On the Minimum Number of Monochromatic Generalized Schur Triples

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Abstract

The solution to the problem of finding the minimum number of monochromatic triples $(x, y, x + ay)$ with $a \geqslant 2$ being a fixed positive integer over any 2-coloring of $[1, n]$ was conjectured by Butler, Costello, and Graham (2010) and Thanathipanonda (2009). We solve this problem using a method based on Datskovsky's proof (2003) on the minimum number of monochromatic Schur triples $(x, y, x + y)$. We do this by exploiting the combinatorial nature of the original proof and adapting it to the general problem.

Keywords: Schur Triples, Ramsey Theory on Integers, Rado Equation, Optimization

1 Introduction

Ramsey theory has a rich history first popularized in 1935 by Erdős and Szekeres in their seminal paper [3]. We investigate a part of the theory that was orginally developed by Issai Schur. The formulation of Schur's theorem was first derived from Van der Waerden's theorem in 1927 [10]. Van der Waerden proved that any r-coloring of $\mathbb{Z}^+$ must admit a monochromatic 3-term arithmetic progression $\{a, a + d, a + 2d\}$ for some $a, d > 1$. A particular choice of x, y and z in terms of a and d admits a monochromatic solution to $x + y = 2z$, on a plane whose coordinates are the positive integers. Hence, a similar question regarding the coloring of monochromatic solutions to a simpler equation can be posed; namely, does there exist a least positive integer $s = s(r)$ such that for any r-coloring of $[1, s]$ there is a monochromatic solution to $x + y = z$? Schur determined that the answer is yes, and we call the solution (x, y, z) to such an equation a Schur triple.

In 1959, Goodman [4] was able to determine the minimum number of monochromatic triangles under a 2-edge coloring of a complete graph on n vertices, which turned out to be the same order as the average, $\frac{n^3}{24}+\mathcal{O}(n^2)$. This motivated Graham [5] to pose the problem of finding the minimum number of monochromatic Schur triples over any 2-coloring of $[1, n]$ at a conference and to offer 100 USD for the result. Graham initially conjectured that the average value should be the minimum at $\frac{n^2}{16} + \mathcal{O}(n)$. However, Zeilberger and his student Robertson [7] used discrete calculus to show that the minimum number must be $\frac{n^2}{22} + \mathcal{O}(n)$ and won the cash prize. Around the same time, Schoen [8], followed four years later by Datskovsky [2], furnished different proofs using Fourier analysis to show that indeed, $\frac{n^2}{22} + \mathcal{O}(n)$ is the correct minimum. Ultimately, their idea had reduced to one in combinatorics. More recently in 2009, Thanatiponanda [9] confirmed the result using a new technique with computer algebra and a greedy algorithm. He conjectured a minimum number of monochromatic Schur triples for all r-colorings and a minimum number of monochromatic triples satisfying $x + ay = z$ for a fixed integer $a \geqslant 2$ over any 2-coloring of $[1, n]$. We solve the latter part of the conjecture in this paper using a purely combinatorial approach.

2 The Minimum Number of Monchromatic Schur Triples

We first show how to find the minimum coloring of $x + y = z$ in an elementary way using the method by Datkovsky [2]. Then, we explore the more general case in the next two sections. We start by employing a 2-coloring on all integers in $[1, n]$ for $n < \infty$ and count the number of monochromatic Schur triples (x, y, z) where $z = x + y$. Denote the colors to be red (R) and blue (B).

The number of Schur triples includes the number of monochromatic Schur triples $|\mathcal{M}(n)|$ and non-monochromatic Schur triples $|\mathcal{N}(n)|$.

Lemma 1.

$$|\textit{Schur Triples}| = |\mathcal{M}(n)| + |\mathcal{N}(n)|.$$

Lemma 2. *The number of Schur triples in* $[1, n]$ *is* $\frac{1}{2}\binom{n}{2}$.

Proof. Observe that a Schur triple can be defined by simply choosing numbers for x and z which gives two triples $(x, z - x, z)$ and $(z - x, x, z)$. □

Next we show $|\mathcal{N}(n)|$ can be written in the form of μ_B, μ_R and $|N^+|$ all of which are defined as follows:

Definition 3. μ_B denotes the number of blue colorings of coordinates on $[1, n]$. Similarly, μ_R denotes the number of red colorings of coordinates on $[1, n]$.

Note that $\mu_B + \mu_R = n$.

Definition 4. The set of non-monochromatic pairs in $[1, n] \times [1, n]$ will be denoted as $N(n)$. In particular, we denote two subsets as follows:

$$N^- = \{(x,y)|\ x+y \leqslant n, x > y\}$$
$$N^+ = \{(x,y)|\ x+y > n, x < y\}$$

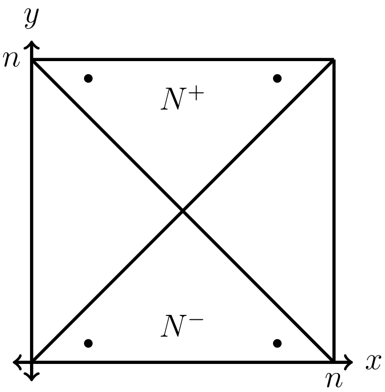


Figure 1: The sets N^- and N^+

Proposition 5. $|\mathcal{N}(n)| = \frac{1}{2}(2\mu_R\mu_B - |N^+|) + \mathcal{O}(n)$.

Proof.

$$\begin{aligned} |\mathcal{N}(n)| &= \frac{1}{2}|\text{non-monochromatic pairs}| \\ &= \frac{1}{2}(|N^+| + |N^-| + |N^-|) + \mathcal{O}(n) \\ &= \frac{1}{2}(2\mu_R\mu_B - |N^+|) + \mathcal{O}(n). \end{aligned}$$

Each non-monochromatic triple gives two non-monochromatic pairs which gives the first equality. To get the second equality, we observe that the pairs in N^- will contribute to two triples but the pairs in N^+ will only contribute to one. For example, in $[1, 10]$, $(5, 3)$ gives the triples $(3, 2, 5)$ and $(5, 3, 8)$. But, in $[1, 10]$, $(8, 9)$ only gives $(1, 8, 9)$. Finally, the last equality comes from the fact that $|N^+| + |N^-| = \mu_R\mu_B + \mathcal{O}(n)$. ☐

By putting together Lemmas 1 and 2 and Proposition 5, we obtain the next lemma.

Lemma 6. *The number of monochromatic Schur triples under a 2-coloring on* $[1, n]$ *is*

$$|\mathcal{M}(n)| = \frac{n^2}{4} - \frac{1}{2}(2\mu_R\mu_B - |N^+|) + \mathcal{O}(n).$$

In order to find the minimum value of $|\mathcal{M}(n)|$, we must obtain the lower bound of $|N^+|$ in terms of μ_B and μ_R. To do this more efficiently, we denote $D := |N^-| - |N^+|$ and find an upper bound on D instead. The proof requires the following notation:

Definition 7. Let S be the set of pairs of the form $\{s, n+1-s\}$ where $1 \leqslant s \leqslant n/2$. Denote by $\mu_{CC'}$ the number of sets S with colorings C and C' in this order. γn is the number of non-monochromatic pairs in S.

