Chromatic Symmetric Functions of Hypertrees

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Abstract

The chromatic symmetric function X_H of a hypergraph H is the sum of all monomials corresponding to proper colorings of H. When H is an ordinary graph, it is known that X_H is positive in the fundamental quasisymmetric functions F_S, but this is not the case for general hypergraphs. We exhibit a class of hypergraphs H — hypertrees with prime-sized edges — for which X_H is F-positive, and give an explicit combinatorial interpretation for the F-coefficients of X_H.

Keywords: symmetric function, quasisymmetric function, chromatic symmetric function, graph coloring, hypergraph, hypertree

1 Introduction

In [13], Stanley defined the *chromatic symmetric function* of a graph G, and since then this invariant has been an object of much study [1,3,4,8,10,11]. A *coloring* of an ordinary graph $G = (V, E)$ is a map $\chi : V \to \mathbb{P} = \{1, 2, \ldots\}$. We say χ is *proper* if $\chi(u) \neq \chi(v)$ whenever $\{u, v\} \in E$. Given $\chi : V \to \mathbb{P}$, we write $x^\chi = \prod_{v \in V} x_{\chi(v)}$ where $x_1, x_2, \ldots$ are commuting indeterminates. We then define the chromatic symmetric function X_G of a finite graph G to be

$$X_G = \sum_{\chi} x^\chi$$

with the sum over all proper colorings χ of G.

The chromatic symmetric function X_G is indeed a symmetric function, since the properness of a coloring is preserved under any permutation of the set of colors $\mathbb{P}$. It

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is possible, then, to consider the expansion of X_G in various bases of the ring of symmetric functions **Sym**, and there are a number of conjectures and open problems concerning positivity of X_G in these bases.

Our present interest is in the larger ring **QSym** of *quasisymmetric functions*. Quasisymmetric functions have been a powerful tool in algebraic combinatorics since they were first investigated by Stanley [12] and Gessel [5] in the 1970s and 1980s. In particular, the *fundamental quasisymmetric functions* F_S^n have have many applications in enumerative combinatorics and representation theory. The precise definition of F_S^n is given in Section 2.

If X is a symmetric function of degree n, then it is also quasisymmetric, so we may consider the coefficients a_S in the expansion $X = \sum_{S\subseteq[n-1]} a_S F_S^n$. If each a_S is nonnegative, we will say that X is F-positive. In [13], Stanley used the theory of P-partitions to show that X_G is always F-positive. The F-coefficients of X_G count linear extensions of posets defined by acyclic orientations of G.

In [15], Stanley presented a generalization of the chromatic symmetric function to *hypergraphs*. Informally, a hypergraph is a graph where the edges are allowed to contain more than two elements. More precisely, a hypergraph is a pair $H = (V, E)$ where V is finite and E is a family of subsets of V such that if $e \in E$ then $|e| > 1$. The elements of E are called *hyperedges*, or just *edges*. A *proper coloring* of the hypergraph H is a map $\chi : V \to \mathbb{P}$ so that no edge $e \in E$ is monochromatic.

Given a hypergraph H, the chromatic symmetric function X_H is defined in the same way as the chromatic symmetric function of an ordinary graph. That is,

$$X_H = \sum_{\chi} x^{\chi}$$

where the sum is taken over all proper colorings χ of H. Again X_H is symmetric, but unlike in the case of ordinary graphs, X_H is not always F-positive. For example, if $H = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2, 3\}, \{2, 3, 4\}\}$ then

$$X_H = 2F_{\{1\}} + 6F_{\{2\}} + 2F_{\{3\}} + 4F_{\{1,2\}} + 8F_{\{1,3\}} + 4F_{\{2,3\}} - 2F_{\{1,2,3\}} \tag{1}$$

is not F-positive. Note that the coefficients in Equation (1) sum to $24 = 4!$. This is not a coincidence. The sum of the F-coefficients in a chromatic symmetric function X_H will always be $n!$ where $n = |V|$, which can be seen by considering the coefficient of $x_1 x_2 \dots x_n$. Thus when X_H is F-positive, we may ask if X_H can be written as a sum of fundamental quasisymmetric functions indexed by permutations. However, for our purposes it is convenient to consider the set of bijections $\pi : V \to [n]$, which we denote L_V, rather than the set of bijections $\sigma : V \to V$ forming the symmetric group $\mathfrak{S}_V$. To avoid confusion, we will call $\pi \in \mathrm{L}_V$ a *labeling* and reserve the word *permutation* for bijections $\sigma \in \mathfrak{S}_V$. If $V = [n]$ the sets coincide and we write $\mathfrak{S}_n$ for $\mathfrak{S}_{[n]}$.

A *hypertree* is a hypergraph generalization of a tree, which will be defined in Section 2. An example of a hypertree is depicted in Figure 1. Our main result is the following fact, appearing as Theorem 9 in Section 5.

Theorem. *Let V be a finite set with $|V| = n$, and let $H = (V, E)$ be a hypertree so that $|e|$ is a prime number for each edge $e \in E$. Then X_H is F-positive. In particular,*

$$X_H = \sum_{\pi \in \mathrm{L}_V} F_{\mathrm{Des}_H(\pi)}$$

where $\mathrm{Des}_H(\pi)$ *is the set of* H-descents *of the labeling* $\pi \in \mathrm{L}_V$*, to be defined in Section 5.*

We note that it is easy to extend Theorem 9 to disjoint unions of hypertrees, or *hyperforests*, but for simplicity we only consider the connected case. We guess (Conjecture A) that the primality condition could be removed, although our proof relies on primality in a crucial way.

In Section 2 we give relevant background needed for the rest of the paper. In Section 3, we use a standardization procedure due to Gessel and Reutenauer [7] to show F-positivity of X_H when H consists of a single prime-sized edge. In Section 4, we combine the result of Section 3 with the theory of *P-partitions* due to Stanley [12] and Gessel [5] to show F-positivity of X_H for a hypertree H with prime-size edges. In Section 5 we describe a combinatorial interpretation of the results in Section 4 by giving the definition of H-descents and proving Theorem 9. We conclude in Section 6 by giving some conjectures and suggestions for further work. In the Appendix we give, for $n = 2, 3, 4$, explicit decompositions of the set of nonconstant colorings $\chi : [n] \to \mathbb{P}$ into inequalities corresponding to the fundamental quasisymmetric functions F_S^n.

2 Background

2.1 Symmetric and quasisymmetric functions

A formal power series of bounded degree X in the variables $x_1, x_2, \ldots$ is called *symmetric* if it is unchanged after any permutation of its variables. Some important symmetric functions include the *power sum symmetric function*

$$p_n = x_1^n + x_2^n + \ldots,$$

the *elementary symmetric function*

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

and the *complete homogeneous symmetric function* h_n

$$h_n = \sum_{i_1 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

A formal power series X in variables $x_1, x_2, \ldots$ is called *quasisymmetric* if the coefficients of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ and $x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$ in X are the same whenever $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$. Sums and products of quasisymmetric functions are also quasisymmetric, so the set of quasisymmetric functions forms a ring **QSym**. An important basis of **QSym** is given by the *fundamental quasisymmetric functions* F^n_S indexed by subsets $S \subseteq [n-1]$. They are defined by

$$F^n_S = \sum_{i_1, i_2, \ldots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

with the sum over all weakly increasing sequences $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n$ of positive integers with the restriction that if $j \in S$ then $i_j < i_{j+1}$. For example, if $n = 4$ and $S = \{1, 3\}$ then F^n_S is the sum of all monomials $x_{i_1} x_{i_2} x_{i_3} x_{i_4}$ with $i_1 < i_2 \leqslant i_3 < i_4$. If $S = [n-1]$ then $F^n_S = e_n$, while if $S = \emptyset$ we have $F^n_S = h_n$. These are the only cases in which F^n_S is actually symmetric. In what follows all the symmetric and quasisymmetric functions are homogeneous of a fixed degree n, so we will write $F_S = F^n_S$ without ambiguity.

