

Infinite excursions of rotor walks on regular trees

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Abstract

A rotor configuration on a graph contains in every vertex an infinite ordered sequence of rotors, each is pointing to a neighbor of the vertex. After sampling a configuration according to some probability measure, a rotor walk is a deterministic process: at each step it chooses the next unused rotor in its current location, and uses it to jump to the neighboring vertex to which it points. Rotor walks capture many aspects of the expected behavior of simple random walks. However, this similarity breaks down for the property of having an infinite excursion. In this paper we study that question for natural random configuration models on regular trees. Our results suggest that in this context the rotor model behaves like the simple random walk unless it is not “close to” the standard rotor-router model.

Keywords: rotor walk; self interacting walk; regular tree; recurrence; transience; multi-type branching process

1 Introduction

1.1 Informal motivation

We consider first rotor walks on \mathbb{N} : on each vertex n there is an infinite *rotor sequence* $\mathbf{a}_n \in \{-1, 1\}^{\mathbb{N}}$ pointing to one of the neighbors. The walk starts in the origin. Inductively,

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the walk being at n follows the direction of the first rotor in \mathbf{a}_n and deletes this rotor. Assume that each \mathbf{a}_n is non-degenerate, i.e. it contains infinitely many -1 's and $+1$'s. The first question that we ask is whether having at each vertex a “local drift” zero implies “recurrence” of the rotor walk; call a rotor sequence \mathbf{a} L -periodic if $\mathbf{a}(x) = \mathbf{a}(x + L)$ for all $x \in \mathbb{N}$ and *balanced* if there are as many -1 's as $+1$'s per period.

- Fix a period L . In optimizing over all L -periodic balanced choices of \mathbf{a}_n , $n \in \mathbb{N}$, what is the maximal number of “infinite excursions” that can be achieved?
- Choose \mathbf{a}_n , $n \in \mathbb{N}$, in an i.i.d. way. What are the conditions which ensure that the rotor walk is recurrent a.s.?

Consider the same model on the binary tree \mathbb{T}_2 . Now, we have three possible directions to choose from. Let 0 describe the direction towards the root and let 1 and 2 stand for pointing towards the two children respectively, see Figure 1. A rotor sequence \mathbf{a} now takes values in $\{0, 1, 2\}^{\mathbb{N}}$. It is well known that the simple random walk on the binary tree is transient. However, the following rotor(-router) walk is recurrent, see [24, 3]. Consider the 3-periodic rotor sequences

$$\mathbf{a}^{(1)} = (0, 1, 2, 0, 1, 2, \dots), \mathbf{a}^{(2)} = (1, 2, 0, 1, 2, 0, \dots) \text{ and } \mathbf{a}^{(3)} = (2, 0, 1, 2, 0, 1, \dots),$$

and choose for each vertex of \mathbb{T}_2 independently one of these three sequences with equal probability. This behavior difference is somewhat surprising, since rotor walks share many properties with the simple random walk.

- Is it a general phenomenon that (periodic and balanced) rotor walks are recurrent on \mathbb{T}_2 ?

We answer this question negatively. In particular, consider the 6-periodic sequences

$$\mathbf{a}^{(1)} = (0, 0, 1, 2, 1, 2, \dots), \mathbf{a}^{(2)} = (1, 1, 2, 0, 2, 0, \dots) \text{ and } \mathbf{a}^{(3)} = (2, 2, 0, 1, 0, 1, \dots),$$

then the corresponding rotor walk in the i.i.d. uniform configuration model is transient a.s.

- Which sequences \mathbf{a} give rise to recurrent rotor walks?
- Are there interesting i.i.d. rotor configurations on \mathbb{T}_d , $d \geq 3$, that are recurrent?

1.2 General introduction and results

A rotor walk on a graph is a deterministic process where a particle is routed through the vertices of a graph. At each vertex the particle is routed to one of the neighboring vertices following a prescribed periodic sequence, called the *rotor sequence*. In the classical model, called *rotor-router walk*, the rotor sequence is a fixed cyclic order of the neighboring vertices. For an overview of this model and its classic properties we refer to the expository paper [15]. In this paper we consider configurations that may arise from

any non-degenerate sequence. By *non-degenerate* we mean that there are infinitely many rotors pointing to every neighboring vertex. In this more general case we speak of *rotor walks*. Note that this model was also introduced in [27] as *stack walks*.

Rotor walks capture in many aspects the expected behavior of simple random walks, but with significantly reduced fluctuations compared to a typical random walk trajectory; for more details see [9, 12, 16, 20]. However, this similarity breaks down when one looks at the property of being recurrent or transient. In fact, the rotor walk may behave differently than the corresponding random walk. We say that a rotor walk which started at the origin with initial rotor configuration ρ is *recurrent* if it returns to the origin infinitely many times; in this case we say that the rotor configuration ρ is recurrent. Otherwise we say that the rotor walk is *transient* or that the rotor configuration ρ is transient. In the recurrent regime all excursions (from the origin) are finite. However, in the transient regime there is a first *infinite excursion*. This excursion eventually leaves every finite ball around the origin and hence when it reaches “infinity” it leaves a well defined rotor configuration. For this reason we can start a new walk after the first infinite excursion in the origin and proceed inductively.

In [4] existence of recurrent initial configurations for the rotor-router walk on many graphs, including \mathbb{Z}^d , and planar graphs with locally finite embeddings is proved. An example of an initial rotor-router configuration on \mathbb{Z}^2 for which the rotor-router walk is recurrent was given earlier in [16, Theorem 5]. See also [11] for initial rotor-router configurations with all rotors aligned on \mathbb{Z}^d . Infinite excursions of rotor-router walks on homogeneous trees were studied in [24] and [23]. On general trees, the issue of transience and recurrence of rotor-router walks was studied in detail in [3]. An extension for random initial configuration of rotor-router walks was made in [18] on directed covers of graphs (periodic trees) and in [17] on Galton-Watson trees.

In this paper we give criteria for recurrence and transience of rotor walks on d -ary trees $\mathbb{T}_d, d \geq 1$. Recurrence of rotor-router walks on \mathbb{Z} (and similarly on $\mathbb{N} = \mathbb{T}_1$) is rather obvious. Indeed, if we start a rotor-router walk on \mathbb{Z} with i.i.d. uniform initial configuration then the walk has a simple structure. It follows the rotors in one direction until it hits a rotor pointing in the opposite direction. The walk reverses its direction and retraces its path entirely and continues until it hits a rotor pointing in the opposite direction and so on. This behavior was presented in [25] as an example of self-organization.

A fundamental tool used in this paper is a connection, observed in [14], between nearest neighbor walks and Galton-Watson processes. In the context of random walks in random environments, its usefulness was demonstrated in the well-known paper [19]. In the special case of the rotor-router model a more specific construction of a Galton-Watson process was used to prove transience criteria in [24] and [3].

Rotor walks can be seen as a special case of excited random walks by regarding the rotors as non-elliptic cookies. This deterministic point of view is already observed and extensively used in [1] and [2], where rotors are called ‘arrows’. In the context of excited random walks on the one-dimensional lattice the relation to Galton-Watson processes was used first in [6]. The case of excited random walks on regular trees and their relations to survival of multi-type Galton-Watson processes was introduced by the same authors in

[7]. The interested reader may find more details on excited random walks in [8] or in the survey [22].

In our setting the relevant process is based on [7] and can be considered as a multi-type Galton-Watson process with *a priori* infinitely many types, see Chapter 3. For general background on multi-type Galton-Watson processes we refer to [5, 13].