Using our new notation, $\gamma n = \mu_{RB} + \mu_{BR}$.

Lemma 8. *Assume, without loss of generality, that $\mu_B \geqslant \mu_R$. Then*

$$D \leqslant \frac{\mu_B^2}{4}.$$

Proof. Assuming $1 \leqslant y < x \leqslant \frac{n}{2}$, denote the sets X and Y as

$$X = \{x, n+1-x\}$$
$$Y = \{y, n+1-y\}.$$

Ordered pairs in $X \times Y$ when colored *differently* are contained in $N^+ \cup N^-$, that is:

$$(x, y), (n+1-x, y) \in N^-$$
$$(x, n+1-y), (n+1-x, n+1-y) \in N^+.$$

We outline all possible colorings of the X and Y sets that contribute to the value of D in the table in Figure 2. We see that with the exception of the first four cases, the contribution to D is 0.

We now obtain an upper bound of D:

$$\begin{aligned} D &= 2\mu_{RR}\mu_{BR} + 2\mu_{BB}\mu_{RB} - 2\mu_{RR}\mu_{RB} - 2\mu_{BB}\mu_{BR} \\ &\leqslant 2\mu_{RR}\mu_{BR} + 2\mu_{BB}\mu_{RB} & (1) \\ &\leqslant 2\mu_{BB}(\mu_{BR} + \mu_{RB}) & (2) \\ &= (\mu_B - \gamma n)\,\gamma n. & (3) \end{aligned}$$

The inequality (2) comes from our assumption that $\mu_{BB} \geqslant \mu_{RR}$. The last equality comes from $\mu_{BB} = \dfrac{\mu_B - \gamma n}{2}$.

Calculus shows that the maximum of (3) occurs when $\gamma = \frac{\mu_B}{2n}$ which simplifies our inequality to

$$D \leqslant \left(\mu_B - \frac{\mu_B}{2}\right)\frac{\mu_B}{2} = \frac{\mu_B^2}{4}.$$ □

Case	x	$n+1-x$	y	$n+1-y$	Non-Mono Pairs	D
Monochromatic X set and Non-monochromatic Y set						
1	red	red	blue	red	(x, y) $(n+1-x, y)$	+2
2	blue	blue	red	blue	(x, y) $(n+1-x, y)$	+2
3	red	red	red	blue	$(x, n+1-y)$ $(n+1-x, n+1-y)$	–2
4	blue	blue	blue	red	$(x, n+1-y)$ $(n+1-x, n+1-y)$	–2
Monochromatic X and Y sets						
5	red	red	red	red	none	0
6	blue	blue	blue	blue	none	0
7	red	red	blue	blue	(x, y) $(n+1-x, y)$ $(x, n+1-y)$ $(n+1-x, n+1-y)$	0
8	blue	blue	red	red	(x, y) $(n+1-x, y)$ $(x, n+1-y)$ $(n+1-x, n+1-y)$	0
Non-monochromatic X set and Monochromatic Y set						
9	red	blue	red	red	$(n+1-x, y)$ $(n+1-x, n+1-y)$	0
10	blue	red	blue	blue	$(n+1-x, y)$ $(n+1-x, n+1-y)$	0
11	red	blue	blue	blue	(x, y) $(x, n+1-y)$	0
12	blue	red	red	red	(x, y) $(x, n+1-y)$	0
Non-monochromatic X and Y sets						
13	red	blue	blue	red	(x, y) $(n+1-x, n+1-y)$	0
14	blue	red	red	blue	(x, y) $(n+1-x, n+1-y)$	0
15	red	blue	red	blue	$(x, n+1-y)$ $(n+1-x, y)$	0
16	blue	red	blue	red	$(x, n+1-y)$ $(n+1-x, y)$	0

Figure 2: Colorings of the elements in sets X and Y

Theorem 9. *Over all 2-colorings of* $[1, n]$*, the minimum number of monochromatic Schur triples is* $\frac{n^2}{22} + \mathcal{O}(n)$*.*

Proof. An upper bound of the minimum can be obtained from a coloring on $[1, n]$. We color $\left[R^{4n/11}, B^{6n/11}, R^{n/11}\right]$ as illustrated in Figure 3. This proportion comes from a brute force computer search first proposed by Zeilberger [7].

Figure 3: The Optimal Coloring for $x + y = z$

This coloring will give us $\frac{n^2}{22} + \mathcal{O}(n)$ monochromatic triples.

Next, we look for the lower bound of the minimum. By using Lemma 8 and the fact that $|N^-| + |N^+| = \mu_R\mu_B + \mathcal{O}(n)$, we get:

$$|N^+| \geqslant \frac{1}{2}\mu_R\mu_B - \frac{\mu_B^2}{8} + \mathcal{O}(n),$$

which, together with Lemma 6, gives

$$|\mathcal{M}(n)| \geqslant \frac{n^2}{4} - \frac{3}{4}\mu_R\mu_B - \frac{\mu_B^2}{16} + \mathcal{O}(n). \tag{4}$$

The right hand side of (4) achieves a maximum when $\mu_R = \frac{5n}{11}$ and $\mu_B = \frac{6n}{11}$. As a result, we get the lower bound for the minimum to be:

$$|\mathcal{M}(n)| \geqslant \frac{n^2}{22} + \mathcal{O}(n).$$

Because the lower and upper bounds match, we have therefore shown the desired result. □

Remark. We can be confident that the bounds for equations (1) and (2) is sharp relative to the optimal coloring because we know that cases 3 and 4 from Figure 2 will not occur and that $\mu_{BR} = 0$.

This method of Datkovsky's also gives the optimal coloring for fixed ratios of μ_B and μ_R.

Corollary 10. *For any fixed* $\mu_B \geqslant \mu_R$*, the coloring on* $[1, n]$ *that gives the minimum number of monochromatic Schur triples is* $\left[R^{\frac{n}{2}-\frac{\mu_B}{4}}, B^{\mu_B}, R^{\frac{n}{2}-\frac{3\mu_B}{4}}\right]$ *for* $\mu_B \leqslant \frac{2n}{3}$ *and* $[R^{\mu_R}, B^{\mu_B}]$ *for* $\mu_B > \frac{2n}{3}$*.*

Proof. We follow the proof of Lemma 8 and use

$$D \leqslant (\mu_B - \gamma n)\gamma n.$$

For the case $\mu_B \leqslant \frac{2n}{3}$, the maximum of D occurs when $\gamma = \frac{\mu_B}{2n}$. So,

$$|\mathcal{M}(n)| \geqslant \frac{n^2}{4} - \frac{3}{4}\mu_R\mu_B - \frac{\mu_B^2}{16} + \mathcal{O}(n).$$

For the case $\mu_B > \frac{2n}{3}$, the maximum of D occurs when $\gamma = \frac{\mu_R}{n}$. With similar calculations,

$$|\mathcal{M}(n)| \geqslant \frac{n^2}{4} - \mu_R\mu_B + \frac{\mu_R^2}{4} + \mathcal{O}(n).$$