2.2 *P*-partitions

One of the first applications of the theory of quasisymmetric functions was to the theory of *P-partitions* of a poset P [5, 12]. Given a poset P on a finite vertex set V, a mapping $\chi : V \to \mathbb{P} = \{1, 2, \ldots\}$ is a P-partition if $x \leqslant_P y$ implies $\chi(x) \leqslant \chi(y)$. If P is the poset $[n]$ with the usual order, then a P-partition is a sequence of increasing integers $\chi(1) \leqslant \chi(2) \leqslant \chi(3) \leqslant \ldots \leqslant \chi(n)$. This is equivalent to the usual definition of a *partition* of the integer $\chi(1) + \chi(2) + \cdots + \chi(n)$. Traditionally a partition of an integer is written in descending order, so what we call P-partitions were called *reverse* P-partitions by Stanley in [12, 16].

Suppose that $\omega \in \mathrm{L}_V$ is a labeling of V. In what follows it will be convenient to identify ω with the total order $<_\omega$ put on the vertices of P where $x <_\omega y$ means that $\omega(x) < \omega(y)$. A *(P, ω)-partition* is a P-partition χ that has strict inequalities where the orders P and ω disagree. That is, if $x <_P y$ then $\chi(x) \leqslant \chi(y)$, and if both the inequalities $x <_P y$ and $x >_\omega y$ occur then $\chi(x) < \chi(y)$.

It is sometimes useful to rephrase the definition of (P, ω)-partitions in terms of *covering relations*. We write $x \lessdot_P y$, and say that *y covers x* in P, if $x <_P y$ and there is no $z \in P$ so that $x <_P z <_P y$. By transitivity, we see that χ is a (P, ω)-partition if and only if χ satisfies the conditions that $\chi(x) \leqslant \chi(y)$ when $x \lessdot_P y$ and that $\chi(x) < \chi(y)$ whenever $x \lessdot_P y$ and $x >_\omega y$.

Given a bijective labeling $\pi \in \mathrm{L}_V$ and a subset $S \subseteq [n-1]$, let $A(\pi, S)$ be the set of $\chi : V \to \mathbb{P}$ satisfying the conditions

$$\chi(\pi^{-1}(1)) \leqslant \chi(\pi^{-1}(2)) \leqslant \ldots \leqslant \chi(\pi^{-1}(n))$$

and $\chi(\pi^{-1}(i)) < \chi(\pi^{-1}(i+1))$ when $i \in S$.

The main result on (P, ω)-partitions we need is the following fact, sometimes called the "fundamental theorem of (P, ω)-partitions". See [14, Lemma 3.15.3] for a proof when (P, ω)-partitions are taken to be order-reversing. It is given without proof in [16, 7.19.4] for (P, ω)-partitions taken to be order-preserving as we do. Recall that the *descent set* $\mathrm{Des}(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is the set of $i < n$ so that $\sigma(i) > \sigma(i+1)$. A *linear extension* π of P is an bijective P-partition $\pi \in \mathrm{L}_V$.

Theorem 1. *Let $P = (V, \leqslant_P)$ be a finite poset with $|V| = n$ and let $\omega : V \to [n]$ be any bijection. Then the set of (P, ω)-partitions is exactly the disjoint union*

$$\biguplus_{\pi} A(\pi, \mathrm{Des}(\omega\pi^{-1}))$$

where the union is taken over all linear extensions $\pi : V \to [n]$ of P.

Note that for any $\pi \in \mathrm{L}_V$ and $S \subseteq [n-1]$, the sum of all monomials corresponding to mappings $\chi \in A(\pi, S)$ is a fundamental quasisymmetric generating function:

$$\sum_{\chi \in A(\pi,S)} x^{\chi} = \sum_{\chi \in A(\pi,S)} x_{\chi(\pi^{-1}(1))} x_{\chi(\pi^{-1}(2))} \cdots x_{\chi(\pi^{-1}(n))} = F_S. \tag{2}$$

Define $K_{P,\omega}$ to be the sum of all monomials x^{χ} associated with (P, ω)-partitions χ. Then from Theorem 1 and Equation (2) we see how to expand $K_{P,\omega}$ into the fundamental quasisymmetric functions.

Corollary 2. *Let $P = (V, \leqslant_P)$ be a finite poset with $|V| = n$ and let $\omega \in \mathrm{L}_V$ be any labeling. Then we have*

$$K_{P,\omega} = \sum_{\pi} F_{\mathrm{Des}(\omega\pi^{-1})}$$

where the sum is taken over all linear extensions π of P.

2.3 Hypertrees

There are a number of closely-related definitions of *hypertree* occurring in the literature. We adopt the definition given in [6]. Let $H = (V, E)$ be a hypergraph. A *path* in H is a nonempty sequence

$$v_1, e_1, v_2, e_2, \ldots, e_m, v_{m+1}$$

of distinct vertices $v_i \in V$ and edges $e_i \in E$ so that $v_i, v_{i+1} \in e_i$ for each i. A sequence of vertices and edges satisfying all the conditions of a path except that $v_1 = v_{m+1}$ and $m > 1$ is a *cycle*. We say that a hypergraph $H = (V, E)$ is *connected* if there is a path from v to v' for any given $v, v' \in V$. A *hypertree* is a hypergraph that is connected and has no cycles. Thus in a hypertree there is a *unique* path between any two distinct vertices. Figure 1 depicts a hypertree as well as a hypergraph that is not a hypertree.

A hypergraph H is called *linear* if $|e \cap e'| \leqslant 1$ for any distinct edges $e, e' \in E$. Any hypertree H is linear, for if e_1, e_2 are distinct edges of H sharing two distinct vertices $v_1, v_2 \in e_1 \cap e_2$ then H has the cycle v_1, e_1, v_2, e_2, v_1. It is not true that X_H is F-positive whenever H is linear. For example, X_H is not F-positive when H consists of the edges $\{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4, 5\}$.

The following lemma provides a characterization of hypertrees that will be useful. It is a generalization of the fact that any ordinary tree can be constructed from a single vertex by adding one leaf at a time.

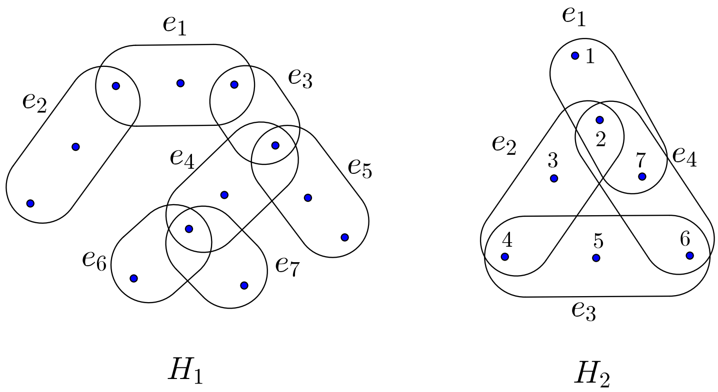

Figure 1: Two hypergraphs, H_1 and H_2, with edges circled. H_1 is a hypertree with an edge-labeling satisfying Equation (3). H_2 is not a hypertree, as it includes the cycles $2, e_2, 4, e_3, 6, e_4, 2$ and $2, e_1, 7, e_4, 2$.