In the first part of paper we give criteria for recurrence for rotor walks on \mathbb{N} , see Theorem 11, Corollary 13 and Theorem 16. Moreover, we observe another phenomenon of self-organization; we consider the case where a rotor sequence at some vertex can be any L -periodic (i.e. has period L) and balanced sequence (i.e., there are as many rotors pointing to the right as pointing to the left in each period). In this case there are at most $L/2$ infinite excursions, see Theorem 8. In other words, while there might be infinite excursions for the first walks the system organizes itself in such a way that after at most $L/2$ infinite excursions it behaves as it “should”, namely it is recurrent.

In the second part we consider rotor configurations on the homogeneous tree $\mathbb{T}_d, d \geq 2$. We give a criterion for recurrence and transience for the general model, see Theorem 23. This criterion is based on the fact that transience of the rotor walk is equivalent to survival of a multi-type Galton-Watson process. We then show that \mathbb{T}_2 can be considered as the critical case in the following sense. We choose uniformly a rotation of a fixed rotor sequence independently in all vertices of \mathbb{T}_2 . In this model, that we call uniform rotation model, the rotor walk may be recurrent or transient, see Theorem 27. However, in higher dimensions, $\mathbb{T}_d, d \geq 3$, the walk is always transient, see Theorem 28. We conjecture that this behavior holds also for another model, the uniform shift model, see Conjectures 29 and 32. In particular, our results generalize recurrence and transience criteria for rotor-router walks on regular trees [3], periodic trees [18], and Galton-Watson trees with bounded degree [17].

1.3 Formal definition.

Let (\mathbb{T}_d, o) be the d -ary rooted tree, $d \geq 1$. For a vertex $v \in \mathbb{T}_d$, we denote by v_0 the parent of v and by v_1, v_2, \dots, v_d its children. We have in mind a planar embedding of the tree where the root is at the top, each generation is located below its ancestors, and the children are ordered from left to right. We add a self-loop to the origin o and define $o_0 = o$. A *rotor sequence* is a function $\mathbf{a} : \mathbb{N} \rightarrow \{0, \dots, d\}$. A rotor sequence \mathbf{a} is called *non-degenerate* if for every $r \in \{0, \dots, d\}$ there are infinitely many $x \in \mathbb{N}$ such that $\mathbf{a}(x) = r$. We denote by $\mathcal{A} \subset \{0, \dots, d\}^{\mathbb{N}}$ the set of all non-degenerate rotor sequences.

For $v \in \mathbb{T}_d$, we consider a rotor sequence \mathbf{a}_v on v by identifying the set $\{0, \dots, d\}$ with $N_v = \{v_0, v_1, v_2, \dots, v_d\}$, the ordered (from left to right) set of v 's neighboring vertices. We refer the reader to Figure 1 for an illustration of these definitions.

To each vertex of \mathbb{T}_d we assign a non-degenerate rotor sequence to get a rotor configuration $\{\mathbf{a}_v\}_{v \in \mathbb{T}_d} \in \mathcal{A}^{\mathbb{T}_d}$. The corresponding *rotor walk* $X = (X_n)_{n \geq 0}$ on \mathbb{T}_d , with a local time $l = (l_n)_{n \geq 0}$ is defined recursively as follows:

$$X_0 := o, \quad l_0 \equiv 0, \quad X_{n+1} = (X_n)_{\mathbf{a}_v}, \quad \text{and} \quad l_{n+1}(v) = l_n(v) + \delta_{X_n}(v),$$

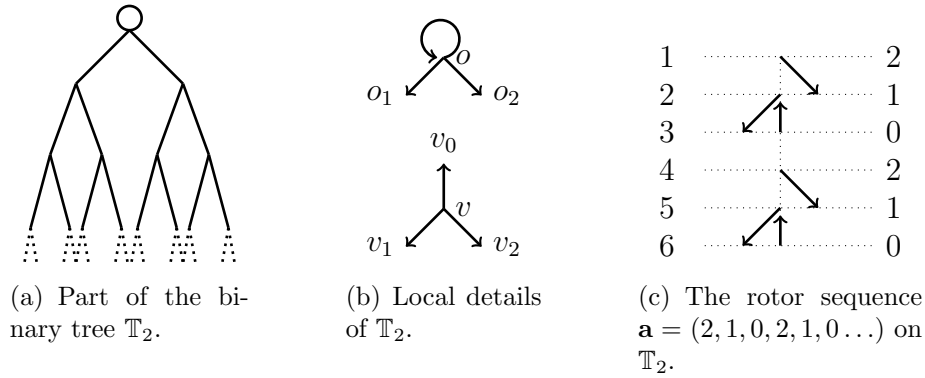


Figure 1: The notations for the binary tree \mathbb{T}_2 .

where $i = \mathbf{a}_{X_n}(1 + l_n(X_n))$. In words, the rotor walk starts in o at time $n = 0$. At time $n = 1$ it follows the direction of the first rotor in the sequence \mathbf{a}_o , i.e. it moves to $o_{\mathbf{a}_o(1)}$. Inductively, assume that the rotor walk is in v at time n , then at time $n+1$ the walk moves in the direction of the first unused rotor in v and moves to v_i , where $i = \mathbf{a}_v(1 + l_n(v))$.

We define a *finite excursion* of X to be a finite sequence $X_m, X_{m+1}, \dots, X_{m+n}$ starting at the root $X_m = o$ and ending with a self loop $X_{m+n-1} = X_{m+n} = o$ such that it does not contain any self loop before that time, that is $X_j \neq o$ if $X_{j-1} = o$ for all $m+1 \leq j \leq m+n-1$. An *infinite excursion* of X is an infinite sequence X_m, X_{m+1}, \dots starting at the root $X_m = o$ so that $X_j \neq o$ whenever $X_{j-1} = o$ for all $j \geq m+1$.

Whenever the rotor configuration is in $\mathcal{A}^{\mathbb{T}_d}$, the walk X is recurrent if and only if every vertex is visited infinitely many times. Indeed, if the walk visits the origin infinitely many times, by non-degeneracy all of its neighbors are visited infinitely often; hence, by induction this is the case for all vertices of the graph. We shall assume throughout the paper that the rotor sequences are always non-degenerate, even whenever it is not mentioned explicitly.

Definition 1 (Finitely supported distribution model). Let p be a probability measure on \mathcal{A} with finite support S . Enumerating the elements in S from 1 to $|S|$ we rewrite p as $p = (p_1, p_2, \dots, p_{|S|})$; the corresponding rotor configurations are written as $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(|S|)} \in \mathcal{A}^{\mathbb{T}_d}$. We call the rotor distribution a *finitely supported distribution* if we sample a rotor sequence in every vertex independently and identically distributed according to p .

The following two models are interesting special cases of the finitely supported distribution model.

Definition 2 (Uniform rotation model). Fix a rotor sequence $\mathbf{a} \in \mathcal{A}$. Let $\pi \in S_{d+1}$ be the permutation on the $d+1$ symbols $\{0, \dots, d\}$ defined by the rotation $\pi : n \mapsto n+1 \pmod{d+1}$. We denote by $\pi\mathbf{a}$ the rotor sequence given by a point-wise use of π on \mathbf{a} : $\pi\mathbf{a}(x) := \pi(\mathbf{a}(x))$, $x \in \mathbb{N}$. We call the rotor distribution a *uniform rotation* if we sample a rotor sequence in every vertex independently and identically distributed according to the uniform distribution on the set of all $d+1$ rotations of \mathbf{a} : $\{\mathbf{a}, \pi\mathbf{a}, \pi^2\mathbf{a}, \dots, \pi^d\mathbf{a}\}$.

