New results on k-independence of graphs

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Abstract

Let $G = (V, E)$ be a graph and $k \geqslant 0$ an integer. A k-independent set $S \subseteq G$ is a set of vertices such that the maximum degree in the graph induced by S is at most k. Denote by $\alpha_k(G)$ the maximum cardinality of a k-independent set of G. For a graph G on n vertices and average degree d, Turán's theorem asserts that $\alpha_0(G) \geqslant \frac{n}{d+1}$, where the equality holds if and only if G is a union of cliques of equal size. For general k we prove that $\alpha_k(G) \geqslant \dfrac{(k+1)n}{d+k+1}$, improving on the previous best bound $\alpha_k(G) \geqslant \dfrac{(k+1)n}{\lceil d\rceil+k+1}$ of Caro and Hansberg [E-JC, 2013]. For 1-independence we prove that equality holds if and only if G is either an independent set or a union of almost-cliques of equal size (an almost-clique is a clique on an even number of vertices minus a 1-factor). For 2-independence, we prove that equality holds if and only if G is an independent set. Furthermore when $d > 0$ is an integer divisible by 3 we prove that $\alpha_2(G) \geqslant \dfrac{3n}{d+3}\left(1+\dfrac{12}{5d^2+25d+18}\right)$.

1 Introduction

Let $G = (V, E)$ be a graph on n vertices and let $k \geqslant 0$ be an integer. A k-independent set $S \subseteq V$ is a set of vertices such that the maximum degree in the graph induced by S is at most k. Let $\alpha_k(G)$ denote the maximum cardinality of a k-independent set of G and call it the k-independence number of G. For $k = 0$ we have $\alpha_0(G) = \alpha(G)$ where $\alpha(G)$ is the independence number of G. Let d be the average degree of graph G. Turán's theorem [Tur41] asserts that $\alpha(G) \geqslant \frac{n}{d+1}$, where equality holds if and only if G is a union of cliques of equal size. Turán's bound was improved by the Caro-Wei bound [Car79, Wei81] which asserts that $\alpha(G) \geqslant \sum_{v\in V} \frac{1}{\deg(v)+1}$, where equality holds if and only if G is a union of cliques (see also [AS08] page 95 for a proof). The following two conjectures were made in [BCHN13]:

Conjecture 1. For a graph G on n vertices and an integer $k \geqslant 1$, we have

$$\alpha_k(G) \geqslant \frac{k+1}{d+k+1} n,$$

where d is the average degree of graph G.

Conjecture 2. For a graph G on n vertices and an integer $k \geqslant 1$, we have

$$\alpha_k(G) \geqslant \sum_{v \in V} \frac{k+1}{\deg(v)+k+1}.$$

Conjecture 1 follows from Conjecture 2 by Jensen's inequality. Hence Conjecture 2 is a strengthening of Conjecture 1. The first result on bounding k-independence was given in [CT91], where it was shown that if $d \geqslant k+1$ then

$$\alpha_k(G) \geqslant \frac{k+2}{2(d+1)} n.$$

Furthermore, an important step in proving Conjecture 1 was made in [CH13, Theorem 18], where it was shown that if the average degree $d \geqslant 0$ is an integer then

$$\alpha_k(G) \geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} n \geqslant \frac{k+1}{d+k+1} n,$$

where t is such that $d \equiv k+1-t \pmod{k+1}$ and $1 \leqslant t \leqslant k+1$. As a corollary they have the result

$$\alpha_k(G) \geqslant \frac{k+1}{\lceil d \rceil + k + 1} n.$$

Now we shall describe our results. An almost-clique is a clique on an even number of vertices minus a 1-factor. In this paper we prove that

$$\alpha_1(G) \geqslant \frac{2}{d+2} n,$$

where equality holds if and only if G is either an independent set or a union of almost-cliques of equal size.

For general k we prove $\alpha_k(G) \geqslant \frac{k+1}{d+k+1} n$, thus solving Conjecture 1. More generally we prove that $\alpha_k(G) \geqslant f\left(\frac{d}{k+1}\right) n$, where

$$f(x) = \frac{1}{1+x} \left(1 + \frac{\{x\}(1-\{x\})}{(\lfloor x \rfloor + 1)(\lfloor x \rfloor + 2)} \right)$$

and where $\lfloor x \rfloor$ is the floor function and $\{x\}$ is the fractional part function.

Notice that [CH13, Theorem 18] gives the same bound when d is an integer. The novelty of our bound is that d does not need to be an integer. Furthermore, we improve the bounds for 2-independence, showing that if $d = 3t$ for some integer $t > 0$ then

$$\alpha_2(G) \geqslant \frac{n}{t+1} \left(1 + \frac{4}{15t^2 + 25t + 6} \right).$$

This improves the bound in [CH13, Problem 23] for $d = 3t$, where the best previous bound was $\alpha_2(G) \geqslant \frac{n}{t+1}$. Finally we make modest progress on Conjecture 2 by showing that

$$\alpha_1(G) \geqslant \sum_{v \in V} \frac{2}{\deg(v) + 2}$$

holds for graphs G of maximum degree at most 4.

All along this paper, we will use the following notation and definitions. Let G be a graph. $V(G)$ denotes the set of vertices of G and $n(G) = |V(G)|$ denotes its cardinality. $E(G)$ denotes the set of edges of G and $e(G)$ denotes its cardinality. For a vertex $v \in V(G)$, $\deg(v) = \deg_G(v)$ is the degree of v in G. By $\Delta(G)$ we denote the maximum degree of G and by $d(G)$ the average degree $\frac{1}{n(G)} \sum_{v \in V(G)} \deg(v)$. The minimum degree of G is denoted by $\delta(G)$. For a subset $S \subseteq V(G)$, we write $G[S]$ for the graph induced by S in G, and $\deg_S(v)$ stands for the degree $\deg_{G[S]}(v)$ of v in $G[S]$. For an integer $m > 1$, mG is the graph consisting of m disjoint copies of G. Lastly, for a vertex $v \in V(G)$, $G - v$ represents the graph G without vertex v and all the edges incident to v.

2 Lower bounds on k-independence as a function of maximum degree

For a graph G, we will denote by $\chi_k(G)$ the k-chromatic number of G, i.e. the minimum number t such that there is a partition $V(G) = V_1(G) \cup V_2(G) \cup \ldots \cup V_t(G)$ of the vertex set $V(G)$ such that $\Delta(G[V_i]) \leqslant k$ for all $1 \leqslant i \leqslant t$. The following theorem is proven in [Lov66].