The colorings mentioned in the statement of the corollary give us upper bounds for the minimum, which happens to match the lower bounds. □

Corollary 11. *For any fixed $\mu_B \geqslant \mu_R$, the coloring on $[1, n]$ that gives the maximum number of monochromatic Schur triples is $\left[R^{\frac{n}{2}-\frac{3\mu_B}{4}}, B^{\mu_B}, R^{\frac{n}{2}-\frac{\mu_B}{4}}\right]$ for $\mu_B \leqslant \frac{2n}{3}$ and $[B^{\mu_B}, R^{\mu_R}]$ for $\mu_B > \frac{2n}{3}$.*

Proof. Using a similar calculation to Lemma 8, we have that

$$D \geqslant -(\mu_B - \gamma n)\gamma n.$$

For the case $\mu_B \leqslant \frac{2n}{3}$, the minimum of D occurs when $\gamma = \frac{\mu_B}{2n}$. So,

$$|\mathcal{M}(n)| \leqslant \frac{n^2}{4} - \frac{3}{4}\mu_R\mu_B + \frac{\mu_B^2}{16} + \mathcal{O}(n).$$

For the case $\mu_B > \frac{2n}{3}$, the minimum of D occurs when $\gamma = \frac{\mu_R}{n}$. Therefore,

$$|\mathcal{M}(n)| \leqslant \frac{n^2}{4} - \frac{\mu_R\mu_B}{2} - \frac{\mu_R^2}{4} + \mathcal{O}(n).$$

The colorings mentioned in the statement of the corollary give us lower bounds of the maximum, which happens to match the upper bounds. □

3 The Minimum Number of Monochromatic Triples $(x, y, x + 2y)$

The technique illustrated in the previous section can be extended to $x + ay = z$ for any fixed integer $a \geqslant 2$. However, the nice symmetry we had previously with the equation $x + y = z$ is no longer there. In this section we deal with the specific case $a = 2$. The general case will be outlined in Section 4.

We parallel the same argument as in Section 2. First, we write the number of non-monochromatic triples $|\mathcal{N}(n)|$ in terms of variables we can optimize.

Definition 12. Denote by μ_{B_1} and μ_{R_1} the number of blue and red colorings respectively on $\left[1, \frac{n}{2}\right]$. Furthermore, denote by μ_{B_2} and μ_{R_2} the number of blue and red colorings respectively on $\left(\frac{n}{2}, n\right]$.

Note that $\mu_{B_1} + \mu_{R_1} = \frac{n}{2}$ and $\mu_{R_1} + \mu_{R_2} = \mu_R$.

Definition 13. The sets of non-monochromatic pairs in $[1, n] \times [1, \frac{n}{2}]$ are defined as follows:

$$\begin{aligned} N_x^- &= \{(x, y) |\, x + 2y \leqslant n, x > 2y\} \\ N_x^+ &= \{(x, y) |\, x + 2y > n, x > 2y\} \\ N_y^- &= \{(x, y) |\, x + 2y \leqslant n, x < 2y\} \\ N_y^+ &= \{(x, y) |\, x + 2y > n, x < 2y\} \end{aligned}$$

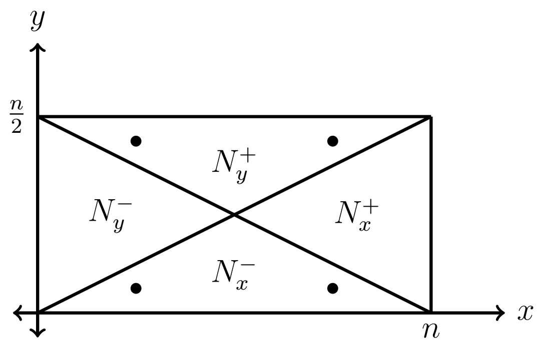


Figure 4: The sets N_x^-, N_x^+ N_y^- and N_y^+.

Proposition 14. $|\mathcal{N}(n)| \leqslant \frac{1}{2}\left(\frac{\mu_R\mu_B}{2} + \mu_R\mu_{B_1} + \mu_B\mu_{R_1} + |N_x^-| - |N_y^+|\right) + \mathcal{O}(n)$.

Proof. We count sets of ordered pairs that come from non-monochromatic triples of type (x, y, z) where $z = x + 2y$ as follows:

$$\begin{aligned} \nu_1 &= |\{\text{non-monochromatic } (x, y) \text{ pairs}\}| = |\,\{(x, y) |\, x + 2y \leqslant n\}\,| \\ \nu_2 &= |\{\text{non-monochromatic } (y, z) \text{ pairs}\}| = |\,\{(x, y) |\, y > 2x\}\,| = |\,\{(x, y) |\, x > 2y\}\,| \\ \nu_3 &= |\{\text{non-monochromatic } (x, z) \text{ pairs}\}| = |\,\{(x, y) |\, 2 \text{ divides } (y - x), x < y\}\,|. \end{aligned}$$

We then have that:

$$\begin{aligned} |\mathcal{N}(n)| &= \frac{1}{2}(\nu_1 + \nu_2 + \nu_3) \\ &\leqslant \frac{1}{2}\left(\frac{\mu_R\mu_B}{2} + 2|N_x^-| + |N_x^+| + |N_y^-|\right) + \mathcal{O}(n) \\ &= \frac{1}{2}\left(\frac{\mu_R\mu_B}{2} + \mu_R\mu_{B_1} + \mu_B\mu_{R_1} + |N_x^-| - |N_y^+|\right) + \mathcal{O}(n). \end{aligned}$$

To obtain the second inequality, observe that $\nu_1 = |N_x^-| + |N_y^-| + \mathcal{O}(n)$, $\nu_2 = |N_x^-| + |N_x^+| + \mathcal{O}(n)$ and $\nu_3 \leqslant \frac{\mu_R\mu_B}{2}$ (refer to Lemma 3 of Thanatipanonda [9]). The last equality comes from $|N_x^-| + |N_x^+| + |N_y^-| + |N_y^+| = \mu_R\mu_{B_1} + \mu_B\mu_{R_1}$ which can be seen from Figure 4. □

The next lemma follows immediately from Proposition 14.

Lemma 15. *The number of monochromatic triples of type* $(x, y, x+2y)$ *under a 2-coloring on* $[1, n]$ *is*

$$|\mathcal{M}(n)| \geqslant \frac{n^2}{4} - \frac{1}{2}\left(\frac{\mu_R \mu_B}{2} + \mu_R \mu_{B_1} + \mu_B \mu_{R_1} + |N_x^-| - |N_y^+|\right) + \mathcal{O}(n).$$

Minimizing $|\mathcal{M}(n)|$ can be reduced to finding the upper bound of $D_2 := |N_x^-| - |N_y^+|$ in terms of $\mu_R, \mu_B, \mu_{R_1}, \mu_{B_1}$. D_2 is much more difficult to count than in the Schur triple case.