A rotor sequence on the d -ary tree \mathbf{a} is said to be L -periodic if $\mathbf{a}(x + L) = \mathbf{a}(x)$ for all $x \in \mathbb{N}$ for some L . We say that an L -periodic rotor sequence \mathbf{a} is balanced if each of the values $\{0, 1, 2, \dots, d\}$ appears $N = L/(d + 1)$ times in the first L rotors.

Definition 3 (Uniform shift model). Let $\mathbf{a}^{(1)} = (a_n)_{n \in \mathbb{N}} \in \mathcal{A}$ be an L -periodic rotor sequence. Let $S^{(i)} : (a_n)_{n \in \mathbb{N}} \mapsto (a_{n+i})_{n \in \mathbb{N}}$ be the shift operator and define $\mathbf{a}^{(i)} = S^{(i-1)}\mathbf{a}^{(1)}$ for $i \in \{2, \dots, L\}$. We call the rotor distribution a *uniform shift* if we sample a rotor sequence in every vertex independently and identically distributed according to the uniform distribution on the set all possible shifts of \mathbf{a} : $\{\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(L)}\}$.

2 The unary tree

In this section we analyze rotor walks on the unary tree \mathbb{N} . We begin in Section 2.1 with the uniform rotation model and show recurrence. We then turn to periodic sequences and bound the number of infinite excursions by half the period in Section 2.2.1. Finally, in Section 2.2.2 we derive a simple criterion for recurrence of the walk. For convenience of the reader and in order to emphasize the structure of $\mathbb{N} = \mathbb{T}_1$ we will in the rest of this section use -1 and $+1$ for 0 and 1 , respectively, in the rotor sequences.

2.1 Uniform rotations: recurrence

We shall show that the walk is recurrent on the unary tree \mathbb{N} , whenever \mathbf{a} is a non-degenerate rotor sequence, and the (two) permutations on \mathbf{a} are chosen uniformly and independently in all vertices.

Theorem 4. *Let \mathbf{a} be a rotor sequence. Then, the rotor walk on \mathbb{N} in the uniform rotation model corresponding to \mathbf{a} (see Definition 2) is recurrent a.s.*

Before we prove this theorem we introduce some more notations and basic facts. For a rotor sequence \mathbf{a} and $x \in \mathbb{N}$ we define

$$T_{\mathbf{a}}(x) = \inf \left\{ t \geq 0 : \sum_{i=1}^t \mathbf{1}_{\mathbf{a}(i)=0} = x \right\}, \quad (1)$$

the number of rotors in \mathbf{a} prior to x rotors pointing to 0 . Define $U_{\mathbf{a}}(0) := 0$ and

$$U_{\mathbf{a}}(x) = T_{\mathbf{a}}(x) - x \quad (2)$$

be the number of $+1$ -rotors in \mathbf{a} prior to x -1 -rotors. Call $U_{\mathbf{a}}$ the U -function of \mathbf{a} .

Lemma 5. [1, Observation 2.8] *Let \mathbf{a} be a rotor sequence. The following hold for the transposition τ of -1 and $+1$.*

1. $U_{\mathbf{a}}(x)$ and $U_{\tau\mathbf{a}}(x)$ are monotonically non-decreasing in x .
2. $U_{\tau\mathbf{a}}(U_{\mathbf{a}}(x)), U_{\mathbf{a}}(U_{\tau\mathbf{a}}(x)) < x$ for every $x \in \mathbb{N}$, and equality holds if $x = 0$.

Fix $\{\mathbf{a}_v\}_{v \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ and $k \in \mathbb{N}$. We are now interested to know whether X has k finite excursions. For that we define a process Z^k on the non-negative integers inductively by the following:

$$Z_0^k := k \text{ and } Z_n^k = U_{\mathbf{a}_{n-1}}(Z_{n-1}^k), \quad n \geq 1. \quad (3)$$

Note that Z_n^k is non-decreasing in k by Lemma 5 (1), but not necessarily in n . We also write Z_n for Z_n^1 . Here is an informal description of Z^k . The variable Z_1^k is the number of rotors in \mathbf{a}_0 pointing away from 0 (i.e., to the right) prior to k rotors pointing back to 0 (i.e., to the left). In other words, the k th excursion is finite if and only if the walk jumps from 1 to 0 exactly Z_1^k times. Assuming the $(k-1)$ st excursion is finite, by induction on n , the k th excursion is finite if and only if for every vertex n the walk jumps exactly a total number Z_n^k times from n to $n-1$ in the first k excursions (which can be infinite time a priori). Eventually, Z_n^k is the number of times the vertex $n-1$ is visited from n before the k th traversal of the loop in the root.

The following Lemma is now evident. It can also be considered as a consequence of Proposition 3.4 in [7] together with the deterministic point of view of Chapters 2.2 and 2.3 of [1].

Lemma 6. *The first k excursions of X are (well defined and) finite if and only if $Z_n^k = 0$ for some n .*

We now can prove Theorem 4.

Proof of Theorem 4. Recurrence follows once we show that all excursions of X are finite a.s. For notation, write $S_2 = \{id, \pi\}$. For each rotor configuration $\{\mathbf{a}_v\}_{v \in \mathbb{T}_d} \in \mathcal{A}^{\mathbb{T}_d}$ we let $Y_n := \text{sign}(\sigma_{n-1})$, where $\sigma_n \in S_2$ is such that $\mathbf{a}_n = \sigma_n \mathbf{a}$, $n \geq 0$. Define $Q_0 = 0$ and $Q_n = \sum_{i=1}^n Y_i$. Then the measure on Q is distributed as a simple symmetric random walk on \mathbb{Z} , started at the origin. For every $k \in \mathbb{N}$ we shall show that the k th excursion is well defined and finite a.s. By Lemma 6, it is enough to show that the process $Z_n^k = 0$ for some finite time $n > 0$ a.s. Since Z_n^k is a Markov chain it follows by repeated use of the strong Markov property that it is enough to show that Z_n^k gets below $k-1$ in finite time a.s. Let N be the first random positive integer $n \geq 1$ such that $Q_n = 0$. We claim that $Z_N^k \leq k-1$. In other words, our goal is to show that the following decomposition of N functions

$$U_{\sigma_{N-1}\mathbf{a}} \circ \cdots \circ U_{\sigma_2\mathbf{a}} \circ U_{\sigma_0\mathbf{a}}(k) \quad (4)$$

is bounded above by $k-1$. We first sketch the proof. By the definition of N , there are exactly $N/2$ appearances of $U_{\pi\mathbf{a}}$ and $N/2$ appearances of $U_{\mathbf{a}}$ in the last decomposition. Therefore, one can regard the ordered sequence (Y_N, \dots, Y_1) as balanced parentheses where the rightmost sign corresponds to $\text{'}'$ and the opposite sign corresponds to '(' . Then, using Lemma 5 we iteratively bound our expression by removing $U_{\sigma_{j+1}\mathbf{a}} \circ U_{\sigma_j\mathbf{a}}$ that corresponds to the right-most couple '(' . In the last iteration, we use Lemma 5 to conclude.

