On the Caccetta-Häggkvist Conjecture with a Forbidden Transitive Tournament

Andrzej Grzesik*

Faculty of Mathematics and Computer Science
Jagiellonian University
ul. St. Łojasiewicza 6
30-348 Krakow, Poland

Andrzej.Grzesik@tcs.uj.edu.pl

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Abstract

The Caccetta-Häggkvist Conjecture asserts that every oriented graph on n vertices without directed cycles of length less than or equal to l has minimum outdegree at most $(n-1)/l$. In this paper we state a conjecture for graphs missing a transitive tournament on $2^k + 1$ vertices, with a weaker assumption on minimum outdegree. We prove that the Caccetta-Häggkvist Conjecture follows from the presented conjecture and show matching constructions for all k and l. The main advantage of considering this generalized conjecture is that it reduces the set of the extremal graphs and allows using an induction.

We also prove the triangle case of the conjecture for $k = 1$ and 2 by using the Razborov's flag algebras. In particular, it proves the most interesting and studied case of the Caccetta-Häggkvist Conjecture in the class of graphs without the transitive tournament on 5 vertices. It is also shown that the extremal graph for the case $k = 2$ has to be a blow-up of a directed cycle on 4 vertices having in each blob an extremal graph for the case $k = 1$ (complete regular bipartite graph), which confirms the conjectured structure of the extremal examples.

In the paper we are considering oriented graphs, i.e., directed graphs without loops, multiple edges and two vertices connected by edges in both directions. By T_m we denote a *transitive tournament* — an oriented graph on m vertices with all possible edges and no directed cycles. By $\vec{C}_l$ we denote a directed cycle on l edges. We also denote the *minimum outdegree* as

$$\delta^+(G) = \min_{v \in G} \deg^+(v).$$

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Whenever the graph G is known from the context we will omit it and write just δ^+. By *blow-up* of an oriented graph H we consider a graph whose vertex set is divided into $v(H)$ equal parts with all the edges between parts placed accordingly to edges in H (between each two parts there are all edges directed in one way or no edges at all) and no edges inside the parts. We call those parts of the blow-up by *blobs*. By *k-iterated* blow-up of an oriented graph H we consider a graph which is a blow-up of $(k-1)$-iterated blow-up of H, where the 1-iterated blow-up of H is just a blow-up of H.

In 1978 Caccetta and Häggkvist [4] stated the following conjecture

Conjecture 1 (Caccetta-Häggkvist Conjecture)**.** *Every oriented graph on n vertices without directed cycles of length less than or equal to l has $\delta^+ \leqslant (n-1)/l$.*

The conjecture was proved for many large values of l by Caccetta and Häggkvist [4], Hamidoune [11], Hoáng and Reed [14], and Shen [25]. Another approach was to force a directed cycle of length at most $l + c$ for some small c. This was proved for $c = 2500$ by Chvátal and Szemerédi [5], for $c = 304$ by Nishimura [17], and for $c = 73$ by Shen [26].

Small values of l are more difficult and received much attention. Even for $l = 3$ the conjecture is still open. We focus more on this case.

Let c be the minimal constant for which each triangle-free graph has $\delta^+ \leqslant cn$. Conjectured value is $c = 1/3 \approx 0.3333$. Caccetta and Häggkvist [4] proved that $c \leqslant (3-\sqrt{5})/2 \approx 0.3820$. Then it was improved by Bondy [3] to $(2\sqrt{6} - 3)/5 \approx 0.3798$, Shen [24] to $3 - \sqrt{7} \approx 0.3542$, Hamburger, Haxell, and Kostochka [10] to 0.3532, and to 0.3465 by Hladký, Král', and Norine [13]. Recently, Sereni and Volec [27] stated even further improvement to 0.3388 using flag algebras in a sophisticated way.

For more results and problems related to the Caccetta-Häggkvist Conjecture see [23].