Theorem 3. *Let G be a graph with maximum degree Δ. If $k_1, k_2, \ldots, k_t \geqslant 0$ are integers such that $\Delta+1 = \sum_{i=1}^{t}(k_i+1)$, then there is a partition $V(G) = V_1(G) \cup V_2(G) \cup \ldots \cup V_t(G)$ of the vertex set of G such that $\Delta(G[V_i]) \leqslant k_i$ for all $1 \leqslant i \leqslant t$.*

A partition as in the theorem above can be found in polynomial time. In [HL97] it is shown how to find such a partition in time $O(n^3)$.

Corollary 4. *If G is a graph of maximum degree Δ then $\chi_k(G) \leqslant \left\lceil \frac{\Delta+1}{k+1} \right\rceil$.*

Now as $\alpha_k(G) \geqslant \frac{n}{\chi_k(G)}$ we get the following bound first proved in [HS86]

Theorem 5. *Let G be a graph of order n and maximum degree Δ. Then*

$$\alpha_k(G) \geqslant \frac{n}{\left\lceil \frac{\Delta+1}{k+1} \right\rceil}.$$

Furthermore, as the partition in Theorem 3 can be found in polynomial time, we also can find the k-independent set in Theorem 5 in polynomial time by picking the largest set in the partition.

The bound in Theorem 5 is tight for $k = 0$ and $k = 1$. For $k = 0$, an example of the sharpness of the bound is a union of cliques of equal size. For $k = 1$, an example of the sharpness of the bound is a union of almost-cliques of equal size, as noted in [CH13].

For $k \geqslant 2$, the bounds of Theorem 5 can be improved in certain ranges by using bounds on dominating sets. Recall that a set $S \subseteq V$ is a dominating set of a graph $G = (V, E)$ if each vertex in V is either in S or is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. The following bound was proven in [Ree96]

Theorem 6. *If G is a graph of order n and minimum degree $\delta(G) \geqslant 3$, then $\gamma(G) \leqslant \frac{3}{8}n$.*

The following Theorem was proven in [HP13, Theorem 3 (ii)].

Theorem 7. *Let $k \geqslant 0$ be a fixed integer. If there is a constant c such that for all n every graph G of order n with minimum degree $\delta(G) \geqslant k + 1$ satisfies $\gamma(G) \leqslant cn$, then every graph G of order n and maximum degree $\Delta(G) \leqslant k + 1$ satisfies $\alpha_k(G) \geqslant (1 - c)n$.*

We will also need the following corollary also stated in [HP13].

Corollary 8. *Let G be a graph of order n with maximum degree $\Delta \leqslant 3$. Then $\alpha_2(G) \geqslant \frac{5}{8}n$.*

Proof. Follows directly from combining Theorem 7 and Theorem 6. □

Generalizing Corollary 8 we shall give improved bounds on $\alpha_2(G)$ in the case where the maximum degree of G is divisible by 3.

Theorem 9. *Let G be a graph of order n with maximum degree $\Delta \leqslant 3t$, for a natural number t. Then $\alpha_2(G) \geqslant \frac{5}{5t+3}n$.*

Proof. We prove the statement by induction on t. The base case $t = 1$ is proven in Corollary 8. Assume that the theorem holds for $t - 1$ and we will prove for t. Let G be a graph of order n and maximum degree $\Delta \leqslant 3t$. By Theorem 3 there is a partition $V(G) = V_1(G) \cup V_2(G)$ of the vertex set of G such that $\Delta(G[V_1]) \leqslant 3t-3$ and $\Delta(G[V_2]) \leqslant 2$. If $|V_2| \geqslant \frac{5}{5t+3}n$ we are done, because $\Delta(G[V_2]) \leqslant 2$. Thus we may assume that $|V_1| \geqslant \frac{5t-2}{5t+3}n$. Applying the induction hypothesis on $G[V_1]$ we conclude that

$$\alpha_2(G[V_1]) \geqslant \frac{5}{5t-2} \cdot \frac{5t-2}{5t+3} \cdot n = \frac{5}{5t+3}n$$

and we are done. □

Corollary 10. *Let G be a graph of order n with maximum degree $\Delta > 0$, where Δ is divisible by 3. Then $\alpha_2(G) \geqslant \frac{15}{5\Delta+9}n$.*

3 Lower bounds on k-independence in terms of average degree

Definition 3.1. Define the function $f(x)$ for real $x \geqslant 0$ in the following manner:

$$f(x) = \frac{1}{1+x}\left(1 + \frac{\{x\}\left(1-\{x\}\right)}{\left(\lfloor x \rfloor + 1\right)\left(\lfloor x \rfloor + 2\right)}\right),$$

where $\lfloor x \rfloor$ is the floor function and $\{x\}$ is the fractional part function.

Lemma 11. *Let $x \geqslant 0$ be a real number. Then $f(x) \geqslant \frac{1}{1+x}$, and equality holds if and only if x is an integer.*

Proof. Notice that for any real $x \geqslant 0$ we have

$$\frac{\{x\}\left(1-\{x\}\right)}{\left(\lfloor x \rfloor + 1\right)\left(\lfloor x \rfloor + 2\right)} \geqslant 0,$$

and the equality holds if and only if x is an integer, and thus we are done. □

In this section we will prove the following theorem:

Theorem 12. *Let $k \geqslant 0$ be an integer. Then for any graph G of order n and average degree d we have*

$$\alpha_k(G) \geqslant f\left(\frac{d}{k+1}\right) n.$$

Corollary 13. *Let $k \geqslant 0$ be an integer. Then for any graph G of order n and average degree d we have*

$$\alpha_k(G) \geqslant \frac{k+1}{d+k+1} n.$$

Proof. This follows from Theorem 12 and Lemma 11, as

$$\frac{1}{1+\frac{d}{k+1}} = \frac{k+1}{d+k+1}.$$ □

We note that Theorem 12 was proven for the case where the average degree d of graph G is an integer in [CH13] (Theorem 3.7). Here we prove it in full generality. The difference between our proof and the proof in [CH13] is that we make the induction on the ranges of the degrees, while in [CH13] the induction is on the average degree as an integer. Before proving Theorem 12 we will need a few more lemmas and definitions.

