(4z-2)^2$. By Lemma 22 we have

$$\alpha\beta = \sqrt{\frac{21}{(n+8)(n+12)}} \in \mathbb{Q}.$$

Then $(n+8)(n+12) = 21p^2/q^2$ for some coprime integers p and q. Furthermore $21p^2/q^2$ should be an integer. Thus $q^2|21$, i.e., $q = 1$. We have $(n+8)(n+12) = 21p^2$ and get the following table for some integers y_1, y_2, y_3.

	$n+5$	$n+8$	$n+12$
(i)	y_1^2	$7y_2^2$	$3y_3^2$
(ii)	$7y_1^2$	y_2^2	$21y_3^2$
(iii)	$3y_1^2$	$21y_2^2$	y_3^2
(iv)	$21y_1^2$	$3y_2^2$	$7y_3^2$

We know that $\gcd(n+5, n+8) = 1$ or 3. If $\gcd(n+5, n+8) = 1$, then $(n+5)(n+8) = 7(4z-2)^2$ implies that $n+5 = y_1^2$, $n+8 = 7y_2^2$ or $n+5 = 7y_1^2$, $n+8 = y_2^2$. If $\gcd(n+5, n+8) = 3$, then $n+5 = 3y_1^2$, $n+8 = 21y_2^2$ or $n+5 = 21y_1^2$, $n+8 = 3y_2^2$.

For case (i), $n+12 = 3y_3^2$ is obtained from $(n+8)(n+12) = 21p^2$. We can similarly get the other three cases in the above table.

(i). $7 = 3y_3^2 - y_1^2$ implies $y_1^2 \equiv 2 \pmod 3$. Impossible.
(ii). $7 = 21y_3^2 - 7y_1^2$ implies $y_1^2 \equiv 2 \pmod 3$. Impossible.
(iii). $3 = 21y_2^2 - 3y_1^2$ implies $y_1^2 \equiv 6 \pmod 7$. Impossible.
(iv). We cannot get a contradiction from a basic observation, but this problem can be formulated as the integral solutions of the following equation:

$$y^2 = (n+5)(n+8)(n+12).$$

By linear transformation $x = n+8$, this equation becomes $y^2 = x^3 + x^2 - 12x$. From the database of elliptic curve with LMFDB label 168.b2, we know that

$$(x, y) = (-4, 0), (0, 0), (3, 0)$$

are all integral solutions of $y^2 = x^3 + x^2 - 12x$, namely, $y^2 = (n+5)(n+8)(n+12)$ has no non-trivial integral solution.

Remark 24.
i). Any elliptic curve over $\mathbb{Q}$ has a Weierstrass equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \qquad (**)$$

They are often displayed as a list $[a_1, a_2, a_3, a_4, a_6]$. More information about the database of elliptic curve is available from:

http://www.lmfdb.org/EllipticCurve/Q

ii). The integral solutions of some elliptic equations of form $(**)$ can be solved using SAGE [18] with the following two commands (the reader should put suitable values for a_1, a_2, a_3, a_4, a_6):

```
E=EllipticCurve(QQ,[a1,a2,a3,a4,a6])
E.integral_points()
```

Case (2): If $9 \leqslant n \leqslant 75$, then $|X| \geqslant |Y| \geqslant 2\binom{n+1}{2}$. Using the first statement in Lemma 22, we see that both $\frac{1-\alpha^2}{\beta^2-\alpha^2}$ and $\frac{1-\beta^2}{\alpha^2-\beta^2}$ are integers. It is easy to check that neither of them is an integer for $9 \leqslant n \leqslant 75$.

Case (3): If $2 \leqslant n \leqslant 8$, $b_{8,T} = 65$ is the unique case with $b_{n,T} \in \mathbb{Z}$. We set up the semidefinite programming (SDP) method on the upper bounds for spherical 4-distance sets with the indicated inner product values. Theorem 25 is what we set up to estimate the upper bounds of spherical 4-distance sets. Such an SDP formula can be obtained from special setting of Bachoc-Vallentin [1, p. 10–11] or generalization of Barg-Yu [3, Theorem 3.1] for spherical 2-distance sets. We choose the positive semidefinite matrices S_k^n with size $(9-k) \times (9-k)$ and linear constraints $\sum_{c_i, c_j \in Y} Q_{n,k}(\langle c_i, c_j \rangle) \geqslant 0$ for $k = 1, \cdots, 8$. (S_k^n is the same notation in [3]). When $n = 8$, the SDP upper bound solved by CVX for the spherical 4-distance set is 50.23.