Definition 16. Let S be the set of pairs of the form $\{s, \frac{n}{2} + 1 - s\}$ where $1 \leqslant s \leqslant n/4$. Denote by $\mu^{(1)}_{CC'}$ the number of sets S with colorings C and C' in this order. The superscript (1) refers to the coloring of pairs on $\left[1, \frac{n}{2}\right]$. Also, denote by $\gamma_1 n$ the number of non-monochromatic pairs in S.

With this notation, $\gamma_1 n = \mu^{(1)}_{RB} + \mu^{(1)}_{BR}$.

Definition 17. Define sets X and Y_1 as follows,

$$X = \{x, n + 1 - x\}, \quad 1 \leqslant x \leqslant \frac{n}{2}$$
$$Y_1 = \left\{y, \frac{n}{2} + 1 - y\right\}, \quad 1 \leqslant y \leqslant \frac{n}{4}.$$

The direct product $\mu_{CC'} \otimes \mu^{(1)}_{EE'}$ is defined by the number of pairs (X, Y_1) where X has the coloring $\{C, C'\}$ and Y_1 has the coloring $\{E, E'\}$ under the condition $2y < x$.

Lemma 18.

$$D_2 = 2\mu_{RR} \otimes \mu^{(1)}_{BR} + 2\mu_{BB} \otimes \mu^{(1)}_{RB} - 2\mu_{RR} \otimes \mu^{(1)}_{RB} - 2\mu_{BB} \otimes \mu^{(1)}_{BR}.$$

Proof. Assuming $1 \leqslant 2y < x \leqslant \frac{n}{2}$, we observe that the ordered pairs in $X \times Y_1$ when colored *differently* are contained in $N_x^- \cup N_y^+$, that is:

$$(x, y), (n + 1 - x, y) \in N_x^-$$
$$\left(x, \frac{n}{2} + 1 - y\right), \left(n + 1 - x, \frac{n}{2} + 1 - y\right) \in N_y^+.$$

The table in Figure 5 shows that there are only four cases that contribute any value to D_2, while the other cases contribute 0, similar to the table in Figure 2.

The result follows immediately. □

The next proposition gives the upper bound for D_2.

Monochromatic X set and Non-monochromatic Y_1 set						
Case	x	$n+1-x$	y	$\frac{n}{2}+1-y$	Non-Mono Pairs	D_2
1	red	red	blue	red	(x,y) $(n+1-x,y)$	+2
2	blue	blue	red	blue	(x,y) $(n+1-x,y)$	+2
3	red	red	red	blue	$\left(x,\frac{n}{2}+1-y\right)$ $\left(n+1-x,\frac{n}{2}+1-y\right)$	–2
4	blue	blue	blue	red	$\left(x,\frac{n}{2}+1-y\right)$ $\left(n+1-x,\frac{n}{2}+1-y\right)$	–2

Figure 5: Colorings of the elements in sets X and Y_1 where $D_2 \neq 0$

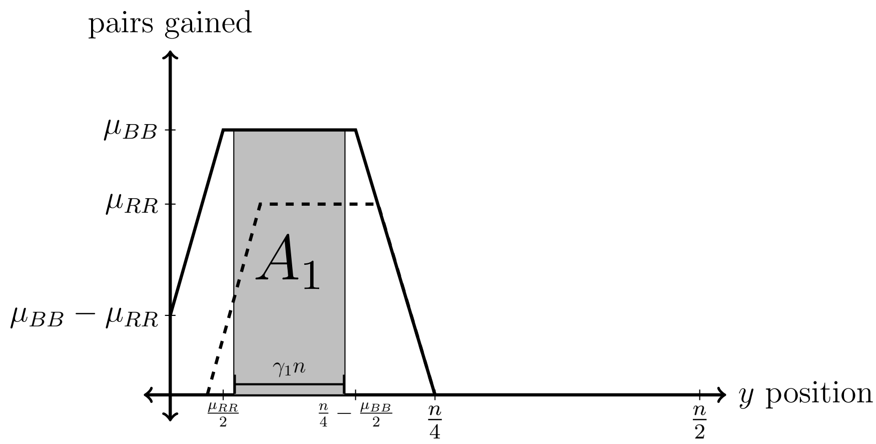


Figure 6: The upper bound of D_2

Proposition 19. *Assume, without loss of generality, that $\mu_B \geqslant \mu_R$ and suppose the number of non-monochromatic pairs in S, $\gamma_1 n$, is fixed. Then*

$$D_2 \leqslant 2A_1,$$

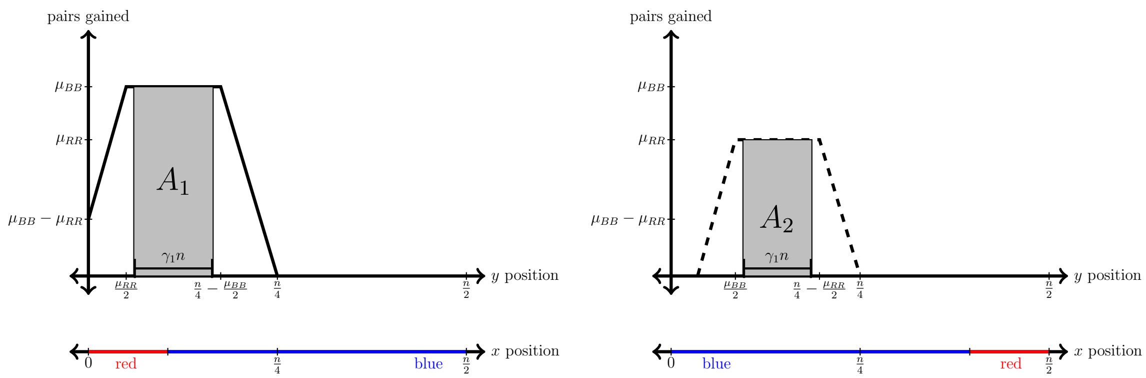
where A_1 is the largest possible area under the curve in Figure 6, with a base of length $\gamma_1 n$ for $\gamma_1 \leqslant \frac{1}{4}$.

Proof. From Lemma 18,

$$\begin{aligned} D_2 &= 2\mu_{RR} \otimes \mu_{BR}^{(1)} + 2\mu_{BB} \otimes \mu_{RB}^{(1)} - 2\mu_{RR} \otimes \mu_{RB}^{(1)} - 2\mu_{BB} \otimes \mu_{BR}^{(1)} \\ &= 2(\mu_{BB} \otimes \mu_{RB}^{(1)} - \mu_{RR} \otimes \mu_{RB}^{(1)}) + 2(\mu_{RR} \otimes \mu_{BR}^{(1)} - \mu_{BB} \otimes \mu_{BR}^{(1)}). \end{aligned}$$

We configure X to gain the maximum of $\mu_{BB} \otimes \mu_{RB}^{(1)} - \mu_{RR} \otimes \mu_{RB}^{(1)}$ by coloring the far left of the interval $\left[1, \frac{n}{2}\right]$ red and the remainder of the interval blue, which is justified by the condition $2y < x$. With this set up for X, we can count the number of pairs gained for every y in Y_1 as shown in Figure 7a.