Definition 3.2. Define a function $g(x)$ for real $x \geqslant 0$ in the following manner:

$$g(x) = \begin{cases} 1, & \text{if } x = 0 \\ \dfrac{2\lceil x \rceil - x}{\lceil x \rceil\left(1 + \lceil x \rceil\right)}, & \text{if } x > 0. \end{cases}$$

where $\lceil x \rceil$ is the ceiling function.

Lemma 14. *For all real $x \geqslant 0$ we have $f(x) = g(x)$.*

Proof. First notice that

$$g(x) = \frac{2\lfloor x \rfloor + 2 - x}{(\lfloor x \rfloor + 1)(\lfloor x \rfloor + 2)}.$$

This follows from the fact that if x is not an integer then $\lceil x \rceil = \lfloor x \rfloor + 1$, while if $x = n$ for some integer $n \geqslant 0$ then $g(n) = f(n) = \frac{1}{n+1}$. We conclude that

$$\begin{aligned}
g(x) - \frac{1}{x+1} &= \frac{2\lfloor x \rfloor + 2 - x}{(\lfloor x \rfloor + 1)(\lfloor x \rfloor + 2)} - \frac{1}{x+1} \\
&= \frac{(2\lfloor x \rfloor + 2 - x)(x+1) - (\lfloor x \rfloor + 1)(\lfloor x \rfloor + 2)}{(x+1)(\lfloor x \rfloor + 1)(\lfloor x \rfloor + 2)} \\
&= \frac{(x - \lfloor x \rfloor) - (x - \lfloor x \rfloor)^2}{(x+1)(\lfloor x \rfloor + 1)(\lfloor x \rfloor + 2)} \\
&= \frac{\{x\}(1 - \{x\})}{(x+1)(\lfloor x \rfloor + 1)(\lfloor x \rfloor + 2)} \\
&= f(x) - \frac{1}{x+1}.
\end{aligned}$$

and we are done. □

Lemma 15. *The function $f(x)$ is continuous, monotonically decreasing and convex on the interval $[0, \infty)$.*

Proof. As $f(x) = g(x)$ for $x \geqslant 0$, it suffices to prove the claim for $g(x)$. The continuity of $g(x)$ for $x \geqslant 0$ follows from the fact that

$$\lim_{x \to 0^+} g(x) = 1 = g(0),$$

and for any integer $n \geqslant 1$

$$\lim_{x \to n^-} g(x) = \lim_{x \to n^+} g(x) = \frac{1}{n+1} = g(n).$$

Now as the function $g(x)$ is continuous and monotonically decreasing on each interval $[n, n+1]$ for any integer $n \geqslant 0$, it follows that $g(x)$ is monotonically decreasing on the interval $[0, \infty)$.

To prove convexity we will use the following fact:

If a function is continuous on an open interval I and possesses a non-decreasing right-derivative on I, then the function is convex on I (this is Theorem 5.3.1 in [JBHUL93]).

Notice that for every integer $n \geqslant 0$ the right-derivative of $g(x)$ on the interval $[n, n+1)$ satisfies

$$g'(x^+) = -\frac{1}{(n+1)(n+2)},$$

hence the right-derivative of g is non-decreasing in the interval $(0, \infty)$. We conclude that $g(x)$ is convex in the interval $(0, \infty)$. Finally, by the continuity of $g(x)$, we can extend the convexity to the interval $[0, \infty)$, and we are done. □

Lemma 16. *Let $k \geqslant 0$ and $r \geqslant 0$ be integers. Let G is a graph of order n with e edges and average degree $d = \frac{2e}{n}$. If $r(k+1) < d \leqslant (r+1)(k+1)$ holds and*

$$\alpha_k(G) \geqslant \frac{2}{r+2}\left(n - \frac{e}{(r+1)(k+1)}\right),$$

then $\alpha_k(G) \geqslant f\left(\frac{d}{k+1}\right) n$.

Proof. Set $t = \frac{d}{k+1}$. As $r(k+1) < d \leqslant (r+1)(k+1)$ we have $r < t \leqslant r+1$. Hence $\lceil t \rceil = r+1$. Thus we have

$$\begin{aligned}
\alpha_k(G) &\geqslant \frac{2}{r+2}\left(n - \frac{e}{(r+1)(k+1)}\right) \\
&= \frac{2}{\lceil t \rceil + 1}\left(n - \frac{dn}{2\lceil t \rceil (k+1)}\right) && (\text{as } r+1 = \lceil t \rceil) \\
&= \frac{2}{\lceil t \rceil + 1}\left(n - \frac{tn}{2\lceil t \rceil}\right) && \left(\text{as } t = \frac{d}{k+1}\right) \\
&= \frac{2n}{\lceil t \rceil + 1}\left(1 - \frac{t}{2\lceil t \rceil}\right) \\
&= n\frac{2\lceil t \rceil - t}{\lceil t \rceil (1 + \lceil t \rceil)} = g(t)n = f(t)n && (\text{by Lemma 14})
\end{aligned}$$

and we are done. □

Lemma 17. *Let $k \geqslant 0$ be an integer. If G is a graph of order n with e edges then*

$$\alpha_k(G) \geqslant n - \frac{e}{k+1}.$$

Proof. Set $G_0 = G$. If there is a vertex $v_0 \in V(G_0)$ such that $\deg_{G_0}(v_0) \geqslant k+1$ remove it from the graph G_0 and call the resulting graph G_1 (that is, $G_1 = G_0 - v_0$).
Now, if there is a vertex $v_1 \in V(G_1)$ such that $\deg_{G_1}(v_1) \geqslant k+1$ remove it from the graph G_1 and call the resulting graph G_2 (that is, $G_2 = G_1 - v_1$).
We can repeat this operation iteratively until we get a graph G_i for some $i \geqslant 0$ such that the maximum degree of G_i satisfies $\Delta(G_i) \leqslant k$. Notice that $i \leqslant \left\lfloor \frac{e}{k+1} \right\rfloor$, as there are e edges in G_0, and in each iteration the number of edges in the resulting graph is decreased by at least $k+1$. Hence if we have reached iteration t for $t = \left\lfloor \frac{e}{k+1} \right\rfloor$, then graph G_t will contain at most $e - (k+1)\left\lfloor \frac{e}{k+1} \right\rfloor \leqslant k$ edges, and thus G_t is a k-independent set, which means that $i \leqslant t$. Since $V(G_i)$ is a k-independent set in G and $|V(G_i)| = n - i \geqslant n - \frac{e}{k+1}$, we are done. □

Corollary 18. *Let $k \geqslant 0$ be an integer. If G is a graph of order n with average degree $0 < d \leqslant k+1$, then $\alpha_k(G) \geqslant f\left(\frac{d}{k+1}\right) n$.*

Proof. This follows from Lemma 17 by setting $r = 0$ in Lemma 16. □

Finally we are ready to prove the main theorem of this section.