We follow the argument in [10, p. 79] to obtain rigorous bound of our semidefinite programming problems. We independently solve the dual problem to avoid numerical issue in CVX and guarantee that our bounds are justified. We checked that the error did not affect our results. i.e. our computational SDP bounds plus the error still strictly less than linear programming bound of tight spherical designs of harmonic index T. Then, such tight designs do not exist.

In our problem, the solved value and the error are $1 + \frac{1}{3}\sum_{i=1}^{4} x_i = 50.23$ and $\epsilon = \sum_{i=1}^{24} x_i \epsilon_i' \leqslant 4.7605 \times 10^{-5}$, where $\epsilon_i' = \max\{\epsilon_i, 0\}$. (Even if we take $\epsilon_i' = |\epsilon_i|$, the error $\epsilon = \sum_{i=1}^{24} x_i \epsilon_i'$ is very small and bounded above by 0.00011592.) Hence our computational SDP bound plus error is still strictly less than the LP bound (=65) for spherical design of harmonic index $\{8, 4\}$. We can conclude there exists no tight spherical design of harmonic index $\{8, 4\}$. ☐

Theorem 25. *Let Y be a spherical 4-distance set with inner products a, b, c and d. Let p be a positive integer. The cardinality $|Y|$ is bounded above by the solution of the following semidefinite programming problem:*

$$1 + \frac{1}{3}\max(x_1 + x_2 + x_3 + x_4)$$

subject to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(x_1 + x_2 + x_3 + x_4) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \sum_{(u,v,t) \in \{a,b,c,d\}} x(u,v,t) \succeq 0$$

$$3 + Q_{n,k}(a)x_1 + Q_{n,k}(b)x_2 + Q_{n,k}(c)x_3 + Q_{n,k}(d)x_4 \geqslant 0, \quad k = 1, 2, \ldots, p$$

$$\begin{aligned} &S_k^n(1,1,1) + S_k^n(a,a,1)x_1 + S_k^n(b,b,1)x_2 + S_k^n(c,c,1)x_3 + S_k^n(d,d,1)x_4 \\ &+ \sum_{(u,v,t) \in \{a,b,c,d\}} S_k^n(u,v,t)x(u,v,t) \succeq 0, \quad k = 0, 1, \ldots, p \end{aligned}$$

$$x_1, x_2, x_3, x_4, x(u,v,t) \geqslant 0,$$

where $S_k(\cdot,\cdot,\cdot)$ are $(p-k+1)\times(p-k+1)$ matrices with entries related to Gegenbauer polynomial and explicit definition can be found in [3] equation (2-3).

In this theorem the variables $x_1,\dots,x_4$ refer to the number of ordered pairs of vectors in Y with inner product a,b,c and d respectively; for instance $(1/3)x_1 = |Y|^{-1}\big|\{(c_1,c_2)\in Y^2 : \langle c_1,c_2\rangle = a\}\big|$. The variables $x(u,v,t)$ refer to the number of triples in Y such that inner products are (u,v,t), where $(u,v,t)\in\{a,b,c,d\}$. The number of (u,v,t) is the combinations with repetition for choosing three times out of 4 elements and it is $\binom{4+3-1}{3}=20$. We use CVX solver in MATLAB to solve the above optimization problems. We set up $p=8$ i.e. S_0^n matrix is an 9×9 matrix. The key inequalities to set up SDP bounds for spherical few distance sets are:

$$\sum_{c_i,c_j\in Y} Q_{n,k}(\langle c_i,c_j\rangle)\geqslant 0$$

and

$$\sum_{c_i,c_j,c_m\in Y} S_k^n(\langle c_i,c_j\rangle,\langle c_i,c_m\rangle,\langle c_j,c_m\rangle)\succeq 0.$$

Therefore, for spherical s-distance sets with $s\geqslant 5$, it is not hard to set up the SDP upper bounds for the cardinality and clarify the feasibility of our SDP bounds.

7 Tight spherical designs of harmonic index $\{6,4\}$, $\{6,2\}$, $\{8,6\}$, $\{8,2\}$, as well as $\{10,6,2\}$, $\{12,8,4\}$

We consider the cases when $T=\{t_1,t_2,\dots,t_\ell\}$ and $L(x)$ takes the minimum value $-c_{n,T,L}$ at ℓ non-negative points, where

$$L(x)=Q_{n,2e}(x)+f_{t_2}Q_{n,t_2}(x)+\cdots+f_{t_\ell}Q_{n,t_\ell}(x).$$

In this section, we will prove the non-existence of tight spherical designs of harmonic index T for some T with $\ell=2$ or 3.