The main obstacle, which makes this problem hard, is the fact that there are many non-isomorphic extremal examples. The same is happening for example in case of the well-known Turán Conjecture. Bondy [3] observed that the class of extremal graphs for the Caccetta-Häggkvist Conjecture is closed under lexicographic product. Later, Razborov [21] generalized the Bondy's construction.

Here we present a new way of proving the Caccetta-Häggkvist Conjecture. The main idea is to define a set of conjectures, which lead to the Caccetta-Häggkvist Conjecture, such that for each of them the set of extremal examples is more restricted and we can use the inductive arguments in parts of the extremal graphs. This way, it will be easier to prove those partial conjectures using the graph limits methods and inductive arguments.

Let us now state the main conjecture.

Conjecture 2. *Every oriented graph on n vertices without directed cycles of length less than or equal to l and without transitive tournament T_{2^k+1} has*

$$\delta^+ \leqslant \left(\frac{1}{l+1} + \frac{1}{(l+1)^2} + \frac{1}{(l+1)^3} + \ldots + \frac{1}{(l+1)^k}\right) n.$$

An example construction of a graph giving the above bound is a k-iterated blow-up of $\vec{C}_{l+1}$. In other words, we take n divisible by $(l+1)^k$ and divide the vertices into $l+1$

equal parts. Between the parts we put all the edges forming a $\vec{C}_{l+1}$. So, between each two parts there is a complete one-way directed bipartite graph or no edges at all. Inside each part we iteratively, $k-1$ times, do the same — divide each of them into $l+1$ equal parts and put edges forming a $\vec{C}_{l+1}$. It can be easily seen there are no cycles of length less than or equal to l. Each transitive tournament can have vertices only in at most two big parts (since in $\vec{C}_{l+1}$ there is no T_3). Similarly in each smaller parts. Hence, the biggest transitive tournament we can find is T_{2^k} and there is no T_{2^k+1}. Outdegree of each vertex is exactly $n(l+1)^{-1} + n(l+1)^{-2} + n(l+1)^{-3} + \ldots + n(l+1)^{-k}$. Each summand comes from the edges between parts of the different size.

In the most interesting case $l = 3$ the conjecture is following.

Conjecture 3. *Every oriented graph on n vertices without $\vec{C}_3$ and T_{2^k+1} has*

$$\delta^+ \leqslant \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^k}\right) n.$$

In this case, for $k = 1$ the set of extremal examples is just the set of all complete regular bipartite graphs. For $k = 2$ we show that it is a blow-up of $\vec{C}_4$ with some extremal graph from the case $k = 1$ in each blob (Theorem 7). This is also conjectured to be true in general.

Conjecture 4. *Every extremal graph on n vertices without $\vec{C}_3$ and T_{2^k+1} having*

$$\delta^+ = \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^k}\right) n$$

is a blow-up of $\vec{C}_4$ having in each blob some graph without $\vec{C}_3$ and $T_{2^{k-1}+1}$ with outdegree inside this blob equal to

$$\left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^{k-1}}\right) n.$$

In other words, every extremal graph for Conjecture 3 is a $(k-1)$-iterated blow-up of $\vec{C}_4$ with some complete regular bipartite graph in every small blob.

Observation 5. *For each l, Conjecture 2 implies the Caccetta-Häggkvist Conjecture. In particular, Conjecture 3 implies the Caccetta-Häggkvist Conjecture in the triangle case.*

Proof. Let us assume that the Caccetta-Häggkvist Conjecture is false, i.e., there exists a graph G on n vertices without any directed cycle of length less than or equal to l having $\delta^+(G) > (n-1)/l$. Since n is an integer, we have $\delta^+(G) \geqslant n/l$. Let k be sufficiently large, so that G does not contain T_{2^k+1}. From the fact that

$$\frac{1}{l+1} + \frac{1}{(l+1)^2} + \frac{1}{(l+1)^3} + \ldots = \frac{1}{l},$$

we have

$$\delta^+(G) \geqslant \frac{n}{l} > \left(\frac{1}{l+1} + \frac{1}{(l+1)^2} + \frac{1}{(l+1)^3} + \ldots + \frac{1}{(l+1)^k}\right) n.$$