Proof of Theorem 12:
Let $k \geqslant 0$ be an integer. Recall that we need to prove that for any graph G of order n and average degree d we have

$$\alpha_k(G) \geqslant f\left(\frac{d}{k+1}\right) n.$$

We will prove by induction on integer $r \geqslant 0$ that for any graph G of order n and average degree $d \leqslant (r+1)(k+1)$ we have:

$$\alpha_k(G) \geqslant f\left(\frac{d}{k+1}\right) n.$$

The base of the induction $r = 0$ was proven in Corollary 18. We will assume that the claim holds for $r - 1 \geqslant 0$ and we will prove for r.

By Lemma 16 and the induction hypothesis, it suffices to prove that that if G is a graph on n vertices, e edges and average degree d satisfying $r(k+1) < d \leqslant (r+1)(k+1)$, then

$$\alpha_k(G) \geqslant \frac{2}{r+2}\left(n - \frac{e}{(r+1)(k+1)}\right).$$

We can assume that both n and e are divisible by $(r+2)(k+1)$. This is because we can build a graph $G' = (r+2)(k+1)G$ (namely, G' is a disjoint union of $(r+2)(k+1)$ copies of G), where $d(G) = d(G')$, and the number of vertices n' and number of edges e' of G' are both divisible by $(r+2)(k+1)$. If $\alpha_k(G') \geqslant n' f\left(\frac{d}{k+1}\right)$ then the original graph G satisfies

$$\alpha_k(G) \geqslant \frac{n'}{(r+2)(k+1)} f\left(\frac{d}{k+1}\right) = nf\left(\frac{d}{k+1}\right)$$

(this observation which greatly simplifies the proof is essentially taken from [CH13, Lemma 16]).

We define parameter t as follows:

$$t = \frac{2e - nr(k+1)}{(r+2)(k+1)}.$$

Notice that t is an integer (because n and e are divisible by $(r+2)(k+1)$), and that $t > 0$ (because $d > r(k+1)$).

Set $G_0 = G$. If there is a vertex $v_0 \in V(G_0)$ such that $\deg_{G_0}(v_0) \geqslant (r+1)(k+1)$, remove it from the graph G_0 and call the resulting graph G_1 (that is, $G_1 = G_0 - v_0$).

Now if $t > 1$ and there is a vertex $v_1 \in V(G_1)$ such that $\deg_{G_1}(v_1) \geqslant (r+1)(k+1)$, remove it from the graph G_1 and call the resulting graph G_2, (that is, $G_2 = G_1 - v_1$).

We repeat this operation iteratively, that is on iteration i (starting with $i = 0$) we first check if $i = t$ or $\Delta(G_i) < (r+1)(k+1)$, and if one of these conditions holds we terminate the process. Otherwise, we pick a vertex $v_i \in V(G_i)$ such that $\deg_{G_i}(v_i) \geqslant (r+1)(k+1)$ and remove it from the graph G_i. We call the resulting graph G_{i+1} (that is $G_{i+1} = G_i - v_i$).

Suppose that the process above terminated on iteration $j \leqslant t$ (that is, the last graph created in the process is G_j). If $j < t$ then $\Delta(G_j) < (r+1)(k+1)$, and thus by Theorem 5 we have $\alpha_k(G_j) \geqslant \frac{n-t}{r+1}$. Now we shall prove that if $j = t$ we also have $\alpha_k(G_j) \geqslant \frac{n-t}{r+1}$.

First we notice that

$$\begin{aligned} n - t &= n - \frac{2e - nr(k+1)}{(r+2)(k+1)} \\ &= \frac{(r+2)(k+1)n + nr(k+1) - 2e}{(r+2)(k+1)} \\ &= \frac{2[n(r+1)(k+1) - e]}{(r+2)(k+1)}. \end{aligned} \tag{3.1}$$

Now we claim that $d(G_t) \leqslant r(k+1)$. Notice that as in each iteration at least $(r+1)(k+1)$ edges were removed we have that $e(G_t)$ (the number of edges in graph G_t) satisfies

$$\begin{aligned} e(G_t) &\leqslant e - t(r+1)(k+1) \\ &= e - \frac{(r+1)(2e - nr(k+1))}{r+2} \\ &= \frac{nr(r+1)(k+1) - er}{r+2} \\ &= \frac{r[n(r+1)(k+1) - e]}{r+2} \\ &= \frac{1}{2}(k+1)r(n-t) && \text{(by Equation 3.1).} \end{aligned}$$

It follows that

$$d(G_t) = \frac{2e(G_t)}{n-t} \leqslant r(k+1).$$

Now as $d(G_t) \leqslant r(k+1)$ we can apply the induction hypothesis on G_t. By the induction hypothesis and Lemma 11 we have

$$\alpha_k(G_t) \geqslant \frac{n-t}{r+1}.$$

We conclude that

$$\begin{aligned} \alpha_k(G) &\geqslant \alpha_k(G_j) \\ &\geqslant \frac{n-t}{r+1} \\ &= \frac{2}{r+2}\left(n - \frac{e}{(r+1)(k+1)}\right) && \text{(by Equation 3.1)} \\ &\geqslant f\left(\frac{d}{k+1}\right)n && \text{(by Lemma 16)} \end{aligned}$$

and that is exactly what we needed to prove. This concludes the proof of the induction. $\square$

The proof of Theorem 12 gives a polynomial time algorithm for finding a k-independent set of cardinality at least $f\left(\frac{d(G)}{k+1}\right)n$ for any graph G on n vertices.

Algorithm I

Input: A graph G on n vertices and e edges.
Output: A k-independent set S of size at least $f\left(\frac{d(G)}{k+1}\right)n$.

1. Set $i=0$, $G_0=G$, $B=f\left(\frac{d(G)}{k+1}\right)n$ and GOTO 2.
2. If $n(G_i)\geqslant\left\lceil\frac{\Delta(G_i)+1}{k+1}\right\rceil B$ then apply Theorem 5 to find a k-independent set S in G_i of size at least B and END. Otherwise GOTO 3.
3. Choose a vertex v of maximum degree in G_i, set $G_{i+1}=G_i-v$, set $i=i+1$ and GOTO 2.