7.1 The non-existence of tight spherical designs of harmonic index $\{6,4\}$

Find f_4 such that

$$Q_{n,6}(x)+f_4Q_{n,4}(x)=ax^2(x^2-\alpha^2)^2-c_{n,T,L}. \tag{15}$$

By comparing the coefficients in (15), we get the following results:

$$\alpha^2=\frac{15}{2(n+8)}-\frac{15}{(n+8)(n+10)}f_4,$$

$$\alpha^4=\frac{45}{(n+6)(n+8)}-\frac{180}{(n+4)(n+8)(n+10)}f_4.$$

Solving for f_4 and α^2 from these two equations gives:

$$f_4 = \frac{n+10}{10(n+4)(n+6)}\left((n+6)(n-12) \pm 2(n+8)\sqrt{-(n+6)(n-4)}\right),$$

$$\alpha^2 = \frac{3\left(2(n+6) \pm \sqrt{-(n+6)(n-4)}\right)}{(n+4)(n+6)}.$$

If $n \geqslant 5$, then f_4 is a complex number. So we cannot find $L(x) = Q_{n,6}(x) + f_4 Q_{n,4}(x)$ satisfying our assumption. It is easy to check that $b_{2,T} = 2$, $b_{3,T} = 3$ (or 0) and $b_{4,T} = 2$. By Remark 11, there exists no tight spherical design of harmonic index $\{6, 4\}$ when $n = 2$ and $|Y| = 2$. When $n = 3$ and $n = 4$, observe that the lower bounds for spherical designs of harmonic index 6 are about 3.41 and 5.29, respectively, which are strictly larger than $b_{3,T}$ and $b_{4,T}$. Note that spherical design of harmonic index $\{6, 4\}$ should also satisfy the condition for harmonic index 6. From the discussion above, there exists no tight spherical design of harmonic index $\{6, 4\}$ for any n.

7.2 The non-existence of tight spherical designs of harmonic index $\{6, 2\}$

The Gegenbauer polynomial $Q_{n,2}(x)$ is

$$Q_{n,2}(x) = \frac{(n+2)(nx^2-1)}{2}.$$

Let $Q_{n,6}(x) + f_2 Q_{n,2}(x) = ax^2(x^2-\alpha^2)^2 - c_{n,T,L}$. Then with similar calculation we have the following results.

$$\begin{aligned} f_2 &= \frac{(n-2)(n+4)(n+10)}{32(n+8)}, \quad \alpha = \pm\sqrt{\frac{15}{2(n+8)}}, \\ b_{n,T} &= \frac{n(n+4)(2n+1)^2}{15(7n-4)} \\ &= \frac{1}{2401 \times 15}(1372n^3 + 7644n^2 + 10199n + 7200) + \frac{1920}{2401(7n-4)}. \end{aligned}$$

Case (1): If $n \geqslant 8817$, i.e., $\left|\frac{1920}{2401(7n-4)}\right| < \frac{1}{2401 \times 15}$, then $b_{n,T}$ is not an integer.
Case (2): Tight spherical design Y of harmonic index $\{6, 2\}$ is regarded as an at most 3-distance set in S^{n-1} with $I(Y) \subseteq \{0, \pm\alpha\}$.
Lemmens-Seidel (1973) proved the following fact.

Theorem 26 ([13, Theorem 3.4]). *If there are $|X|$ equiangular lines with angle* arccos α *in Euclidean n-dimensional space $\mathbb{R}^n$, and if $|X| > 2n$, then $1/\alpha$ is an odd integer.*

Then we can give a weaker condition for α with 3-distance set as follows.

Theorem 27. *If $X \subseteq S^{n-1}$ is a spherical 3-distance set with $I(X) = \{0, \pm\alpha\}$ and $|X| > 2n$, then $1/\alpha$ is an integer.*

Proof. Let X be a set of unit vectors whose mutual inner product set is $\{0, \pm\alpha\}$, and let G be the Gram matrix of such vectors. Then,

$$G = \begin{pmatrix} 1 & & x \\ & \ddots & \\ x & & 1 \end{pmatrix}, \quad A = \frac{1}{\alpha}(G - I) = \begin{pmatrix} 0 & & x' \\ & \ddots & \\ x' & & 0 \end{pmatrix},$$

where $x \in \{0, \pm\alpha\}$ and $x' \in \{0, \pm 1\}$.