Lemma 3. *Let $H = (V, E)$ be a hypergraph. Then H is a hypertree if and only if there is an ordering of its edges so that $E = \{e_1, e_2, \ldots, e_k\}$ with*

$$|(e_1 \cup e_2 \cup \cdots \cup e_i) \cap e_{i+1}| = 1 \tag{3}$$

for $i = 1, \ldots, k-1$.

Proof. First suppose that $H = (V, E)$ has an edge-ordering $E = \{e_1, \ldots, e_k\}$ satisfying Equation (3). Let $H_i = (V_i, E_i)$ where $V_i = e_1 \cup \cdots \cup e_i$ and $E_i = \{e_1, \ldots, e_i\}$. Let $\{v\} = e_{i+1} \cap V_i$. No cycle of H_{i+1} can use the edge e_{i+1}, since the vertex v would need to be repeated. Inductively assuming that H_i is a hypertree, H_{i+1} must be a hypertree as well. So H is a hypertree.

Now suppose that H is a hypertree. It is enough to find an $f \in E$ and $v \in f$ so that $e \cap f \subseteq \{v\}$ for any $e \in E$ with $e \neq f$. The edge f can be seen as an analog for a leaf in an ordinary tree. Once such an f is found, let $H' = (V', E')$ with $V' = V \backslash f \cup \{v\}$, $E' = E \backslash \{f\}$. Then H' is a hypertree with $k-1$ edges, so we may assume inductively that H' has an ordering $e_1, e_2, \ldots, e_{k-1}$ of E' satisfying the desired condition. Setting $e_k = f$, we see that $e_1, \ldots, e_k$ is the desired order of E.

To find such an f and $v \in f$, let $v_1, f_1, v_2, f_2, \ldots, v_l, f_l, v_{l+1}$ be a path of maximal length l in H. We claim that the choice $f_l = f$, $v = v_l$ satisfies the desired property. Suppose that there is $e \in E$ with $e \cap f_l \not\subseteq \{v_l\}$. Then there is $u \in e \cap f_l$ with $u \neq v$. Choosing any $u' \in e$ with $u' \neq u$, we claim that

$$v_1, f_2, \ldots, v_l, f_l, u, e, u' \tag{4}$$

is a path. For if $u = v_i$ for some i then we would have a cycle $v_i, f_i, \ldots, f_l, v_i$. Similarly, for any i we cannot have $e = f_i$ or $u' = v_i$ without creating a cycle. Thus (4) is a path of length $l + 1$, contradicting the maximality of l. □

3 The single edge case

Let $H = (V, E)$ be a hypergraph with the single edge $e = V$. Then the set of proper colorings of H is the set of all colorings that are not constant. Thus if $|V| = n$ then $X_H = p_1^n - p_n$. In the case that n is prime, we will show that there is a partition of the set of all nonconstant colorings into subsets of the form $A(\pi, S)$. This will show that $X_H = p_1^n - p_n$ is F-positive. Furthermore, we will see in Section 4 that if such a partition can be found for each edge e in a hypertree then there is a similar partition of the set of proper colorings of that hypertree. It will then follow that chromatic symmetric functions of hypertrees with prime-sized edges are F-positive. (For those readers familiar with the *Schur functions*, we note that it is not hard to show that $p_1^n - p_n$ is Schur-positive for any n using the Murnaghan-Nakayama rule, for example. However, this method does not extend to hypertrees in general.)

We say that $c \in \mathfrak{S}_V$ is *cyclic* if there is an ordering of V so that $V = \{v_1, v_2, \dots, v_n\}$ where $c(v_i) = v_{i+1}$ for $i < n$, with $c(v_n) = v_1$. Note that in this definition, a cyclic permutation has no fixed points. The goal of this section is to prove the following fact.

Theorem 4. *Let V be a set with $|V| = n$ prime and let $c : V \to V$ be an arbitrary cyclic permutation of V. Then the set of nonconstant colorings $\chi : V \to \mathbb{P}$ is the disjoint union*

$$\biguplus_{\pi \in \mathrm{L}_V} A(\pi, \mathrm{Des}(\pi c \pi^{-1})). \tag{5}$$

Theorem 4 immediately gives the F-expansion of the symmetric function $p_1^n - p_n$ when n is prime. We have

$$p_1^n - p_n = \sum_{\pi \in \mathfrak{S}_n} F_{\mathrm{Des}(\pi c \pi^{-1})}. \tag{6}$$

The conjugation $\pi c \pi^{-1}$ is equivalent to the cyclic permutation c after its elements have been relabeled via π. Thus Equation 6 states that $p_1^n - p_n$ is the sum of the fundamental quasisymmetric functions corresponding to descent sets of relabelings of c.

In the Appendix we give the sets $A(\pi, \mathrm{Des}(\pi c \pi^{-1}))$ in the decomposition (5) for the cases $V = \{1, 2\}$ and $V = \{1, 2, 3\}$.

To prove Theorem 4, we will describe a technique due to Gessel and Reutenauer [7]. It will be convenient in what follows to use the language of *words* rather than colorings. A word w on an alphabet A of length n is an n-tuple $w = w(1) \cdots w(n) \in A^n$. There is a natural action of the cyclic group C_n on A^n given by identifying C_n with the rotations $r_i : A^n \to A^n$ with $r_i(w(1) \cdots w(n)) = w(i) \cdots w(n) w(1) \cdots w(i-1)$. A *necklace* of length n is an orbit of this action. If $w \in A_n$ is such that that $r^i(w) \neq w$ for $1 \leqslant i < n$, we say that the orbit of w is a *primitive necklace*. For example, the orbit of 1211 is primitive, but the orbit of 1212 is not. When n is prime, every necklace is primitive unless it consists only of a single letter $a \in A$ repeated n times. In particular, the words w whose orbit is primitive are equivalent to *proper colorings* of the single-edge hypergraph.

Gessel and Reutenauer describe a bijection between words on a the alphabet $\mathbb{P} = \{1, 2, 3, \ldots\}$ and *multisets of primitive necklaces*. In particular, they show that

$$\sum_v x^v = \sum_c F_{\mathrm{Des}(c)}. \tag{7}$$

where the sum on the left-hand side is taken over all primitive necklaces v of length n and the sum on the right-hand side is taken over all cyclic permutations $c : [n] \to [n]$. When n is prime, the orbit of any nonconstant word $w \in A^n$ is a primitive necklace and each orbit has size n, so Equation (7) reduces to

$$\frac{1}{n}(p_1^n - p_n) = \sum_c F_{\mathrm{Des}(c)}. \tag{8}$$

The symmetric function in Equation (8) is also interesting from an algebraic standpoint: it is the Frobenius characteristic of the $\mathfrak{S}_n$-representation given by the degree-n multilinear part of the free Lie algebra on a set of size n. Equations (6) and (8) are equivalent, since every cyclic permutation will appear exactly n times in the sum (6).