The correctness of the algorithm follows directly from the proof of Theorem 12. Notice however that at certain points in the proof of Theorem 12 we take our graph and expand it to a union of several disjoint copies of the graph. This does not affect the validity of Algorithm 1 because a removal of a vertex v in algorithm I corresponds to the removal of the same vertex v from each copy in the extended graph until it is exhausted from all copies (as it remains a vertex of maximum degree). Finally as the procedure in Theorem 5 has running time $O(n^3)$, Algorithm I has running time of $O(n^3)$ too.

4 The case of equality for the 1-independence bound

Given a graph G of order n with average degree d and e edges, the combination of Theorem 12 and Lemma 11 implies that

$$\alpha_1(G)\geqslant f\left(\frac{d}{2}\right)\geqslant\frac{2}{d+2}n=\frac{n^2}{n+e}\qquad\left(\text{as } d=\frac{2e}{n}\right)\tag{4.1}$$

Recall that an almost-clique is a clique on an even number of vertices minus a 1-factor. Let t be an even integer. We denote by J_t an almost clique on t vertices. In this section we shall show that $\alpha_1(G)=\frac{2}{d+2}n$ if and only if G is either an independent set or a union of almost-cliques of equal size.

First of all it is clear that for all even integers $t\geqslant 2$ we have

$$\alpha_1(J_t)=2=\frac{2}{d\left(J_t\right)+2}t.$$

This was first stated in [CH13, Theorem 17]. Now we shall prove that if $\alpha_1(G)=\frac{2}{d+2}n$ then G is either an independent or a union of almost-cliques of equal size. We will do so in a series of lemmas.

Lemma 19. *Let G be a graph of order n and average degree d. If $\alpha_1(G) = \frac{2}{d+2}n$ then d is an even integer.*

Proof. This follows from Theorem 12 and Lemma 11. $\square$

Lemma 20. *Let $k \geqslant 0$ be integer. Let G be a graph of order n with average degree $d(G) = 2k$. If $\alpha_1(G) = \frac{n}{k+1}$ then G is $2k$-regular (that is all its vertices are of degree $2k$).*

Proof. The claim is trivial for $k = 0$ and for the case $n \leqslant 2$. Henceforth we assume that $k \geqslant 1$ and $n \geqslant 3$. Suppose (by contradiction) that G is not $2k$-regular. Then from the fact that $d(G) = 2k$ we know that G contains a vertex v of degree at least $2k+1$. Remove vertex v from G and call the resulting graph G' (that is, $G' = G - v$).

Thus we have

$$\begin{aligned}
\alpha_1(G) &\geqslant \alpha_1(G') \\
&\geqslant \frac{(n-1)^2}{e(G') + n - 1} && \text{(by Equation 4.1)} \\
&\geqslant \frac{(n-1)^2}{kn - 2k - 1 + n - 1} && \text{(as } e(G') \leqslant kn - (2k+1)) \\
&= \frac{(n-1)^2}{(n-2)(k+1)} \\
&> \frac{n}{k+1} && \text{(as } (n-1)^2 > n(n-2))
\end{aligned}$$

We got a contradiction and the proof follows. $\square$

Recall that C_n is a cycle on n vertices and P_n is a path on n vertices. Notice that the almost clique on 4 vertices is simply a cycle on 4 vertices, that is $J_4 = C_4$.

Lemma 21. *Let G be a graph of order n and maximum degree $\Delta(G) \leqslant 2$. If $\alpha_1(G) = \frac{n}{2}$ then G is a union of copies of C_4.*

Proof. Each connected component of G is either a path or a cycle. Now notice that for all integers $n > 0$ we have $\alpha_1(P_n) \geqslant \frac{2}{3}n$, and thus for all $n \geqslant 3$ we have $\alpha_1(C_n) \geqslant \frac{2}{3}(n-1)$. It follows that $\alpha_1(P_n) > n/2$ for all $n > 0$, and that $\alpha_1(C_n) = n/2$ if and only if $n = 4$. This concludes the proof. $\square$

Lemma 22. *Let $k \geqslant 1$ be integer. Let G be a graph of order n and maximum degree $\Delta(G) \leqslant 2k$. If $\alpha_1(G) = \frac{n}{k+1}$ then G is a union of copies of J_{2k+2}.*

Proof. We will prove the claim by induction on k. The base of the induction $k = 1$ is simply Lemma 21. Assume that the claim holds for $k - 1$ and we will prove it for k.

Let G be a graph of order n and maximum degree $\Delta(G) \leqslant 2k$. Assume that $\alpha_1(G) = \frac{n}{k+1}$. By Theorem 3 there is a partition $V(G) = V_1(G) \cup V_2(G)$ of the vertex set of G such that $\Delta(G[V_1]) \leqslant 2k - 2$ and $\Delta(G[V_2]) \leqslant 1$.

Let $G_1 = G[V_1]$ be a graph on n_1 vertices and $G_2 = G[V_2]$ be a graph on n_2 vertices. Observe that $n_2 \leqslant \frac{n}{k+1}$, as G_2 is a 1-independent set. If $n_1 > \frac{k}{k+1}n$, then by Theorem 5 and the fact that $\Delta(G_1) \leqslant 2k-2$, we have $\alpha_1(G_1) > \frac{n}{k+1}$. Thus $n_1 \leqslant \frac{k}{k+1}n$.

As $n_1 \leqslant \frac{k}{k+1}n$, $n_2 \leqslant \frac{1}{k+1}n$ and $n_1+n_2 = n$, we conclude that $n_1 = \frac{k}{k+1}n$ and $n_2 = \frac{1}{k+1}n$. Since $n_1 = \frac{k}{k+1}n$ and $\Delta(G_1) \leqslant 2k-2$ we have by Theorem 5 that $\alpha_1(G_1) \geqslant \frac{n}{k+1}$. Thus we have

$$\frac{n}{k+1} = \alpha_1(G) \geqslant \alpha_1(G_1) \geqslant \frac{n}{k+1}$$

and we conclude that $\alpha_1(G_1) = \frac{n}{k+1}$.

Finally, as G_1 has exactly $n_1 = \frac{k}{k+1}n$ vertices, maximum degree $\Delta(G_1) \leqslant 2k-2$ and $\alpha_1(G_1) = \frac{n}{k+1}$, we have by the induction hypothesis that G_1 is a union of copies of J_{2k}. Let the copies of J_{2k} in G_1 be $U_1, U_2, \ldots, U_t$ where $t = \frac{n}{2(k+1)}$ (that is, $G_1 = U_1 \cup U_2 \cup \ldots \cup U_t$). Let v be an arbitrary vertex of G_2. We claim that v (as a vertex of G) is adjacent to all the $2k$ vertices of U_i for some i. Suppose (for the sake of contradiction) that this does not hold. Then there are two cases.