G is a symmetric and positive semidefinite matrix of order $|X|$. It has the smallest eigenvalue 0 of multiplicity $m \geqslant |X| - n$. Therefore, A has the smallest eigenvalue $-1/\alpha$ of multiplicity $m \geqslant |X| - n$. Moreover, $-1/\alpha$ is an algebraic integer since A is an integer matrix, and every algebraic conjugate of $-1/\alpha$ is also an eigenvalue of A with multiplicity m. If $|X| > 2n$, then $m > \frac{|X|}{2}$. Note A cannot have more than one eigenvalue of multiplicity m because A is a $|X| \times |X|$ matrix. Therefore $-1/\alpha$ is rational, since it is also an algebraic integer, hence $-1/\alpha$ is an integer. □

There is an example of X in S^{n-1} with $I(X) = \{0, \pm\alpha\}$ and $|X| > 2n$ so that $1/\alpha$ is an even integer.

Example 28. The E_8 root system consists of 240 points in S^7 with inner products $0, \pm 1/2, -1$. An example of the above is a half of E_8-roots, which consists of 120 points in S^7 by choosing one of antipodal pairs.

Assume X is a spherical 3-distance set with $I(X) = \{0, \pm\alpha\}$ such that $|X| \geqslant |Y|$. If $5 \leqslant n \leqslant 8816$, then $|X| \geqslant |Y| > 2n$. By Theorem 27, we know that $1/\alpha \in \mathbb{Z}$. And it is easy to check that $\frac{1}{\alpha} = \sqrt{\frac{2(n+8)}{15}} \in \mathbb{Z}$ and $b_{n,T} \in \mathbb{Z}$ cannot hold simultaneously for $5 \leqslant n \leqslant 8816$.

Case (3): If $2 \leqslant n \leqslant 4$, then $b_{2,T} = 2$ is the unique integral case. By Remark 11, $Y = \{(1,0), (0,1)\}$ is a tight spherical design of harmonic index $\{6, 2\}$ in $S^1 \subseteq \mathbb{R}^2$.

7.3 The non-existence of tight spherical designs of harmonic index $\{8, 6\}$

Find f_6 such that

$$Q_{n,8}(x) + f_6 Q_{n,6}(x) = a(x^2 - \alpha^2)^2 (x^2 - \beta^2)^2 - c_{n,T,L}. \tag{16}$$

Set $\gamma = \alpha^2 + \beta^2$, $\xi = \alpha^2\beta^2$. By comparing the coefficients in (16), we get the following results:

$$\gamma = \frac{14}{n+12} - \frac{28}{(n+12)(n+14)} f_6, \tag{17}$$

$$\gamma^2 + 2\xi = \frac{210}{(n+10)(n+12)} - \frac{840}{(n+8)(n+12)(n+14)} f_6, \tag{18}$$

$$\gamma\xi = \frac{210}{(n+8)(n+10)(n+12)} - \frac{1260}{(n+6)(n+8)(n+12)(n+14)} f_6. \tag{19}$$

Using Eq. (17) and Eq. (19), we have the following relations:

$$f_6 = \frac{n+14}{2}\left(1 - \frac{n+12}{14}\gamma\right), \quad \gamma = \frac{420}{(n+10)\big(45 - (n+6)(n+8)\xi\big)}.$$

Therefore Eq. (18) and the above two relations imply that

$$\begin{aligned}
&(n+6)^2(n+8)^3(n+10)^2\xi^3 \\
&+15(n+6)(n+8)^2(n+10)(n-18)\xi^2 \\
&-225(n+8)(n+10)(5n-6)\xi + 1575(11n-2) = 0.
\end{aligned} \tag{20}$$

We denote the LHS of (20) as $g(\xi)$, then $g(\xi)$ must have at least one non-negative root since we define $\xi = \alpha^2\beta^2$. However we can prove that $g(\xi)$ only has one real root and that root is negative. $g(\xi)$ is a degree 3 polynomial with positive leading coefficient and has two critical points:

$$\xi_1 = -\frac{5(5n-6)}{(n+6)(n+8)(n+10)}, \qquad \xi_2 = \frac{15}{(n+6)(n+8)}.$$

Therefore, for $n \geqslant 2$, $g(\xi)$ has local maximum at $\xi_1 < 0$ and local minimum at $\xi_2 > 0$. We have $g(\xi_2) = \frac{7200(n+3)(n-4)}{n+6} > 0$ if $n > 4$. Then, $g(\xi)$ only has one real root which is negative for $n \geqslant 5$.

From the calculation, we obtain the lower bound for $|Y|$ is

$$b_{n,T} = \frac{1}{252 \times 9.427094401\cdots} \times n^4(1+o(1)).$$

If $2 \leqslant n \leqslant 4$, it is easy to check that $b_{4,T} = 2$ is the unique case when $b_{n,T} \in \mathbb{Z}$. However, the lower bound for spherical design of harmonic index 6 is about 5.29, which is strictly larger than $b_{4,T}$. Therefore, there is no tight spherical design of harmonic index $\{8, 6\}$.