It means that graph G also contradicts Conjecture 2. $\square$

In this paper we prove Conjecture 3 in the case $k = 2$ together with the theorem describing the structure of the extremal graphs. This indicates that proving Conjecture 2 might be easier and it might be the way to prove the Caccetta-Häggkvist Conjecture.

The main tool used in the proofs is the flag algebra calculus developed by Razborov [19]. Since there are already many results using this method (see for example [1, 2, 6, 7, 8, 9, 12, 18, 22]), here we omit the whole presentation of the theory. For unexperienced readers, we recommend a Razborov's survey [20] or recent book of Lovász [16] with introduction to the theory and examples of results obtained using this techniques.

In this paper we use the standard flag algebra notation from [19]. By $[\![\cdot]\!]$ we understand the averaging operator from flags with one labeled vertex to nonlabeled flags. In the drawings of flags we use unfilled circle to denote the rooted vertex. In the text, we identify each graph (e.g., $\vec{C}_3$) with unlabeled flag representing its density.

Theorem 6. *In the case $k = 2$ Conjecture 3 is true, i.e., every oriented graph on n vertices without $\vec{C}_3$ and T_5 has $\delta^+ \leqslant \frac{5}{16}n$.*

Proof. To prove this statement, we run a flag algebra programme in the space of flags on 5 vertices without $\vec{C}_3$ and T_5, which proves that if  $\geqslant 5/16$, then

$\left[\!\!\left[\left(\right.\right.\right.$  $\left.-\frac{5}{16}\right)\cdot$  $\left.\right]\!\!\right] \leqslant 0.$

The above statement implies the theorem. Assume there exists a graph contradicting Theorem 6. It will be still a counterexample if we remove vertices with indegree 0, so we can assume that each vertex has positive indegree. By considering blow-ups we can construct an infinite sequence of counterexamples and thus get a counterexample for the above statement.

Using basic flag algebraic calculus we derive that

$96\left[\!\!\left[\left(\right.\right.\right.$  $\left.-\frac{5}{16}\right)\cdot$  $\left.\right]\!\!\right] = 16$  $+\,16$  $-\,15$  . (1)

Now, one can use publicly available Flagmatic software developed by Vaughan (see [28]) to maximize the right-hand side of (1) and prove that it is really 0. It is enough to use flags of size 5 and just three types — triangle, edge on 3 vertices, and the single vertex — the respecting matrices are of the sizes 24, 20 and 14. On the webpage http://www2.im.uj.edu.pl/AndrzejGrzesik/CH we provide the commands for Flagmatic to reproduce the result, as well as the certificate to verify the calculations, and non-positive coefficients of the final inequality. □

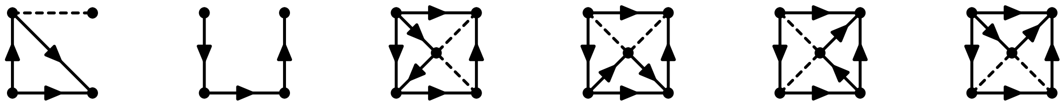
To our knowledge this is the first proof of the triangle case of the Caccetta-Häggkvist Conjecture in the class of T_5-free graphs.

The case $k = 1$ of Conjecture 3, after forgetting the orientations of edges, means that a triangle-free graph has edge density less than half, which is the well-known Mantel's Theorem. We know that in the undirected case, the only extremal case is a complete balanced bipartite graph, hence with orientations, we have that the set of the extremal graphs is exactly the set of complete regular bipartite graphs. In the case $k = 2$ we have the following theorem.