Similarly, the configuration of X to gain the maximum of $\mu_{RR} \otimes \mu_{BR}^{(1)} - \mu_{BB} \otimes \mu_{BR}^{(1)}$ is when we color the far left of the interval $\left[1, \frac{n}{2}\right]$ blue and the remainder of the interval red. With this set up for X, we can count the number of pairs gained for every y in Y_1 as shown in Figure 7b.



(a) (b)

Figure 7: Counting D_2 with different colorings of X.

Finally, since $\mu_{RB}^{(1)} + \mu_{BR}^{(1)} = \gamma_1 n$, and the number of pairs gained shown in Figure 7b is clearly dominated by the number of pairs gained in Figure 7a, the result follows. ☐

We now find the upper bound for $|\mathcal{N}(n)|$.

Proposition 20. *Over all 2-colorings of* $[1, n]$*, the maximum number of non-monochromatic triples satisfying* $x + 2y = z$ *is*

$$|\mathcal{N}(n)| \leqslant \frac{5n^2}{22} + \mathcal{O}(n).$$

Proof. Assume, without loss of generality, that $\mu_B \geqslant \mu_R$. The proof ultimately depends on determining the area under the curve of Figure 6, which we break down into three cases according to the value of γ_1. The three cases are illustrated in Figure 8.
We complete the proof of this proposition as follows. For each case, we write A_1 in terms of the variables μ_R, μ_B, γ and γ_1. Then, we optimize γ, γ_1 with respect to μ_{B_1}, μ_{R_1} and μ_{R_2}. Finally, we use the optimal γ and γ_1 values to maximize

$$\Delta := \frac{1}{2}\left(\frac{\mu_R \mu_B}{2} + \mu_R \mu_{B_1} + \mu_B \mu_{R_1} + 2A_1\right).$$

Denote this maximum to be Δ_{max}. Propositions 14 and 19 show that Δ_{max} will be the upper bound for $|\mathcal{N}(n)|$. The optimization of Δ has been done using Maple and for curious readers, the code can be found at Thanatipanonda's website. We note that Δ can

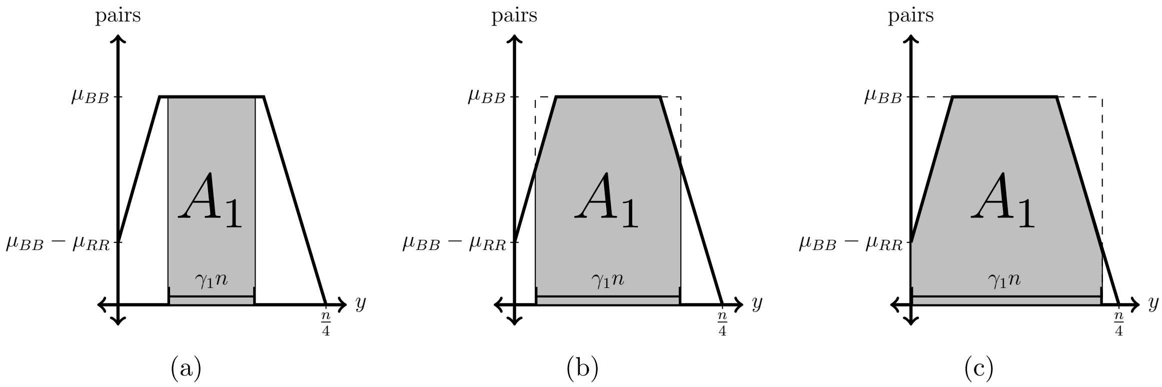


Figure 8: The Three Cases of Proposition 20

ultimately be written as a function of only two variables μ_R and μ_{R_1}. In our calculations, we use the following lower bound of γn and upper bound of $\gamma_1 n$:

$$\gamma_1 n \leqslant \min(\mu_{R_1}, \mu_{B_1})$$
$$\gamma n \geqslant |\mu_{R_1} - \mu_{R_2}|$$

Maple's current technology does not allow us to optimize with absolute value and minimum functions. Thus, we separate each case into the following pieces. The subcases are summarized in the table in Figure 9. It is important to note that subcase D can be ignored in our calculation because it only produces one pair, namely $\mu_{R_1} = \frac{n}{4}$ and $\mu_R = \frac{n}{2}$ (recall that $\mu_R \leqslant \frac{n}{2}$).

Subcase	Conditions on μ_{R_1}, μ_{B_1} and μ_{R_2}
A	$\mu_{B_1} \leqslant \mu_{R_1}$ and $\mu_{R_1} \geqslant \mu_{R_2}$
B	$\mu_{B_1} \geqslant \mu_{R_1}$ and $\mu_{R_1} \geqslant \mu_{R_2}$
C	$\mu_{B_1} \geqslant \mu_{R_1}$ and $\mu_{R_1} \leqslant \mu_{R_2}$
D	$\mu_{B_1} \leqslant \mu_{R_1}$ and $\mu_{R_1} \leqslant \mu_{R_2}$

Figure 9: The Four Subcases

Case 1: $\gamma_1 n < \dfrac{\gamma n}{2}$. This case is illustrated in Figure 8a.

$$\begin{aligned} A_1 &= \mu_{BB} \cdot \gamma_1 n \\ &= \frac{\mu_B - \gamma n}{2} \cdot \gamma_1 n. \end{aligned}$$

In order to maximize A_1, we maximize $\gamma_1 n$ and minimize γn. To be able to determine the values of γ_1 and γ, we consider two further subcases as follows:

Case 1.1: $\min(\mu_{R_1}, \mu_{B_1}) < \frac{|\mu_{R_1} - \mu_{R_2}|}{2}$

The optimal values of $(\gamma_1 n, \gamma n)$ is $(\min(\mu_{R_1}, \mu_{B_1}), |\mu_{R_1} - \mu_{R_2}|)$ as shown in Figure 10a.

Case 1.2: $\min(\mu_{R_1}, \mu_{B_1}) \geqslant \frac{|\mu_{R_1} - \mu_{R_2}|}{2}$

The optimal values of $(\gamma_1 n, \gamma n)$ lies on the line $\gamma n = 2\gamma_1 n$ as shown in Figure 10b. This gives

$$A_1 = \frac{(\mu_B - \gamma n) \cdot \gamma n}{4}.$$

Thus, A_1 attains its maximum value at $\gamma n = \frac{\mu_B}{2}$.

The calculations for Δ_{max} for all subcases are summarized in the table in Figure 11 and the admissible regions for subcases A, B, and C are shown in Figure 14a. In this case, Δ_{max} occurs under subcase A.