7.4 The non-existence of tight spherical designs of harmonic index $\{8, 2\}$

We want to determine f_2 such that

$$Q_{n,8}(x) + f_2 Q_{n,2}(x) = a(x^2-\alpha^2)^2(x^2-\beta^2)^2 - c_{n,T,L}.$$

From calculation we have the following results:

$$f_2 = \frac{(n-2)(n+4)(n+5)(n+6)(n+14)}{90(n+12)^2},$$

$$\alpha^2,\beta^2=\frac{7}{n+12}\pm\frac{\sqrt{42(n+5)(n+10)}}{(n+10)(n+12)}.$$

Then we have

$$\begin{aligned} b_{n,T} &= \frac{n(n+6)(n+5)(n^2+15n+8)^2}{168(n^3+27n^2+356n-240)} \\ &= \frac{1}{168}(n^4+14n^3-133n^2+2638n-10584) \\ &\quad +\frac{-4032n^2+26208n-15120}{n^3+27n^2+356n-240}. \end{aligned}$$

Let $p(n)=-4032n^2+26208n-15120$ and $q(n)=n^3+27n^2+356n-240$. Then $|\frac{p(n)}{q(n)}|<\frac{1}{168}$ gives a condition for $b_{n,T}$ not being an integer. This implies that $b_{n,T}$ cannot be an integer if $n\geqslant 677343$. We can check the remaining cases where $b_{n,T}$ is an integer, and obtain $b_{2,T}=2$, $b_{4,T}=9$, $b_{9,T}=96$. The case $b_{2,T}=2$ is eliminated by Remark 11.

If $n=4$, the SDP upper bound for 4-distance set is 8.9981 (<9).

For $n=9$, we assume X is a spherical 4-distance set with $I(X)=\{\pm\alpha,\pm\beta\}$ such that $|X|\geqslant|Y|$. Then $|X|\geqslant|Y|=b_{n,T}\geqslant 2\binom{n+1}{2}$. But neither $\frac{1-\alpha^2}{\beta^2-\alpha^2}$ nor $\frac{1-\beta^2}{\alpha^2-\beta^2}$ is an integer. By Lemma 22, the case where $n=9$ is also impossible.

7.5 The non-existence of tight spherical designs of harmonic index $\{10,6,2\}$.

Consider the case where $L(x)=Q_{n,10}(x)+f_6Q_{n,6}(x)+f_2Q_{n,2}(x)$ takes minimum value $-c_{n,T,L}$ at three non-negative points $\{0,\alpha,\beta\}$, i.e.,

$$L(x)=Q_{n,10}(x)+f_6Q_{n,6}(x)+f_2Q_{n,2}(x)=ax^2(x^2-\alpha^2)^2(x^2-\beta^2)^2-c_{n,T,L}.$$

We can solve for f_6 and f_2 as follows.

$$f_6=\frac{(n-2)(n+8)(n+18)(13n+28)}{1344(n-8)(n+16)},$$

$$f_2=\frac{(n-2)(n+4)(n+8)(n+14)(n+18)(37n^3-742n^2+1792n+20256)}{129024(n-8)^2(n+12)(n+16)}.$$

Then we can get α^2,β^2 and the lower bound $b_{n,T}$.

$$\alpha^2,\beta^2=\frac{45(n-8)(n+12)\pm\sqrt{15(n-8)(n+12)(43n^2-244n-1952)}}{4(n-8)(n+12)(n+16)},$$

$$b_{n,T}=\frac{n(n+4)(n+8)(4n^3-10n^2-143n-84)^2}{45(781n^4-9548n^3+10128n^2+160960n-108032)}.$$

The following theorem is very useful to consider the existence of spherical design of harmonic index $\{10,6,2\}$.

Theorem 29 ([16, Theorem 5.3]). *Let X be an antipodal s-distance set on S^{n-1} where s is an even integer at least 4. Let $I(X) = \{-1, \beta_1 = 0, \pm\beta_2, \dots, \pm\beta_{\frac{s}{2}}\}$.*

(1) Let $N = \binom{n+s-3}{s-2}$. If $|X| \geqslant 4N$, then for each $i = 1, \dots, s/2$,

$$k_i := \prod_{j=1,\dots,\frac{s}{2}, j\neq i} \frac{1-\beta_j^2}{\beta_i^2 - \beta_j^2}$$

must be an integer with $|k_i| \leqslant U(N)$.