Gessel and Reutenauer make use of a form of *standardization*, a map from multisets of necklaces to permutations. Since are working with colorings instead of multisets of necklaces, we give a variant of their argument that suffices to prove Theorem 4. In the proof it will be convenient to assume without loss of generality that $V = [n]$ and c is the particular cyclic permutation $c(i) = i + 1$ for $1 \leqslant i < n$ with $c(n) = 1$.

Recall the *lexicographic order* on words. If $v = v(1)v(2)\cdots v(n)$,$w = w(1)w(2)\cdots w(n)$ are words on the alphabet $\mathbb{P}$, we write $v \leqslant_{\mathrm{lex}} w$ if $v = w$, or else either

$$v(1) < w(1) \tag{9}$$

or, inductively,

$$v(1) = w(1) \text{ and } v(2)\cdots v(n) \leqslant_{\mathrm{lex}} w(2)\cdots w(n). \tag{10}$$

Assuming n is prime and w is a nonconstant word, define the *cyclic standardization* of w, which we'll denote $\mathrm{cstd}(w)$, to be the permutation obtained by ordering the rotations of w lexicographically. That is, we say $\pi = \mathrm{cstd}(w)$ if π is the unique permutation in $\mathfrak{S}_n$ so that $r_x(w) <_{\mathrm{lex}} r_y(w)$ whenever $\pi(x) < \pi(y)$. We find $\pi = \mathrm{cstd}(w)$ by setting $\pi(i) = j$ when $r_i(w)$ is the jth smallest rotation of w. For example, if $w = 2114132$ then $\pi = \mathrm{cstd}(w) = 4137265$. We have $\pi(2) = 1$ since $r_2(w) = 1141321$ is the least rotation of w lexicographically, $\pi(5) = 2$ since $r_5(w) = 1322114$ is the next smallest, etc.

By the primality of n, the set of words $w \in \mathbb{P}^n$ that are not constant is the disjoint union

$$\biguplus_{\pi \in \mathfrak{S}_n} \{w \in \mathbb{P}^n : \mathrm{cstd}(w) = \pi\}.$$

Thus the proof of Theorem 4 is immediate from the following lemma.

Lemma 5. *Let n be prime and let $w \in \mathbb{P}^n$ be a nonconstant word. Then* $\operatorname{cstd}(w) = \pi$ *if and only if* $w \in A(\pi, S)$ *where* $S = \operatorname{Des}(\pi c \pi^{-1})$.

Proof. First, suppose that $\operatorname{cstd}(w) = \pi$. To show that $w \in A(\pi, S)$, we must first show that $w(\pi^{-1}(i)) \leqslant w(\pi^{-1}(i+1))$ for any $1 \leqslant i < n$. Suppose $\pi(x) = i$ and $\pi(y) = i+1$. Then $r_y(w) = w(y)w(y+1)\cdots$ is the next largest rotation of w in lexicographic order after $r_x(w) = w(x)w(x+1)\cdots$, where we take $n+1 = 1$, $n+2 = 2$, etc. In particular, we must have $w(x) \leqslant w(y)$, or $w(\pi^{-1}(i)) \leqslant w(\pi^{-1}(i+1))$ as desired.

Now suppose that i is a descent of $\pi c \pi^{-1}$. We must show that $w(x) < w(y)$. We have $\pi c \pi^{-1}(i) > \pi c \pi^{-1}(i+1)$, so that $\pi(x+1) > \pi(y+1)$. Then we have

$$r_{x+1}(w) = w(x+1)w(x+2)\cdots >_{\text{lex}} r_{y+1}(w) = w(y+1)w(y+2)\cdots .$$

But we also know that $w(x)w(x+1)\cdots <_{\text{lex}} w(y)w(y+1)\cdots$, and the only way both of these lexicographic inequalities can occur is if $w(x) < w(y)$. Thus $w \in A(\pi, S)$.

Conversely, suppose that $w \in A(\pi, S)$. To show that $\operatorname{cstd}(w) = \pi$, we will show that $\pi(x) < \pi(y)$ implies $r_x(w) <_{\text{lex}} r_y(w)$. Since n is prime, all rotations of w are distinct, and so it is enough to show that $\pi(x) < \pi(y)$ implies $r_x(w) \leqslant_{\text{lex}} r_y(w)$. We will proceed inductively. For any word $v = v(1)v(2)\cdots v(n) \in \mathbb{P}^n$, let $v|_m$ be the truncation $v(1)\cdots v(m)$. We will show that for each m, if $\pi(x) < \pi(y)$ we must have $r_x(w)|_m \leqslant_{\text{lex}} r_y(w)|_m$. By transitivity, it is enough to assume that $\pi(x) = i$ and $\pi(y) = i+1$. If $m = 1$, the truncations $r_x(w)|_m, r_y(w)|_m$ are the single-character words $w(x), w(y)$. We know $w(\pi^{-1}(i)) \leqslant w(\pi^{-1}(i+1))$ since $w \in A(\pi, S)$, and so $w(x) \leqslant w(y)$.

Now suppose that $\pi(x) < \pi(y)$ implies that $r_x(w)|_m \leqslant_{\text{lex}} r_y(w)|_m$ for any x, y. We will show that $r_x(w)|_{m+1} \leqslant_{\text{lex}} r_y(w)|_{m+1}$ in the case $\pi(x) = i$, $\pi(y) = i+1$. By the argument for the base case, we know $w(x) \leqslant w(y)$. If $w(x) < w(y)$ we are done by property (9) in the definition of lexicographic order, so assume $w(x) = w(y)$. Since $w \in A(\pi, S)$, i must be an ascent of $\pi c \pi^{-1}$. Thus $\pi c \pi^{-1}(i) < \pi c \pi^{-1}(i+1)$, or equivalently, $\pi(x+1) < \pi(y+1)$. By our inductive hypothesis, we know that $r_{x+1}(w)|_m \leqslant r_{y+1}(w)|_m$. Since $w(x) = w(y)$, it follows by property (10) that $w(x)(r_{x+1}(w)|_m) \leqslant_{\text{lex}} w(y)(r_{y+1}(w)|_m)$, or equivalently $r_x(w) \leqslant_{\text{lex}} r_y(w)$. □

4 Proof of F-positivity

With Theorem 4 established, we can now prove the F-positivity of X_H when $H = (V, E)$ is a hypertree with prime-sized edges. The primality gives us the decomposition described in Theorem 4 for each edge $e \in E$, and the hypertree structure will enable us to put these decompositions together to get a similar decomposition of the set of all proper colorings of H. The tool we need is the theory of *P-partitions*.

The key fact we will use about hypertrees is that posets on different edges are compatible with each other.