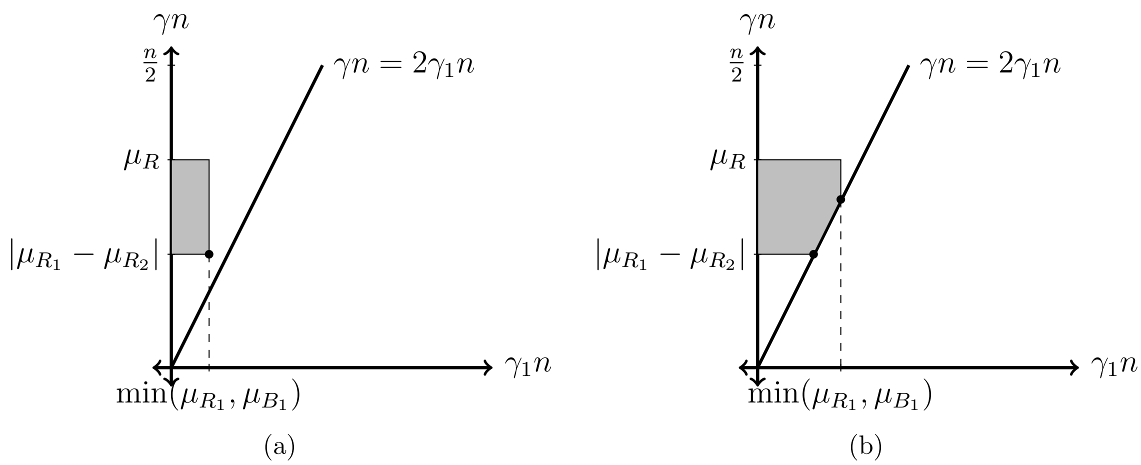


Figure 10: Finding the optimal values of $\gamma_1 n$ and γn in Case 1

Case	Subcase	Optimal $\gamma_1 n$	Optimal γn	Δ_{max}	(μ_{R_1}, μ_R)
1.1	A	μ_{B_1}	$\mu_{R_1} - \mu_{R_2}$	$\frac{2n^2}{9}$	$\left(\frac{n}{3}, \frac{n}{3}\right)$
	B	μ_{R_1}	$\mu_{R_1} - \mu_{R_2}$	0	(0,0)
	C	μ_{R_1}	$\mu_{R_2} - \mu_{R_1}$	$\frac{13n^2}{64}$	$\left(\frac{n}{8}, \frac{n}{2}\right)$
1.2	A	$\frac{\mu_B}{4}$	$\frac{\mu_B}{2}$	$\frac{2n^2}{9}$	$\left(\frac{n}{3}, \frac{n}{3}\right)$
	B	$\frac{\mu_B}{4}$	$\frac{\mu_B}{2}$	$\frac{5n^2}{24}$	$\left(\frac{n}{4}, \frac{n}{3}\right)$
	C	$\frac{\mu_B}{4}$	$\frac{\mu_B}{2}$	$\frac{9n^2}{44}$	$\left(\frac{5n}{22}, \frac{5n}{11}\right)$

Figure 11: Results for Case 1

Case 2: $\frac{\gamma n}{2} \leqslant \gamma_1 n < \frac{\mu_R}{2}$. This case is illustrated in Figure 8b.

In this case, we take the area of the rectangle but subtract those pairs that are outside of the region. Thus, we have that:

$$\begin{aligned} A_1 &= \mu_{BB} \cdot \gamma_1 n - \frac{1}{2}\left(\gamma_1 n - \frac{\gamma n}{2}\right)^2 \\ &= \frac{(\mu_B - \gamma_1 n) \cdot \gamma_1 n}{2} - \frac{(\gamma n)^2}{8}. \end{aligned}$$

In order to maximize A_1, we want to make $\gamma_1 n$ as close to $\frac{\mu_B}{2}$ as possible and minimize γn. We break this case down into subcases depending on whether or not the upper bound of $\gamma_1 n$ is less than $\frac{\mu_R}{2}$.

Case 2.1 $\min(\mu_{R_1}, \mu_{B_1}) < \frac{\mu_R}{2}$

The optimal value of $(\gamma_1 n, \gamma n)$ is $(\min(\mu_{R_1}, \mu_{B_1}), |\mu_{R_1} - \mu_{R_2}|)$.

Case 2.2 $\min(\mu_{R_1}, \mu_{B_1}) \geqslant \frac{\mu_R}{2}$

The optimal value of $(\gamma_1 n, \gamma n)$ is $(\frac{\mu_R}{2}, |\mu_{R_1} - \mu_{R_2}|)$.

The calculations for Δ_{max} for all subcases are summarized in the table in Figure 12 and the admissible regions for subcases A, B, and C are shown in Figure 14b. In this case, Δ_{max} occurs under subcase A.

Case	Subcase	Optimal $\gamma_1 n$	Optimal γn	Δ_{max}	(μ_{R_1}, μ_R)
2.1	A	μ_{B_1}	$\mu_{R_1} - \mu_{R_2}$	$\frac{9n^2}{40}$	$\left(\frac{3n}{10}, \frac{2n}{5}\right)$
	B	μ_{R_1}	$\mu_{R_1} - \mu_{R_2}$	$\frac{2n^2}{9}$	$\left(\frac{2n}{9}, \frac{4n}{9}\right)$
	C	μ_{R_1}	$\mu_{R_2} - \mu_{R_1}$	$\frac{2n^2}{9}$	$\left(\frac{2n}{9}, \frac{4n}{9}\right)$
2.2	A	$\frac{\mu_R}{2}$	$\mu_{R_1} - \mu_{R_2}$	$\frac{9n^2}{40}$	$\left(\frac{3n}{10}, \frac{2n}{5}\right)$
	B	$\frac{\mu_R}{2}$	$\mu_{R_1} - \mu_{R_2}$	$\frac{43n^2}{192}$	$\left(\frac{n}{4}, \frac{5n}{12}\right)$
	C	$\frac{\mu_R}{2}$	$\mu_{R_2} - \mu_{R_1}$	$\frac{2n^2}{9}$	$\left(\frac{2n}{9}, \frac{4n}{9}\right)$

Figure 12: Results for Case 2

Case 3: $\frac{\mu_R}{2} < \gamma_1 n$. This case is illustrated in Figure 8c.

In this final case, A_1 indicated by nearly the entire region under the graph. We take the area of the rectangle but subtract those pairs that are outside of the region. Thus, we have that

$$\begin{aligned} A_1 &= \mu_{BB} \cdot \gamma_1 n - \left(\frac{\mu_{RR}}{2}\right)^2 - \left(\gamma_1 n - \frac{\mu_{RR}}{2} - \frac{\gamma n}{2}\right)^2 \\ &= \left(\frac{n}{2} - \gamma_1 n\right) \cdot \gamma_1 n - \frac{1}{4}\left((\gamma n + \mu_{RR})^2 + \mu_{RR}^2\right) \\ &= \left(\frac{n}{2} - \gamma_1 n\right) \cdot \gamma_1 n - \frac{(\gamma n)^2}{8} - \frac{\mu_R^2}{8}. \end{aligned}$$

In order to maximize A_1, we want to make $\gamma_1 n$ as to close to $\frac{n}{4}$ as possible and minimize γn. The calculations for Δ_{max} for all subcases are summarized in the table in Figure 13 and the admissible regions for subcases A, B, and C are shown in Figure 14c. Once again, Δ_{max} occurs under subcase A.