(2) Let $N = \binom{n+s-4}{s-3}$. If $|X| \geqslant 4N + 2$, then for each $i = 1, \dots, s/2$,

$$k_i := \frac{1}{\beta_i} \prod_{j=2,\dots,\frac{s}{2}, j\neq i} \frac{1-\beta_j^2}{\beta_i^2 - \beta_j^2}$$

must be an integer with $|k_i| \leqslant \lfloor \sqrt{2N^2/(N+1)} \rfloor$.

Similarly, tight spherical design of harmonic index $\{10, 6, 2\}$ is regarded as an at most 5-distance set $Y \in S^{n-1}$ with $I(Y) \subset \{0, \pm\alpha, \pm\beta\}$. We construct an antipodal set $X' = Y \cup (-Y)$ with $I(X') \subset \{-1, 0, \pm\alpha, \pm\beta\}$.

In this subsection, we assume X is an antipodal spherical 6-distance set with $I(X) = \{-1, 0, \pm\alpha, \pm\beta\}$ such that $|X| \geqslant |X'| = 2|Y|$.

Case (1): If $n \geqslant 170$, then $|X|/2 \geqslant |Y| \geqslant 2\binom{n+6-3}{6-2} + 1$. Theorem 21 implies that $\alpha, \beta \in \mathbb{Q}$. Namely, $\alpha\beta = \sqrt{\frac{(n-8)(n+12)(n+16)}{15(23n-172)}} \in \mathbb{Q}$. So there exists an integer u such that

$$15(n-8)(n+12)(n+16)(23n-172) = u^2.$$

Assume that $n - 8 = Ay_1^2$, $n + 12 = By_2^2$, $n + 16 = Cy_3^2$, $23n - 172 = Dy_4^2$. Let $p(n_1, n_2)$ be a prime divisor of $\gcd(n_1, n_2)$. Since $\gcd(n-8, n+12)$ can be one of $1, 2, 4, 5, 10, 20$, then $p(n-8, n+12) = 2$ or 5. And with similar argument, we obtain the following result.

$$p(n-8, n+12) = 2 \text{ or } 5, \qquad p(n-8, n+16) = 2 \text{ or } 3,$$

$$p(n-8, 23n-172) = 2 \text{ or } 3, \qquad p(n+12, n+16) = 2,$$

$$p(n+12, 23n-172) = 2 \text{ or } 7, \quad p(n+16, 23n-172) = 2,\ 3, \text{or } 5.$$

Then we get 16 elliptic equations

$$(n-8)(n+12)(n+16) = Ey^2$$

with $E \mid 2 \times 3 \times 5 \times 7$. We can obtain the integral solutions of these equations.

Multiply both sides of the equation $(n-8)(n+12)(n+16) = Ey^2$ by E^3 and make linear transformation $M = E^2 y$ and $N = E(n+12)$. Then $(n-8)(n+12)(n+16) = Ey^2$ becomes $N(N-20E)(N+4E) = M^2$ with coefficients of M^2 and N^3 being 1.

If $E = 2$, with the linear transformation $N = 2n + 24$ and $M = 4y$, the equation becomes $N(N-40)(N+8) = M^2$. It has only three integral solutions $(N, M) = (0, 0)$, $(40, 0)$ and $(-8, 0)$.

In the table below, we list all the cases when the equation $(n-8)(n+12)(n+16) = Ey^2$ has non-trivial integral solutions.

E	solution(n, y)
5	$(33, 105)$
7	$(16, 32), (68, 240)$
6	$(20, 48), (48, 160)$
10	$(24, 48), (38, 90)$
15	$(308, 1440)$
21	$(9, 5), (488, 2400)$
42	$(12, 8), (128, 240)$
210	$(44, 24)$

There are two values for $n \geqslant 170$, i.e., $n = 308, 488$. But in these two cases, $\alpha \notin \mathbb{Q}$.

Case (2): If $19 \leqslant n \leqslant 169$, then the second statement in Theorem 29 implies that $\frac{1-\beta^2}{\alpha(\alpha^2-\beta^2)}$ must be an integer. And it is easy to check this cannot be satisfied.

Case (3): If $2 \leqslant n \leqslant 18$, then $b_{8,T} = 8$ is the only integral case. Let

$$m_1 = \alpha^2 + \beta^2,\ m_2 = (\alpha^2 + \beta^2)^2 + 2\alpha^2\beta^2,\ m_3 = \alpha^2\beta^2(\alpha^2 + \beta^2).$$

Then $m_1(m_2 - m_1^2) - 2m_3 = 0$. However, comparing the coefficients in $L(x)$ and $R(x)$, we obtain

$$\tfrac{1}{2}m_1(m_2 - m_1^2) - m_3 = \tfrac{1}{(n+12)(n+14)(n+16)^2}\left(\tfrac{18900(n-8)}{(n+8)(n+18)}f_6 - \tfrac{225(13n+28)(n-2)}{16(n+16)}\right).$$

If $n = 8$, then the RHS of the above equation is $-\frac{15}{8192} \neq 0$. This is a contradiction. And it implies that, when $n = 8$, we cannot find a function $L(x)$ satisfying our assumption.