Lemma 6. *Let $H = (V, E)$ be a hypertree with $E = \{e_1, \ldots, e_k\}$. Suppose that each edge $e_i \in E$ has an associated poset P_i with vertex set e_i and relation $<_i$. Define the relation*

$<$ *on* V *by taking the transitive closure of all the relations* $<_e$*, so that* $x < y$ *in* V *if there is a chain* $x = v_1 <_{i_1} v_2 <_{i_2} \cdots <_{i_l} v_l = y$*. Then* $P = (V, <)$ *is a poset.*

Proof. Form a directed graph G on V by setting $x \to y$ when there is an edge $e \in E$ with $x, y \in e$ and $x <_e y$. Then G is easily seen to be acyclic since H is a hypertree, and any directed acyclic graph determines a poset after extending transitively. □

Theorem 7. *Let* $H = (V, E)$ *be a hypertree so that* $|e|$ *is prime for each* $e \in E$*. Then* X_H *is* F*-positive.*

Proof. Say $E = \{e_1, \ldots, e_k\}$. For each edge $e_i \in E$ fix a particular cyclic permutation $c_i : e_i \to e_i$. Since each edge $e_i \in |E|$ has $|e_i|$ prime, by Theorem 4 the set of nonconstant colorings $\chi_i : e_i \to \mathbb{P}$ is the disjoint union

$$\biguplus_{\pi \in \mathrm{L}_{e_i}} A(\pi, \mathrm{Des}(\pi c_i \pi^{-1})). \tag{11}$$

Let $PC(H)$ be the set of proper colorings $\chi : V \to \mathbb{P}$ of H. Note that $\chi \in PC(H)$ if and only if the restriction $\chi|_{e_i}$ is not constant for every edge $e_i \in E$. Given a tuple $(\pi_1, \ldots, \pi_k) \in \mathrm{L}_{e_1} \times \cdots \times \mathrm{L}_{e_k}$, define the set

$$A(\pi_1, \pi_2, \ldots, \pi_k) = \{\chi : \mathbb{P} \to V : \chi|_{e_i} \in A(\pi_i, \mathrm{Des}(\pi_i c_i \pi_i^{-1})) \text{ for all } i\}.$$

We note that for each coloring $\chi \in PC(H)$, there is a unique k-tuple $(\pi_1, \ldots, \pi_k)$ so that $\chi \in A(\pi_1, \ldots, \pi_k)$. We must take π_i to be the unique $\pi_i \in S_n$ so that $\chi|_{e_i} \in A(\pi, \mathrm{Des}(\pi_i c_i \pi_i^{-1})$. Thus the set of all proper colorings of H has the partition

$$PC(H) = \biguplus_{\pi_1, \ldots, \pi_k} A(\pi_1, \pi_2, \ldots, \pi_k) \tag{12}$$

where the union is taken over all k-tuples $(\pi_1, \ldots, \pi_k) \in \mathrm{L}_{e_1} \times \cdots \times \mathrm{L}_{e_k}$. We will observe that for any choice of $\pi_1, \ldots, \pi_k$, the set $A(\pi_1, \ldots, \pi_k)$ is actually the set of (P, ω)-partitions for a particular $P = P_{\pi_1, \ldots, \pi_k}$ and $\omega = \omega_{\pi_1, \ldots, \pi_k}$. The proof will then be complete after applying Theorem 1 to each set $A(\pi_1, \ldots, \pi_k)$.

Fix a choice of $(\pi_1, \ldots, \pi_k) \in \mathrm{L}_{e_1} \times \cdots \times \mathrm{L}_{e_k}$. For each i, define a poset P_{π_i} on the vertex set e_i by the rule that $x <_{P_{\pi_i}} y$ when $\pi_i(x) < \pi_i(y)$. (Note that P_{π_i} is in fact a total order.) Let $\omega_{\pi_i} : e_i \to [m]$ be the labeling $\pi_i c_i$. Then π_i is the unique linear extension of P_{π_i}, so by Theorem 1 the set of $(P_{\pi_i}, \omega_{\pi_i})$-partitions is exactly the set $A(\pi_i, \mathrm{Des}(\omega_{\pi_i} \pi_i^{-1})) = A(\pi_i, \mathrm{Des}(\pi_i c_i \pi_i^{-1}))$. Thus $A(\pi_1, \pi_2, \ldots, \pi_k)$ is the set of colorings χ so that $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$-partition for each i.

By Lemma 6 there are well-defined posets $P = P_{\pi_1, \ldots, \pi_k}$ and $Q = Q_{\pi_1, \ldots, \pi_k}$ given by taking the transitive closure of the relations of the posets P_{π_i} and ω_{π_i} respectively. Let $\omega = \omega_{\pi_1, \ldots, \pi_k}$ be an arbitrary linear extension of Q. (In general there may be many linear extensions ω, but we fix a particular one.) We claim that $A(\pi_1, \pi_2, \ldots, \pi_k)$ is exactly the set of (P, ω)-partitions.

Let χ be a (P, ω)-partition. We will show that $\chi \in A(\pi_1, \ldots, \pi_k)$ by showing $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$-partition for each i. If $u, v \in e_i$ with $u <_{P_{\pi_i}} v$, then since χ is a P-partition we must have $\chi(u) < \chi(v)$, so $\chi|_{e_i}$ is a P_{π_i}-partition. If in addition $u >_{\omega_i} v$, then $u >_\omega v$ since ω is a linear extension of $Q_{\pi_1,\ldots,\pi_k}$. Thus we must have $\chi(u) < \chi(v)$ since χ is a (P, ω)-partition. So $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$-partition.

Conversely, suppose that $\chi \in A(\pi_1, \ldots, \pi_k)$. We will show that χ is a (P, ω)-partition. If $u \lessdot_P v$ is a covering relation in P, we must have $u, v \in e_i$ for some i. Then by the definition of P, we have $u <_{P_{\pi_i}} v$, and so $\chi(u) \leqslant \chi(v)$ since $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$-partition. This shows that χ is a P-partition. Similarly, if $u <_\omega v$ then we have $u <_{\omega_i} v$ since ω is a linear extension of Q. Thus $\chi(u) < \chi(v)$ since $\chi|_{e_i}$ is a $(P_{\pi_i}, \omega_{\pi_i})$-partition.

Now for every $(\pi_1, \ldots, \pi_k) \in \mathrm{L}_{e_1} \times \cdots \times \mathrm{L}_{e_k}$ fix some choice of linear extension $\omega_{\pi_1,\ldots,\pi_k}$ of $Q(\pi_1, \ldots, \pi_k)$. Since $A(\pi_1, \ldots, \pi_k)$ is the set of $(P_{\pi_1,\ldots,\pi_k}, \omega_{\pi_1,\ldots,\pi_k})$-partitions, we may apply Theorem 1 along with Equation (12) to get

$$PC(H) = \biguplus_{\pi_1,\ldots,\pi_k} \biguplus_{\pi} A(\pi, \mathrm{Des}(\omega_{\pi_1,\ldots,\pi_k} \sigma^{-1})) \tag{13}$$

where the union is taken over all tuples $(\pi_1, \ldots, \pi_k) \in \mathrm{L}_{e_1} \times \cdots \times \mathrm{L}_{e_k}$ and linear extensions $\pi : V \to [n]$ of $P_{\pi_1,\ldots,\pi_k}$. Finally, the F-positivity of X_H follows by applying Equation (2) to each set $A(\sigma, \mathrm{Des}(\omega_{\pi_1,\ldots,\pi_k} \sigma^{-1}))$. □

By rewriting Equation (13) in a simpler form we can give an expression for X_H as a sum of fundamental quasisymmetric functions indexed by labelings $\pi \in \mathrm{L}_V$. Note that $\pi \in \mathrm{L}_V$ is a linear extension of $P_{\pi_1,\ldots,\pi_k}$ if and only if π_i lists the elements of e_i in the same order they are listed in in π. It follows that every $\pi \in \mathrm{L}_V$ appears exactly once in the union (13). We can also define the posets $Q = Q_{\pi_1,\ldots,\pi_k}$ strictly in terms of a linear extension π. The poset Q is the transitive closure of the relations $x <_Q y$ in V if x, y share an edge e_i and $\pi c_i(x) < \pi c_i(y)$. We summarize this in the following corollary.