Case	Subcase	Optimal $\gamma_1 n$	Optimal γn	Δ_{max}	(μ_{R_1}, μ_R)
3	A	μ_{B_1}	$\mu_{R_1} - \mu_{R_2}$	$\frac{5n^2}{22}$	$\left(\frac{3n}{11}, \frac{4n}{11}\right)$
	B	μ_{R_1}	$\mu_{R_1} - \mu_{R_2}$	$\frac{29n^2}{128}$	$\left(\frac{n}{4}, \frac{3n}{8}\right)$
	C	μ_{R_1}	$\mu_{R_2} - \mu_{R_1}$	$\frac{2n^2}{9}$	$\left(\frac{2n}{9}, \frac{4n}{9}\right)$

Figure 13: Results for Case 3

In all three cases, we can see that Δ_{max} occurs in Case 3 when $\mu_{R_1} = \frac{3n}{11}$ and $\mu_R = \frac{4n}{11}$ with $\Delta_{max} = \frac{5n^2}{22} + \mathcal{O}(n)$. □

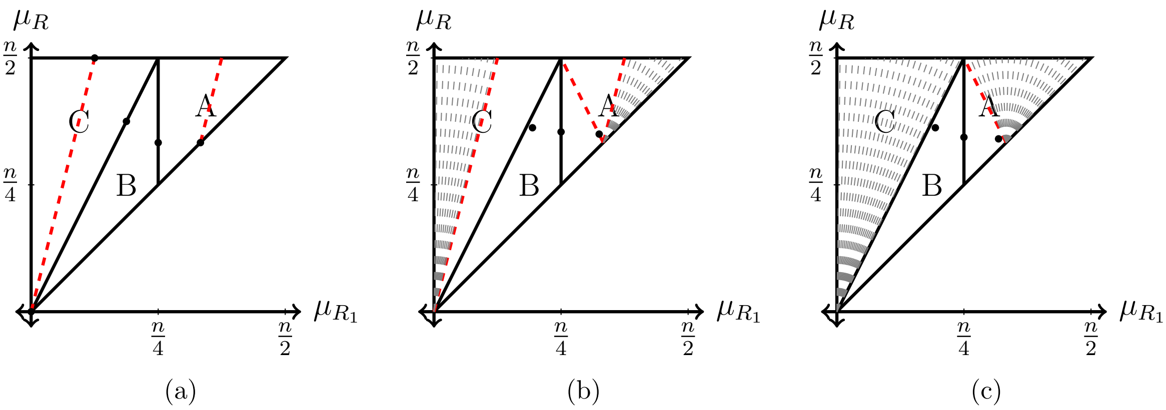


Figure 14: Admissible Regions of Cases in Proposition 20

And now, we are ready to present the main result of this paper.

Theorem 21. *Over all 2-colorings of* $[1, n]$*, the minimum number of monochromatic triples satisfying* $x + 2y = z$ *is* $\frac{n^2}{44} + \mathcal{O}(n)$*.*

Proof. An upper bound of the minimum can be obtained from a coloring on $[1, n]$. We color $\left[R^{3n/11}, B^{7n/11}, R^{n/11}\right]$ as illustrated in Figure 15. This solution was discovered in Butler, Costello, and Graham [1] and in Thanathipanonda [9].

This coloring gives us $\frac{n^2}{44} + \mathcal{O}(n)$ monochromatic triples.

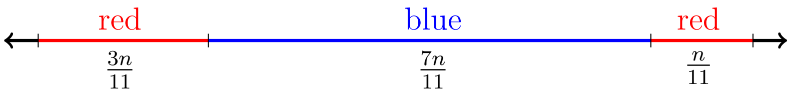


Figure 15: The Optimal Coloring for $x + 2y = z$

Next, we look for the lower bound of the minimum. From Lemma 15 and Proposition 20, we immediately get that

$$\begin{aligned}|\mathcal{M}(n)| &\geqslant \frac{n^2}{4} - \frac{1}{2}\left(\frac{\mu_R\mu_B}{2} + \mu_R\mu_{B_1} + \mu_B\mu_{R_1} + |N_x^-| - |N_y^+|\right) + \mathcal{O}(n) \\ &\geqslant \frac{n^2}{4} - \frac{5n^2}{22} + \mathcal{O}(n) \\ &= \frac{n^2}{44} + \mathcal{O}(n).\end{aligned}$$

Because the lower and upper bounds match, we have therefore shown the desired result. □

4 The General Case $x + ay = z$, $a \geqslant 2$

We now generalize our result.

Theorem 22. *Over all 2-colorings of $[1, n]$, the minimum number of monochromatic triples satisfying $x + ay = z$, $a \geqslant 2$ is $\frac{n^2}{2a(a^2+2a+3)} + \mathcal{O}(n)$.*

The set up of this proof is similar to the set up in Section 3. We will outline it here.

Definition 23. Denote by μ_{B_1} and μ_{R_1} the number of blue and red colorings respectively on $\left[1, \frac{n}{a}\right]$. Furthermore, denote by μ_{B_2} and μ_{R_2} the number of blue and red colorings respectively on $\left(\frac{n}{a}, n\right]$.

Definition 24. The sets of non-monochromatic pairs in $[1, n] \times [1, \frac{n}{a}]$ are defined as follows:

$$\begin{aligned}N_x^- &= \{(x, y) |\, x + ay \leqslant n, x > ay\} \\ N_x^+ &= \{(x, y) |\, x + ay > n, x > ay\} \\ N_y^- &= \{(x, y) |\, x + ay \leqslant n, x < ay\} \\ N_y^+ &= \{(x, y) |\, x + ay > n, x < ay\}\end{aligned}$$

It is now easy to adapt this notation to prove the following analog to Proposition 14.

Proposition 25. $|\mathcal{N}(n)| \leqslant \frac{1}{2}\left(\frac{\mu_R\mu_B}{a} + \mu_R\mu_{B_1} + \mu_B\mu_{R_1} + |N_x^-| - |N_y^+|\right) + \mathcal{O}(n)$.

Definition 26. Define by S the set of pairs of the form $\{s, \frac{n}{a} + 1 - s\}$ where $1 \leqslant s \leqslant \frac{n}{2a}$.

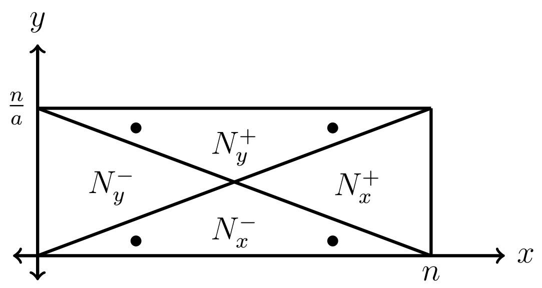


Figure 16: The sets N_x^-, N_x^+ N_y^- and N_y^+.

The values of $D_a := |N_x^-| - |N_y^+|$ can be bounded the same way as in the previous section.

Proposition 27. *Assume, without loss of generality, that $\mu_B \geqslant \mu_R$ and suppose the number of non-monochromatic pairs in S, $\gamma_1 n$, is fixed. Then*

$$D_a \leqslant 2A_1,$$

where A_1 is the largest possible area that can be placed under the curve in Figure 17, with a base of length $\gamma_1 n$ for $\gamma_1 \leqslant \frac{1}{2a}$.