Case 1: For all $i \geqslant 1$, v is adjacent to at most $2k-2$ vertices of U_i.
In this case we can pick from each U_i two vertices that are not adjacent to v (this is possible as $k \geqslant 2$). Call the resulting set S. Notice that $S \cup \{v\}$ is a 1-independent set in G of size $2t+1 > \frac{n}{k+1}$. We get a contradiction and thus this case is done.

Case 2: There are $i, j \geqslant 1$ such that v is adjacent to $2k-1$ vertices in U_i, is adjacent to at most one vertex of U_j, and v is not adjacent to any other vertex in G_1.

In this case we pick arbitrarily two vertices from each U_k, where $k \neq i$ and $k \neq j$. Call the resulting set S. Now we pick a vertex u_1 from U_i which is not adjacent to v. As J_{2k} is a clique minus a 1-factor there is a vertex u_2 in U_i such that u_1 and u_2 are not adjacent. We add u_1 and u_2 to the set S. Finally we pick vertices u_3 and u_4 in U_j such that u_3 and u_4 are not adjacent to v. This is possible as $k \geqslant 2$ and v is adjacent to at most one vertex in U_j. Add u_3 and u_4 to the set S. Notice that $S \cup \{v\}$ is a 1-independent set in G of size $2t+1 > \frac{n}{k+1}$. Once again we got a contradiction.

We have shown that each vertex v in G_2 is adjacent to all the $2k$ vertices of U_i for some i (dependent on v). We conclude that G is a union of copies of J_{2k+2}, and thus we are done. □

Finally we are ready to prove the main theorem of this section.

Theorem 23. *Let G be a graph of order n and average degree d. If $\alpha_1(G) = \frac{2}{d+2}n$ then G is either an independent set or a union of almost-cliques of equal size.*

Proof. The case $d = 0$ is trivial as then the graph G must be an independent set. We will assume henceforth that $d > 0$. By Lemma 19 the average degree d must be an even integer. Let $d = 2k$ for some integer $k \geqslant 1$. By Lemma 20 we have that G is in fact $2k$-regular. Finally by Lemma 22 graph G is a union of copies of J_{2k+2}, and we are done. □

5 Caro-Wei type bound for 1-independence

Let $f(x)$ be the function from Definition 3.1, that is for real $x \geqslant 0$

$$f(x) = \frac{1}{1+x}\left(1 + \frac{\{x\}\,(1-\{x\})}{(\lfloor x\rfloor + 1)\,(\lfloor x\rfloor + 2)}\right).$$

For all integers $k \geqslant 0$ and real $x \geqslant 0$ define $f_k(x) = f\left(\frac{x}{k+1}\right)$.

Lemma 24. *For any integer $k \geqslant 0$ and for x in the interval $[0, \infty)$, the function $f_k(x)$ is continuous, monotonically decreasing and convex.*

Proof. The proof of continuity and that the function is monotonically decreasing is almost identical to the proof of Lemma 15 and thus omitted. The convexity of $f_k(x)$ follows from the convexity of $f(x)$ as proven in Lemma 15, combined with the fact that the composition of a convex function with an affine function is convex. □

Corollary 25. *For all integers $k \geqslant 0$ and $n \geqslant 1$ we have $f_k(n-1) - f_k(n) \geqslant f_k(n) - f_k(n+1)$.*

Proof. This inequality follows from the convexity of the function $f_k(x)$ on the interval $[0, \infty)$. □

In this section we shall make the following conjecture:

Conjecture 26. Let G be a graph of order n with degree sequence $d_1, \ldots, d_n$. Then

$$\alpha_k(G) \geqslant \sum_{i=1}^{n} f_k\left(d_i\right)$$

For $k = 0$ the conjecture above is simply the Caro-Wei bound. Thus the first open case is the case when $k = 1$. This conjecture seems to be significantly harder than Conjecture 1. Notice that Conjecture 26 implies Conjecture 2 (by Lemma 11). Furthermore, by Jensen's inequality, Theorem 12 follows directly from Conjecture 26. In this section we shall make a very modest progress by proving that the $k = 1$ case of Conjecture 26 holds for graphs of maximum degree 4.

For integers the following formulation of the function f_1 is easier to work with.

Lemma 27. *For all integers $n \geqslant 0$ we have*

$$f_1(n) = \begin{cases} \frac{2}{n+2}, & \text{if } n \text{ is even} \\ \frac{1}{n+1} + \frac{1}{n+3}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Follows directly from Definition 3.1. □

The first few values of f_1 are: $f_1(0) = 1$, $f_1(1) = \frac{3}{4}$, $f_1(2) = \frac{1}{2}$, $f_1(3) = \frac{5}{12}$ and $f_1(4) = \frac{1}{3}$.

Definition 5.1. Given a graph G. Define

$$s(G) = \sum_{v \in V(G)} f_1\left(\deg_G(v)\right).$$

We are ready to prove the main result of this section.

Theorem 28. *If G is a graph of order n with maximum degree $\Delta(G) \leqslant 4$ then $\alpha_1(G) \geqslant s(G)$*

Proof. We proceed by induction on n. For $n = 1$ the claim is trivial. Now we will assume that the claim holds for graphs on at most $n - 1$ vertices, and we will prove the claim for graphs on n vertices.

Assume we have a graph G with $n \geqslant 2$ vertices. If $\Delta(G) \leqslant 1$ the claim holds trivially as G is a 1-independent set in this case. Hence we may assume that $2 \leqslant \Delta(G) \leqslant 4$. Furthermore, we may assume that graph G is connected, for otherwise we can apply the inductive hypothesis to each connected component. Now we will consider 3 cases.