Corollary 8 (of the proof of Theorem 7)**.** *Let $H = (V, E)$ be a hypertree with edges $e_1, \ldots, e_k$. For each i, let $c_i : e_i \to e_i$ be an arbitrary cyclic permutation. For each $\pi \in \mathrm{L}_V$, let $Q(\pi)$ be the poset on V generated by the relations $x <_{Q(\pi)} y$ when x, y share an edge e_i and $\pi c_i(x) < \pi c_i(y)$. For each $\pi \in \mathrm{L}_V$, fix some choice of linear extension ω_π of $Q(\pi)$. Then*

$$X_H = \sum_{\pi \in \mathrm{L}_V} F_{\mathrm{Des}(\omega_\pi \pi^{-1})}. \tag{14}$$

In Corollary 8 the choice of the linear extension ω_π is arbitrary. Every poset has a linear extension, and this fact is enough to prove F-positivity. From a combinatorial standpoint, however, it would be desirable to find a specific choice of ω_π that is natural in some sense. In the next section we do so, giving a simple combinatorial interpretation to the F-coefficients of X_H.

5 Combinatorial interpretation

Let $H = (V, E)$ be a hypertree. We now define the H-descents $\mathrm{Des}_H(\pi)$ of a labeling $\pi \in \mathrm{L}_V$. This definition will assume that H has an edge ordering $E = \{e_1, \ldots, e_k\}$ satisfying the conclusion of Lemma 3. For our purposes it does not matter which of these edge-orderings we choose, but the definitions that follow will depend on this choice. Thus for the rest of this section we will assume that the hypertree H is equipped with a fixed choice of edge-ordering satisfying Equation (3).

To make the notation clearer, we first assume that $V = [n]$ so that V is equipped with the usual order $<$. Let $H = (V, E)$ be a hypertree with edges $E = \{e_1, \ldots, e_k\}$ satisfying Equation (3). Fix a choice of cyclic permutation $c_i : e_i \to e_i$. Since H is a hypertree, for each i there is a *unique* path from i to $i + 1$ with distinct vertices and edges. Suppose

$$i = v_1, e_{j_1}, v_2, e_{j_2}, \ldots, e_{j_l}, v_{l+1} = i + 1$$

is this path and $j_r = \min(j_1, j_2, \ldots, j_l)$. We say that i is an H-descent if $c_{j_r}(v_r) > c_{j_r}(v_{r+1})$.

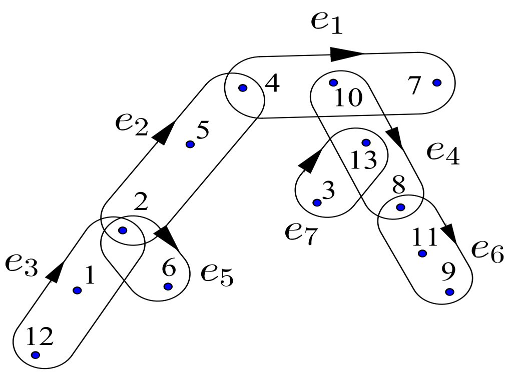

Figure 2: A hypertree with labeled vertices, a suitable ordering of its edges, and a cyclic permutation of each edge.

For example, let H be the hypergraph in Figure 2. The cyclic permutations c_i are given by reading along the indicated direction in cycle notation, so that c_3 is $(12, 1, 2)$ with $12 \mapsto 1 \mapsto 2 \mapsto 12$. The unique path from 1 to 2 is just $1, e_3, 2$ since 1 and 2 are both contained in e_3. Then $c_3(1) = 2 < c_3(2) = 12$, so 1 is not an H-descent. The unique path from 2 to 3 is given by $2, e_2, 4, e_1, 10, e_4, 13, e_7, 3$ and the edge with the smallest index occurring in this path is e_1. Then $c_1(4) = 10 > c_1(10) = 7$ and so 2 is an H-descent. Continuing, we find the H-descents are $\{2, 6, 8, 10, 12\}$.

Now fix an arbitrary (unordered) set V. Let $H = (V, E)$ be a hypertree with an edge-ordering satisfying Equation (3) and a cyclic permutation c_i of each edge e_i. Given a labeling $\pi \in \mathrm{L}_V$, we identify v with $\pi(v)$ to get an isomorphic hypergraph on $[n]$ where $n = |V|$. We then denote the corresponding set of H-descents by $\mathrm{Des}_H(\pi)$ and call them the H-descents of π. With these definitions in hand, we state our main theorem.

Theorem 9. *Let $H = (V, E)$ be a hypertree so that $|e|$ is prime for each edge $e \in E$. Fix an ordering of the edges so $E = \{e_1, \ldots, e_k\}$ with the property that $|(e_1 \cup \cdots \cup e_i) \cap e_{i+1}| = 1$*

for all $1 \leqslant i < k$*, and also fix a choice of cyclic permutation* $c_i : e_i \to e_i$ *of each edge* $e_i \in E$*. Then*

$$X_H = \sum_{\pi \in \mathrm{L}_V} F_{\mathrm{Des}_H(\pi)}$$

where $\mathrm{Des}_H(\pi)$ *is the set of* H*-descents of* π *with respect to the chosen edge-ordering and cyclic permutations.*

Note that if H consists of a single edge e with a cyclic permutation $c : e \to e$, $\mathrm{Des}_H(\pi)$ is exactly $\mathrm{Des}(\pi^{-1}c\pi)$, so Theorem 9 reduces to Corollary 6.

To prove Theorem 9, we will need a systematic way of combining total orders together. Let U, V be totally ordered sets (U, ω_U), (V, ω_V) where U, V share a single element, say $U \cap V = \{x\}$. We define $\omega = \omega_U \leftarrow \omega_V$ to be the total order of the union $U \cup V$ given by "inserting" V with its total order ω_V into the place of x in U. That is, ω is the unique total order agreeing with ω_U, ω_V on U, V so that when $u \in U, v \in V$ we have $u <_\omega v$ if and only if $u <_{\omega_U} x$. For example, if $U = \{x <_{\omega_U} b <_{\omega_U} y\}$, $V = \{a <_{\omega_V} b <_{\omega_V} c\}$ are totally ordered sets then $\omega = \omega_U \leftarrow \omega_V$ is the total order $x <_\omega a <_\omega b <_\omega c <_\omega y$. Note that $\omega_U \leftarrow \omega_V$ is not the same as $\omega_V \leftarrow \omega_U$.

We now consider the total orders that arise from repeated insertion.