Theorem 7. *Every graph on n vertices without $\vec{C}_3$ and T_5 having $\delta^+ = (\frac{1}{4} + \frac{1}{4^2})n = \frac{5}{16}n$ is a blow-up of $\vec{C}_4$ with a complete regular bipartite graph in every blob.*

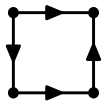
Proof. Consider a graph G satisfying the assumptions. If there exists a vertex of a higher outdegree than δ^+ then, by removing some edges going from this vertex, we obtain a graph which also satisfies the assumptions. Notice also that adding any edge to the conjectured structure (blow-up of $\vec{C}_4$ with a complete regular bipartite graph in every blob) is constructing a $\vec{C}_3$ or T_5, so, it is enough to prove this theorem if the degree of every vertex is equal to δ^+. In particular, we can assume that the set of out-neigbours of any vertex is not contained in the set of out-neigbours of any other vertex.

In the proof of Theorem 6, if the coefficient of the final inequality corresponding to a particular flag is strictly negative, then this flag cannot appear in any extremal graph as an induced subgraph. Otherwise, by considering blow-ups we could construct an infinite sequence of extremal graphs that gives strict inequality, which contradicts the outdegree assumption. Thus, from the proof of Theorem 6, we have that in every extremal graph there are no subgraphs isomorphic to any of the following graphs, for each possibility of putting edges (including non-edge) on the dashed segments:



(A) (B)

One can notice, that nonexistance of the last four graphs means that in the extremal graph there is no induced subgraph isomorpic to



(C)

with one of the edges belonging to a transitive triangle with edges w_1w_2, w_2w_3, w_1w_3 as the w_1w_3 edge.

Consider any extremal graph and pick a transitive triangle with vertices v_1, v_2, v_3 and edges v_1v_2, v_2v_3, v_1v_3 (there exists, because the edge density is above $1/2$). Since the graph (A) is forbidden, we have that any other vertex of the graph is connected to v_1 or v_3. Lets create the following groups of vertices:

G_1 — Out-neigbours of v_3 which are not connected to v_1.

G_2 — In-neigbours of v_1 which are not connected to v_3.

Other out-neighbours of v_3 are contained in out-neighbours of v_1 (otherwise it creates a directed triangle) and other in-neighbours of v_1 are contained in in-neighbours of v_3, so all the remaining vertices are out-neighbours of v_1 or in-neighbours of v_3.

We will show now that the sets G_1 and G_2 are not empty. If G_1 is empty then all out-neighbours of v_3 are out-neigbours of v_1, contradicting the assumption from the first paragraph of this proof. Assume G_2 is empty and take any g_1 from G_1 and any out-neigbour v of g_1. We argue that v is an out-neighbour of v_3. If v is in G_1, then it is an out-neighbour of v_3. Otherwise, since all the remaining vertices are out-neighbours of v_1 or in-neighbours of v_3, to avoid directed triangle with v_3 and g_1, it needs to be an out-neighbour of v_1. There cannot be the edge vv_3, because it creates a directed triangle with the vertex g_1. If there is no edge between v and v_3 then, together with g_1 and v_1 it creates the forbidden graph (C), and so there must be the edge v_3v. Thus, all out-neighbours of g_1 are out-neighbour of v_3, contradicting the assumption from the first paragraph of this proof.

Take any vertex g_1 from G_1 and g_2 from G_2. If g_1 and g_2 are not connected, then the tuple g_2, v_1, v_3, g_1 forms the forbidden graph (B), if there is the edge g_2g_1, the same tuple forms the forbidden graph (C), so there must be the edge g_1g_2. Thus, from every vertex in G_1 we have edges to every vertex in G_2.