Case 1: $\Delta(G) = 4$
Let v be a vertex of maximum degree in G. Define graph G' as $G' = G - v$. Let v_1, v_2, v_3, v_4 be the neighbors of v in G.

$$s(G') = s(G) - f_1(\deg_G(v)) + \sum_{i=1}^{4} \left[f_1(\deg_G(v_i) - 1) - f_1(\deg_G(v_i))\right].$$

Hence to prove that $s(G') \geqslant s(G)$ it suffices to show that

$$\sum_{i=1}^{4} \left[f_1(\deg_G(v_i) - 1) - f_1(\deg_G(v_i))\right] \geqslant f_1(\deg_G(v)),$$

This follows as

$$\begin{aligned}
\sum_{i=1}^{4} \left[f_1(\deg_G(v_i) - 1) - f_1(\deg_G(v_i))\right] &\geqslant \sum_{i=1}^{4} \left[f_1(3) - f_1(4)\right] && \text{(by Corollary 25)} \\
&= 4\left(\frac{5}{12} - \frac{1}{3}\right) \\
&= \frac{1}{3} = f_1(4) = f_1(\deg_G(v)).
\end{aligned}$$

Hence, we obtain $s(G') \geqslant s(G)$. Combined with $\alpha_1(G) \geqslant \alpha_1(G')$ and $\alpha_1(G') \geqslant s(G')$, which is given by the induction hypothesis, the claim follows.

Case 2: $\Delta(G) = 3$
If G is a 3-regular graph then by Theorem 5 we have $\alpha_1(G) \geqslant \frac{1}{2}n > \frac{5}{12}n = s(G)$ (as $f_1(3) = \frac{5}{12}$) and we are done.

The remaining case is when G is a connected graph of maximum degree 3, which contains a vertex of degree 3 and also a vertex of degree at most 2. Thus G contains a vertex v of degree 3 with at least one neighbor of degree at most 2. Let v_1, v_2, v_3 be the neighbors of v in G and assume that $\deg_G(v_1) \leqslant 2$. Define graph G' as $G' = G - v$ and notice that

$$s(G') = s(G) - f_1(\deg_G(v)) + \sum_{i=1}^{3} \left[f_1(\deg_G(v_i) - 1) - f_1(\deg_G(v_i)) \right].$$

Now

$$\begin{aligned}\sum_{i=1}^{3} \left[f_1(\deg_G(v_i) - 1) - f_1(\deg_G(v_i)) \right] &\geqslant (f_1(1) - f_1(2)) + 2(f_1(2) - f_1(3)) \\ &= \frac{5}{12} = f_1(3) = f_1(\deg_G(v)).\end{aligned}$$

Hence $s(G') \geqslant s(G)$, and thus the claim follows by the induction hypothesis.

Case 3: $\Delta(G) = 2$
Let v be a vertex of maximum degree in G. Define graph G' as $G' = G - v$. Let v_1, v_2 be the neighbors of v in G.

$$s(G') = s(G) - f_1(\deg_G(v)) + \sum_{i=1}^{2} \left[f_1(\deg_G(v_i) - 1) - f_1(\deg_G(v_i)) \right].$$

Notice that

$$\begin{aligned}\sum_{i=1}^{2} \left[f_1(\deg_G(v_i) - 1) - f_1(\deg_G(v_i)) \right] &\geqslant \sum_{i=1}^{2} \left[f_1(1) - f_1(2) \right] \\ &= 2\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{1}{2} = f_1(2) = f_1(\deg_G(v)).\end{aligned}$$

Hence $s(G') \geqslant s(G)$, and thus the claim follows by the induction hypothesis. $\square$

6 Improved bounds for 2-independence

In this section we will improve the bounds for 2-independence for graphs in which the average degree is an integer divisible by 3. It was proven in [CH13] that a graph G of order n and average degree $d = 3t$ for an integer $t \geqslant 0$ satisfies $\alpha_2(G) \geqslant \frac{n}{t+1}$. In this section we improve this bound, proving the following.

Theorem 29. *Let $t > 0$ be an integer. If G is a graph of order n and average degree $d = 3t$ then*

$$\alpha_2(G) \geqslant \frac{n}{t+1} \left(1 + \frac{4}{15t^2 + 25t + 6} \right).$$

Proof. Define function the $h(x) = \dfrac{1}{x+1}\left(1 + \dfrac{4}{15x^2+25x+6}\right)$. We will prove that $\alpha_2(G) \geqslant nh(t)$.

We can assume that n is divisible by $l = 15t^2 + 25t + 6$. This assumption holds as we can build a graph $G' = lG$ (that is G' is a disjoint union of l copies of G). Notice that $d(G) = d(G') = 3t$ and that the number of vertices n' of G' is divisible by l. Now if $\alpha_2(G') \geqslant n'h(t)$ then by the pigeonhole principle the original graph G satisfies $\alpha_2(G) \geqslant \frac{n'}{l}h(t) = nh(t)$.

We define parameter s as follows:

$$s = \frac{6tn}{15t^2 + 25t + 6}. \tag{6.1}$$

Notice that s is an integer as n is divisible by $15t^2 + 25t + 6$. Furthermore, $s > 0$.
Set $G_0 = G$. If there is a vertex $v_0 \in V(G_0)$ such that $\deg_{G_0}(v_0) \geqslant 3t + 1$, remove it from the graph G_0 and call the resulting graph G_1 (that is $G_1 = G_0 - v_0$).

Now if $s > 1$ and there is a vertex $v_1 \in V(G_1)$ such that $\deg_{G_1}(v_1) \geqslant 3t + 1$, remove v_1 from the graph G_1 and call the resulting graph G_2, that is $G_2 = G_1 - v_1$.
We repeat this operation iteratively. In iteration i (starting with $i = 0$) we first check if $i = s$ or $\Delta(G_i) \leqslant 3t$, and if one of these conditions holds we terminate the process. Otherwise, we pick a vertex $v_i \in V(G_i)$ such that $\deg_{G_i}(v_i) \geqslant 3t + 1$ and remove it from the graph G_i. We call the resulting graph G_{i+1} (that is, $G_{i+1} = G_i - v_i$).
Suppose that the process above terminated on iteration $j \leqslant s$, that is, the last graph created in the process is G_j. If $j < s$ then $\Delta(G_j) \leqslant 3t$ and thus we have

$$\begin{aligned}
\alpha_2(G_j) &\geqslant (n-s)\frac{5}{5t+3} && \text{(by Theorem 9)}\\
&= n\frac{5}{5t+3}\left(1 - \frac{6t}{15t^2+25t+6}\right)\\
&= n\frac{5}{5t+3}\,\frac{15t^2+19t+6}{15t^2+25t+6}\\
&= n\frac{5}{5t+3}\,\frac{(5t+3)(3t+2)}{15t^2+25t+6}\\
&= n\frac{5(3t+2)}{15t^2+25t+6}\\
&= \frac{n}{t+1}\left(1 + \frac{4}{15t^2+25t+6}\right)
\end{aligned}$$

and that is what we wanted to prove.