Lemma 10. *Let* $H = (V, E)$ *be a hypertree with* $E = \{e_1, \ldots, e_k\}$ *so that Equation* (3) *holds. Suppose there is a total order* ω_i *on each* e_i*, and define a total order* ω *on* V *by*

$$\omega = (\cdots(\omega_1 \leftarrow \omega_2) \leftarrow \cdots) \leftarrow \omega_k.$$

Then for any distinct $x, y \in V$*, the pair satisfies* $x <_\omega y$ *if and only* $v_r <_{\omega_{j_r}} v_{r+1}$ *where*

$$x = v_1, e_{j_1}, v_2, e_{j_2}, \ldots, v_l, e_{j_l}, v_{l+1} = y \tag{15}$$

is the unique path from x *to* y *in* H *and* $j_r = \min(j_1, j_2, \ldots, j_l)$*.*

Proof. We proceed by induction on the number of edges of H. If H has only one edge, the statement is trivial, so suppose that the statement holds for hypertrees with fewer than k edges and that H has exactly k edges. Let $H' = (V', E')$ be the hypertree with $V' = e_1 \cup \cdots \cup e_{k-1}$ and $E' = E \backslash \{e_k\}$, and let ω' be the total order on V' given by

$$\omega' = (\cdots(\omega_1 \leftarrow \omega_2) \leftarrow \cdots) \leftarrow \omega_{k-1},$$

so that $\omega = \omega' \leftarrow \omega_k$. If both x and y are in V' then we are done by the inductive hypothesis. Similarly if $x, y \in e_k$ then there is nothing to show. So assume that $x \in e_k \backslash V'$ and $y \in V' \backslash e_k$ and let the tuple (15) be the path from x to y. Then we must have $j_1 = k$, so that $\{v_2\} = V' \cap e_k$. From the definition of the insertion $\omega' \leftarrow \omega_k$, we have $x <_\omega y$ if and only if $v_2 <_{\omega'} y$. Then

$$v_2, e_{j_2}, \ldots, v_l, e_{j_l}, v_{l+1}$$

is the unique path from v_2 to y in H'. Furthermore,

$$j_r = \min(j_1, j_2, \ldots, j_l) = \min(j_2, \ldots, j_l)$$

since $j_1 = k$ is the highest index of any edge in H. Thus by our inductive hypothesis we see that $v_2 <_{\omega'} y$ is equivalent to $v_r <_{\omega_{j_r}} v_{r+1}$. ☐

Proof of Theorem 9. Given a bijection $\pi : V \to [n]$, let $Q(\pi)$ be as in the statement of Corollary 8. For each i let ω_i be the total order of e_i given by restricting Q to e_i, so that $x <_{\omega_i} y$ in e if $\pi c_i(x) < \pi c_i(y)$. Then define ω_π to be the total order on V given by repeated insertion, so $\omega_\pi = (\cdots(\omega_1 \leftarrow \omega_2) \leftarrow \cdots) \leftarrow \omega_k$. Then ω_π is a linear extension of $Q(\pi)$, and by Lemma 10 we see that i is a descent of $\omega_\pi \pi^{-1}$ if and only if $v_r >_{\omega_{j_r}} v_{r+1}$, that is, $\pi c_{j_r}(v_r) > \pi c_{j_r}(v_{r+1})$ where j_r is the least-index edge in the path $\pi^{-1}(i) = v_1, e_{j_1}, \ldots, e_{j_l}, v_{l+1} = \pi^{-1}(i+1)$ from $\pi^{-1}(i)$ to $\pi^{-1}(i+1)$. After identifying v with $\pi(v)$, the descents of $\omega_\pi \pi^{-1}$ become the H-descents, so that $\mathrm{Des}_H(\pi) = \mathrm{Des}(\omega\pi^{-1})$. Applying Corollary 8 then finishes the proof. $\square$

6 Suggestions for future work

6.1 Removing the primality condition

It is likely that the condition that the edges have prime size could be removed. A closer examination of the proof of F-positivity (Theorem 7) reveals that it does not depend on primality *per se*, but only on the existence of partitions of colorings into disjoint sets of the form $A(\pi, S)$.

Theorem 11. *Suppose that $H = (V, E)$ is a hypertree with $E = \{e_1, \ldots, e_k\}$ and let $n_i = |e_i|$. Furthermore, suppose that for each e_i and $\pi \in \mathrm{L}_{e_i}$ there is a set $D_i(\pi) \subseteq [n_i - 1]$ so that the set of non-constant colorings $\chi : e_i \to \mathbb{P}$ is the disjoint union*

$$\biguplus_{\pi \in \mathrm{L}_{e_i}} A(\pi, D_i(\pi)). \tag{16}$$

Then X_H is F-positive.

Proof. For each i and $\pi \in \mathrm{L}_{e_i}$, choose some $\omega_{\pi_i} \in \mathrm{L}_{e_i}$ so that $\mathrm{Des}(\omega_{\pi_i}\pi_i^{-1}) = D_i(\pi)$. Then apply the proof of Theorem 7 with the maps ω_{π_i} playing the role of the maps $\pi_i c_i$. $\square$

Lemma 4 shows that a partition of the form (16) exists when $n = |e_i|$ is prime. In fact, such a partition of the nonconstant colorings of a set of $n = 4$ elements does exist as well. It was found and verified using the software package Sage [17]. We give this partition in the appendix. Finding such a partition for each n would then constitute a proof of the following.

Conjecture A. *Let H be a hypertree. Then X_H is F-positive.*

We can rephrase this idea in terms of *simplicial complexes*. A simplicial complex Δ is a family of subsets of a finite vertex set V so that if $F \in \Delta$ and $F' \subseteq F$ then $F' \in \Delta$. If Δ is a simplicial complex and $S \subseteq \Delta$ is any subset of Δ, then S is a *partial simplicial complex* and we say that S is *partitionable* if S is a disjoint union

$$S = \biguplus_i [G_i, F_i]$$

where the F_i are maximal faces (also called facets) of Δ, with subsets $G_i \subseteq F_i$, and where the sets $[G_i, F_i]$ are defined to be $\{F \in \Delta : G_i \subseteq F \subseteq F_i\}$.

The existence of a partition of the nonconstant colorings of the form (16) when $|e_i| = n$ is equivalent to the statement a certain partial simplicial complex $\Delta_n \backslash \emptyset$ is partitionable. The complex Δ_n is the well-known *Coxeter complex* of type A_{n-1}, a simplicial complex which among other properties has its facets in bijection with permutations $\pi \in \mathfrak{S}_n$. The intervals $[G, F]$ are then equivalent to the sets of colorings $A(\pi, S)$. The problem of partitionability for a partial simplicial complex $S \subseteq \Delta_n$ is discussed by Breuer and Klivans in [2], where S is thought of as a *scheduling problem.*

6.2 Schur positivity

The *Schur functions* form an important basis of **Sym** with deep connections to representation theory of the symmetric and general linear groups. It is not the case in general that X_H is Schur-positive when H is a hypertree, or even an ordinary tree. For example, if $C = (V, E)$ is the "claw" with $V = \{1, 2, 3, 4\}, E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ then X_C is not Schur-positive. Stanley has conjectured in [13] that if G is *clawfree* then X_G is Schur-positive, where a graph G is clawfree when it has no induced subgraphs isomorphic to the claw C.

It would be interesting to generalize Stanley's conjecture to hypergraphs, but we do not attempt that here. Instead we offer a more modest conjecture. We say that a hypergraph H is an *interval hypergraph* if it is isomorphic to a hypergraph (V, E) where $V = [n]$ and each edge $e \in E$ is an interval $e = \{i, i + 1, \ldots, j\}$. Recall that a hypergraph H is linear if $|e \cap e'| \leqslant 1$ for each pair of distinct edges e, e'. For example, if $V = [9]$ and $E = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6\}, \{6, 7, 8, 9\}\}$ then $H = (V, E)$ is an interval hypergraph that is linear.