More formally, let J be the first j so that $Y_j \neq Y_{j+1}$. Due to Lemma 5 every factor of the form $U_{\pi\mathbf{a}} \circ U_{\mathbf{a}}(y)$ or $U_{\mathbf{a}} \circ U_{\pi\mathbf{a}}(y)$ is at most $y-1$. In particular,

$$U_{\sigma_{J+1}\mathbf{a}} \circ U_{\sigma_J\mathbf{a}} \circ U_{\sigma_{J-1}\mathbf{a}} \circ \cdots \circ U_{\sigma_1\mathbf{a}}(k) \leq U_{\sigma_{J-1}\mathbf{a}} \circ \cdots \circ U_{\sigma_1\mathbf{a}}(k).$$

By Lemma 5(1) every decomposition of some of the functions $U_{\sigma_j \mathbf{a}}$ is also monotone. Therefore, the whole expression in (4) is bounded by

$$U_{\sigma_N \mathbf{a}} \circ \cdots \circ U_{\sigma_{J+2} \mathbf{a}} \circ U_{\sigma_{J-1} \mathbf{a}} \circ \cdots \circ U_{\sigma_1 \mathbf{a}}(k) \leq U_{\sigma_N \mathbf{a}} \circ \cdots \circ U_{\sigma_{J+2} \mathbf{a}} \circ U_{\sigma_{J-1} \mathbf{a}} \circ \cdots \circ U_{\sigma_1 \mathbf{a}}(k).$$

By considering the right end side of the last inequality instead of (4), after a suitable update of the indices, we can iterate the argument above $N/2 - 1$ times to get

$$Z_N^k = U_{\sigma_{N-1} \mathbf{a}} \circ \cdots \circ U_{\sigma_2 \mathbf{a}} \circ U_{\sigma_0 \mathbf{a}}(k) \leq U_{\sigma_{j+m} \mathbf{a}} \circ U_{\sigma_j \mathbf{a}}(k).$$

for some $j, m \geq 1$ with different signs $Y_j \neq Y_m$. Using Lemma 5 again, the last expression is bounded above by $k - 1$. \square

2.2 Periodic balanced configurations: a criterion

In this section we consider configurations on the unary tree where for each vertex there is an L -periodic and balanced rotor sequence.

2.2.1 Deterministic properties.

The following properties of the U -functions (see the definition in (2)) are immediate consequences of periodicity together with being balanced.

Lemma 7. *Let \mathbf{a} be an L -periodic balanced sequence, U its U -function, and $N = L/2$. Then, for $x = \alpha N + \beta$ with $\alpha, \beta \in \mathbb{N}$ and $1 \leq \beta \leq N$ we have that*

$$U(x) = \alpha N + U(\beta) \text{ and } U(\beta) \leq N.$$

For the proof of the next theorem we consider the notion of *leftover environments*. We start with a rotor configuration $\{\mathbf{a}_n\}_{n \in \mathbb{N}}$. If the rotor walk is transient then the total number of rotors $l(n)$ used in n by the walk is finite. The rotor configuration after the first infinite excursion is well-defined and noted by LO . In other words, $LO(n, i) := \mathbf{a}_n(l(x) + i)$, $n \in \mathbb{N}$. Now another walker starts from the origin and moves according to the rotors in the leftover configuration. This procedure can be sequentially executed as long as the walkers are transient.

Theorem 8 (Number of infinite excursions). *Let $L \in \mathbb{N}$. For any vertex $n \in \mathbb{N}$ choose (deterministically) an L -periodic and balanced rotor sequence \mathbf{a}_n . Then, the number of infinite excursions is bounded from above by $L/2$.*

Proof. Let us first give an intuition for the proof. For each infinite excursion the number of right crossings of an edge is the number of left crossing of this edge plus 1. Assume now that there have been $N = L/2$ infinite excursions and denote by R the right-most position of all finite excursions prior to this time. For each vertex $n > R$ the number of $+1$'s consumed in n is the number of -1 used in $n+1$ plus N . This means that the leftover environment has a "local drift" to the left for all $n > R$. This turns out to be enough to

prevent the next excursion to be infinite. By induction, all subsequent excursions will be finite as well.

To make this argument precise denote by $U_n^{(N)}$ the U -function in vertex n corresponding to the leftover configuration after the N th infinite excursion. In other words, if $l^N(n)$ is the total number of rotors that were used by the first N walkers (see the paragraph before Theorem 8) then U corresponds to the rotor configuration \mathbf{a}_n^N which is given by $\mathbf{a}_n^N(i) := \mathbf{a}_n(i + l^N(x))$.

The U -functions U_n for $n > R$ are not changed by the finite excursions. In order to control the changes made by the infinite excursions we denote by ℓ_n (resp. r_n) the total number of left (resp. right) rotors used by the rotor walk in n after the N th infinite excursion. For all $n > R$ and every $x \in \mathbb{N}$ we now have

$$U_n^{(N)}(x) = U_n(x + \ell_n) - r_n = U_n(x + \ell_n) - \ell_{n+1} - N, \quad (5)$$

since $r_n = \ell_{n+1} + N$. After the N th infinite excursion we start another rotor walk from the origin and show that its excursion is finite. Note that whenever a configuration is L -periodic (and balanced) then so is its leftover configuration. Therefore, due to Lemma 7 we have that $Z_R \leq N$, where Z is the process defined in (3) corresponding to the $U_n^{(N)}$'s. There are two cases: either the rotor walk never visit R , or it does. In the first case the excursion is finite by non-degeneracy. Hence it remains to consider the case $Z_R > 0$. From (5) we have in this case that

$$Z_{R+1} = U_{R+1}^{(N)}(Z_R) = U_{R+1}(Z_R + \ell_{R+1}) - \ell_{R+2} - N.$$

Let $\alpha \in \mathbb{N}$ and $1 \leq \beta \leq N$ such that

$$Z_R + \ell_{R+1} = \alpha N + \beta.$$

By Lemma 7 and (5),

$$Z_{R+1} = U_R(\alpha N + \beta) - \ell_{R+2} - N = (\alpha - 1)N + U_R(\beta) - \ell_{R+2}. \quad (6)$$

Now,

$$\begin{aligned} Z_{R+2} &= U_{R+2}^{(N)}(Z_{R+1}) \\ &= U_{R+2}(Z_{R+1} + \ell_{R+2}) - \ell_{R+3} - N \\ &= U_{R+2}((\alpha - 1)N + U_{R+1}(\beta) - \ell_{R+2} + \ell_{R+2}) - \ell_{R+3} - N \\ &= (\alpha - 2)N + U_{R+2} \circ U_{R+1}(\beta) - \ell_{R+3}, \end{aligned}$$

where we assume that all the terms are non-negative (otherwise Z_{R+2} equals zero by definition), the second equality uses (5), and the third and fourth equalities follow from Lemma 7. Inductively, as long as $Z_{R+j} > 0$ we have

$$\begin{aligned} Z_{R+j+1} &= U_{R+j+1}^{(N)}(Z_{R+j}) \\ &= (\alpha - j - 1)N + U_{R+j+1} \circ \cdots \circ U_{R+2} \circ U_{R+1}(\beta) - \ell_{R+j+2}. \end{aligned}$$

moreover, since $k^* = \infty$ then for every $k \in \{1, \dots, N\}$ there is some $i_k \in \{1, \dots, \ell\}$ such that $U_{i_k}(k) \leq k - 1$. By monotonicity of the U -functions we have

$$U_{i_1} \circ U_{i_2} \circ \dots \circ U_{i_N}(N) \leq U_{i_1} \circ U_{i_2} \circ \dots \circ U_{i_{N-1}}(N-1) \leq \dots \leq U_{i_1}(1) = 0.$$

Using this together with Lemma 7 we have that $Z_{m+N}^x \leq \alpha N$ if $(\mathbf{a}_m, \dots, \mathbf{a}_{m+N}) = (\mathbf{a}_{i_N}, \dots, \mathbf{a}_{i_1})$. Since the last event has a positive probability, by the i.i.d. assumption and the Borel-Cantelli Lemma we have that there are infinitely many such m 's a.s., denote them by m_1, m_2, \dots . Eventually, we have $Z_{m_\alpha+N}^x = 0$. \square

Example 12 (Rotor-router walk). Consider the 2-periodic sequences $\mathbf{a}^{(1)} = (-1, +1, \dots)$ and $\mathbf{a}^{(2)} = (+1, -1, \dots)$, and let $p = (p_1, 1 - p_1)$ with $p_1 \in (0, 1)$. Then, $U_1(x) = x - 1$ and $U_2(x) = x$ for all $x \in \mathbb{N}$ and hence $k^* = \infty$ which implies that the rotor walk is recurrent a.s.