Consider now a vertex w being any out-neighbour of v_1. To prevent the tuple g_1, g_2, v_1, w, for any $g_1 \in G_1$, $g_2 \in G_2$, from forming (B) we must have the edge g_2w (reversed edge would make a directed triangle), the edge wg_1, or the edge g_1w. If the first two possibilities do not occur, but the last one, then we can find the forbidden graph (C) in the tuple v_1, v_3, g_1, w (if w and v_3 are not connected), or in the tuple g_1, g_2, v_1, w (if there is the edge v_3w, and so the edge v_1w belongs to the transitive triangle with edges v_1v_3, v_3w), or a directed triangle in the triple w, v_3, g_1 (if there is the edge wv_3). Vertex w cannot have an out-neighbour in G_1 and an in-neighbour in G_2, because it creates a directed triangle, thus w has directed edges to all vertices in G_1, or from all vertices in G_2.

Similarly, by considering now a vertex w' being any in-neighbour of v_3 and taking any $g_1 \in G_1$, $g_2 \in G_2$, we get that there must be the edge g_2w' or the edge $w'g_1$. Otherwise, to avoid (B), we must have the edge g_1w' (which forms a directed triangle) or the edge $w'g_2$ (which, as previously, dependently on the possible edge between v_1 and w' forms (C) in w', g_2, v_1, v_3, or in w', v_3, g_1, g_2, or a directed triangle w', g_2, v_1). Vertex w' cannot have an out-neighbour in G_1 and an in-neighbour in G_2, because it creates a directed triangle, thus w' has directed edges to all vertices in G_1, or from all vertices in G_2.

Since, all the vertices, except vertices in G_1 and G_2, are out-neighbours of v_1 or in-neighbours of v_3, we can split all the remaining vertices to the following groups:

G_3 — Vertices with edges from every vertex in G_2.

G_4 — Vertices with edges to every vertex in G_1.

In particular, $v_1 \in G_3$ and $v_3 \in G_4$.

There are no directed triangles, so there are no edges from vertices in G_3 to vertices in G_1. Assume there is an edge g_1g_3 for some $g_1 \in G_1$, $g_3 \in G_3$. Since all vertices in G_3

are out-neighbours of v_1 or in-neighbours of v_3, and the last case is making a directed triangle in the triple g_3, v_3, g_1, we have the edge v_1g_3. Now, the tuple v_1, v_3, g_1, g_3 forms the forbidden graph (C). Thus there are no edges between vertices in G_1 and vertices in G_3. The same argument shows that there are no edges between vertices in G_2 and vertices in G_4.

To avoid the forbidden graph (B), each vertex in G_3 and each vertex in G_4 need to be connected by some edge. Assume there is an edge g_4g_3 for some $g_3 \in G_3$, $g_4 \in G_4$. Take any $g_1 \in G_1$ and $g_2 \in G_2$. To prevent making the forbidden graph C from the tuple g_4, g_1, g_2, g_3, the edge g_2g_3 cannot appear in the transitive triangle with edges g_2v_1 and v_1g_3, and so there is no edge v_1g_3. The edge g_3v_1 would make a directed triangle with g_4, thus v_1 and g_3 are not connected, and so there is the edge g_3v_3. Simmilarily, the edge g_4g_1 cannot appear in the transitive triangle with edges g_4v_3 and v_3g_1, and so v_3 and g_4 are not connected, and there is the edge v_1g_4. But now the tuple v_1, g_4, g_3, v_3 forms the forbidden graph (C). Hence, all the vertices in G_3 have edges to all the vertices in G_4.

We cannot have a transitive triangle inside any of the groups, because it would make the forbidden graph (A) with any vertex from the non-neighbouring group.

Lets pick in every group a vertex with minimal outdegree inside this group. This minimum is at most a quarter of the size of this group, since the groups cannot contain triangles. By summing up outdegrees of the chosen 4 vertices, we get at most $5n/4$. On the other hand, each of them has outdegree at least $5n/16$, so they sum up to at least $5n/4$. It is possible only when groups have equal sizes and the traingle-free graph inside each group is a complete regular bipartite graph. □

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