7.6 Tight spherical designs of harmonic index $\{12, 8, 4\}$.

Consider the case where $L(x) = Q_{n,12}(x) + f_8 Q_{n,8}(x) + f_4 Q_{n,4}(x)$ takes minimum value at three positive points $\{\alpha, \beta, \gamma\}$, i.e.,

$$F(x) = Q_{n,12}(x) + f_8 Q_{n,8}(x) + f_4 Q_{n,4}(x) + c_{n,T,L}$$

$$= a(x^2 - \alpha^2)^2(x^2 - \beta^2)^2(x^2 - \gamma^2)^2 = R(x),$$

where $c_{n,T,L} = -\min_{-1 \leqslant x \leqslant 1} L(x)$.

Let $Z_1 = \alpha^2 + \beta^2 + \gamma^2$, $Z_2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$ and $Z_3 = \alpha^2\beta^2\gamma^2$. Comparing the coefficients in $F(x)$ and $R(x)$, we can obtain the following results.

$$m_1 = Z_1,\ m_2 = Z_1^2 + 2Z_2,\ m_3 = Z_1Z_2 + Z_3,\ m_4 = Z_2^2 + 2Z_1Z_3,\ m_5 = Z_2Z_3,$$

where $-2am_1$, am_2, $-2am_3$, am_4 and $-2am_5$ are the coefficients of x^{10}, x^8, x^6, x^4, x^2 in polynomial $R(x)$, respectively. Note that m_i is a linear combination of f_8 and f_4. Substituting $Z_2 = \frac{m_2-m_1^2}{2}$ and $Z_3 = m_3 - Z_1Z_2 = m_3 - m_1(\frac{m_2-m_1^2}{2})$ into m_4 and m_5, we can solve for f_8 and f_4 as follows.

$$f_8 = \frac{(n+8)(n+9)(n+22)}{180(n+20)},$$

$$f_4 = \frac{(n+4)(n+5)(n+8)(n+9)(n+18)(n+22)}{7200(n+16)(n+20)}.$$

Immediately we have

$$Z_1 = \alpha^2 + \beta^2 + \gamma^2 = \frac{33}{n+20},$$

$$Z_2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 = \frac{231}{(n+16)(n+20)},$$

$$Z_3 = \alpha^2\beta^2\gamma^2 = \frac{231}{(n+12)(n+16)(n+20)},$$

$$b_{n,T,F} = \frac{1}{27720}(n+1)(n+2)(n+5)(n+6)(n+9)(n+10).$$

We should remark that there is another test function $F_1(x) = Q_{n,12}(x) + f_8'Q_{n,8}(x) + f_4'Q_{n,4}(x) + f_0^{(1)}$, where

$$f_8' = -\frac{(n+16)(n+22)(n^3-19n^2-1564n-4064)}{220(n-4)(n+8)(n+20)},$$

$$f_4' = \frac{(n+4)(n+18)(n+22)(73n^6-1238n^5-31603n^4+304204n^3+5781440n^2+22288384n+25661440)}{61600(n-4)^2(n+8)(n+20)^2}.$$

And in this case we have the following lower bound.

$$\begin{aligned} b_{n,T,F_1} \quad = \quad & \tfrac{1}{216}(n+2)(n+6)(n+10)(n^6+39n^5+407n^4+989n^3+56556n^2+279424n+357888)^2\times \\ & (955n^9-48483n^8+1094513n^7+5943943n^6-158073240n^5+1292462960n^4+ \\ & 29035332352n^3+132584060928n^2+227750248448n+126509383680)^{-1}. \end{aligned}$$

It is easy to check that $b_{n,T,F_1} > b_{n,T,F}$ if and only if $30 \leqslant n \leqslant 38$. Then

$$b_{n,T} = \begin{cases} b_{n,T,F_1} & \text{if } 30 \leqslant n \leqslant 38, \\ b_{n,T,F} & \text{otherwise.} \end{cases}$$

Assume $X \in S^{n-1}$ is an antipodal spherical 7-distance set with $I(X) = \{-1, \pm\alpha, \pm\beta, \pm\gamma\}$ such that $|X| \geqslant 2|Y|$, where Y is a tight spherical design of harmonic index $\{12, 8, 4\}$.