Corollary 13 (Shifts of a balanced sequence). *Let \mathbf{a} be an L -periodic balanced sequence. Choose an i.i.d. configuration such that each of its shifts has a strictly positive probability. Then, the rotor walk is recurrent a.s.*

Lemma 14. *Assume that $a_i \in \{-1, 1\}$, $i = 1, 2, \dots, 2n$ are such that*

$$\sum_{i=1}^{2n} a_i = 0. \tag{8}$$

For $k > 2n$ define $a_k = a_i$ whenever $i \equiv k \pmod{2n}$. Then there is a starting point $j \in \{1, 2, \dots, 2n\}$ so that $\sum_{i=j}^{j-1+k} a_i \leq 0$ for all $k \geq 1$.

One proof of the Lemma is based on a simple variation of the well known Cycle Lemma [10]. We shall supply an even shorter proof due to one of the referees.

Proof. Consider the a_i as the increments of a nearest neighbor walk path. The index at which the walk reaches a maximum gives the desired starting point. \square

The short proof of the next Lemma is due to the same referee as above.

Lemma 15. *Assume that $\mathbf{a} = (a_1, a_2, \dots)$ with $a_i \in \{-1, 1\}$ such that $\sum_{i=1}^k a_i \leq 0$ for all $k \geq 0$. Let U be the corresponding U -function. Then $U(x) < x$ for all $x > 0$.*

Proof. Remember that $T_{\mathbf{a}}(x)$ is the location of the x th -1 (see (1)). Using the assumption for $k = T_{\mathbf{a}}(x) - 1$ we have $0 \geq \sum_{i=1}^{T_{\mathbf{a}}(x)-1} a_i = U(x) - (x - 1)$. \square

Proof of Corollary 13. Lemma 14 gives us a starting point j such that the equality in the lemma holds. Then by Lemma 15 for $\mathbf{a}^{(j-1)} = S^{j-1}\mathbf{a}$, it holds that $U_{\mathbf{a}^{(j-1)}}(x) < x$ for all $x \geq 1$. Therefore $k^* = \infty$ and by Theorem 11 the rotor walk is recurrent a.s. \square

Theorem 16. *(Shift model: the sequence is balanced if and only if the walk is recurrent) Let \mathbf{a} be an L -periodic sequence. Choose an i.i.d. configuration such that each of the shifts has a strictly positive probability. If the number of +1's per period is strictly larger than the numbers of -1's, then the rotor walk is transient a.s. Otherwise it is recurrent a.s.*

Proof. If the number of -1 's per period is larger than the number of $+1$'s, we can compare the sequence with a periodic sequence by arbitrarily changing a few -1 's to $+1$'s in the period so that the resulted sequence is balanced (if L is odd, look at $2L$ instead). Note that the U -function corresponding to the original sequence is not more than the one corresponding to the new one. Using the proof of Corollary 13 for the resulted sequence, we get that there is at least one U -function U such that $U(x) < x$ for all $x > 0$. Therefore $k^* = \infty$ and by Theorem 11 the rotor walk is recurrent a.s.

For the other case, denote by ν the number of $+1$'s per period, and by $\zeta = L - \nu$ the number of -1 's per period. We assume that $\nu > \zeta$. Let U be the U -function corresponding to an arbitrary (but fixed) shift. Observe that by definition of U ,

$$U(\zeta + 1) \geq \#1\text{'s in the first period} = \nu \geq \zeta + 1.$$

Therefore $k^* \leq \zeta + 1 < \infty$ and by Theorem 11 the rotor walk is transient a.s. □

Remark 17. By considering right excursions and left excursions separately, the theorems in this chapter gives corresponding results on rotor walks on \mathbb{Z} . In particular, for a periodic sequence, choosing each one of the shifts with positive probability, we get that the rotor walk on \mathbb{Z} is recurrent a.s. if the sequence is balanced, transient to the right a.s. if there are strictly larger numbers of $+1$'s than -1 's per period, and transient to the left a.s. otherwise.

3 The d -ary tree, $d \geq 2$

In this section we consider rotor walks on the homogeneous tree $\mathbb{T}_d, d \geq 2$. We give a criterion for recurrence and transience, see Theorem 23. This criterion is based on generalizations of the U -functions defined in the previous section and the fact that transience of the rotor walk is equivalent to survival of a multi-type Galton-Watson process, as we shall see below. We then show that the uniform rotation configuration may be recurrent or transient on \mathbb{T}_2 and classify the recurrent rotor sequences, see Theorem 27. In higher dimensions, $\mathbb{T}_d, d \geq 3$, the uniform rotation configuration is always transient, see Theorem 28. We conjecture that a similar behavior holds also for the shift configuration, see Conjectures 29 and 32.

3.1 Notations

For a rotor sequence \mathbf{a} on the homogeneous tree $\mathbb{T}_d, d \geq 2$, i.e. \mathbf{a} is taking values in $\{0, 1, 2, \dots, d\}^{\mathbb{N}}$, we define the U -functions as follows:

$$U_{\mathbf{a}}(x) = \begin{pmatrix} U_{\mathbf{a}}^{(1)}(x) \\ U_{\mathbf{a}}^{(2)}(x) \\ \vdots \\ U_{\mathbf{a}}^{(d)}(x) \end{pmatrix},$$

where $U_{\mathbf{a}}^{(i)}(x)$ is the number of i 's that appear prior to x 0's in the sequence \mathbf{a} . For a finite sequence of rotors $\{a_1, \dots, a_\ell\}$ of length ℓ we write $\overline{(a_1, \dots, a_\ell)}$ for the periodic rotor sequence $(a_1, \dots, a_\ell, a_1, \dots, a_\ell, \dots)$.

Example 18 (rotor-router walk). On \mathbb{T}_2 let $\mathbf{a}^{(1)} = \overline{(0, 1, 2)}$, $\mathbf{a}^{(2)} = \overline{(1, 2, 0)}$ and $\mathbf{a}^{(3)} = \overline{(2, 0, 1)}$. The corresponding U -functions are given by

$$U_1(x) = \binom{x-1}{x-1}, U_2(x) = \binom{x}{x} \text{ and } U_3(x) = \binom{x-1}{x}.$$

Example 19. On \mathbb{T}_2 we consider $\mathbf{a}^{(1)} = \overline{(0, 1, 0, 1, 2, 2)}$, $\mathbf{a}^{(2)} = \overline{(1, 2, 1, 2, 0, 0)}$ and $\mathbf{a}^{(3)} = \overline{(2, 0, 2, 0, 1, 1)}$. The corresponding U -functions are given by

$$U_1(x) = \binom{x-1}{2 \lfloor \frac{x-1}{2} \rfloor}, U_2(x) = \binom{2 \lceil \frac{x-1}{2} \rceil}{2 \lceil \frac{x-1}{2} \rceil} \text{ and } U_3(x) = \binom{2 \lfloor \frac{x-1}{2} \rfloor}{2 \lfloor \frac{x-1}{2} \rfloor}.$$

Given $(\mathbf{a}_v)_{v \in \mathbb{T}_2}$ we define a process Z^k on the tree \mathbb{T}_2 by the following: $Z_o^k := k$, and inductively for $v = v_{0_i}, i \in \{1, \dots, d\}$,

$$Z_v^k = U_{\mathbf{a}_v}^{(i)}(Z_{v_0}^k). \tag{9}$$

Recall that $v = v_{0_i}$ means that v is the i th child of its parent v_0 . This is a generalization of (3). In words, $Z_{o_i}^k$ is the number jumps from o to o_i before the k th traversal of the loop in o , that is before the end of the k th excursion. Inductively, Z_v^k is the number of times the vertex v_0 is visited coming from its children v before the k th traversal of the loop in the root.