The remaining case is $j = s$. In this case $e(G_s) \leqslant \frac{3tn}{2} - (3t+1)s$, as in each iteration at least $3t + 1$ edges were removed. Thus $d(G_s) \leqslant \frac{3tn-2(3t+1)s}{n-s}$. We shall prove this case by applying Theorem 12. Hence, because of Lemma 24, we will assume that

$$d(G_s) = \frac{3tn - (6t+2)s}{n-s}. \tag{6.2}$$

Now we shall prove that $3(t-1) < d(G_s) \leqslant 3t$.

Claim 1: $d(G_s) \leqslant 3t$

This follows as

$$d(G_s) = \frac{3tn - (6t+2)s}{n-s} \leqslant 3t.$$

Claim 2: $d(G_s) > 3(t-1)$

We need to prove that

$$d(G_s) = \frac{3tn - (6t+2)s}{n-s} > 3(t-1)$$

which is equivalent to

$$3tn - (6t+2)s > 3(t-1)(n-s). \tag{6.3}$$

Now Equation 6.3 holds if and only if

$$3n - 5s - 3ts > 0. \tag{6.4}$$

and Equation 6.4 holds if and only if $s < \frac{3n}{5+3t}$. Finally, as $t > 0$ in an integer, we have

$$s = \frac{6tn}{15t^2 + 25t + 6} < \frac{3n}{5+3t}$$

and thus Claim 2 follows.

We have shown that $3(t-1) < d(G_s) \leqslant 3t$, and hence $\lceil d(G_s)/3 \rceil = t$. Now we apply Theorem 12 to get

$$\begin{aligned}
\alpha_2(G_s) &\geqslant f\left(\frac{d(G_s)}{3}\right)(n-s) \\
&= g\left(\frac{d(G_s)}{3}\right)(n-s) && \text{(by Lemma 14)} \\
&= \frac{n-s}{(t+1)t}\left(2t - \frac{d(G_s)}{3}\right) \\
&= \frac{2(n-s)}{t+1} - \frac{3tn-(6t+2)s}{3t(t+1)} && \text{(by Equation 6.2)} \\
&= \frac{6t(n-s) - 3tn + (6t+2)s}{3t(t+1)} \\
&= \frac{3tn+2s}{3t(t+1)} \\
&= \frac{n}{t+1} + \frac{2}{3t(t+1)}\frac{6tn}{15t^2+25t+6} && \text{(by Equation 6.1)} \\
&= \frac{n}{t+1}\left(1 + \frac{4}{15t^2+25t+6}\right).
\end{aligned}$$

This completes the proof. □

Corollary 30. *If G is a graph of order n and average degree $d > 0$ then $\alpha_2(G) > \frac{3}{d+3}n$.*

Proof. If $\frac{d}{3}$ is not an integer then we have by Theorem 12 and Lemma 11 that $\alpha_2(G) > \frac{3}{d+3}n$.

If $\frac{d}{3}$ is an integer then by Theorem 29 we have $\alpha_2(G) > \frac{3}{d+3}n$ for $d > 0$, and we are done. □

7 Discussion

The two most important open questions remaining are the following:

1. Prove Conjecture 2.

2. Improve the bounds of Theorem 12 for $k \geqslant 2$. Even an improvement of the bounds of Theorem 29 would be interesting.

One direction of trying to prove Conjecture 2 would be to find a probabilistic argument, similar in spirit to that used for finding a large independent set (see [AS08] page 95): there one considers a random permutation on the graph vertices, and adds a vertex v to the independent set if in the permutation it precedes all its neighbors.

As for improving the bound of Theorem 29, one could use the following Theorem [SSYH09] to gain a slight improvement for the case where the average degree is 3.

Theorem 31. *If G is a graph of order n and $\delta(G) \geqslant 2$, then $\gamma(G) \leqslant \frac{3n+|V_2|}{8}$, where V_2 denotes the set of vertices of degree 2 in G.*

This improvement is very slight (details are omitted).

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References

[AS08] Noga Alon and Joel H. Spencer. *The Probabilistic Method.* Wiley, third edition, 2008.

[BCHN13] Asen Bojilov, Yair Caro, Adriana Hansberg, and Nedyalko Nenov. Partitions of graphs into small and large sets. *Discrete Applied Mathematics*, 161(13-14):1912–1924, September 2013.

[Car79] Yair Caro. New results on the independence number. *Tech. Report, Tel-Aviv University*, 1979.

[CH13] Yair Caro and Adriana Hansberg. New approach to the k-independence number of a graph. *Electronic Journal of Combinatorics*, 20(1):#P33, 2013.

[CT91] Yair Caro and Zsolt Tuza. Improved lower bounds on k-independence. *Journal of Graph Theory*, 15(1):99–107, March 1991.

[HL97] Magnus M. Halldórsson and Hoong Chuin Lau. Low-degree graph partitioning via local search with applications to constraint satisfaction, max cut, and coloring. *Journal of Graph Algorithms and Applications*, 1(3):49–61, 1997.

[HP13] Adriana Hansberg and Ryan Pepper. On k-domination and j-independence in graphs. *Discrete Applied Mathematics*, 161(10-11):1472–1480, 2013.

[HS86] Glenn Hopkins and William Staton. Vertex partition and k-small subsets of graphs. *Ars Combinatoria*, 22:19–24, 1986.

[JBHUL93] Jean-Baptiste, Hiriart-Urruty, and Claude Lemaréchal. *Convex Analysis and Minimization Algorithms I.* Springer-Verlag, 1993.

[Lov66] László Lovász. On decompositions of graphs. *Studia Sci. Math Hungar.*, 1:237–238, 1966.

[Ree96] Bruce Reed. Paths, stars and the number three. *Combinatorics Probability and Computing*, 5(3):277–295, 1996.

[SSYH09] Er Fang Shan, Moo Young Sohn, Xu Dong Yuan, and Michael A. Henning. Domination number in graphs with minimum degree two. *Acta Mathematica Sinica (English Series)*, 25:1253–1268, 2009.

[Tur41] Paul Turán. On an extremal problem in graph theory (hungarian). *Math. Fiz. Lapok*, 48:436–452, 1941.

[Wei81] V. K. Wei. A lower bound on the stability number of a simple graph. *Bell Laboratories Technical Memorandum, 81-11217-9, Murray Hill, NJ*, 1981.