Conjecture B. *If H is a linear interval hypergraph then X_H is Schur-positive.*

Note that connected linear interval hypergraphs are hypertrees, so they are at least F-positive when they have prime-sized edges. If $H = G$ is a linear interval hypergraph that is in an ordinary graph, G is just a disjoint union of paths. In that case X_G was shown to be e-positive by Stanley [13] and hence Schur-positive.

Conjecture B was motivated by the study of *formal group laws.* A one-dimensional, commutative formal group law in characteristic 0 is equivalent to a formal power series of the form

$$f(f^{-1}(x_1) + f^{-1}(x_2) + \cdots) \tag{17}$$

when $f(x)$ is a formal power series in one variable x with $f(0) = 0, f'(0) = 1$. In [18] and [19], the author gives a number of examples of generating functions $f(x)$ for which the formal group law (17) can be written as a sum of chromatic symmetric functions of certain linear interval hypergraphs.

Note. While this paper was under review, Conjecture B was proved by Brendan Pawlowski [9]. In fact, his method generalizes to show Schur-positivity for any hypertree with the property that no three edges intersect.

Appendix: Partitions of nonconstant colorings

We give here explicit partitions of the set of nonconstant colorings $\chi : [n] \to \mathbb{P}$ into the disjoint sets $A(\pi, S)$ when $n = 2, 3, 4$. We write the set $A(\pi, S)$ in the form of the inequality

$$\chi(i_1)\, R_1\, \chi(i_2)\, R_2\, \cdots\, R_{n-1}\, \chi(i_n)$$

where $i_j = \pi^{-1}(j)$ and the relation R_j is $<$ when $j \in S$ and $\leqslant$ when $j \notin S$. The prime cases $n = 2$, $n = 3$ use the partition given in Theorem 4, while the case $n = 4$ was found and verified using Sage.

The only admissible partition for $n = 2$ is

$$\begin{gathered} \chi(1) < \chi(2) \\ \chi(2) < \chi(1). \end{gathered}$$

There are two partitions for $n = 3$, with each attained from the other by reversing the roles of $\leqslant$ and $<$. One is

$$\begin{gathered} \chi(1) \leqslant \chi(2) < \chi(3) \\ \chi(1) < \chi(3) \leqslant \chi(2) \\ \chi(2) < \chi(1) \leqslant \chi(3) \\ \chi(2) \leqslant \chi(3) < \chi(1) \\ \chi(3) \leqslant \chi(1) < \chi(2) \\ \chi(3) < \chi(2) \leqslant \chi(1). \end{gathered}$$

There are many partitions for $n = 4$. One example is

$$\begin{array}{ll} \chi(1) < \chi(2) < \chi(3) < \chi(4) & \chi(3) < \chi(1) \leqslant \chi(2) < \chi(4) \\ \chi(1) < \chi(2) \leqslant \chi(4) \leqslant \chi(3) & \chi(3) < \chi(1) \leqslant \chi(4) \leqslant \chi(2) \\ \chi(1) \leqslant \chi(3) \leqslant \chi(2) < \chi(4) & \chi(3) \leqslant \chi(2) < \chi(1) \leqslant \chi(4) \\ \chi(1) \leqslant \chi(3) < \chi(4) \leqslant \chi(2) & \chi(3) \leqslant \chi(2) < \chi(4) < \chi(1) \\ \chi(1) < \chi(4) < \chi(2) \leqslant \chi(3) & \chi(3) < \chi(4) < \chi(1) \leqslant \chi(2) \\ \chi(1) < \chi(4) \leqslant \chi(3) < \chi(2) & \chi(3) \leqslant \chi(4) \leqslant \chi(2) < \chi(1) \\ \chi(2) \leqslant \chi(1) < \chi(3) \leqslant \chi(4) & \chi(4) \leqslant \chi(1) < \chi(2) \leqslant \chi(3) \\ \chi(2) \leqslant \chi(1) < \chi(4) < \chi(3) & \chi(4) \leqslant \chi(1) \leqslant \chi(3) < \chi(2) \\ \chi(2) < \chi(3) \leqslant \chi(1) \leqslant \chi(4) & \chi(4) < \chi(2) \leqslant \chi(1) < \chi(3) \\ \chi(2) < \chi(3) \leqslant \chi(4) < \chi(1) & \chi(4) < \chi(2) \leqslant \chi(3) \leqslant \chi(1) \\ \chi(2) \leqslant \chi(4) \leqslant \chi(1) < \chi(3) & \chi(4) \leqslant \chi(3) < \chi(1) \leqslant \chi(2) \\ \chi(2) \leqslant \chi(4) < \chi(3) \leqslant \chi(1) & \chi(4) < \chi(3) < \chi(2) < \chi(1). \end{array}$$

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References

[1] José Aliste-Prieto and José Zamora. Proper caterpillars are distinguished by their chromatic symmetric function. *Discrete Math.*, 315:158–164, 2014.

[2] Felix Breuer and Caroline J. Klivans. Scheduling problems. *J. Comb. Theory Ser. A*, 139(C):59–79, April 2016.

[3] Vesselin Gasharov. Incomparability graphs of $(3+1)$-free posets are s-positive. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 193–197, 1996.

[4] David D. Gebhard and Bruce E. Sagan. A chromatic symmetric function in noncommuting variables. *J. Algebraic Combin.*, 13(3):227–255, 2001.

[5] Ira M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[6] Ira M. Gessel and Louis H. Kalikow. Hypergraphs and a functional equation of Bouwkamp and de Bruijn. *J. Combin. Theory Ser. A*, 110(2):275–289, 2005.

[7] Ira M. Gessel and Christophe Reutenauer. Counting permutations with given cycle structure and descent set. *J. Combin. Theory Ser. A*, 64(2):189–215, 1993.

[8] Jeremy L. Martin, Matthew Morin, and Jennifer D. Wagner. On distinguishing trees by their chromatic symmetric functions. *J. Combin. Theory Ser. A*, 115(2):237 – 253, 2008.

[9] Brendan Pawlowski. Personal communication.

[10] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration spaces*, volume 14 of *CRM Series*, pages 433–460. Ed. Norm., Pisa, 2012.

[11] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions. *Adv. Math.*, 295:497 – 551, 2016.

[12] Richard P. Stanley. *Ordered structures and partitions.* American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 119.

[13] Richard P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995.

[14] Richard P. Stanley. *Enumerative combinatorics. Volume 1.* Cambridge University Press, 2nd edition, 1997.

[15] Richard P. Stanley. Graph colorings and related symmetric functions: ideas and applications: a description of results, interesting applications, & notable open problems. *Discrete Math.*, 193(1-3):267–286, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).

[16] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[17] W. A. Stein et al. *Sage Mathematics Software (Version 6.1.1)*. The Sage Development Team, 2014. http://www.sagemath.org.

[18] Jair Taylor. Formal group laws and chromatic symmetric functions of hypergraphs. In *Proceedings of the 27th International Conference on Formal Power Series and Algebraic Combinatorics*, 2015. http://fpsac.combinatorics.kr/program.

[19] Jair Taylor. *Formal group laws and hypergraph colorings*. PhD thesis, University of Washington, 2016. http://hdl.handle.net/1773/36757.