Similarly to Lemma 6 in the case of the unary tree, the following Lemma is a consequence of Proposition 3.4 in [7], together with the deterministic point of view of Chapters 2.2 and 2.3 of [1].

Lemma 20. *The first k th excursions of X are finite if and only if $Z_v^k > 0$ for only a finite number of vertices v .*

Let us expand on the relation of Z_v^k to a multi-type Galton-Watson process. The *type* of a vertex v is defined as Z_v^k . Given the rotor sequence $\mathbf{a}_v(\cdot)$ the types of $v_i, i \in \{1, \dots, d\}$, are defined deterministically by Equation (9). In the finitely supported distribution model, see Definition 1, we can give a more probabilistic description. The starting point of the construction is the d -ary tree \mathbb{T}_d , which itself can be seen as the genealogical tree of the branching process where each particle (vertex) has a.s. d offspring particles (vertices). To each vertex v in \mathbb{T}_d we assign inductively a type. To start let k be the type of the root o . In other words, we start the branching process with one particle of type k at time 0. Now, we choose a rotor configuration for the root at random (according to p) and the types of the children o_1, \dots, o_d of o are given by the values $Z_{o_1}^k, \dots, Z_{o_d}^k$ following Equation (9). If the type of a particle is 0 we declare the particle and all its descendants as dead. By induction this procedure either continues until all particles are dead, i.e. the process dies out, or continues indefinitely, i.e. the process survives. We denote by $\xi_n^k(i)$ the number of

particles of type i at time n and write ξ^k for the whole branching process. Due to the definition of ξ^k we have that ξ^k dies out if and only if $Z_v^k > 0$ for only a finite number of v .

For $v \in \mathbb{T}_d$ we denote by $|v|$ the level of v , i.e. its graph distance from the root o . The next lemma guarantees that in order to prove transience of the rotor walk for i.i.d. configurations it is sufficient that the process Z_v^k survives with positive probability for some k .

Lemma 21. *Assume that the rotor configuration is i.i.d. Then,*

$$\mathbb{P}_o[X_n \neq o \ \forall n > 0] > 0 \quad \implies \quad \mathbb{P}_o[X_n \neq o \text{ for all } n \text{ large enough}] = 1.$$

Proof. The proof is an adaptation of the proof of [21, Lemma 8]. Let $K_i \in \mathbb{N} \cup \{\infty\}$ be the largest level passed in the i th excursion, defined to be ∞ if either the excursion is infinite, or if $K_{i-1} = \infty$. Let $J \subset \mathbb{N}$ be the set of all ‘tanned’ indices, i.e. indices j such that $K_j < \infty$ and $K_j > K_i$ for all $i < j$. Denote $p = \mathbb{P}_o[X_n \neq o \ \forall n > 0] > 0$. By the i.i.d. assumption, for every $j \in J$ we have $\mathbb{P}_{|X_0|=j}[|X_n| > j \ \forall n > 0] = p > 0$ independently of the past of the walk. Therefore, J is stochastically dominated by a geometric random variable (with parameter p) and therefore a.s. finite. This implies that on the event $\{X_n = o \text{ i.o.}\}$ the range $\{X_n, n \in \mathbb{N}\}$ is finite. By non-degeneracy, the latter happens with probability zero. Therefore $\mathbb{P}_o[X_n = o \text{ i.o.}] = 0$. \square

3.2 Periodic balanced configurations

Let us begin with an immediate but important property of periodic balanced rotor sequences, generalizing Lemma 7 for any d -ary tree.

Lemma 22. *Let \mathbf{a} be a periodic and balanced rotor configuration \mathbf{a} with period $L = (d+1)N$ and let $U_{\mathbf{a}} = (U_{\mathbf{a}}^{(1)}, \dots, U_{\mathbf{a}}^{(d)})$ be its U -function. Then, for $x = \alpha N + \beta$ where $\alpha \in \mathbb{N}$, $1 \leq \beta \leq N$ and for $i \in \{1, \dots, d\}$ we have that*

$$U_{\mathbf{a}}^{(i)}(x) = \alpha N + U_{\mathbf{a}}^{(i)}(\beta) \text{ and } U_{\mathbf{a}}^{(i)}(\beta) \leq N.$$

A rotor configuration $\{\mathbf{a}_v\}_{v \in \mathbb{T}_d} \in \mathcal{A}^{\mathbb{T}_d}$ is called L -periodic (balanced) if all the rotor sequences \mathbf{a}_v , $v \in \mathbb{T}_d$ are L -periodic (balanced).

We consider the finitely supported distribution model, see Definition 1. A consequence of Lemma 22 is that, in the periodic and balanced setting, Z_v^k can be seen as a finite-type Galton-Watson process. In fact, let $k = \alpha N + \beta$ where $\alpha \in \mathbb{N}$, $1 \leq \beta \leq N$ and for $i \in \{1, \dots, d\}$, then $\xi_n^k(y) = 0$ (a.s.) for all $n \in \mathbb{N}$ and all $y > (\alpha + 1)N$.

Theorem 23 (Periodic balanced rotor configuration on \mathbb{T}_d , $d \geq 3$). *In the above setting the rotor walk is transient a.s. if and only if the process ξ^N survives with positive probability.*

Proof. Due to Lemma 20 and the discussion above it remains to prove that the process ξ^k dies out a.s. for all k if and only if it dies out a.s. for $k = N$. Let $k = \alpha N + \beta$ with

$1 \leq \beta \leq N$ and assume that the process ξ^N dies out a.s. The latter together with Lemma 22 implies that a.s. there exists a (random) level n_1 such that all particles at generation n_1 have a type of at most $\alpha_1 N + \beta_1$ with $\alpha_1 = \alpha - 1$ and $1 \leq \beta_1 \leq N$. By induction, a.s. there exists a (random) level n_α such that all vertices will have type of at most N . By the assumption that ξ^N dies out a.s. all of these particles will a.s. have only a finite number of descendants. \square

It is well known, see e.g. [5, Chapter V] that survival of a multi-type Galton-Watson process only depends on the first moment matrix M of the process. Usually one assumes the multi-type Galton-Watson process to be non-singular and positive regular. A branching process is called singular if every particle has exactly one offspring. In this paper all non-trivial examples give rise to non-singular processes, particularly in the finitely supported distribution model with $|S| > 1$. Positive regularity, i.e. strict positivity of the first moment matrix, is a condition that is not always satisfied in our cases. However, using standard theory of non-negative matrices, e.g. [26, Chapter 1], it follows that the multi-type branching processes ξ^N defined above will survive with positive probability if and only if the largest eigenvalue of the first moment matrix is strictly larger than 1.