Case (1): If $n \geqslant 439$, then $|X|/2 \geqslant |Y| = b_{n,T} \geqslant 2\binom{n+7-3}{7-2} + 1$. Theorem 21 implies that

$$\alpha\beta\gamma = \sqrt{\frac{231}{(n+12)(n+16)(n+20)}} \in \mathbb{Q}.$$

Equivalently, we have $(n+12)(n+16)(n+20) = 231y^2$ for some integer y. With linear transformation $M = 231^2 y$ and $N = 231(n+16)$, we have $N(N^2 - 462^2) = M^2$. It has integral solutions $(N, M) = (-528, 17424)$, $(-252, 14112)$, $(1617, 53361)$, $(3388, 189728)$. However, this gives no positive integral solution for $n = N/231 - 16$.

Case (2): If $34 \leqslant n \leqslant 438$, then $|X|/2 \geqslant |Y| = b_{n,T} \geqslant 2\binom{n+7-4}{7-3} + 1$. By Theorem 29, $\frac{(1-\beta^2)(1-\gamma^2)}{\alpha(\alpha^2-\beta^2)(\alpha^2-\gamma^2)}$ must be an integer. And it is easy to check this cannot be satisfied.

Case (3): If $n \leqslant 33$, then $b_{5,T} = 35$, $b_{6,T} = 64$, $b_{9,T} = 285$, $b_{13,T} = 1311$, $b_{16,T} = 3315$, $b_{20,T} = 9425$, $b_{23,T} = 18560$ are the integral cases. But LP upper bounds for spherical 6-distance set corresponding to $n = 5, 6, 20, 23$ are 30.2656, 59.8173, 9405.11, 17926.1, respectively. When $n = 9, 13, 16$, we set up the SDP method on the upper bounds for spherical 6-distance sets. However, the upper bounds coincide with our lower bounds $b_{n,T}$. We conclude that there exists no tight spherical design in S^{n-1} of harmonic index $\{12, 8, 4\}$ if $n \neq 9, 13, 16$. The existence of such tight designs for $n = 9, 13, 16$ is still open.

8 Concluding remarks

In this paper we considered mainly spherical designs of harmonic index $T = \{t\}$, or $T = \{t_1, t_2\}$. For some $T = \{t_1, t_2, \ldots, t_\ell\}$ (with $t_1 = 2e > t_2 > \cdots > t_\ell$ and all t_i are even), it seems that the general interesting case is where $L(x) = Q_{n,2e}(x) + f_{t_2}Q_{n,t_2}(x) + f_{t_3}Q_{n,t_3}(x) + \cdots + f_{t_\ell}Q_{n,t_\ell}(x)$ and the minimum value of $L(x)$ is at ℓ non-negative points $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. Thus, further studies along this line would be interesting.

As we have shown in Section 5 as well as in previous sections, it seems remarkable that a spherical design of harmonic index $t = 2e$ has a Fisher type lower bound $|Y| \geqslant (\text{constant})\cdot n^e$, which is the same order as for spherical $2e$-design. So, all harmonic index T-designs are between harmonic index $2e$-designs and spherical $2e$-designs. It seems that considering tight T-designs have some meaning, although it seems tight harmonic index T-designs rarely exist.

As it is discussed in Bannai-Okuda-Tagami [2, Proposition 4.1] that to some extent,

$$b_n := \lim_{e\to\infty} b_{n,2e}$$

is also studied. The result is explained in terms of Bessel functions. As some special cases are mentioned in [2, p. 10], b_n becomes greater than $n(n+1)/2$ if $n = 7, 8, 9, 10$. This implies that tight spherical designs of harmonic index $2e$ do not exist, if $t = 2e$ becomes large, say for $n = 7, 8, 9, 10$. On the other hand if $n \leqslant 6$, it seems possible to determine the non-existence of tight designs in these cases, but it is not clear how we can show the non-existence of such harmonic index $2e$-designs whose sizes are close to the Fisher type lower bound. It seems that this remains as an interesting open problem.

In concluding this paper, we remark that the theory as well as the concept of harmonic index T-designs in Q-polynomial association schemes exactly go parallel with the spherical case. The concept of T-design for an arbitrary subset T of the index set of nontrivial relations $\{1, 2, \dots, d\}$ is already defined in Delsarte [5, Section 3.4] (1973). On the other hand, it seems that any systematic study on some specific choices of T, beyond the case $T = \{1, 2, \dots t\}$ has not begun, even for the case $T = \{t\}$. We hope to discuss more on this topic in a separate paper.

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