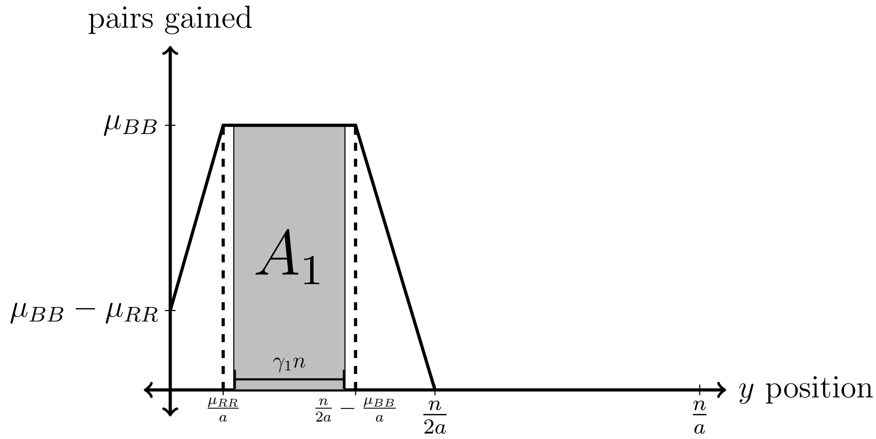


Figure 17: The upper bound of D_a

By combining Propositions 25 and 27, we obtain the upper bound for $|\mathcal{N}(n)|$ as follows:

Proposition 28. *Over all 2-colorings of $[1, n]$, the maximum number of non-monochromatic triples satisfying $x + ay = z$, $a \geqslant 2$ is*

$$|\mathcal{N}(n)| \leqslant \frac{n^2}{2a} - \frac{n^2}{2a(a^2 + 2a + 3)} + \mathcal{O}(n).$$

Proof. Without loss of generality, we assume $\mu_B \geqslant \mu_R$. Suppose

$$\Delta := \frac{1}{2}\left(\frac{\mu_R\mu_B}{a} + \mu_R\mu_{B_1} + \mu_B\mu_{R_1} + 2A_1\right).$$

Propositions 25 and 27 show that optimizing Δ will give us the upper bound for $|\mathcal{N}(n)|$. We will call this optimum Δ_{max}. In order to find Δ_{max}, we consider three different cases to compute A_1. Like before, each case will be subjected to the conditions listed in Figure 9.

Case 1: $\gamma_1 n < \frac{\gamma n}{a}$.

Case 2: $\frac{\gamma n}{a} \leqslant \gamma_1 n < \frac{\mu_R}{a}$.

Case 3: $\frac{\mu_R}{a} \leqslant \gamma_1 n$.

Here, we show only the details for Case 3A which will give us the best upper bound like in the previous section.

$$\begin{aligned} A_1 &= \mu_{BB} \cdot \gamma_1 n - \frac{a}{2} \cdot \left(\frac{\mu_{RR}}{a}\right)^2 - \frac{a}{2} \cdot \left(\gamma_1 n - \frac{\gamma n}{a} - \frac{\mu_{RR}}{a}\right)^2 \\ &= \frac{a}{2}\left(\frac{n}{a} - \gamma_1 n\right) \cdot \gamma_1 n - \frac{1}{2a}\left((\gamma n + \mu_{RR})^2 + \mu_{RR}^2\right) \\ &= \frac{a}{2}\left(\frac{n}{a} - \gamma_1 n\right) \cdot \gamma_1 n - \frac{(\gamma n)^2}{4a} - \frac{\mu_R^2}{4a}. \end{aligned}$$

Similar to before, we want $\gamma_1 n$ to be as close to $\frac{n}{2a}$ as possible and γn should be as small as possible. This is achieved by setting $\gamma_1 n = \mu_{B_1}$ and $\gamma n = \mu_{R_1} - \mu_{R_2}$. Then

$$\Delta_{max} = \frac{n^2}{2a} - \frac{n^2}{2a(a^2+2a+3)} + \mathcal{O}(n),$$

which is attained when $\mu_{R_1} = \frac{a+1}{a^2+2a+3}$ and $\mu_R = \frac{a+2}{a^2+2a+3}$. □

Proof of Theorem 22. An upper bound of the minimum can be obtained from a coloring on $[1, n]$. We color $[R, B, R]$ with the ratio $\left[1, a + \frac{1}{a+1}, \frac{1}{a+1}\right]$ as illustrated in Figure 18, which was discovered in [1] and [9].

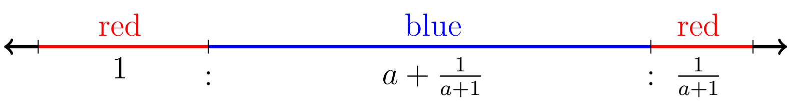


Figure 18: The Optimal Coloring for $x + ay = z$

This coloring gives us $\frac{n^2}{2a(a^2+2a+3)} + \mathcal{O}(n)$ monochromatic triples, $(x, y, x + ay)$.

For the lower bound of the minimum, we use Proposition 28 to get

$$\begin{aligned} |\mathcal{M}(n)| &= \frac{n^2}{2a} - |\mathcal{N}(n)| \\ &\geqslant \frac{n^2}{2a(a^2+2a+3)} + \mathcal{O}(n). \end{aligned}$$

Because the lower and upper bounds match, we have therefore shown the desired result. □

5 Conjectures

In this section, we present conjectures on variations of Graham's original problem. Denote by R, B, G the colors red, blue, and green respectively.

1. Equation: $ax + by = az$ where a, b are integers.

   (a) Case 1: $a > b \geqslant 2, \quad gcd(a, b) = 1$
   The coloring that gives the minimum number of monochromatic solutions over any 2-coloring of $[1, n]$ is
   $$\left[(R^{a-1}, B)^{\frac{n}{a}}\right].$$

   (b) Case 2: $b > a \geqslant 2, \quad gcd(a, b) = 1$
   The coloring that gives the minimum number of monochromatic solutions over any 2-coloring of $[1, n]$ is
   $$\left[(R^{a-1}, B)^{\frac{n}{b}}, \quad R^{(\frac{b-a}{b})n}\right].$$

   This has also been conjectured in [1, p. 409].

2. Equation: $x + y + w = z$
   The coloring that gives the minimum number of monochromatic solutions over any 2-coloring of $[1, n]$ is
   $$\left[R^{\frac{3(10-\sqrt{3})n}{97}}, B^{\frac{(6+\sqrt{3})(10-\sqrt{3})n}{97}}, R^{\frac{(10-\sqrt{3})n}{97}}\right],$$
   with the number of monochromatic solutions to be
   $$\frac{n^3}{12(10+\sqrt{3})^2} + \mathcal{O}(n^2).$$

3. Equation: $x + y = z$
   The coloring that gives the maximum number of rainbow solutions over any 3-coloring of $[1, n]$ is
   $$\left[(R, B)^{\frac{n}{5}}, (G, B)^{\frac{3n}{10}}\right],$$
   with the number of rainbow solutions to be
   $$\frac{n(n+1)}{10}.$$

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