In our case the first moment matrix is given as follows. Assume that v has type k , then the mean number $m(k, \ell)$ of offspring of v of type ℓ is given by

$$m(k, \ell) = \sum_{i=1}^{|S|} p_i \left(\mathbf{1}_{U_{\mathbf{a}_i}^{(1)}(k)=\ell} + \mathbf{1}_{U_{\mathbf{a}_i}^{(2)}(k)=\ell} + \cdots + \mathbf{1}_{U_{\mathbf{a}_i}^{(d)}(k)=\ell} \right). \quad (10)$$

Denote its largest eigenvalue $\rho = \rho(M)$ then the process ξ^k survives with positive probability if and only if $\rho > 1$. Together with Theorem 23 this gives a useful criterion for recurrence and transience. We now illustrate how to utilize the theorem by giving a few examples. We note that Theorem 27 covers parts of the examples below.

Example 24 (rotor-router walk, $(\mathbf{0}, \mathbf{1}, \mathbf{2})$). In the model given in Example 18 we choose each configuration with probability $1/3$. We start the rotor walk with $k = N = 1$ particle or equivalently we start the (1-type) Galton-Watson process ξ^1 with one particle of type 1. The mean number of offspring of the Galton-Watson process is

$$m = \frac{1}{3}(0 + 0) + \frac{1}{3}(1 + 1) + \frac{1}{3}(0 + 1) = 1.$$

Hence, the process ξ^1 a.s. dies out and the rotor walk is a.s. recurrent. Now, more generally we choose $\mathbf{a}^{(1)} = (0, 1, 2)$ with probability p_1 , $\mathbf{a}^{(2)} = (1, 2, 0)$ with probability p_2 , and $\mathbf{a}^{(3)} = (2, 0, 1)$ with probability p_3 . For mean number of offspring we obtain $2p_2 + p_3$. Hence, the rotor walk is recurrent if and only if $2p_2 + p_3 \leq 1$.

Example 25 ($(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{2})$). We analyze further Example 19 where we choose each configuration with equal probability. We start the rotor with $k = N = 2$ particles or equivalently we start the 2-type Galton-Watson process ξ^2 with one particle of type 2. We obtain the following first moment matrix:

$$M = \begin{pmatrix} 1/3(1 + 0 + 0) & 1/3(0 + 2 + 0) \\ 1/3(1 + 0 + 0) & 1/3(0 + 2 + 1) \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 1 \end{pmatrix}.$$

As the largest eigenvalue of M is $\frac{1}{3}(2 + \sqrt{3}) > 1$ the rotor walk is a.s. transient. We shall see in the next chapter that Example 25 also serves as an example for a more specific criterion for transience vs. recurrence for uniform rotations.

To end the chapter we analyze the recurrence vs. transience regimes when we change the probabilities in Example 25.

Example 26. Choose $\mathbf{a}^{(1)} = \overline{(0, 1, 0, 1, 2, 2)}$ with probability p_1 , $\mathbf{a}^{(2)} = \overline{(1, 2, 1, 2, 0, 0)}$ with probability p_2 , and $\mathbf{a}^{(3)} = \overline{(2, 0, 2, 0, 1, 1)}$ with probability p_3 . We obtain the first moment matrix

$$M = \begin{pmatrix} p_3 & 2p_2 \\ p_1 & 2p_2 + p_3 \end{pmatrix}$$

with largest eigenvalue equal to $p_2 + p_3 + \sqrt{p_2(2p_1 + p_2)}$.

3.3 Criterion for uniform rotations on the binary tree

Let \mathbf{a} be a rotor sequence. We call a finite rotor sequence a *piece*. We say that

$$\alpha = (\underbrace{0, \dots, 0}_m, \underbrace{1, \dots, 1}_m, \underbrace{2, \dots, 2}_m)$$

and each of its rotations $\pi^i \alpha$, $i \in \{0, 1, 2\}$, are *m-standard pieces*. A piece is called *standard* if it is *m-standard* for some m . A sequence \mathbf{a} is the *concatenation* of standard pieces if $\mathbf{a} = (\alpha_1, \alpha_2, \dots)$ for some standard pieces $\alpha_1, \alpha_2, \dots$

Theorem 27 (Criterion for uniform rotation). *Fix a rotor sequence \mathbf{a} on \mathbb{T}_2 . The rotor walk in the uniform rotation model corresponding to \mathbf{a} is recurrent a.s. if and only if \mathbf{a} is a concatenation of standard pieces.*

Proof. We shall first check that for any m -standard piece α the first moment matrix M_α corresponding to $\overline{(\alpha)}$ has spectral radius 1. The case $m = 1$ gives rise to an 1×1 matrix with entry 1, see Example 24. In the case $m > 1$ one checks that the first moment matrix is the $m \times m$ matrix where all columns are the 0-vector except for the m th column which is the 1-vector. Therefore its eigenvalues are 0 and 1.

The general case where $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \dots)$ is a concatenation of standard pieces α_i is in the same spirit. However if \mathbf{a} is not periodic the multi-type Galton-Watson process ξ may have infinitely many types and the first moment matrix becomes a non-negative operator $M = (m(i, j))_{i, j \in \mathbb{N}}$. Let $|\alpha_i|$ be the length of α_i . For $k \in \mathbb{N}$ we define

$$J(k) = \inf\{j \in \mathbb{N} : \sum_{i=1}^j |\alpha_i| \geq k\} \text{ and } e(k) = \sum_{i=1}^{J(k)} |\alpha_i|.$$

Note that $e(k) \geq k$ by definition. Since all $\alpha_i, i \in \mathbb{N}$, are balanced we have that

$$U_{\mathbf{a}}^{(i)}(x) \leq \frac{e(k)}{3} \text{ for all } x \leq \frac{e(k)}{3}, i \in \{1, \dots, d\}. \quad (11)$$

Fix $k \in \mathbb{N}$. In order to determine whether the first k excursions are finite a.s. it suffices to consider the first moment matrix $M_k = (m(i, j))_{i, j \leq e(k)}$ of the multi-type Galton-Watson process with $e(k)/3$ types. Moreover, Equation (11) implies that M_k is a block diagonal matrix consisting of $J(k)$ blocks M_{α_i} , $1 \leq i \leq J(k)$, with additional entries in the lower diagonal part. Since $\rho(M_{\alpha_i}) = 1$ for all i we have that $\rho(M_k) = 1$ and so the first k excursions are finite a.s. Since k was arbitrary, recurrence follows.

Let us now prove that every sequence \mathbf{a} that is not a concatenation of standard pieces gives rise to a transient rotor walk. We decompose $\mathbf{a} = (\beta, \mathbf{b})$ where β is a piece which is a concatenation of standard pieces and \mathbf{b} is a rotor sequence which has, without loss of generality, the form

$$\mathbf{b} = (\underbrace{0, \dots, 0}_r, \underbrace{1, \dots, 1}_s, \underbrace{2, \dots, 2}_t, i, \dots), \quad i \neq 2.$$

In other words, we have that s and t are not both equal to r . We treat here the case where $r < s$. The remaining cases ($r = s \neq t$ and $r > s$) are done analogously. We define $\mathbf{b}^{(1)} = \mathbf{b}$, $\mathbf{b}^{(2)} = \tau^1 \mathbf{b}$, $\mathbf{b}^{(3)} = \tau^2 \mathbf{b}$, and

$$\begin{aligned} \mathbf{c}_1 &= (\underbrace{0, \dots, 0}_r, \underbrace{1, \dots, 1}_s, 0, \dots), \\ \mathbf{c}_2 &= (\underbrace{1, \dots, 1}_r, \underbrace{2, \dots, 2}_s, 0, \dots), \\ \mathbf{c}_3 &= (\underbrace{2, \dots, 2}_r, \underbrace{0, \dots, 0}_s, 0, \dots). \end{aligned}$$

That is, $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ are the same as $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(3)}$ respectively in the indices 1 to $r + s$ and are 0 in the larger indices. The first moment matrix $M_{\mathbf{c}}$ of the uniform rotor configuration consisting of $\mathbf{c}_1, \mathbf{c}_2$ and \mathbf{c}_3 (that means that each of them chosen with probability $1/3$) is given by

$$\frac{1}{3} \left(\underbrace{\begin{pmatrix} 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 \end{pmatrix}}_s \right) \left. \vphantom{\begin{pmatrix} 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 \end{pmatrix}} \right\} \begin{matrix} r \\ s \end{matrix}$$

which has spectral radius strictly larger than 1. Denote by q the length of β . The original first moment matrix M restricted to the first $q + r + s + t$ -block is minorated by a block diagonal matrix consisting of a first block of size q and a second block which is $M_{\mathbf{c}}$. The fact that $\rho(M_{\mathbf{c}}) > 1$ implies that $\rho(M_{\mathbf{b}}) > 1$, hence also $\rho(M) > 1$ and the rotor walk is transient a.s. \square

3.4 Transience for uniform rotations on $\mathbb{T}_d, d \geq 3$

The following theorem states that in the uniform rotation model on the d -ary tree, $d \geq 3$, the walk is always transient.

Theorem 28. *Let \mathbf{a} be a rotor sequence on $\mathbb{T}_d, d \geq 3$. Then, the rotor walk in the uniform rotation corresponding to \mathbf{a} is transient a.s.*

Proof. Without loss of generality we assume that \mathbf{a} starts with 0. Let $\pi \in S_{d+1}$ be the rotation mapping $n \mapsto n + 1 \pmod{d+1}$. Let $m \geq 1$ be such that $\mathbf{a}(1) = \mathbf{a}(2) = \dots = \mathbf{a}(m) = 0$ and $x := \mathbf{a}(m+1) \neq 0$. For $j = d - x$ we have $\pi^j(x) = 0$. The set of rotor sequences $\{\pi^i \mathbf{a} : i \notin \{0, j\}\}$ has $d - 1 \geq 2$ elements. We shall now show that the first excursion is infinite with positive probability. To do this we prove that the branching process starting with one particle of type 1, i.e. $k = 1$, will survive with positive probability. We consider first particles of type 1. We have that

$$\begin{aligned} U_{\mathbf{a}}^{(i)}(1) &= 0 \quad \forall i \in \{1, \dots, d\}, \\ U_{\pi^j \mathbf{a}}^{(i)}(1) &= m\delta_{ij} \quad \forall i \in \{1, \dots, d\}, \\ U_{\pi^\ell \mathbf{a}}^{(i)}(1) &\geq m\delta_{i\ell} + \delta_{i(\ell+x)} \quad \forall i \in \{1, \dots, d\} \forall \ell \notin \{0, j\}. \end{aligned}$$

Now, by monotonicity of the U -functions we also have that for all $k \geq 2$

$$\begin{aligned} U_{\pi^j \mathbf{a}}^{(i)}(k) &\geq m\delta_{ij} \quad \forall i \in \{1, \dots, d\}, \\ U_{\pi^\ell \mathbf{a}}^{(i)}(k) &\geq m\delta_{i\ell} + \delta_{i(\ell+x)} \quad \forall i \in \{1, \dots, d\} \forall \ell \notin \{0, j\}. \end{aligned}$$

Hence, the expected number of children of a particle is $d/(d+1) + 2(d-1)/(d+1) = (3d-2)/(d+1) > 1$, since $d \geq 3$. Denote by $\zeta_n = \sum_i \xi_n(i)$ the number of particles at time n in the original multi-type Galton-Watson process. A standard coupling argument gives that $(\zeta_n)_{n \geq 1}$ can be stochastically bounded below by a Galton-Watson process with offspring distribution $q(0) = q(1) = 1/(d+1), q(2) = (d-1)/(d+1)$. As the latter survives with positive probability, so does the multi-type Galton-Watson process. \square

3.5 Conjectures for uniform shifts on the d -ary tree, $d \geq 2$

Let \mathbf{a} be an L -periodic rotor sequence on \mathbb{T}_2 . Denote by \mathcal{A}_{rec}^{conj} the set consisting of sequences of the form

$$\overline{(0, i_1, j_1, 0, i_2, j_2, \dots, 0, i_N, j_N)} \text{ with } \{i_\ell, j_\ell\} = \{1, 2\}, 1 \leq \ell \leq N,$$

and all its possible shifts.

Similar to the first part of the proof of Theorem 27 one checks that the rotor walk in the uniform shift model corresponding to \mathbf{a} is recurrent a.s. if $\mathbf{a} \in \mathcal{A}_{rec}^{conj}$.

Conjecture 29. Let \mathbf{a} be an L -periodic rotor sequence on \mathbb{T}_2 . The rotor walk in the uniform shift model corresponding to \mathbf{a} is recurrent a.s. if and only if $\mathbf{a} \in \mathcal{A}_{rec}^{conj}$.

Remark 30. The above conjecture holds for $L \leq 12$. Indeed, we calculated (using a computer) the largest eigenvalues of the first moment matrices for all possible sequences.

The notion of being A_{rec}^{conj} can naturally be generalized to d -ary trees.

Lemma 31. *Let \mathbf{a} be a non-degenerate L -periodic rotor sequence on $\mathbb{T}_d, d \geq 3$. The rotor walk in the uniform shift model corresponding to \mathbf{a} is transient a.s. if $\mathbf{a} \in \mathcal{A}_{rec}^{conj}$.*

Proof. Let \mathbf{a} be such that \mathbf{a} and all its shifts are concatenations of 1-standard pieces and let $N = L/(d + 1)$. Without loss of generality we assume that \mathbf{a} starts with 0. Then \mathbf{a} is of the form

$$\mathbf{a} = \overline{(0, i_{1,1}, i_{1,2}, \dots, i_{1,d}, 0, i_{2,1}, i_{2,2}, \dots, i_{2,d}, \dots, 0, i_{N,1}, i_{N,2}, \dots, i_{N,d})}$$

with $\{i_{i,1}, \dots, i_{i,d}\} = \{1, 2, \dots, d\}$ for all $i \in \{1, 2, \dots, N\}$. Now, $U_{2+k(d+1)}^{(i)}(1) \geq 1$ for all $i \in \{1, \dots, d\}$ and $k \in \{0, \dots, N - 1\}$. Moreover, $U_{3+k(d+1)}^{(i)}(1) \geq 1$ for all $i \in \{1, \dots, d\} \setminus \{i_{1,1}\}$ and $k \in \{0, \dots, N - 1\}$. Continuing this observation along all shifts leads that the number of U -functions verifying $U(1) \geq 1$ is equal to $Nd(d + 1)$. Since $U(1) \geq 1$ implies that $U(x) \geq 1$ for all $x \leq N$ the mean number of offspring (of any type) of a particle (of any type) in the multi-type Galton-Watson process is at least $Nd(d + 1)/L = d/2 > 1$. Hence, the branching process survives with positive probability and the rotor walk is transient a.s. \square

Conjecture 32. For any L -periodic rotor sequence in $\mathbb{T}_d, d \geq 3$, the rotor walk in the corresponding uniform shift model is transient a.s.

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